

INDEX THEORY FOR SCALAR CURVATURE ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. We extend results of Llarull and Goette-Semmelmann to manifolds with boundary.

1. INTRODUCTION

Llarull showed that the sphere has no Riemannian metric that is greater than the standard round metric and also has a larger scalar curvature [11, 12]. Goette and Semmelmann gave an extension of Llarull's result in which the sphere is replaced by a manifold with nonnegative curvature operator [6].

In [8], Gromov discussed questions about scalar curvature, including an extension of these results to manifolds with boundary. We first give an extension of the Goette-Semmelmann result. Let R denote scalar curvature and let H denote mean curvature.

Theorem 1.1. *Let N and M be compact connected even dimensional Riemannian manifolds with boundary. Let $f : N \rightarrow M$ be a smooth spin map and let $\partial f : \partial N \rightarrow \partial M$ denote the restriction to the boundary. Suppose that*

- *f is Λ^2 -nonincreasing and ∂f is distance-nonincreasing,*
- *M has nonnegative curvature operator and ∂M has nonnegative second fundamental form,*
- *$R_N \geq f^*R_M$ and $H_{\partial N} \geq (\partial f)^*H_{\partial M}$,*
- *M has nonzero Euler characteristic and*
- *$\int_N \hat{A}(N)f^*[M, \partial M] \neq 0$.*

*Then $R_N = f^*R_M$ and $H_{\partial N} = (\partial f)^*H_{\partial M}$.*

Furthermore,

- *If $0 < \text{Ric}_M < \frac{1}{2}R_M g_M$ then f is a Riemannian submersion.*
- *If $\text{Ric}_M > 0$ and f is distance-nonincreasing then f is a Riemannian submersion.*
- *If M is flat then N is Ricci-flat.*

Here the map f is spin if $TN \oplus f^*TM$ admits a spin structure. The class $[M, \partial M] \in H^m(M, \partial M; o_M)$ is the fundamental class in cohomology twisted by the real orientation line bundle o_M , so $f^*[M, \partial M] \in H^m(N, \partial N; o_N)$. Also $\hat{A}(N) \in H^*(N; \mathbb{Q})$ is the $\hat{A}$ -class and $\int_N$ denotes pairing with the fundamental class in

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$H_n(N, \partial N; o_N)$. The quantity $\int_N \widehat{A}(N) f^*[M, \partial M]$ is called the $\widehat{A}$ -degree in [9]. When $\partial N = \partial M = \emptyset$, Theorem 1.1 recovers the Goette-Semmelmann result.

When specialized to the case $\dim(N) = \dim(M)$, we obtain the following extension of Llarull's result from [12].

Corollary 1.2. *Let N and M be compact connected Riemannian manifolds with boundary of the same even dimension. Let $f : N \rightarrow M$ be a smooth spin map and let $\partial f : \partial N \rightarrow \partial M$ denote the restriction to the boundary. Suppose that*

- f is Λ^2 -nonincreasing and ∂f is distance-nonincreasing,
- M has nonnegative curvature operator and ∂M has nonnegative second fundamental form,
- $R_N \geq f^*R_M$ and $H_{\partial N} \geq (\partial f)^*H_{\partial M}$,
- M has nonzero Euler characteristic and
- f has nonzero degree.

Then $R_N = f^*R_M$ and $H_{\partial N} = (\partial f)^*H_{\partial M}$.

Furthermore,

- If $0 < \text{Ric}_M < \frac{1}{2}R_M g_M$ then f is a Riemannian covering map.
- If $\text{Ric}_M > 0$ and f is distance-nonincreasing then f is a Riemannian covering map.
- If M is flat then N is Ricci-flat.

Gromov proved the first part of Corollary 1.2 when M is a ball in Euclidean space [7, Section 2], [8, Section 3.6]. (His interest in this case came from an application to hypersurfaces in Euclidean space.) Gromov's proof used a geometric doubling of N and a limiting procedure, to apply the Goette-Semmelmann result. We apply index theory directly to a Dirac-type operator on N , with local boundary conditions. In general there are topological obstructions to the existence of local boundary conditions for Dirac-type operators, but in our case the obstruction vanishes. The proof of Theorem 1.1 effectively uses an analytic doubling argument.

Gromov asked what happens to the Dirac-type operator in his argument when one passes to the limit. Presumably one recovers the operator that we use.

In addition to local boundary conditions, one can consider nonlocal Atiyah-Patodi-Singer boundary conditions [2]. This leads to the following result.

Theorem 1.3. *Let N and M be compact connected Riemannian manifolds with boundary of the same even dimension. Let $f : N \rightarrow M$ be a smooth spin map and let $\partial f : \partial N \rightarrow \partial M$ denote the restriction to the boundary. Suppose that*

- f is Λ^2 -nonincreasing,
- ∂f is an isometry and preserves the second fundamental forms,
- M has nonnegative curvature operator and ∂M has vanishing mean curvature, and
- $R_N \geq f^*R_M$.

Then $R_N = f^*R_M$ and $H_{\partial N} = (\partial f)^*H_{\partial M}$.

Furthermore,

- If $0 < \text{Ric}_M < \frac{1}{2}R_M g_M$ then f is an isometry.
- If $\text{Ric}_M > 0$ and f is distance-nonincreasing then f is an isometry.
- If M is flat then N is Ricci-flat.

Comparing Corollary 1.2 and Theorem 1.3, one difference is that Corollary 1.2 assumes nonnegativity of the second fundamental form of M , while Theorem 1.3

assumes vanishing of its trace. In Corollary 1.2 the boundary map is assumed to be distance nonincreasing, while in Theorem 1.3 it is actually an isometry and it preserves the second fundamental form.

2. PROOF OF THEOREM 1.1

2.1. Bochner-type argument. For simplicity, we assume that N and M are spin; the general case is similar. Put $E = S_N \otimes f^* S_M$, a Clifford module on N . (This Clifford module exists in the general case.) We take the inner product $\langle \cdot, \cdot \rangle$ on E to be $\mathbb{C}$ -linear in the second slot and $\mathbb{C}$ -antilinear in the first slot.

Let $\omega_{\beta\gamma}^\alpha$ be the connection 1-forms with respect to a local orthonormal framing $\{e_\alpha\}_{\alpha=1}^n$ on N . Let $\widehat{\omega}_{b\gamma}^a$ be the pullbacks under f of connection 1-forms with respect to a local orthonormal framing $\{e_a\}_{a=1}^m$ of M .

Let $\{\gamma^\alpha\}_{\alpha=1}^n$ be generators of the Clifford algebra on $\mathbb{R}^n$, satisfying $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\delta^{\alpha\beta}$. Let $\{\widehat{\gamma}^a\}_{a=1}^m$ be the analogous generators of the Clifford algebra on $\mathbb{R}^m$. The covariant derivative on E has the local form

$$(2.1) \quad \nabla_\sigma^N = e_\sigma + \frac{1}{8} \omega_{\alpha\beta\sigma} [\gamma^\alpha, \gamma^\beta] + \frac{1}{8} \widehat{\omega}_{ab\sigma} [\widehat{\gamma}^a, \widehat{\gamma}^b].$$

The Dirac operator on $C^\infty(N; E)$ is $D^N = -i \sum_{\sigma=1}^n \gamma^\sigma \nabla_\sigma^N$.

We will take the orthonormal frame $\{e_\alpha\}$ at a point in ∂N so that e_n is the inward-pointing unit normal vector there. Let dvol_N denote the Riemannian density on N , and similarly for $\text{dvol}_{\partial N}$. Given $\psi_1, \psi_2 \in C^\infty(N; E)$, we have

$$(2.2) \quad \int_N \langle D^N \psi_1, \psi_2 \rangle \text{dvol}_N - \int_N \langle \psi_1, D^N \psi_2 \rangle \text{dvol}_N = -i \int_{\partial N} \langle \psi_1, \gamma^n \psi_2 \rangle \text{dvol}_N.$$

The Lichnerowicz formula implies

$$(2.3) \quad (D^N)^2 = (\nabla^N)^* \nabla^N + \frac{R_N}{4} - \frac{1}{4} [\gamma^\sigma, \gamma^\tau] \left(\frac{1}{8} \widehat{R}_{ab\sigma\tau} [\widehat{\gamma}^a, \widehat{\gamma}^b] \right).$$

We now extend some computations in [13, Proof of Lemma 4.1]. Suppose that $D^N \psi = 0$. Then (2.3) implies that

$$(2.4) \quad 0 = \int_N |\nabla^N \psi|^2 \text{dvol}_N + \int_{\partial N} \langle \psi, \nabla_{e_n}^N \psi \rangle + \frac{1}{4} \int_N R_N |\psi|^2 \text{dvol}_N - \frac{1}{32} \int_N \widehat{R}_{ab\sigma\tau} \langle \psi, [\gamma^\sigma, \gamma^\tau] [\widehat{\gamma}^a, \widehat{\gamma}^b] \psi \rangle.$$

Now $D^N \psi = 0$ implies that on ∂N , we have

$$(2.5) \quad \begin{aligned} \nabla_{e_n}^N \psi &= -\gamma^n \sum_{\mu=1}^{n-1} \gamma^\mu \nabla_\mu^N \psi \\ &= -\gamma^n \sum_{\mu=1}^{n-1} \gamma^\mu \left(\nabla_\mu^{\partial N} \psi + \frac{1}{4} \omega_{n\beta\mu} \gamma^n \gamma^\beta \psi + \frac{1}{4} \widehat{\omega}_{mb\mu} \widehat{\gamma}^m \widehat{\gamma}^b \psi \right) \\ &= D^{\partial N} \psi + \frac{H_{\partial N}}{4} \psi - \frac{1}{4} \gamma^n \gamma^\mu \widehat{\gamma}^m \widehat{\gamma}^b \widehat{A}_{b\mu} \psi, \end{aligned}$$

where

$$(2.6) \quad D^{\partial N} = -\gamma^n \sum_{\mu=1}^{n-1} \gamma^\mu \nabla_\mu^{\partial N}$$

is the Dirac operator on ∂N coupled to $(\partial f)^* S_M$, $\hat{A}$ is the second fundamental form of M and $\hat{A}_{b\mu} = \hat{A}(\hat{e}_b, (\partial f)_*(e_\mu))$. Hence

$$(2.7) \quad 0 = \int_N |\nabla \psi|^2 \, \text{dvol}_N + \frac{1}{4} \int_N R_N |\psi|^2 \, \text{dvol}_N - \frac{1}{32} \int_N \hat{R}_{ab\sigma\tau} \langle \psi, [\gamma^\sigma, \gamma^\tau][\hat{\gamma}^a, \hat{\gamma}^b] \psi \rangle \, \text{dvol}_N + \int_{\partial N} \langle \psi, D^{\partial N} \psi \rangle \, \text{dvol}_{\partial N} + \frac{1}{4} \int_{\partial N} H_{\partial N} |\psi|^2 \, \text{dvol}_{\partial N} - \frac{1}{4} \int_{\partial N} \hat{A}_{b\mu} \langle \psi, \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b \psi \rangle \, \text{dvol}_{\partial N}.$$

From [6, Section 1.1],

$$(2.8) \quad \frac{1}{32} \hat{R}_{ab\sigma\tau} [\gamma^\sigma, \gamma^\tau][\hat{\gamma}^a, \hat{\gamma}^b] \leq \frac{1}{4} f^* R_M \text{Id}_E.$$

Lemma 2.1. *If $\hat{A} \geq 0$ then*

$$(2.9) \quad \hat{A}_{b\mu} \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b \leq (\partial f)^* H_{\partial M} \text{Id}_E.$$

Proof. We use the method of proof of [6, Section 1.1]. Put $\hat{L} = \sqrt{\hat{A}}$. Then

$$(2.10) \quad \hat{A}_{b\mu} \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b = \hat{A}_{bc} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b = \hat{L}_{ab} \hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b,$$

so

$$(2.11) \quad \hat{A}_{b\mu} \gamma^n \gamma^\mu \hat{\gamma}^m \hat{\gamma}^b = \frac{1}{2} \sum_a \left[\left(\hat{L}_{ab} \hat{\gamma}^m \hat{\gamma}^b + \hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu \right)^2 - \left(\hat{L}_{ab} \hat{\gamma}^m \hat{\gamma}^b \right)^2 - \left(\hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu \right)^2 \right].$$

Now $\hat{L}_{ab} \hat{\gamma}^m \hat{\gamma}^b + \hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu$ is skew-Hermitian, so has nonpositive square. Also,

$$(2.12) \quad \left(\hat{L}_{ab} \hat{\gamma}^m \hat{\gamma}^b \right)^2 = -\hat{L}_{ab}^2 = -\hat{A}_{bb} = -(\partial f)^* H_{\partial M}$$

and

$$(2.13) \quad \left(\hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \gamma^n \gamma^\mu \right)^2 = -\left(\hat{L}_{ac} \langle e_c, (\partial f)_*(e_\mu) \rangle \right)^2 = -\hat{A}_{cd} \langle e_c, (\partial f)_*(e_\mu) \rangle \langle e_d, (\partial f)_*(e_\mu) \rangle = -\hat{A}((\partial f)_*(e_\mu), (\partial f)_*(e_\mu)) \geq -(\partial f)^* H_{\partial M},$$

using the fact that ∂f is distance-nonincreasing.

This proves the lemma. $\square$

Proposition 2.2. *Let N and M be compact connected even dimensional Riemannian manifolds with boundary. Let $f : N \rightarrow M$ be a smooth spin map and let $\partial f : \partial N \rightarrow \partial M$ denote the restriction to the boundary. Suppose that*

- *f is Λ^2 -nonincreasing and ∂f is distance-nonincreasing,*
- *M has nonnegative curvature operator and ∂M has nonnegative second fundamental form,*
- *$R_N \geq f^* R_M$ and $H_{\partial N} \geq (\partial f)^* H_{\partial M}$, and*
- *There is a nonzero $\psi \in C^\infty(N; E)$ with $D^N \psi = 0$ on N and $\int_{\partial N} \langle \psi, D^{\partial N} \psi \rangle \, \text{dvol}_{\partial N} \geq 0$ on ∂N .*

Then $R_N = f^ R_M$, $H_{\partial N} = (\partial f)^* H_{\partial M}$ and ψ is parallel.*

Proof. This follows from (2.7), (2.8) and Lemma 2.1. $\square$

Proposition 2.3. *Suppose that the assumptions of Proposition 2.2 hold.*

- (1) *If $0 < \text{Ric}_M < \frac{1}{2}R_M g_M$ then f is a Riemannian submersion.*
- (2) *If $\text{Ric}_M > 0$ and f is distance-nonincreasing then f is a Riemannian submersion*
- (3) *If M is flat then N is Ricci-flat.*

Proof. Part (1) follows from the computation in [6, Section 1.2]. Part (2) follows from [6, Remark 1.2]. For part (3), we know that S_M has a flat unitary connection. Around a point $p \in N$, we can write $\psi = \sum_a \psi^a s_a$, where $\{s_a\}$ is a parallel basis of f^*S_M and ψ^a is a local section of S_N . Then $\nabla\psi = 0$ implies that $R_{\alpha\beta\sigma\tau}^N[\gamma^\alpha, \gamma^\beta]\psi^a = 0$ for each a . As some ψ^a is nonzero, it follows from [3, Corollary 2.8] that N is Ricci-flat near p . $\square$

Remark 2.4. Under the assumptions of Proposition 2.2, if M is flat and spin then we can say more precisely that the universal cover of N admits a nonzero parallel spinor field.

2.2. Local boundary conditions. We represent the generators of the Clifford algebra as

$$(2.14) \quad \gamma^n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$

where $\{\sigma^\mu\}_{\mu=1}^{n-1}$ are generators for the Clifford algebra on $\mathbb{R}^{n-1}$. Put

$$(2.15) \quad \not{D}^{\partial N} = -i \sum_{\mu=1}^{n-1} \sigma^\mu \nabla_\mu^{\partial N}.$$

With respect to the $\mathbb{Z}_2$ -grading on E coming from S_N , we have

$$(2.16) \quad D^{\partial N} = \begin{pmatrix} -\not{D}^{\partial N} & 0 \\ 0 & \not{D}^{\partial N} \end{pmatrix}.$$

As both S_N and S_M are $\mathbb{Z}_2$ -graded, there is a total $\mathbb{Z}_2$ -grading on E and a bigrading

$$(2.17) \quad \begin{aligned} E_+ &= E_{++} \oplus E_{--}, \\ E_- &= E_{+-} \oplus E_{-+}. \end{aligned}$$

Given $\psi \in C^\infty(N; E)$, we decompose it as $\psi = \psi_{++} + \psi_{--} + \psi_{+-} + \psi_{-+}$. Define boundary conditions by

$$(2.18) \quad \psi_{++} = \psi_{--}, \psi_{+-} = \psi_{-+}.$$

Note that the boundary conditions do not mix E_+ and E_- .

Lemma 2.5. *The operator D^N is formally self-adjoint under the boundary conditions (2.18).*

Proof. Using (2.2), it suffices to show that if $\psi^1, \psi^2 \in C^\infty(N; E)$ satisfy the boundary conditions then $\int_{\partial N} \langle \psi^1, \gamma^n \psi^2 \rangle \text{dvol}_{\partial N} = 0$. For $i \in \{1, 2\}$, let us write

$$(2.19) \quad \psi^i = \psi_{++}^i + \psi_{+-}^i + \psi_{-+}^i + \psi_{--}^i.$$

Then

$$(2.20) \quad \langle \psi^1, \gamma^n \psi^2 \rangle = i (\langle \psi_{-+}^1, \psi_{++}^2 \rangle + \langle \psi_{--}^1, \psi_{+-}^2 \rangle) - i (\langle \psi_{++}^1, \psi_{-+}^2 \rangle + \langle \psi_{+-}^1, \psi_{--}^2 \rangle) = 0.$$

This proves the lemma. $\square$

One can check that (2.18) gives elliptic boundary conditions in the sense of [1]. Hence there is a Fredholm operator D_+^N , with domain consisting of the H^1 -sections ψ of E_+ so that the boundary trace of ψ_{++} equals the boundary trace of ψ_{--} . The domain of the adjoint $(D_+^N)^*$ consisting of the H^1 -sections ψ of E_- so that the boundary trace of ψ_{+-} equals the boundary trace of ψ_{-+} . In theory one can compute the index of D_+^N using the procedure described in [1]. However, in this case we can use a more direct approach. First, we can deform the metric on N so that it is a product near ∂N , without changing the metric on ∂N , and similarly for M . Next, we can deform f , while fixing ∂f , so that in a product neighborhood $[0, \delta) \times \partial N$ of ∂N , the map f takes the form $f(t, x) = (t, (\partial f)(x))$. These deformations do not change the index.

Let us discuss spinors on the double of a manifold; c.f. [4, Section 4.4]. Suppose that N is spin and its metric is a Riemannian product near ∂N . Let DN be the double of N . As DN is the boundary of $I \times N$, it inherits a spin structure. We can extend the structure group from $\text{Spin}(n)$ to $\text{Pin}_+(n)$. The involution on DN lifts to an involution T on sections of S_N , that commutes with the Dirac operator D^{DN} . Writing a product neighborhood of $\partial N \subset DN$ as $(-\delta, \delta) \times \partial N$, the involution acts on a spinor field by $(T\psi)(t, x) = i\epsilon\gamma^n\psi(-t, x)$, where ϵ is the $\mathbb{Z}_2$ -grading operator. (In Minkowski space the time reversal operator involves complex conjugation, but that is not the case here.) In terms of the $\mathbb{Z}_2$ -grading on S_N , we can write $i\epsilon\gamma^n = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Note that it anticommutes with ϵ . The T -invariant H^1 -regular spinors on DN can then be identified with the H^1 -regular spinors ψ on N so that the boundary trace of ψ_+ equals the boundary trace of ψ_- . Note that these boundary conditions mix chiralities, so there is not a well-defined index problem. This is a reflection of the fact that in general, Dirac-type operators on manifolds with boundary do not admit local boundary conditions for index problems.

In our case, we can pass to doubles and extend f to a spin map $F : DN \rightarrow DM$. There is an involution T on $S_{DN} \otimes F^*S_{DM}$ that commutes with the twisted Dirac operator D^{DN} . On a neighborhood of ∂N , it acts on sections of $S_{DN} \otimes F^*S_{DM}$ by $(T\psi)(t, x) = -\epsilon\gamma^n\widehat{\epsilon}\gamma^m\psi(-t, x)$. The T -invariant H^1 -regular sections on DN can be identified with the H^1 -regular sections of E , on N , that satisfy the boundary conditions (2.18). Because of the $\mathbb{Z}_2$ -grading on E , we do obtain local boundary conditions for an index problem.

Thus the index of D_+^N is the same as the index of D_+^{DN} when acting on the T -invariant sections on the double. We can think of $DN/\mathbb{Z}_2$ as N with an orbifold structure, so we effectively have an index problem on the orbifold. From [10], the index is $\int_N \widehat{A}(N) \text{ch}(f^*S_M)$. (Since ∂N is odd dimensional and the characteristic forms have even degree, there is no boundary contribution.) Here S_M is $\mathbb{Z}_2$ -graded, and $\text{ch}(S_M)$ equals the Euler form of TM . From our present assumptions about a product structure near the boundary, $\text{ch}(S_M)$ vanishes near ∂M and represents $\chi(M)[M, \partial M] \in H^m(M, \partial M; o_M)$. Hence the index is $\chi(M) \int_N \widehat{A}(N) f^*[M, \partial M]$.

When combined with Propositions 2.2 and 2.3, this proves Theorem 1.1.

3. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. We begin more generally. Let N and M be compact connected even dimensional Riemannian manifolds with boundary. Let

$f : N \rightarrow M$ be a smooth spin map and let $\partial f : \partial N \rightarrow \partial M$ denote the restriction to the boundary. Suppose that

- f is Λ^2 -nonincreasing,
- M has nonnegative curvature operator and ∂N has vanishing mean curvature, and
- $R_N \geq f^* R_M$.

To avoid confusion with the previous section, we now write the Dirac-type operator D^N as $\tilde{D}^N$. Define a boundary Dirac-type operator by

$$(3.1) \quad \tilde{D}^{\partial N} = -\gamma^n \sum_{\mu=1}^{n-1} \gamma^\mu \nabla_{e_\mu}^N = -\gamma^n \sum_{\mu=1}^{n-1} \gamma^\mu \left(\nabla_\mu^{\partial N} + \frac{1}{4} \hat{\omega}_{mb\mu} \hat{\gamma}^m \hat{\gamma}^b \right),$$

where the last equality uses the fact that $H_{\partial N} = 0$; c.f. equation (2.5). If $\tilde{D}^N \psi = 0$ then (2.7) becomes

$$(3.2) \quad 0 = \int_N |\nabla^N \psi|^2 \, \text{dvol}_N + \frac{1}{4} \int_N R_N |\psi|^2 \, \text{dvol}_N - \frac{1}{32} \int_N \hat{R}_{ab\sigma\tau} \langle \psi, [\gamma^\sigma, \gamma^\tau][\hat{\gamma}^a, \hat{\gamma}^b] \psi \rangle \, \text{dvol}_N + \int_{\partial N} \langle \psi, \tilde{D}^{\partial N} \psi \rangle \, \text{dvol}_{\partial N}.$$

Hereafter we use the $\mathbb{Z}_2$ -grading (2.17). In terms of it, we can write

$$(3.3) \quad \tilde{D}^{\partial N} = \begin{pmatrix} -\tilde{\not{D}}^{\partial N} & 0 \\ 0 & \tilde{\not{D}}^{\partial N} \end{pmatrix}$$

for an elliptic self-adjoint operator $\tilde{\not{D}}^{\partial N}$. (Here we use γ^n to implicitly identify $E_+|_{\partial N}$ and $E_-|_{\partial N}$.) Let $P^{>0}$ denote projection onto the subspace spanned by eigenvectors of $\tilde{\not{D}}^{\partial N}$ with positive eigenvalue, and similarly for $P^{\leq 0}$. Given $\psi \in C^\infty(N; E)$, let $\psi_\pm^{\partial N}$ be the components of its boundary restriction, relative to the $\mathbb{Z}_2$ -grading. We impose the boundary conditions

$$(3.4) \quad P^{>0} \psi_+^{\partial N} = P^{\leq 0} \psi_-^{\partial N} = 0.$$

Lemma 3.1. *The operator $\tilde{D}^N$ is formally self-adjoint under the boundary conditions (3.4).*

Proof. Using (2.2), it suffices to show that if $\psi^1, \psi^2 \in C^\infty(N; E)$ satisfy the boundary conditions then $\int_{\partial N} \langle \psi^1, \gamma^n \psi^2 \rangle \, \text{dvol}_{\partial N} = 0$. In terms of the $\mathbb{Z}_2$ -grading on E , we can write $\gamma^n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Then

$$(3.5) \quad \langle \psi^1, \gamma^n \psi^2 \rangle = i \langle \psi_-^1, \psi_+^2 \rangle - i \langle \psi_+^1, \psi_-^2 \rangle.$$

The boundary conditions imply that

$$(3.6) \quad \int_{\partial N} \langle \psi_-^1, \psi_+^2 \rangle \, \text{dvol}_{\partial N} = \int_{\partial N} \langle \psi_+^1, \psi_-^2 \rangle \, \text{dvol}_{\partial N} = 0.$$

This proves the lemma. $\square$

The boundary conditions (3.4) make the differential operator $\tilde{D}^N$ into an elliptic self-adjoint operator, which we write as $\tilde{D}^{N,APS}$. Its domain is the set of H^1 -regular sections ψ of E whose boundary trace satisfies (3.4). The conditions (3.4) differ slightly from the Atiyah-Patodi-Singer boundary conditions [2], which would be $P^{\geq 0}\psi_+^{\partial X} = P^{< 0}\psi_-^{\partial X} = 0$, but the boundary conditions (3.4) work just as well.

From (3.3), the boundary conditions imply that $\int_{\partial N} \langle \psi, \tilde{D}^{\partial N} \psi \rangle \text{dvol}_{\partial N} \geq 0$. We conclude from (3.2) that if $\psi \in C^\infty(N; E)$ is a nonzero solution of $\tilde{D}^N \psi = 0$, satisfying the boundary conditions (3.4), then $R_N = f^* R_M$ and ψ is parallel.

Lemma 3.2. *If $R_N \neq f^* R_M$ then the kernel of $\tilde{D}^N$ vanishes.*

Suppose that $R_N = f^ R_M$. The kernel of $\tilde{D}_+^{N,APS}$ is isomorphic to the vector space of parallel sections of E_+ . The kernel of $\tilde{D}_-^{N,APS}$ vanishes.*

Proof. We have already proved the first statement of the lemma. For the second statement, by elliptic regularity, an element of the kernel is smooth on N . Suppose that $R_N = f^* R_M$. If $\tilde{D}^{N,APS} \psi = 0$ then ψ is parallel. From the definition of $\tilde{D}^{\partial N}$, it follows that $\tilde{D}^{\partial N} \psi = 0$. Writing $\psi = \psi_+ + \psi_-$, the boundary conditions (3.4) imply that $\psi_- = 0$. On the other hand, if ψ_+ is a parallel section of E_+ then $\tilde{D}^N \psi_+ = 0$ and ψ_+ satisfies the boundary condition (3.4). Thus ψ_+ is in the kernel of $\tilde{D}_+^{N,APS}$. $\square$

Using Lemma 3.2, the task now is to find situations which guarantee that $\tilde{D}_+^{N,APS}$ has a nonzero kernel or, equivalently for us, that it has a nonzero index. One situation that is easy to analyze is when $N = M$ and f is the identity map. Then E is isomorphic to $\Lambda^*(TM)$, with the $\mathbb{Z}_2$ -grading distinguishing even and odd forms. As the constant function is always a nonzero parallel section of E , it lies in the kernel of $\tilde{D}_+^{N,APS}$.

To motivate the assumptions of Theorem 1.3, if we deform the Riemannian metric on N , allowing the boundary metric to also change, then the index of $\tilde{D}_+^{N,APS}$ can change. The change is determined by the spectral flow of $\tilde{\partial}^{\partial N}$, which could be hard to compute. However, if $\tilde{\partial}^{\partial N}$ doesn't change in the deformation then the index doesn't change. Such is the case when the metric and the second fundamental form of ∂N do not change in the deformation.

To return to the proof of Theorem 1.3, let us write $N' = M$ for the case when N is the same as M , with $f' : N' \rightarrow M$ being the identity map. Let N and f be as in the statement of Theorem 1.3. We can assume that $\partial f = \partial f'$. By the Atiyah-Patodi-Singer index theorem [2] and its extension to the case of a nonproduct structure near the boundary [5], since the boundary data is the same for N and N' , the difference Δ of the indices is

$$(3.7) \quad \Delta = \int_N \hat{A}(TN) f^* \text{ch}(S_M) - \int_{N'} \hat{A}(TN') (f')^* \text{ch}(S_M).$$

Here $\text{ch}(S_M)$ denotes the explicit Chern form. It has top degree and equals the Euler form $e(M)$. Hence

$$(3.8) \quad \Delta = (\deg(f) - \deg(f')) \int_M e(M).$$

Since ∂f and $\partial f'$ are spin diffeomorphisms, it follows that $\deg(f) = \deg(f') = 1$. Thus $\Delta = 0$ and $\tilde{D}_+^{N,APS}$ has a positive index.

This proves the first part of Theorem 1.3. The second part follows from Proposition 2.3.

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