The Implicit Graph Conjecture is False

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***Abstract*—An efficient implicit representation of an n-vertex graph G in a family $\mathcal{F}$ of graphs assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency between every pair of vertices can be determined only as a function of their codes. This function can depend on the family but not on the individual graph. Every family of graphs admitting such a representation contains at most $2^{O(n\log(n))}$ graphs on n vertices, and thus has at most factorial speed of growth.**

The Implicit Graph Conjecture states that, conversely, every hereditary graph family with at most factorial speed of growth admits an efficient implicit representation. We refute this conjecture by establishing the existence of hereditary graph families with factorial speed of growth that require codes of length $n^{\Omega(1)}$.

***Index Terms*—Implicit Graph Conjecture, Hereditary Graph Family, Universal Graph, Adjacency Labeling Scheme**

I. Introduction

This article aims to refute the implicit graph conjecture with a short and simple proof. The implicit graph conjecture, posed as a question in 1988 by Kannan, Naor, and Rudich [KNR88] and reformulated to a conjecture by Spinrad [Spi03], is one of the most well-known problems regarding efficient representations of graphs. Informally speaking, it states that if a hereditary family of graphs does not contain too many graphs (i.e., its speed is $2^{O(n\log(n))}$), then every graph in the family has a (local) implicit representation that requires storing only $O(\log(n))$ bits on each node.

Many natural families of graphs are small and thus fall in the framework of this conjecture: forests, planar graphs, bounded degree graphs, intersection graphs, disk graphs, and more generally all semi-algebraic families of graphs. The latter is a rich class that includes most natural geometric constructions of graphs on bounded-dimensional real spaces.

The implicit graph conjecture speculates the existence of space-efficient representations for *all* small hereditary graph families. Such representations typically allow answering edge queries in only poly-logarithmic time in the size of the network. These space and time efficiencies are highly desirable for the design of networks. Therefore, the implicit graph conjecture has seen much attention by both mathematicians and computer scientists [KNR92], [Spi03], [Alo17], [FK16], [AKTZ19], and it has been verified for numerous restricted classes of hereditary graph families [ADK17], [Sch99], [AR02], [GL07], [ACLZ15], [DEG+21], [BGK+21], [HWZ22], [MM13].

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The notion of an implicit representation was formally defined by Müller [Mul88] and Kannan, Naor, and Rudich [KNR88], [KNR92]. Let $\mathcal{F}$ be a family of (labeled) finite graphs, and let $\mathcal{F}_n$ denote the n-vertex graphs in $\mathcal{F}$ with vertex set $[n] := \{1, \ldots, n\}$. For a function $w : \mathbb{N} \to \mathbb{N}$, an *implicit representation* with code-length $w(n)$ for an n-vertex graph G in $\mathcal{F}$ is an assignment of a $w(n)$-bit code to each vertex of G, together with a *decoder* function that takes two vertex-codes and returns whether or not they are adjacent in G. Here, the decoder function may depend on $\mathcal{F}$, but it is independent of the individual graph G.

Note that even the family of all graphs can be represented using codes of length $n + \lceil \log n \rceil$, since every vertex can be labeled with the corresponding row of the adjacency matrix along with the index of the row. This can be improved to code-length $\frac{n}{2} + O(1)$, which is essentially optimal [Alo17], [Moo65]. We call an implicit representation *efficient* if its code-length is $w(n) = O(\log n)$. The literature often refers to efficient implicit representations simply as implicit representations. However, in this article, we are interested in obtaining lower bounds on the code-length, and therefore, we have chosen to allow implicit representations to have arbitrary code-lengths.

The original definitions of [Mul88], [KNR92] require that the decoder function is computable by a polynomial-time algorithm. However, as these computational matters are irrelevant to our lower bound, we do not require them here.

Example I.1. *Let $\mathcal{T}$ be the family of all forests. Given any $G \in \mathcal{T}_n$, we can turn each connected component of G to a rooted tree by choosing an arbitrary root, and assign to every vertex of G, the index of its parent (if it exists). Note that the adjacency of a pair of vertices can be decided given their $\lceil \log n \rceil$-length codes, and thus $\mathcal{T}$ has an efficient implicit representation.*

A sequence $(U_n)_{n\in\mathbb{N}}$ of graphs is said to be a sequence of *universal graphs* for a graph family $\mathcal{F}$, if for every n and $F \in \mathcal{F}_n$, there exists a graph embedding $\pi_F : V(F) \to V(U_n)$. The family $\mathcal{F}$ is said to be representable by polynomial-size *universal graphs* if these universal graphs satisfy $|V(U_n)| = n^{O(1)}$. Universal graphs were initially defined with the purpose of representation of the family of all graphs [Rad64], [Moo65], and were extensively studied for various families of graphs [CG78], [CG79], [BCE+82], [Wu85], [BCLR89], [Mul88], [KNR92], [BT81], [AKTZ19], [BEGS21]. Efficient implicit representations are equivalent to polynomial-size universal graphs [Mul88], [KNR92]. To see the equivalence, note that given an efficient implicit representation, one can take $V(U_n)$ to be the set of all $O(\log(n))$-bits codes and define the edges of U_n according to the decoder function. For the converse direction, one can assign a unique $O(\log(n))$-bit code to each of the $n^{O(1)}$ vertices of U_n, and use the functions π_F to assign the codes of the vertices of each $F \in \mathcal{F}_n$ accordingly.

A simple counting argument shows that families of graphs with efficient implicit representations cannot be very large. Indeed, if $\mathcal{F}$ has an efficient implicit representation, then every graph in $\mathcal{F}_n$ can be described using $O(n \log n)$ bits, and thus we must have $|\mathcal{F}_n| \leq 2^{O(n \log n)}$. In the terminology of [Ale97], [BBW00] such families are said to have at most *factorial* speed of growth.

A graph family $\mathcal{F}$ is called *hereditary* if it is closed under taking *induced* subgraphs. More precisely, for every $m \leq n$, injection $\sigma : [m] \to [n]$, and $G \in \mathcal{F}_n$, the graph F with the vertex set $[m]$ and the edge set $\{ij : \sigma(i)\sigma(j) \in E(G)\}$ belongs to $\mathcal{F}$. Note that taking $m = n$ in this definition ensures that $\mathcal{F}$ is closed under graph isomorphism. The *hereditary closure* of a family $\mathcal{F}$, denoted by $\mathrm{cl}(\mathcal{F})$, is the set of all induced subgraphs of every $G \in \mathcal{F}$. It is the smallest hereditary family that contains $\mathcal{F}$.

Next, we discuss why in the study of implicit representations, it is more natural to consider hereditary graph families. Let $\mathcal{F}$ be representable by the sequence of polynomial-size universal graphs U_n. Let $\epsilon > 0$ be a fixed constant, and let U'_n be the disjoint union $U'_n = U_1 \cup \ldots \cup U_n$. Let $\mathcal{F}_\epsilon \subseteq \mathrm{cl}(\mathcal{F})$ be obtained from $\mathcal{F}$ by including all induced subgraphs of size at least n^ϵ of every $G \in \mathcal{F}_n$ for every n. Since the size of U'_n is polynomial n^ϵ, the family $\mathcal{F}_\epsilon$ is representable by the universal graphs U'_n. This observation shows that for $\mathcal{F}$ to have an efficient implicit representation, not only $\mathcal{F}$ must have at most factorial speed of growth, but more generally, the families $\mathcal{F}_\epsilon$ must also have at most factorial speed of growth.

Example I.2. *(See also [Spi03]) Let $\mathcal{F}$ be the set of all graphs with at least $n - \sqrt{n}$ isolated vertices. Note that the speed of $\mathcal{F}$ is at most $\binom{n}{\sqrt{n}} 2^{\binom{\sqrt{n}}{2}} = 2^{O(n)}$, which is subfactorial. On the other hand, $\mathcal{F}_{1/2}$ contains all graphs and thus has the super-factorial growth speed of $2^{\Theta(n^2)}$. Hence $\mathcal{F}$ does not admit an efficient implicit representation despite having at most factorial growth speed.*

Note that $\mathcal{F}_\epsilon \subseteq \mathrm{cl}(\mathcal{F})$. In light of these observations, the *implicit graph conjecture* speculates that if $\mathrm{cl}(\mathcal{F})$ has at most factorial speed of growth, then $\mathcal{F}$ admits an efficient implicit representation.

Conjecture I.3 (Implicit Graph Conjecture [KNR88], [KNR92], [Spi03])**.** *Every hereditary graph family of at most factorial speed has an efficient implicit representation.*

We disprove this three-decades-old conjecture via a probabilistic argument.

Theorem I.4 (Main Theorem)**.** *For every $\delta > 0$, there is a hereditary graph family $\mathcal{F}$ that has at most factorial speed of growth, but every implicit representation of $\mathcal{F}$ must use codes of length $\Omega(n^{\frac{1}{2}-\delta})$.*

Theorem I.4 not only refutes the existence of $O(\log(n))$-bit codes, but it shows that there are factorial hereditary graph families that require code-lengths that are polynomial in n.

Proof overview.

First, note that the number of polynomial-size universal graphs is $2^{n^{O(1)}}$, and each such graph can represent at most $n^{O(n)}$ graphs of size n. Since there are $2^{\Omega(n^2)}$ graphs on n vertices, a simple counting argument shows that, with high probability, a random collection of $n^{\omega(1)}$ graphs on n vertices cannot be represented by any universal graph of size $n^{O(1)}$. It is thus tempting to construct $\mathcal{F}$ by randomly selecting $n^{\omega(1)}$ graphs for each n. Then, as desired, with high probability, $\mathrm{cl}(\mathcal{F})$ does not have a polynomial-size universal graph representation, but unfortunately, taking the hereditary closure will expand the family to contain all graphs and thus $\mathrm{cl}(\mathcal{F})$ will have growth speed $2^{\Omega(n^2)}$. Indeed for every n, for sufficiently large m, even a single random graph on m vertices will, with high probability, contain all graphs on n vertices as induced subgraphs.

The key idea to overcome this issue is to consider slightly sparser random graphs. We will select the initial graphs $\mathcal{F}$ from the set of graphs that have $n^{2-\epsilon}$ number of edges for some $\epsilon > 0$. There are still many such graphs; thus, the same counting argument shows that, with high probability, polynomial-size universal graphs cannot represent this family. Regarding the growth speed of $\mathrm{cl}(\mathcal{F})$, note that every n-vertex graph $F \in \mathrm{cl}(\mathcal{F})$ is an induced subgraph of some randomly selected graph $G \in \mathcal{F}_m$ where $m \geq n$. If m is much larger than n, then F is a small subgraph of the random sparse graph G, and thus it has a simple structure. We will show that due to these structural properties, the number of such graphs is at most factorial. On the other hand, if m is not much larger than n, there are only m^n possible ways of choosing F inside G, and thus the number of such graphs F can be upper-bounded by the sum of $m^n|\mathcal{F}_m|$ for $m = n^{O(1)}$. Since also $|\mathcal{F}_m|$ is not very large as a function of m, this sum will be at most $2^{O(n \log(n))}$.

II. Preliminary lemmas

Denote by $B(n, m)$ for $0 \leq m \leq n^2$, the set of all bipartite graphs with equal parts $\{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\}$, and exactly m edges. Recall that a graph is said to be *k-degenerate* if all its induced subgraphs have a vertex of degree at most k.

Lemma II.1. *Suppose $0 \leq \epsilon' < \epsilon < 1$ are fixed constants. Consider the random graph G chosen uniformly at random from $B(n, m)$ with $m = \lfloor n^{2-\epsilon} \rfloor$. Then, for sufficiently large n, with probability $1 - o(1)$, every induced subgraph of G with at most $n^{\epsilon'}$ vertices is $(c-1)$-degenerate where $c = \frac{4}{\epsilon - \epsilon'}$.*

Proof. Let F be an induced subgraph of G with at most $n^{\epsilon'}$ vertices. Let A and B denote the set of the vertices of F in the first part and the second part, respectively, and denote $a = |A|$ and $b = |B|$. The case when $a = 0$ or $b = 0$ is trivial, so we may assume $a, b > 0$. We have $a, b \leq n^{\epsilon'}$. We call F *bad* if its minimum degree is at least c.

The probability that every vertex in A has degree at least c is bounded from above by

$$\frac{\binom{b}{c}^a \binom{n^2-ac}{m-ac}}{\binom{n^2}{m}} \leq \binom{b}{c}^a \left(\frac{m}{n^2}\right)^{ac} \leq \left(\frac{mb}{n^2}\right)^{ac} = \left(\frac{b}{n^\epsilon}\right)^{ac}.$$

Similarly the probability that every vertex in B has degree at least c is at most $\left(\frac{a}{n^\epsilon}\right)^{bc}$. Thus, by a union bound, the probability that a bad F with parts of size a and b, with $a < b \leq n^{\epsilon'}$, exists is at most

$$\begin{aligned}\binom{n}{a}\binom{n}{b}\left(\frac{a}{n^\epsilon}\right)^{bc} &\leq n^{2b}\left(\frac{a}{n^\epsilon}\right)^{bc} = \left(n^2 \cdot \frac{a^c}{n^{c\epsilon}}\right)^b \\ &\leq \left(\frac{n^2}{n^{c(\epsilon-\epsilon')}}\right)^b = o(n^{-2}),\end{aligned}$$

where we used the assumption that $c = \frac{4}{\epsilon-\epsilon'}$. A similar bound holds for the case $b < a \leq n^{\epsilon'}$. Thus, by a union bound, since there are at most n^2 choices for a and b, the probability that G has a bad induced subgraph with parts of size $a, b \leq n^{\epsilon'}$ is $o(1)$.

It follows that with probability $1 - o(1)$, every induced subgraph of G with at most $n^{\epsilon'}$ vertices contains a vertex of degree less than c. Consequently, every induced subgraph of G with $n^{\epsilon'}$ many vertices is $(c-1)$-degenerate. ☐

Lemma II.2. *The number of c-degenerate n-vertex graphs is at most $2^{O(cn\log n)}$.*

Proof. By deleting the vertices of degree at most c in turn, one obtains an ordering of the vertices such that every vertex has at most c neighbours in the subsequent vertices. Hence, the number of such graphs is at most

$$O(n!(n^c)^n) = 2^{O(cn\log n)}.$$

☐

III. Existence of counter-examples

We are now ready to prove Theorem I.4. Call an n-vertex bipartite graph *good* if it is a positive instance of Lemma II.1 with $n' = \frac{n}{2}$, $\epsilon = \frac{1}{2} + \frac{\delta}{2}$, and $\epsilon' = \frac{1}{2}$. Let $\mathcal{G}$ be the family of all good graphs. Note that all good graphs are bipartite graphs with even number of vertices, and bipartition $\{1, \ldots, \frac{n}{2}\}$ and $\{\frac{n}{2} + 1, \ldots, n\}$. We consider hereditary families that are constructed by picking small subsets $\mathcal{M}_n \subseteq \mathcal{G}_n$ and taking the hereditary closure of $\mathcal{M} = \cup_{n\in\mathbb{N}}\mathcal{M}_n$. To this end, we first prove that as long as $\mathcal{M}_n$ are not too large, the resulting hereditary family will have at most factorial speed.

Claim III.1. *For every n, let $\mathcal{M}_n \subseteq \mathcal{G}_n$ be a subset with $|\mathcal{M}_n| \leq 2^{\sqrt{n}}$. The hereditary closure of $\cup_{n\in\mathbb{N}}\mathcal{M}_n$ has at most factorial speed.*

Proof. Let $\mathcal{F}$ denote the hereditary closure of $\cup_n\mathcal{M}_n$. For an n-vertex graph $G \in \mathcal{F}$, let m be the smallest integer such that $G \in \mathrm{cl}(\mathcal{M}_m)$. We consider two cases, based on whether $n \leq \sqrt{m/2}$. If $n \leq \sqrt{m/2}$, then since the graphs in $\mathcal{M}_m$ are good, G is c-degenerate for $c = \frac{4}{\epsilon-\epsilon'} = 8\delta$. By Lemma II.2, the number of graphs G of this type is bounded by $2^{O(n\log n)}$. The number of graphs G, with $n > \sqrt{m/2}$ is bounded by

$$\begin{aligned}\sum_{m=n}^{2n^2} m^n|\mathcal{M}_m| &= \sum_{m=n}^{2n^2} m^n 2^{\sqrt{m}} \\ &\leq 2n^2 \cdot (2n^2)^n \cdot 2^{\sqrt{2n^2}} \\ &= 2^{O(n\log n)}.\end{aligned}$$

☐

We will focus our attention to even n. Define $k_n = \lceil 2^{\sqrt{n}} \rceil$. We will show that there is an $\mathcal{M}_n$ as in the assumption of Claim III.1 that does not have a universal graph representation of size $u \leq 2^{n^{\frac{1}{2}-\delta}}$. Combined with Claim III.1, the closure of $\cup_{n\in\mathbb{N}}\mathcal{M}_n$ is a hereditary family of at most factorial speed that does not have a $2^{n^{\frac{1}{2}-\delta}}$-size universal graph representation. Equivalently there is no implicit representation of this family with codes of length at most $n^{\frac{1}{2}-\delta}$, refuting the Implicit Graph Conjecture.

Suppose U is a u-vertex graph. The number of n-vertex graphs that can be represented by U is at most u^n. Thus the number of collections of k_n graphs on n-vertices that are simultaneously represented by U is at most u^{nk_n}.

Since the number of distinct u-vertex graphs U is at most 2^{u^2}, the number of collections of k_n graphs on n-vertices that have a u-vertex universal graph is at most

$$2^{u^2} \cdot u^{nk_n}. \tag{1}$$

On the other hand, let us first estimate $|\mathcal{G}_n|$. The number of graphs in the support of $B(\frac{n}{2}, \lfloor(\frac{n}{2})^{2-\epsilon}\rfloor)$ is

$$\binom{\frac{n^2}{4}}{\lfloor(\frac{n}{2})^{2-\epsilon}\rfloor} \geq 2^{\Omega(n^{2-\epsilon}\log n)}.$$

Hence by Lemma II.1, $|\mathcal{G}_n|$ is also at least $2^{\Omega(n^{2-\epsilon}\log n)}$. As a result, the number of choices of $\mathcal{M}_n \subseteq \mathcal{G}_n$ with $|\mathcal{M}_n| = k_n$ is at least

$$2^{\Omega(k_n n^{2-\epsilon}\log n)}.$$

We finally observe that for sufficiently large n, this is larger than (1):

$$\begin{aligned}\log(2^{u^2} \cdot u^{nk_n}) &= u^2 + nk_n\log(u) \\ &\leq 2^{2n^{\frac{1}{2}-\delta}} + n\lceil 2^{\sqrt{n}}\rceil n^{\frac{1}{2}-\delta} \\ &= O(2^{\sqrt{n}} n^{\frac{3}{2}-\delta}),\end{aligned}$$

while

$$\begin{aligned}\log(2^{\Omega(k_n n^{2-\epsilon}\log n)}) &= \Omega(k_n n^{2-\epsilon}\log n) \\ &= \Omega(2^{\sqrt{n}} n^{2-\epsilon}\log n) \\ &= \Omega(2^{\sqrt{n}} n^{\frac{3}{2}-\frac{\delta}{2}}\log n).\end{aligned}$$

Thus, there exist collections $\mathcal{M}_n \subseteq \mathcal{G}_n$ with $|\mathcal{M}_n| \leq k_n$ such that $\mathcal{M}_n$ cannot be represented by any universal graph of size u. This combined with Claim III.1 concludes the proof of Theorem I.4.

IV. Concluding remarks

Theorem I.4 shows that for any $\delta > 0$, there is a hereditary graph family $\mathcal{F}$ with speed $2^{O(n \log n)}$ such that any implicit representation of $\mathcal{F}$ must have code-length $\Omega(n^{\frac{1}{2}-\delta})$. A natural question is whether this lower-bound can be improved beyond $\sqrt{n}$. In fact, regarding upper bounds, we do not even know whether there exists a universal constant $\epsilon > 0$ such that every hereditary graph family with speed $2^{O(n \log n)}$ admits an implicit representation with code-length $O(n^{1-\epsilon})$.

Shortly after we posted an earlier preprint of this paper on arXiv, Noga Alon proved that if we allow ϵ to depend on the speed, such a bound holds even under weaker assumptions.

Theorem IV.1 (Alon [Alo22])**.** *For every growth function $f(n) = 2^{o(n^2)}$, there exists $\epsilon_f > 0$ such that every hereditary family $\mathcal{F}$ of graphs with speed $|\mathcal{F}_n| \leq f(n)$ admits an implicit representation with code-length $O(n^{1-\epsilon_f})$.*

The refutation of the Implicit Graph Conjecture leaves the problem of determining whether there is a simple characterization of hereditary graph properties with efficient implicit representations wide open. A problem of similar nature in the area of communication complexity is to characterize the hereditary classes of Boolean matrices that have randomized communication complexity $O(1)$. Indeed, [HWZ22] used a derandomization argument to show that every such class must admit an efficient implicit representation. These connections to communication complexity, as explored in [HWZ22], [HHH21], provided the initial inspiration for our work.

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References

[ACLZ15] A. Atminas, A. Collins, V. Lozin, and V. Zamaraev. Implicit representations and factorial properties of graphs. *Discrete Math.*, 338(2):164–179, 2015.

[ADK17] Stephen Alstrup, Søren Dahlgaard, and Mathias Bæk Tejs Knudsen. Optimal induced universal graphs and adjacency labeling for trees. *Journal of the ACM (JACM)*, 64(4):Art. 27, 22, 2017.

[AKTZ19] Stephen Alstrup, Haim Kaplan, Mikkel Thorup, and Uri Zwick. Adjacency labeling schemes and induced-universal graphs. *SIAM J. Discrete Math.*, 33(1):116–137, 2019.

[Ale97] V. E. Alekseev. On lower layers of a lattice of hereditary classes of graphs. *Diskretn. Anal. Issled. Oper. Ser. 1*, 4(1):3–12, 89, 1997.

[Alo17] Noga Alon. Asymptotically optimal induced universal graphs. *Geometric and Functional Analysis*, 27(1):1–32, 2017.

[Alo22] Noga Alon. Implicit representation of sparse hereditary families. *ArXiv*, 2022. arXiv:2201.00328.

[AR02] Stephen Alstrup and Theis Rauhe. Small induced-universal graphs and compact implicit graph representations. In *The 43rd Annual IEEE Symposium on Foundations of Computer Science (FOCS), 2002. Proceedings.*, pages 53–62. IEEE, 2002.

[BBW00] József Balogh, Béla Bollobás, and David Weinreich. The speed of hereditary properties of graphs. *Journal of Combinatorial Theory. Series B*, 79(2):131–156, 2000.

[BCE+82] L. Babai, F. R. K. Chung, P. Erdős, R. L. Graham, and J. H. Spencer. On graphs which contain all sparse graphs. In *Theory and practice of combinatorics*, volume 60 of *North-Holland Math. Stud.*, pages 21–26. North-Holland, Amsterdam, 1982.

[BCLR89] Sandeep N. Bhatt, F. R. K. Chung, F. T. Leighton, and Arnold L. Rosenberg. Universal graphs for bounded-degree trees and planar graphs. *SIAM J. Discrete Math.*, 2(2):145–155, 1989.

[BEGS21] Marthe Bonamy, Louis Esperet, Carla Groenland, and Alex Scott. Optimal labelling schemes for adjacency, comparability, and reachability. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 1109–1117, 2021.

[BGK+21] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width II: small classes. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1977–1996. SIAM, 2021.

[BT81] Béla Bollobás and Andrew Thomason. Graphs which contain all small graphs. *European J. Combin.*, 2(1):13–15, 1981.

[CG78] F. R. K. Chung and R. L. Graham. On graphs which contain all small trees. *Journal of Combinatorial Theory. Series B*, 24(1):14–23, 1978.

[CG79] F. R. K. Chung and R. L. Graham. On universal graphs. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 136–140. New York Acad. Sci., New York, 1979.

[DEG+21] Vida Dujmović, Louis Esperet, Cyril Gavoille, Gwenaël Joret, Piotr Micek, and Pat Morin. Adjacency labelling for planar graphs (and beyond). *Journal of the ACM (JACM)*, 68(6):1–33, 2021.

[FK16] Pierre Fraigniaud and Amos Korman. An optimal ancestry labeling scheme with applications to XML trees and universal posets. *Journal of the ACM (JACM)*, 63(1):Art. 6, 31, 2016.

[GL07] Cyril Gavoille and Arnaud Labourel. Shorter implicit representation for planar graphs and bounded treewidth graphs. In *European Symposium on Algorithms*, pages 582–593. Springer, 2007.

[HHH21] Lianna Hambardzumyan, Hamed Hatami, and Pooya Hatami. Dimension-free bounds and structural results in communication complexity. *Israel J. Math.*, 2021. to appear.

[HWZ22] Nathaniel Harms, Sebastian Wild, and Viktor Zamaraev. Randomized communication and implicit graph representations. In Stefano Leonardi and Anupam Gupta, editors, *STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 - 24, 2022*, pages 1220–1233. ACM, 2022.

[KNR88] Sampath Kannan, Moni Naor, and Steven Rudich. Implicit representation of graphs. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, STOC '88, page 334–343, New York, NY, USA, 1988. Association for Computing Machinery.

[KNR92] Sampath Kannan, Moni Naor, and Steven Rudich. Implicit representation of graphs. *SIAM J. Discrete Math.*, 5(4):596–603, 1992.

[MM13] Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory. Series B*, 103(1):114–143, 2013.

[Moo65] J. W. Moon. On minimal n-universal graphs. *Proc. Glasgow Math. Assoc.*, 7:32–33 (1965), 1965.

[Mul88] John H. Muller. *Local structure in graph classes*. PhD thesis, Georgia Institute of Technology, USA, 1988.

[Rad64] R. Rado. Universal graphs and universal functions. *Acta Arithmetica*, 9:331–340, 1964.

[Sch99] Edward R. Scheinerman. Local representations using very short labels. *Discrete Math.*, 203(1-3):287–290, 1999.

[Spi03] Jeremy P. Spinrad. *Efficient graph representations*, volume 19 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2003.

[Wu85] Angela Y Wu. Embedding of tree networks into hypercubes. *Journal of Parallel and Distributed Computing*, 2(3):238–249, 1985.