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STATIC MARKOWITZ MEAN-VARIANCE PORTFOLIO SELECTION MODEL WITH LONG-TERM BONDS

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Abstract. We propose a static Markowitz mean-variance portfolio selection model suitable for long-term zero-coupon bonds. The model uses a multi-factor term structure model of Vasicek (Ornstein-Uhlenbeck) type to compute the portfolio's expected return and its variance in the model. German Government zero-coupon bonds with short to very long time to maturity are considered; the data spans August 2002 to December 2020. The main investment assumption is the re-investment of cash flows of zero-coupon bonds with maturities less than the planning horizon at the current spot interest rate. Solutions for the zero-coupon holding vector and the tangency portfolio are obtained in closed form. Model parameters are estimated under an assumption of modeling ambiguity which takes the form of Knightian uncertainties at the level of the latent factors, allowing the use of a Kalman filter. Different investment strategies are examined on various risk portfolios. Results show that one- and two-factor Vasicek models produce attractive out-of-sample portfolio predictions in terms of the Sharpe ratio especially on long-term investments. It is also noted that a small number of risky bonds can adequately produce very attractive portfolio risk-return profiles.

1. **Introduction.**

The optimal portfolio allocation of long-term securities and liabilities in finance and risk insurance is associated with valuation problems (DeMiguel et al., 2009). These challenges emanate from uncertainty surrounding financial instruments such as long-term interest rates. Many fluctuations may happen along the way to maturity, for example a significant change in interest rates due to political ambiguity and economic rumor. The quotes for bonds with very long maturities, say over 30 years, are not very liquid, a reality which can be interpreted as partial observation, or as a data quality issue. Market models based on these data may not be very accurate and estimating model parameters can be difficult, with correspondingly uncertain marking to market, and more generally modeling risk due to ambiguity.

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Construction of an asset portfolio in an ambiguous environment involves mathematical modeling challenges. There will be a compromise on the level of transparency of the model's probability distribution. We take this up in this paper, as we analyze a portfolio selection optimization problem for zero-coupon bonds where an investor decides on a basket of various risky-asset bonds with different maturities, some of which are very long. The main objectives are to derive, exploit, and analyze the dynamics of an optimal portfolio of risky long-term bonds in a financial market where bond trading only takes place at discrete time epochs.

Our portfolio selection model is constructed based on Markowitz's mean-variance portfolio selection framework (Markowitz, 1952), which has for decades been applied to various risky asset portfolio optimization questions, including the discounted risk-free bonds and other theoretical and empirical research (Korn and Koziol, 2006; Wilhelm, 1992; Puhle, 2008; Fabozzi, 2004; DeMiguel et al., 2004; Svoboda, 2004); also see Zhang et al. 2018; Bjork et al., 2014; Chang, 2015; Bessler et al., 2017; and references therein. One may maximize an investment's expected return subject to a certain level of investment risk; or one may minimize the investment's risk subject to a desired level of expected return. Estimation of errors in parameters, high transaction costs and portfolio reallocation when applied in asset management can result in poor portfolio performance (Michaud 1989).

Vasicek (1977) was the first to introduce dynamic term structure models; short rate models named after Vasicek typically include a linear time series model or stochastic process (e.g. the Ornstein-Uhlenbeck process) featuring mean reversion. Brennan and Schwartz (1980) discovered that dynamic term structure models can be used for bond portfolio selection problem, reducing the data requirement burden. Wilhem (1992) proposed a Markowitz bond portfolio selection model through the use of dynamic term structure models. Unfortunately it was discovered that the model in such contexts is very complex to solve practically, hence one is led to consider a static model. We adopt such a strategy as well, with Vasicek multi-factor dynamics, and a static portfolio selection with mean-variance optimization. The important investment assumption we follow is that cash flows of bonds maturing before the investment horizon are re-invested at the current spot interest rate up to the planning horizon.

The specific literature context for our paper includes the following works. Bajeux-Besnainon and Portait (1998) analyse the portfolio strategies by allowing continuous re-balancing between the current data and the investment period. Korn and Koziol (2006) apply Markowitz's portfolio selection method to German government bond portfolios. They estimate expected portfolio returns, portfolio variances and co-variance using the multi-factor Vasicek term structure model. Results obtained therein show that a small number of risky bonds are sufficient to obtain attractive predicted risk-return profiles. Their assumptions in this term-structure framework do not include the possibility of modeling uncertainty. Puhle (2008) introduced a static bond portfolio optimization based on Markowitz's mean-variance framework. He derives the adjusted portfolio selection problem of zero-coupon bonds and applies it to the Hull-White (1990) and Vasicek models. The assumption of his model is similar to that of Wilhem (1992). In another study, DeMiguel et al. (2009) make comparison between mean-variance strategies and a naïve diversified $\frac{1}{N}$-weighted portfolio on different stock data sets to investigate which of these methods out-performs the other. Report indicate that the naïve portfolio consistently out-performs mean-variance strategies in an out-of-sample setting. Wu et al. (2014)

propose a discrete-time Markowitz mean-variance portfolio selection problem using uncertain time-horizon. They derived the efficient frontier and the optimal investment strategy explicitly using an embedding method. Zhang et al. (2018) review several techniques that sufficiently improve the Markowitz mean-variance model's performance, including robust portfolio optimization, portfolio optimization with practical factors, dynamic portfolio optimization and fuzzy portfolio optimization.

Our paper has contributions to the literature by empirically analyzing Markowitz's mean-variance portfolio selection model performance in an out-of-sample setting using German Government market data for the period August 2002 to December 2020. Our paper proposes the use of a multi-factor dynamic term structure model of Vasicek type, using the Kalman filter for a parameter estimation method that takes full advantage of the affine term structure of Vasicek models, simultaneously providing a clear interpretation of modeling ambiguity. We tackle zero-coupon bond prices and yields. Our first objective is to construct a mean-variance portfolio selection of zero-coupon bonds and obtain the solutions for the minimized variance-covariance of the model given the initial and expected terminal wealth constraints. The solution for the tangency portfolio is also obtained. As mentioned, the main assumption of the model is that there is re-balancing of cash flows of bonds with times to maturity below the planning horizon at the current spot interest rate until the planning horizon. Secondly, we show how to estimate the parameters of a multi-factor Vasicek, and further how to use them to compute moments of the mean-variance portfolio model (expected return, variance-covariance of both the bond prices and investor's terminal wealth). The multi-factor Vasicek model constructed is also suitable for long-term maturity zero-coupon bonds, which are also included in our analysys. Thirdly, we investigate the effect of different investment strategies (short-term, medium-term, and long-term planning horizons) on various risk portfolios and times to maturity for zero-coupon bonds. For this, we compute the out-of-sample portfolio predictions using three of our multifactor Vasicek model specifications, with one factor, two factors, or three factors.

2. **Problem formulation.** We formulate a static Markowitz portfolio selection model for an investor who is skeptical about modeling risk due to interest rate dependent long-term horizon securities. The investor seeks to select a mean-variance efficient portfolio with high expected return with a given minimum variance of returns. The portfolio model is static meaning that the investor is presumed to build a portfolio today and sell it at the investment horizon (Puhle, 2008; Bjork, 2009). That is between today (the day of investment) and the investment horizon, the investor remains inactive. Another assumption is that the investor only cares about the expected terminal portfolio wealth, expected portfolio return and the variance of the portfolio return. Therefore, the main objective of the investor is to minimize the variance of this terminal wealth given an expected terminal wealth, initial wealth and the self-financing budget constraint. She considers a financial market consisting of zero-coupon bonds with finite investment horizon maturities $T_i < \infty$. We also assume that the maturity time $T < \infty$ for a risk-free zero-coupon bond matches closely with the planning horizon of the investor's portfolio selection problem. Let us consider a financial market where time is divided into periods of length Δt and where financial transaction only takes place at the discrete points in time $n\Delta t$, $n = 0, 1, 2, ...$ We suppose that the bond prices in the financial market are affected by two state factors given by the following:

$$\begin{cases} r(t) = & \text{the short interest rate at time } t \\ y(t,T) = & \text{the yield to maturity for some fixed maturity } T. \end{cases}$$

In this paper, we construct interest rates models leading to analytically tractable and accurate optimal portfolio strategies for long-term risky bonds. We formulate state space representations of interest rate models to quantify long-term risk and modeling uncertainty, by using a filtering-theoretic approach, (see Mawonike et al., 2021) for more details on the topic of how to use Kalman filtering for estimating state space models for long-term bonds). The challenge with modeling long-term bond markets is that there is no classical single-factor stochastic short rate modeling framework which can adequately represent the modeling uncertainty inherent in these markets. No amount of statistical uncertainty on single-factor short rate model parameters can deliver the required model ambiguity. In this sense, we need to use a Knightian uncertainty framework, where we cannot expect any quantifiable knowledge about some possible occurrences, as opposed to statistical uncertainty, where this knowledge can be quantified. Specifically, we make use of multifactor short rate models that capture our desired long-term features (illiquidity and partial observation, which are sources of model ambiguity), together with the suitable filtering framework developed in Mawonike et al., 2021) and (Babbs and Nowman, 1999; Bolder, 2001), such that the more liquid quotes of the interest rate models (short to medium term) will be matched with a high signal-to-noise ratio (nearly perfect signal), whereas the long-term part of the term structure model involves instead a filtering problem, seeing less liquid quotes as more noisy observations. In particular, a linear state-space representation of the continuous-time infinite horizon short rate models and the Kalman filter are used to jointly estimate the current term structure and its dynamics for markets with an illiquid long-term bond. We take the measurement process to be the observed yields in the market, modeled as a linear function of a multi-factor state variable, plus a noise term given by a Gaussian white noise with an unknown time-varying covariance matrix to capture the embedded long-term model risks. The state process driving the yields, on the other hand, is the short rate process, which we assume is a sum of several unobserved factor which are modeled by Gaussian Vasicek processes. The white noises driving these Vasicek processes need to be correlated in specific ways to be consistent with the term structure of bond markets; they depend on a limited number of maturity-dependent variance parameters which must be estimated.

2.1. **Optimal portfolio of short, medium, and long term bonds.** We let τ be the longest maturity of all zero-coupon bonds available in the bond market. Therefore, there exists one bond for each maturity date $1, 2, 3, ..., \tau$. First, we present a case where the portfolio is set initially at time $t = 0$ and held until maturity (planning horizon) $t = T$ without rebalancing (see, e.g., Korn and Koziol, 2006). The trading strategy is characterised by $\hbar = \{\hbar_0, \hbar_T\}$, where $\hbar_0$ represents the initial wealth of the investor invested in the τ zero-coupon bonds and $\hbar_T$ denotes the investor's terminal wealth at the end of the investment horizon. At time $t = 0$, the investor allocates his initial wealth $\hbar_0$ to the τ zero-coupon bonds.

$$\hbar_0 = \sum_{T_i=T}^{\tau} \mathcal{N}_{T_i} P(0, T_i), \tag{1}$$

where $\mathcal{N}_{T_i}$ is the number of zero-coupon bonds purchased with maturity date T_i at current price $P(0,T_i)$. The investor is only permitted to invest in one risk-free bond and risky bonds over the finite investment planning horizon $[0,T]$. That is, τ zero-coupon bonds are divided into one risk-free and $\tau-1$ risky bonds. A T maturity zero-coupon bond invested in the investment horizon T is risk-free. Therefore, holding any bond which matures later than the investment horizon is risky. Now combining the risk-free and risky bonds in the holdings vector $\widetilde{\mathcal{N}}$ and the $\widetilde{\mathcal{P}}_0$ price vector gives us:

$$\hbar_0 = \widetilde{\mathcal{N}}'\widetilde{\mathcal{P}_0} + \mathcal{N}_T P(0,T), \tag{2}$$

with

$$\widetilde{\mathcal{N}}' = (\mathcal{N}_{T+1}, \mathcal{N}_{T+2}, ..., \mathcal{N}_\tau),$$
$$\widetilde{\mathcal{P}}_0 = (P(0,T+1), P(0,T+2), ..., P(0,\tau)).$$

where $\mathcal{N}_T$ is the number of T- maturity zero-coupon bonds purchased at time $t=0$. An investment $\mathcal{N}_T P(0,T)$ of T - maturity zero-coupon bonds is risk-free. Holdings of zero-coupon bonds with time to maturities greater than the planning horizon are at time T value the sum of the prices of individual zero-coupon bonds (see, Puhle, 2008). In addition, holdings of zero-coupon bonds with time to maturity less than T are more complex to value at time T. In the investor's selection problem, zero-coupon bonds with maturities less than the investment horizon T are not incorporated into her model. However, she is allowed to reinvest cash flows in the current zero-coupon bonds with maturities greater than T at the current yield to maturity, $y(t,T)$. Hence, the investor's terminal wealth becomes:

$$\hbar_T = \mathcal{N}_T + \sum_{T_i=T+1}^{\tau} \mathcal{N}_{T_i} e^{-(T_i-T)y(T,T_i)}, \tag{3}$$

with

$$y(t,T) = -\frac{1}{T-t}\ln[P(t,T)],$$

it follows that:

$$\hbar_T = \mathcal{N}_T + \sum_{T_i=T+1}^{\tau} \mathcal{N}_{T_i} P(T,T_i). \tag{4}$$

We let

$$\widetilde{\mathcal{P}}_T = (P(T,T+1), P(T,T+2), ..., P(T,\tau))',$$

then presenting $\hbar_T$ in vector form gives:

$$\hbar_T = \mathcal{N}_T + \widetilde{\mathcal{N}}'\widetilde{\mathcal{P}}_T. \tag{5}$$

(see e.g., Wilhelm, 1992 p217; Puhle, 2008 p44): As mentioned, our investor's portfolio selection problem adopts the mean-variance framework of Markowitz (1952). Our investor only uses her expected return and variance of her return, at the start of the investment period, $t=0$. We compute the expectation and variance at time $t=0$ of the terminal wealth $\hbar_T$ in Equation (5):

$$\mathbb{E}^{\mathbb{P}}[\hbar_T] = \mathcal{N}_T + \widetilde{\mathcal{N}}'\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T], \tag{6}$$

$$\begin{aligned} var^{\mathbb{P}}[\hbar_T] &= \sum_{T_i=T+1}^{\tau} \sum_{T_{i'}=T+1}^{\tau} \mathcal{N}_{T_i}\mathcal{N}_{T_i'} Cov\left(P(T,T_i), P(T,T_{i'})\right) \\ &= \widetilde{\mathcal{N}}'\Sigma\widetilde{\mathcal{N}} \end{aligned} \tag{7}$$

where $\Sigma = Cov\left(P(T,T_i), P(T,T_{i'})\right)$ for $T_i = T+1, T+2, ..., \tau$ and $T_{i'} = T+1, T+2, ..., \tau$ is the covariance matrix of the vector of bond prices maturing after T. With these expressions in (6) and (7) plus the condition from Equation (2) on the initial wealth, the Markowitz bond portfolio selection problem is formulated as:

$$\min_{\widetilde{\mathcal{N}}} \quad \frac{1}{2}\widetilde{\mathcal{N}}'\Sigma\widetilde{\mathcal{N}} \tag{8}$$

$$\text{subject to:} \quad \mathcal{N}_T + \widetilde{\mathcal{N}}'\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] = \widetilde{\hbar}_T \tag{9}$$

$$\text{and to:} \quad \mathcal{N}_T P(0,T) + \widetilde{\mathcal{N}}'\widetilde{\mathcal{P}}_0 = \hbar_0 \tag{10}$$

Combining Equation (9) and Equation (10), the investor's bond portfolio selection problem becomes:

$$\min_{\widetilde{N}} \quad \frac{1}{2}\widetilde{\mathcal{N}}'\Sigma\widetilde{\mathcal{N}} \tag{11}$$

$$\text{subject to:} \quad \frac{\hbar_0}{P(0,T)} + \widetilde{\mathcal{N}}'\left(\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] - \frac{\widetilde{\mathcal{P}}_0}{P(0,T)}\right) = \widetilde{\hbar}_T \tag{12}$$

Equation (12) involves both the risk-free initial wealth and the vector of risk premia (see, also, Puhle, 2008 for a detailed explanation). Solving the quadratic minimisation problem in Equation (11) is classical via Lagrange multipliers, to compute the mean-variance efficient portfolio.

In the second case, we consider the more general scenerio where the portfolio is constructed at time $t = 0$ and held until maturity $t = T$ and there is an assumption of rolling over. That is, the face value of each zero-coupon bond which matures before the investment horizon T is re-invested at time $t < T$ at the current yield $y(t,T)$ until the investment horizon. Hence, the holdings vector $\widetilde{\mathcal{N}}$ and the price vector $\widetilde{\mathcal{P}}_0$ must now include elements before T, beyond what was given in (2). They are thus given by:

$$\widetilde{\mathcal{N}}' = (\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_{T-1}, \mathcal{N}_{T+1}, ..., \mathcal{N}_\tau)$$

$$\widetilde{\mathcal{P}}_0 = \left(P(0,1), P(0,2), ..., P(0,T-1), P(0,T+1), ..., P(0,\tau)\right).$$

Therefore, the terminal wealth from (3) must be modified to become:

$$\begin{aligned} \hbar_T &= \sum_{T_i=1}^{T-1} \mathcal{N}_{T_i} e^{(T-T_i)y(T_i,T)} + \mathcal{N}_T + \sum_{T_i=T+1}^{\tau} \mathcal{N}_{T_i} e^{-(T_i-T)y(T,T_i)} \\ &= \sum_{T_i=1}^{T-1} \mathcal{N}_{T_i} \frac{1}{P(T_i,T)} + \mathcal{N}_T + \sum_{T_i=T+1}^{\tau} \mathcal{N}_{T_i} P(T,T_i). \end{aligned} \tag{13}$$

since bonds which have already matured at time T will continue accruing interest at the corresponding spot rates. Letting

$$\widetilde{\mathcal{P}}_T = \left(\frac{1}{P(1,T)}, ..., \frac{1}{P(T-1,T)}, P(T,T+1), P(T,T+2), ..., P(T,\tau)\right)',$$

then the terminal wealth $\hbar_T$ in vector form becomes:

$$\hbar_T = \mathcal{N}_T + \widetilde{\mathcal{N}}'\widetilde{\mathcal{P}}_T. \tag{14}$$

Now the expectation and variance of $\hbar_T$ become:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\hbar_T] &= \sum_{T_i=1}^{T-1} \mathcal{N}_{T_i}\mathbb{E}^{\mathbb{P}}\left[\frac{1}{P(T_i,T)}\right] + \mathcal{N}_T + \sum_{T_i=T+1}^{\tau} \mathcal{N}_{T_i}\mathbb{E}^{\mathbb{P}}\left[P(T,T_i)\right] \\ &= \mathcal{N}_T + \widetilde{\mathcal{N}}'\,\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T].\end{aligned} \tag{15}$$

$$\begin{aligned}var^{\mathbb{P}}[\hbar_T] &= \sum_{T_i=1}^{T-1}\sum_{T_{i'}=1}^{T-1} \mathcal{N}_{T_i}\mathcal{N}_{T_{i'}} Cov\left(\frac{1}{P(T_i,T)}, \frac{1}{P(T_{i'},T)}\right) \\ &\quad + \sum_{T_i=T+1}^{\tau}\sum_{T_{i'}=T+1}^{\tau} \mathcal{N}_{T_i}\mathcal{N}_{T_{i'}} Cov\left(P(T,T_i), P(T,T_{i'})\right) \\ &\quad + 2\sum_{T_i=1}^{T-1}\sum_{T_{i'}=T+1}^{\tau} \mathcal{N}_{T_i}\mathcal{N}_{T_{i'}} Cov\left(\frac{1}{P(T_i,T)}, P(T,T_{i'})\right) \\ &= \widetilde{\mathcal{N}}'\,\Sigma\widetilde{\mathcal{N}}\end{aligned} \tag{16}$$

where Σ is defined as:

$$\Sigma = \left\{\begin{array}{lll} Cov\left(\frac{1}{P(T_i,T)}, \frac{1}{P(T_{i'},T)}\right) & for & T_i = 1,...,T-1; i' = 1,...,T-1 \\ Cov\left(\frac{1}{P(T_i,T)}, P(T,T_{i'})\right) & for & T_i = 1,...,T-1; i' = T+1, T+2,...,\tau \\ Cov\left(P(T,T_i), \frac{1}{P(T_{i'},T)}\right) & for & T_i = T+1, T+2,...,\tau; i' = 1,...,T-1 \\ Cov\left(P(T,T_i), P(T,T_{i'})\right) & for & T_i = T+1, T+2,...,\tau; T_{i'} = T+1, T+2,...,\tau \end{array}\right\}, \tag{17}$$

In (20), $\mathbb{E}^{\mathbb{P}}[P(T,T_i)]$ is the expectation at time 0 of the discount factor at time T for the maturity $T_i > T$, and $\mathbb{E}^{\mathbb{P}}[\frac{1}{P(T_i,T)}]$ is expected accrual factor from time T_i to time T when $T_i < T$ (see, e.g. Puhle, 2008, pp47). Now our investor's constrained variance minimization problem of interest is of the same form as in (8), (9) and (10), and can thus be formulated as:

$$\begin{aligned} \min_{\widetilde{N}} \quad & \frac{1}{2}\widetilde{\mathcal{N}}'\,\Sigma\widetilde{\mathcal{N}} \\ \text{subject to:} \quad & \mathcal{N}_T + \widetilde{\mathcal{N}}'\,\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] = \widetilde{\hbar}_T \\ & \mathcal{N}_T P(0,T) + \widetilde{\mathcal{N}}'\,\widetilde{\mathcal{P}}_0 = \hbar_0 \end{aligned}$$

with the new definitions of $\widetilde{\mathcal{N}}, \widetilde{\mathcal{P}}_T$, and $\widetilde{\mathcal{P}}_0$ in this section. As in the previous section, equivalently this is:

$$\begin{aligned} \min_{\widetilde{N}} \quad & \frac{1}{2}\widetilde{\mathcal{N}}'\,\Sigma\widetilde{\mathcal{N}} \\ \text{subject to:} \quad & \frac{\hbar_0}{P(0,T)} + \widetilde{\mathcal{N}}'\left(\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] - \frac{\widetilde{\mathcal{P}}_0}{P(0,T)}\right) = \widetilde{\hbar}_T, \end{aligned}$$

using the expectation and variance of the terminal wealth given in (20), (21), and (22). The solution to this constrained quadratic minimization problem is obtained identically to the solution steps detailed in section (2.1), since the functional to be minimized and the constraint functional are indifferent to how time is interpreted in the allocation vector.

2.2. **Model uncertainty specifications.** In this section, we present the Vasicek term structure model for bond prices with maturity dependent model uncertainty specifications.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a time horizon $[0, T]$ with filtration, $\{\mathcal{F}_t\}_{t\geq 0}$ generated by n-dimensional correlated Brownian motions

$$\{W_1(t), W_2(t), ..., W_n(t)\}, t \geq 0,$$

where therefore the probability measure $\mathbb{P}$ on Ω can be taken as the n-dimensional standard correlated Wiener measure, with correlation matrix ρ, on the space of continuous functions with values in $\mathbb{R}^n$. Any decision made at time any time $t > 0$, and any zero-coupon bond determined at such a time, must be based solely on $\mathcal{F}_t$, the information available until time t; in other words, every random variable $P(t, t')$ is $\mathcal{F}_t$-measurable. As mentioned below, since we will use a so-called affine term-structure model based on short-rate modeling, $\mathcal{F}_t, t \geq 0$ can also be understood as the filtration generated by the short rate process $r(t), t \geq 0$. We assume the investor can trade discretely in the financial market over time with no friction (no taxes or other transaction costs). The investment planning horizon T of the market is defined as the maturity date T_n of the n^{th} bond. The price process of the i^{th} zero-coupon bond, denoted by $P(t, T_i), t \geq 0$, is given by:

$$P(t, T_i) = \mathbb{E}^{\mathbb{Q}_{\text{rn}}}\left[\exp\left(-\int_t^{T_i} r(s)ds\right)|\mathcal{F}_t\right], \quad i = 1, 2, 3, ..., n \tag{18}$$

where $\mathcal{F}_t$ is generated by the short interest rate process $r(s), s \in [0, t]$, and $\mathbb{E}^{\mathbb{Q}_{\text{rn}}}$ denotes the expectation under a risk-neutral probability measure for the bond market. We posit that the dynamics of the short rate process r are given by:

$$r(t) = \mu - \sum_{j=1}^{J} X_j(t) \tag{19}$$

where μ is the long-term mean, and each economic factor $X_j(t)$ is assumed to follow an Ornstein-Uhlenbeck (OU) process with zero long-term mean. Each factor exhibits a constant risk premium λ_j. The short rate $r(t)$ and factors $X_j(t)$ are governed by (19) and the following SDEs under an objective (real, not risk-neutral) probability measure: for each $j = 1, \cdots, J$,

$$dX_j(t) = -\xi_j X_j(t)dt + c_j dW_j(t) \tag{20}$$

where ξ_j is a mean-reversion rate, c_j is the factor's local volatility and $W_j(t)$ are correlated standard Brownian motions. It is known (see Bjork (2006)) that the model can also be specified under a risk-neutral probability measure as:

$$dX_j(t) = -\xi_j(\lambda_j c_j + X_j(t))dt + c_j dZ_j(t) \tag{21}$$

where $Z_j(t)$ are uncorrelated standard Brownian motions and the λ_j's are the aforementioned risk premia. Under the objective (real) market probability measure, we assume that the market is liquid enough with sufficient data available for accurate calibration of asset models, particularly the volatilities and risk premia c_j, λ_j.

Equivalently to the previous formulation, the version of equation (25) with modeling ambiguity under $\mathbb{P}$ is:

$$dX_j(t) = -(\xi_j + \Delta\xi_j)X_j(t)dt + c_j dW_j(t) \tag{22}$$

The OU process in Equation (22) for $X_j(t)$ is well known to have an explicit solution (e.g. see Mastro (2013)):

$$X_j(t) = e^{-(\xi_j+\Delta\xi_j)t}X_j(0) + c_j e^{-(\xi_j+\Delta\xi_j)t}\int_0^t e^{(\xi_j+\Delta\xi_j)s}dW_j(s), \tag{23}$$

The expectation and variance of the j^{th} factor $X_j(t)$ at $t = T$ in Equation (23) are also well known (as can be computed readily from the known normal distribution of Wiener stochastic integrals with deterministic integrands) resulting in the following:

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{P}}(X_j(T)) = X_j(0)e^{-(\xi_j+\Delta\xi_j)\cdot T}\\
&Var_0^{\mathbb{P}}(X_j(T)) = \frac{c_j^2}{2(\xi_j+\Delta\xi_j)}\left(1-e^{-2(\xi_j+\Delta\xi_j)\cdot T}\right)\\
&Cov_0^{\mathbb{P}}\left(X_j(T), X_{j'}(T_i)\right)\\
&= 2\frac{\rho_{j,j'}c_jc_{j'}}{(\xi_j+\Delta\xi_j)+(\xi_{j'}+\Delta\xi_{j'})}\left(1-e^{-((\xi_j+\Delta\xi_j)+(\xi_{j'}+\Delta\xi_{j'}))(min[T,T_i])}\right), \quad j<j'
\end{aligned} \tag{24}$$

As is commonly done, we abuse the notation slightly by using the same letter P for this deterministic function, known as a pricing function, as follows:

$$P(t,T) = P\left(X(t), t, T_i\right), \tag{25}$$

where P is not a function of r directly, but of the vector X of all J factors. The Markovian dynamics also imply, by a straightforward application of Ito's formula, that the zero-coupon bond price dynamics from Equation (25) are: It is known that Equation (23) together with the terminal condition $P(T_i, T_i; X) = 1$ has a unique closed form solution for a zero-coupon bond pricing function given by:

$$P(t, T_i; X_j) = P(t,T) = e^{A(t,T)-\mu\cdot(T-t)-\sum_{j=1}^J B_j(t,T)X_j(t)} \tag{26}$$

with

$$B_j(t,T) := \frac{1-e^{-(\xi_j+\Delta\xi_j)\cdot(T-t)}}{\xi_j+\Delta\xi_j}$$

$$\begin{aligned}
A(t,T) := &-\sum_{j=1}^J\left(\left((\mu+\lambda_jc_j)-\frac{c_j^2}{2(\xi_j+\Delta\xi_j)^2}\right)\cdot(T-t)\right)\\
&+\sum_{j=1}^J\left(\frac{1}{\xi_j+\Delta\xi_j}\left(1-e^{-(\xi_j+\Delta\xi_j)\cdot(T-t)}\right)\cdot\left((\mu+\lambda_jc_j)-\frac{c_j^2}{2(\xi_j+\Delta\xi_j)^2}\right)\right)\\
&-\sum_{j=1}^J\left(\frac{c_j^2}{4(\xi_j+\Delta\xi_j)^3}\left(1-e^{-(\xi_j+\Delta\xi_j)\cdot(T-t)}\right)^2\right).
\end{aligned}$$

Let us call $\epsilon_\tau(T)$ to be the source of this modeling uncertainty in the zero-coupon bond prices at the planning horizon T for any given maturity $\tau \neq T$ (see, Puhle, 2008). We signify how this uncertainty comes in by introducing it directly at the affine level in Equation (22) by adding it as an additive noise term next to the affine function of the factors: for instance, for bond prices,

$$P(T,\tau) = e^{A(T,\tau)-\mu\cdot(\tau-T)-\sum_{j=1}^J B_j(T,\tau)X_j(T)+\epsilon_\tau(T)}. \tag{27}$$

A similar definition of $\epsilon_\tau(T)$ is used for accrual factors. The error terms $\epsilon_\tau(T)$ are assumed to be independent and normally distributed random variables with mean zero and variance $\sigma^2(\epsilon_\tau)$ as τ changes. We have already determined that the state variables $X_j(T)$ are jointly normally distributed with mean $\mathbb{E}(X_j(T))$ and covariance matrix $\text{Cov}(X_j(T), X_{j'}(T))$. Since the short rate $r(T)$ is a linear combination of these state variables (see (19)), it is normally[1] distributed as well (see, e.g., Puhle, 2008. pp.50). Now, $\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T]$ and the variance-covariance Σ of terminal returns can be constructed as in Korn and Koziol, (2006).

Using the classical Laplace transform of the Gaussian distribution, elementary calculations similar to what was performed in (see e.g. Bjork et al., 2014, Bolder,2001,and Cox et al.,1985), result in formulas for the expectations, variances, and covariances of the zero-coupon bond prices and corresponding accrual factors. First, the expected zero-coupon bond price at time T to maturity T_i, under the objective probability measure, is

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] &= \mathbb{E}_0^{\mathbb{P}}\left(P(T,T_i)\right)\\ &= \mathbb{E}\left[e^{A(T,T_i)-\mu\cdot(T_i-T)-\sum_{j=1}^{J}B_j(T,T_i)X_j(T)+\epsilon_\tau(T_i)}\right] \\ &=: e^{\mu^{(1)}(T_i)+\frac{1}{2}\sigma^{(1)}(T_i)^2}\end{aligned} \tag{28}$$

The natural expressions for $\mu^{(1)}(T_i)$ and $\sigma^{(1)}(T_i)^2$ are stated explicitly below, under equation (30). The expected accrual factors from time $T_i < T$ to investment horizon T are given by:

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\frac{1}{\widetilde{\mathcal{P}}_T}\right]\\ &= \mathbb{E}_0^{\mathbb{P}}\left(\frac{1}{P(T_i,T)}\right)\\ &= \mathbb{E}\left[e^{-\left\{A(T_i,T)-\mu\cdot(T-T_i)-\sum_{j=1}^{J}B_j(T_i,T)X_j(T)+\epsilon_\tau(T_i)\right\}}\right]\\ &= e^{-A(T_i,T)+\mu(T-T_i)+\sum_{j=1}^{J}B_j(T_i,T)\mathbb{E}[X_j(T)]+\frac{1}{2}\sum_{j=1}^{J}B_j(T_i,T)^2 Var[X_j(T)]+\sigma^2(\epsilon_{T_i})}\end{aligned} \tag{29}$$

$$\begin{aligned}var^{\mathbb{P}}[\widetilde{\mathcal{P}}_T] &= \text{Cov}_0^{\mathbb{P}}\left(P(T,T_i),P(T,T_i)\right)\\ &= \mathbb{E}\left[P(T,T_i)\right]\cdot\mathbb{E}\left[P(T,T_i)\right]\cdot\left(e^{\sum_{j=1}^{J}B_j(T,T_i)^2\text{Var}[X_j(T)]+\sigma^2(\epsilon_{T_i})}-1\right)\\ &= e^{2\mu^{(1)}(T_i)+\sigma^{(1)}(T_i)^2}\cdot\left(e^{\sigma^{(1)}(T_i)^2}-1\right)\end{aligned} \tag{30}$$

The covariances between discount factors at time T for maturities $T_i > T$ and $T_{i'} > T$ can be expressed as:

$$\begin{aligned}\Sigma_{11} &:= \Sigma_{11}^{ii'} := \text{Cov}_0^{\mathbb{P}}\left(P(T,T_i),P(T,T_{i'})\right)\\ &= \mathbb{E}\left[P(T,T_i)\right]\cdot\mathbb{E}\left[P(T,T_{i'})\right]\\ &\quad\cdot\left(e^{\sum_{j=1}^{J}(B_j(T,T_i)+B_j(T,T_{i'}))^2\cdot\text{var}[X_j(T)]+\sigma^2(\epsilon_{T_i})+\sigma^2(\epsilon_{T_{i'}})}-1\right)\\ &=: e^{\mu^{(2)}(T_i,T_{i'})+\sigma^{(2)}(T_i,T_{i'})^2}-e^{\mu^{(1)}(T_i)+\mu^{(1)}(T_{i'})+\frac{1}{2}\left(\sigma^{(1)}(T_i)^2+\sigma^{(1)}(T_{i'})^2\right)}\end{aligned} \tag{31}$$

with

$$\mu^{(1)}(T_i) \quad := \quad A(T,T_i)-\mu\cdot(T_i-T)-\sum_{j=1}^{J}\left\{B_j(T,T_i)\cdot\mathbb{E}_0^{\mathbb{P}}(X_j(T))\right\}$$

[1] If y is Gaussian distributed, then classically, $E[e^y] = e^{E[y]+\frac{1}{2}var[y]}$ and $var[e^y] = E[e^y]^2\cdot\left(e^{var[y]}-1\right)$

$$
\begin{aligned}
\sigma^{(1)}(T_i) &:= \sqrt{\sum_{j=1}^{J}\left\{B_j(T,T_i)^2 \cdot Var_0^{\mathbb{P}}(X_j(T)) + \sigma^2(\epsilon_{T_i})\right\}} \\
\mu^{(2)} &:= A(T,T_i) + A(T,T_{i'}) - \mu \cdot (T_i + T_{i'} - 2T) \\
&\quad - \sum_{j=1}^{J}\left\{(B_j(T,T_i) + B(T,T_{i'})) \cdot \mathbb{E}_0^{\mathbb{P}}(X_j(T))\right\}
\end{aligned}
$$

and

$$
\sigma^{(2)}(T_i,T_{i'}) := \sqrt{\sum_{j=1}^{J}\left[B_j(T,T_i) + B(T,T_{i'})\right]^2 \cdot Var_0^{\mathbb{P}}(X_j(T)) + \sigma^2(\epsilon_{T_i}) + \sigma^2(\epsilon_{T_{i'}})}
$$

The covariances between accrual factors for maturity T at times $T_i < T$ and $T_{i'} < T$ can be expressed as follows:

$$
\begin{aligned}
\Sigma_{22} = \Sigma_{22}^{ii'} &= Cov^{\mathbb{P}_0}\left(\frac{1}{P(T_i,T)}, \frac{1}{P(T_{i'},T)}\right) \\
&= \mathbb{E}\left[\frac{1}{P(T_i,T)}\right] \cdot \mathbb{E}\left[\frac{1}{P(T_{i'},T)}\right] \\
&\quad \cdot \left(e^{\sum_{j=1}^{J}(B_j(T_i,T)+B_j(T_{i'},T))^2 \cdot \mathrm{Cov}[X_j(T),X_j(T_i)] + \sigma^2(\epsilon_{T_i}) + \sigma^2(\epsilon_{T_{i'}})} - 1\right)
\end{aligned}
\tag{32}
$$

The covariances between the discount factors at time T for maturity T_i and accrual factors at time $T_{i'} < T$ for maturity T can be computed as:

$$
\begin{aligned}
\Sigma_{12} := \Sigma_{12}^{ii'} &:= \mathrm{Cov}^{\mathbb{P}_0}\left(P(T_i,T), \frac{1}{P(T_{i'},T)}\right) \\
&= \mathbb{E}\left[P(T_i,T)\right] \cdot \mathbb{E}\left[\frac{1}{P(T_{i'},T)}\right] \\
&\quad \cdot \left(e^{-\sum_{j=1}^{J}(B_j(T,T_i)+B_j(T,T_{i'}))^2 \cdot \mathrm{Cov}[X_j(T),X_j(T_i)] + \sigma^2(\epsilon_{T_i}) + \sigma^2(\epsilon_{T_{i'}})} - 1\right)
\end{aligned}
\tag{33}
$$

The covariances between the accrual factors at time $T_i < T$ for maturity T and discount factors at time T for maturity $T_i > T$ can be computed as:

$$
\begin{aligned}
\Sigma_{21} &:= \Sigma_{21}^{ii'} := \mathrm{Cov}^{\mathbb{P}_0}\left(\frac{1}{P(T_i,T)}, P(T,T_{i'})\right) \\
&= \mathbb{E}\left[\frac{1}{P(T_i,T)}\right] \cdot \mathbb{E}\left[P(T,T_{i'})\right] \\
&\quad \cdot \left(e^{-\sum_{j=1}^{J}(B_j(T_i,T)+B_j(T,T_{i'}))^2 \cdot \mathrm{Cov}[X_j(T),X_j(T_i)] + \sigma^2(\epsilon_{T_i}) + \sigma^2(\epsilon_{T_{i'}})} - 1\right) \\
&= \left(e^{-A(T_i,T) + \mu(T-T_i) + \sum_{j=1}^{J} B_j(T_i,T)\mathbb{E}[X_j(T)] + \frac{1}{2}\sum_{j=1}^{J} B_j(T_i,T)^2 Var[X_j(T)] + \sigma^2(\epsilon_{T_i})}\right) \\
&\quad \times \left(e^{A(T,T_{i'}) - \mu(T_{i'}-T) - \sum_{j=1}^{J} B_j(T_{i'},T)\mathbb{E}[X_j(T)] + \frac{1}{2}\sum_{j=1}^{J} B_j(T_{i'},T)^2 Var[X_j(T)] + \sigma^2(\epsilon_{T_{i'}})}\right) \\
&\quad \times \left(e^{-\sum_{j=1}^{J}(B_j(T_i,T)+B_j(T,T_{i'}))^2 \cdot \mathrm{Cov}[X_j(T),X_j(T_i)] + \sigma^2(\epsilon_{T_i}) + \sigma^2(\epsilon_{T_{i'}})} - 1\right)
\end{aligned}
\tag{34}
$$

The $\mathbb{E}[X_j(T)]$, $\mathrm{Var}[X_j(T)]$ and $\mathrm{Cov}(X_j(T), X_j(T_i))$ of the above moments are given in Equation (24). The values of u and $\widetilde{\mathcal{N}}$ can now be determined as described in Sections 2.1, see formulas (16) and (17). Thus the portfolio selection problem of the investor is solved via these formulas and the expectations, variances, and the matrix Σ given in this section; the mean-variance efficient portfolios can also be determined (see, e.g., Merton,1972; Huang and Litzenberger, 1988; Kallberg and Ziemba, 1983; Korn and Koziol, 2006; Puhle, 2008; Wilhelm, 1992; Elton et al., 2003; Zhang et al., 2018).

3. **Empirical results.** The previous two sections show that the Markovitz portfolio optimization problem only requires us to compute the means of the zero-coupon bonds maturing before and after T, and the covariances of these discount factors and their inverses (accrual factors), depending on whether their maturities are before or after the planning horizon T. However, it is very challenging to estimate prices consistently for medium-to-long term bonds under a single specification of interest rate model. To address this, we follow the state-space bond market model developed in (Mawonike et al., 2021) that incorporates uncertainty in the underlying interest rate parameters. The state space model coupled with the complementary Kalman filter provides us with a model uncertainty configuration designed in a consistent fashion for parameter estimation with medium to long-term bond prices. In particular, we estimate parameters of a state-space bond market model formulated on the multi-factor Vasicek interest rate process. There are two main results presented in this paper. The first result is the formulation of optimal portfolio selection consisting of short, medium and long-maturity bonds within a consistent modeling and valuation framework. To do this, we follow the state-space bond market model developed in (Mawonike et al., 2021) that incorporates maturity dependent uncertainty in the underlying interest rate parameters we assume that the underlying short rate follows the mean-reverting multi-factor Vasicek model. The second result is the implementation of the optimal portfolio selection algorithm using observed yield data.

3.1. **A consistent state-space model of bond prices.** In this section we present in compact form the relationship between the multi-factor model X and the vector of bond prices which we detailed in the previous sections, including the maturity dependent error terms which represent our modeling uncertainty. Because our factors in X satisfy the OU (Vasicek) dynamics, and the term structure has an affine form with respect to X, it is most convenient to work with bond yields. As we have already hinted, as we are about to see, the relationship between yields y and factors X is a linear (affine) one, and the presence of maturity dependent uncertainty results in a multivariate linear model, for which classical statistical estimation methods can apply. Because of the dynamic nature of the data, as we will see in Section 3, it is most appropriate to use Kalman filtering for part of the estimation.

The relationship between yields and factors is sometimes known as a set of *measurement equations*, whereas the time dynamics of the factors may be known as *transition equations*. In order to lighten the notation, it will be convenient to denote $\tau_i = T_i - t$ for all i: this is the remaining time to maturity at any given time $t > 0$. The value of t is thus implicitly contained in τ_i.

The Measurement Equations

Following basic accounting principles in continuous time, the relationship between the zero-coupon bond yield and the zero-coupon bond price, over the interval $[t, T]$, for a bond written at time 0 and maturing at $T > 0$, is simply:

$$y(t,T) = -\frac{\log P(t,T)}{T-t} \tag{35}$$

Let $\{T_i\}_{i=0}^{n}$ be any given sequence of n bond maturities such that $0 \leq T_0 \leq T_1 \leq ... \leq T_n$. Our market observations consist of these n bonds, and specifically of their price processes, each of them observed discretely and at N equally spaced times t_k, so that $\delta t := t_k - t_{k-1}$ does not depend on k, and is proportional to N^{-1} (Bolder, 2001; Mawonike et al. 2021). Then, the measurement equations for all n bonds, with each component of the n-dimensional yield vector y_{t_k} at time t_k being the yield $y(t_k, t_k + \tau_i)$ for $k = 1, ..., N$, and $i = 1, ..., n$, is given by a multivariate linear regression model:

$$y_{t_k} = \gamma_{t_k} X_{t_k} + \psi_{t_k} + \epsilon_{t_k}, \quad t_k < T_i = t_k + \tau_i, \tag{36}$$

Indeed, according to the bond pricing formula with modeling ambiguity given in (36), by applying the logarithm in (44), we find that its discretisation produces the following linear system of N measurement equations:

$$\underbrace{\begin{bmatrix} y(t_k,\tau_1) \\ y(t_k,\tau_2) \\ \vdots \\ y(t_k,\tau_N) \end{bmatrix}}_{y_{t_k}} = \underbrace{\begin{bmatrix} \frac{-A(t_k,\tau_1)+\mu(\tau_1)}{\tau_1} \\ \frac{-A(t_k,\tau_2)+\mu(\tau_2)}{\tau_2} \\ \vdots \\ \frac{-A(t_k,\tau_N)+\mu(\tau_N)}{\tau_N} \end{bmatrix}}_{\psi_{t_k}}$$
$$+ \underbrace{\begin{bmatrix} \frac{B_1(t_k,\tau_1)}{\tau_1} & \frac{B_2(t_k,\tau_1)}{\tau_1} & \dots & \frac{B_J(t_k,\tau_1)}{\tau_1} \\ \frac{B_1(t_k,\tau_2)}{\tau_2} & \frac{B_2(t_k,\tau_2)}{\tau_2} & \dots & \frac{B_J(t_k,\tau_2)}{\tau_2} \\ \vdots & \vdots & \dots & \vdots \\ \frac{B_1(t_k,\tau_N)}{\tau_N} & \frac{B_2(t_k,\tau_N)}{\tau_N} & \dots & \frac{B_J(t_k,\tau_N)}{\tau_N} \end{bmatrix}}_{\gamma_{t_k}} \underbrace{\begin{bmatrix} X_1(t_k) \\ X_2(t_k) \\ \vdots \\ X_J(t_k) \end{bmatrix}}_{X_{t_k}}$$
$$+ \underbrace{\begin{bmatrix} \epsilon_1(t_k) \\ \epsilon_2(t_k) \\ \vdots \\ \epsilon_N(t_k) \end{bmatrix}}_{\epsilon_{t_k}}$$

where ϵ_{t_k} is assumed to be a vector of mean-zero independent normal statistical error terms, sharing a common unknown variance.

The Transition Equation

The transition equations describe the evolution of the stochastic factors over discrete time intervals of fixed length $t_k - t_{k-1} = \delta t$. Since the factors follow OU (mean-zero Vasicek) dynamics, the following transition equations result:

$$\begin{aligned} X_j(t_k) = {} & \underbrace{(\lambda_j c_j)\left(1 - e^{-(\xi_j+\Delta\xi_j)\delta t}\right)}_{D_j} + \underbrace{e^{-(\xi_j+\Delta\xi_j)\delta t} X_j(t_{k-1})}_{\mathbf{F}_{jj} X_j(t_{k-1})} \\ & + \underbrace{c_j e^{-(\xi_j+\Delta\xi_j)\delta t} \int_{t_{k-1}}^{t_k} e^{(\xi_j+\Delta\xi_j)s} dW_j(s),}_{\eta_j(t_{k-1}) \sim N(0,\mathbf{Q})} \end{aligned} \tag{37}$$

where $\mathbf{Q}$ represents the $J \times J$ covariance matrix between the noise terms W_j under the objective (real) market probability measure. Here D is also J-dimensional, and while $\mathbf{F}$ could be a matrix in principle, it is diagonal in our case. However, the factors are correlated via $\mathbf{Q}$. Thus for $k = 1, ..., N$, the dynamics of the J-dimensional state vector of factors X from t_{k-1} to t_k is given by:

$$X(t_k) = D + \mathbf{F}X(t_{k-1}) + \eta(t_{k-1}). \tag{38}$$

3.2. **Data description and numerical implementations.** In this study, we analyse the investment strategies using market data obtained from German Government bonds. We preferred German bonds to other European bonds because its bond market is very large (Lynch, 2002) and consists of bond maturities covering all years from as little as one year to more than 30 years. German Government bonds are issued by the Deutsche Bundesbank based on the approach in Svensson (1994) (see Korn and Kozoil, 2006, section 3). In our analysis, we used two different sets of bonds depending on their maturities. The set consists of bonds with maturities; (a): $1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10, and we call this set 'short - medium maturity'. (b): $20, 21, 22, 23, 24, 25, 26, 27, 28, 29$ and 30, and we call this set 'long-term maturity'. Thus we consider 21 German Government bonds for our analysis. We obtained daily data from Germany's central bank; there are missing values in this daily data, especially during weekends. We obtained monthly average data, to avoid this problem of missing data. The data span from August 2002 to December 2020, making it 220 months per each bond type, which is an amply adequate number of data points per bond type in terms of our statistical needs. Considering both the time series and cross-sectional data, we have 2200 data points for short-medium maturity and 2420 data points for long-term maturity. We use the yield spot rates from the yield curve provided by the Bundesbank.

While bonds with multiple coupons are transacted in the bond market, we base our analysis on zero-coupon bonds, as the most intuitive way to analyze the effect of diversification with respect to different times to maturity (see, e.g., Korn and Kozoil, 2006; Puhle, 2008). We consider two different investment strategies per set for our investor (she). Firstly, the investor considers one year investment horizon (very short investment). In this case the one-year zero coupon bond becomes a riskless bond and there is no re-balancing strategy. Secondly, she considers a medium term investment of 5 years. That is all zero-coupon bonds with maturities below the investment horizon are reinvested as all cash flows that are received before the investment horizon are reinvested at the current spot rate until T. On the second set of long-term bond maturities, the investor considers 20-year and 25-year investment horizons.

To determine the values of the expected return and variance-covariance matrix of returns on zero coupon bonds, the investor uses the multi-factor term structure model of the OU (Vasicek) type analyzed in Section 3.1. The computation of these risk-return portfolio moments depends on the unknown model parameters in the term structure model. To consistently estimate these parameters accross bond maturirties, we combine maximum likelihood estimator (MLE) with the Kalman filter derived in (Mawonike et al., 202) which incorporates the observed time series data. The OU (Vasicek) model is expressed in state space form (see Equation (36) and (37). The State space formulation allows the use of the Kalman filter algorithm in order to extract/reconstruct the latent factors X (see e.g. Babbs and Nowman, 1999; Bolder, 2001, Mawonike et al., 202).

TABLE 1. Vasicek: parameter estimation based on one- two- and three-factor models with bond maturities: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

Parameter	One-factor	two-factor	Three-factor
μ	0.0255	0.0292	0.0243
ξ_1	0.3245	0.2554	0.2966
ξ_2		0.3017	0.3288
ξ_3			0.2860
c_1	0.0256	0.0208	0.0211
c_2		0.0228	0.0232
c_3			0.0191
λ_1	0.1172	0.1151	0.1124
λ_2		0.1073	0.0963
λ_3			0.0954
ρ_1		0.9396	0.9531
ρ_2			0.8077
ρ_3			0.8247
L	1033.4	1048.8	1047.0
$\sigma_{\epsilon 2}$	0.00267	0.00019	0.00018
$\sigma_{\epsilon 3}$	0.00268	0.00021	0.00019
$\sigma_{\epsilon 4}$	0.00266	0.00021	0.00018
$\sigma_{\epsilon 5}$	0.00263	0.00021	0.00020
$\sigma_{\epsilon 6}$	0.00259	0.00020	0.00019
$\sigma_{\epsilon 7}$	0.00092	0.00020	0.00019
$\sigma_{\epsilon 8}$	0.00251	0.00020	0.00018
$\sigma_{\epsilon 9}$	0.00248	0.00021	0.00018
$\sigma_{\epsilon 10}$	0.00246	0.00021	0.00019

Table 1 shows estimates of the Vasicek model parameters. The long-term mean rate increases when the number of economic factors increases by one and then drops afterwards. The reversion rate decreases as the number of factors increases. The reversion rate is significantly relevant to our mean-variance model as it provides some degree of state dependence and predictability of expected returns. The volatility of the Vasicek model also decreases with an increase in the number of state variables. The latent factors X are highly positively correlated with the minimum correlation being 0.8247. In addition, the maximum likelihood function L increases when more state factors are incorporated into the model. The standard deviation of the measurement errors on the risky zero-coupon bonds indicates that the yield basis points (hundredths of a percent) fluctuate between 1.8 and 27. This means that the rise and fall of bond yield rates is minimal.

Table 2 shows estimates of the multi-factor Vasicek model. Long-term zero-coupon bonds with time to maturity: $20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30$ were used. Standard deviations of measurement errors were also obtained. The likelihood function values L for long-term bonds are slightly higher than those in Table 1 but the statistical error basis points have dropped slightly, and are thus even more minimal.

3.3. **Long-term investment of** 25 **years.** Here the investor's problem is to invest in government zero-coupon bonds with very long maturities over a 25-year investment period, selecting from the following zero-coupon bond maturities: $20, 21, 22, \ldots, 30$. That is, the investor selects from eleven zero-coupon bonds, among which

TABLE 2. Vasicek: Parameter estimation based on one-two-and three-factor models with bond maturities: $20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30$. Here the shorthand notation $\sigma_{\epsilon i}$ represents the estimated standard deviation $\sigma(\epsilon_{T_i})$.

Parameter	**One-factor**	**two-factor**	**Three-factor**
μ	0.0353	0.0405	0.0369
ξ_1	0.3245	0.2551	0.2523
ξ_2		0.3011	0.3297
ξ_3			0.2846
c_1	0.0226	0.0184	0.0165
c_2		0.0200	0.0203
c_3			0.0167
λ_1	0.1172	0.1152	0.1125
λ_2		0.1072	0.0999
λ_3			0.0941
ρ_1		0.9396	0.9515
ρ_2			0.8050
ρ_3			0.8209
L	1042	1057	1052
$\sigma_{\epsilon 21}$	0.00236	0.00019	0.00018
$\sigma_{\epsilon 22}$	0.00236	0.00019	0.00019
$\sigma_{\epsilon 23}$	0.00236	0.00018	0.00018
$\sigma_{\epsilon 24}$	0.00236	0.00018	0.00020
$\sigma_{\epsilon 25}$	0.00235	0.00019	0.00019
$\sigma_{\epsilon 26}$	0.00237	0.00019	0.00019
$\sigma_{\epsilon 27}$	0.00237	0.00018	0.00018
$\sigma_{\epsilon 28}$	0.00238	0.00019	0.00018
$\sigma_{\epsilon 29}$	0.00237	0.00019	0.00019
$\sigma_{\epsilon 30}$	0.00238	0.00019	0.00018

ten are risky and one is risk-free (maturity 25 years). The investor is fully aware of the probability distribution of long-term bonds and the risk associated with them. In Table 9 are spot prices of long-term maturity zero-coupon bonds, based on our market-calibrated parameters, in our three models. These spot prices diminish with further maturities, as it should be according to ordinary term-structure intuition. More specifically, these bond prices are much smaller than those in Table 3 for much shorter maturities, a reflection, out of our modeling, of the higher levels of uncertainty, the realities of partial observation in the long-term bond markets, as well as economic distortion of prices and yields in these markets. Pricing models of long-term maturity zero-coupon bonds should be able to filter the noise coming from the attenuation of signals and economic and political ambiguity, with an acknowledgement of the additional aggregate uncertainty incorporated into the prices.

Table 10 presents the expected zero-coupon bond prices and standard deviations of long-term maturity bonds invested over our 25-year planning horizon. We note that expected bond prices are slightly higher than those in the one-year and five-year investment periods. The two-factor Vasicek model gives us highest expected values and smallest standard deviations as compared to other model variants. This is consistent with the one-year and five-year investment horizons.

Table 11 presents the Vasicek expected zero-coupon holding period returns over 25-year investment period of long-term maturity bonds. These returns are very high as bonds are invested for a long period. The least expected percentage zero bond return is 448.5% and the highest is 679.3%. That means if an investor decided to invest in a 20-maturity zero-coupon bond today over a 25-year investment period, she would be expecting to be rewarded over 450% at the end of the $25th$ period under the one-factor Vasicek model. Results show that the two-factor Vasicek model gives higher returns as compared to other the two other models, showing again that one is best advised to take a balanced view of how many economic indicators should be used in portfolio allocation in the bond markets.

We also calculated tangency portfolios for the three Vasicek models on risky long-term maturity zero bonds. Here a 25-year zero bond is considered a free-risk bond.

$$\mathcal{N}_{tan} = \underbrace{\begin{pmatrix} 13.6741 \\ 4.6604 \\ 31.3523 \\ -22.8920 \\ 23.1610 \\ -2.3390 \\ -10.6770 \\ -0.5735 \\ -79.2575 \\ 19.3764 \end{pmatrix}}_{\textbf{1-factor}}, \underbrace{\begin{pmatrix} 587.5 \\ 513.3 \\ 102.8 \\ -1216.2 \\ 1272.3 \\ -677.2 \\ 717.1 \\ 29.4 \\ 345.0 \\ -756.9 \end{pmatrix}}_{\textbf{2-factor}}, \underbrace{\begin{pmatrix} -223.0498 \\ 66.4082 \\ -33.1601 \\ 189.2082 \\ 82.8295 \\ 476.8828 \\ -489.6733 \\ 381.7088 \\ -698.8905 \\ 239.9164 \end{pmatrix}}_{\textbf{3-factor}}$$

Results from the tangency portfolio shows high figures especially on the two- and three-factor models, particularly the 23, 24, 27 and 30-year zero bonds under the two-factor model. Over a 25-year investment period of long-term maturity zero bonds, we expect higher values both positives (buying) and negatives (short-selling) on the tangency portfolio because the investment decision is made today and the investment spans over a very long time. We strictly adhere to the model assumption that there is no re-balancing to those zero-coupon bonds with maturities greater than the planning horizon; that assumption of remaining static amplifies the market leverage in optimal allocation under a long-term planning horizon.

In Table 12, we present four different zero-coupon risky portfolios. In each bond portfolio, the risky-free bond (25-year maturity zero bond) is included. We start with one risky zero bond with time maturity of 27, followed by two risky bonds with maturities 24 and 30, the average of their maturities is equal to one risky bond in the first portfolio, to make fair comparisons as noted earlier in this section. The same holds for the two other portfolios, with maturities 24, 27 and 30, and with all risky bonds in our basket. Table 12 reports expected portfolio returns, expected portfolio terminal wealth, portfolio Sharpe Ratio and the portfolio risk (standard deviation).

Results from Table 12 indicate that the expected returns from a 25-year planning horizon, which vary from 491.6% to 655.1%, depend largely on what model is used, the two-factor model being preferred, and are otherwise highly consistent across portfolios. The same holds for the portfolio terminal wealth. Its increase over the

Table 3. Vasicek: Spot prices of long-term maturity zero coupon bonds

1-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$P(0,T_i)$	0.2454	0.2281	0.2120	0.1970	0.1831	0.1702	0.1582	0.1471	0.1367	0.1270	0.1181
2-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$P(0,T_i)$	0.2106	0.1939	0.1785	0.1643	0.1512	0.1392	0.1281	0.1180	0.1086	0.100	0.0920
3-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$P(0,T_i)$	0.2477	0.2297	0.2130	0.1975	0.1832	0.1698	0.1575	0.1460	0.1354	0.1255	0.1164

Table 4. Vasicek: Expected zero coupon bond prices and standard deviations over 25-year investment period

1-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$\mathbb{E}_0[P(25,T_i)]$	1.3689	1.2749	1.1894	1.1136	1.0495	1.000	0.9566	0.9016	0.8441	0.7875	0.7334
$\mathrm{std}[P(25,T_i)]$	0.0038	0.0033	0.0028	0.0025	0.0022	0.000	0.0018	0.0016	0.0014	0.0012	0.0011
2-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$\mathbb{E}_0[P(25,T_i)]$	1.4032	1.3004	1.2085	1.1277	1.0585	1.000	0.9488	0.8910	0.8319	0.7735	0,7170
$\mathrm{std}[P(25,T_i)]$	0.0042	0.0030	0.0019	0.0010	0.0003	0.000	0.0002	0.0006	0.0009	0.0011	0.0011
3-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
$\mathbb{E}_0[P(25,T_i)]$	1.3587	1.2711	1.1918	1.1206	1.0573	1.000	0.9500	0.8972	0.8445	0.7926	0.7421
$\mathrm{std}[P(1,T_i)]$	0.0114	0.0087	0.0063	0.0043	0.0028	0.000	0.0022	0.0028	0.0032	0.0034	0.0034

TABLE 5. Vasicek: Expected zero coupon bond holding period returns over 25-year investment horizon

1-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
ExpReturn(%)	457.8	458.9	461.0	465.3	473.2	487.5	504.7	512.9	517.5	520.0	521.0
2-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
ExpReturn(%)	566.3	570.7	577.0	586.4	600.0	618.4	640.7	655.1	666.0	673.5	679.3
3-factor											
T_i	20	21	22	23	24	25	26	27	28	29	30
ExpReturn(%)	448.5	453.4	459.5	467.4	477.1	488.9	503.2	514.5	523.7	531.6	537.5

TABLE 6. Expected returns, Expected terminal wealth, Sharpe Ratios, and risk, over 25-year investment period

Risky bonds(yrs)	**Exp Returns (%)**	$E[\hbar_{25}]$	**Sharpe Ratio**	**Risk (%)**
		1-factor		
27	500.2	5.894	2.23	5.7
24;30	497.1	5.882	1.68	5.7
24;27;30	497.3	5.892	1.72	5.7
20;21;22;23;24;26;27;28;29;30	489.2	5.868	0.26	6.5
		2-factor		
27	655.1	7.206	4.89	7.5
24;30	639.7	7.198	3.80	6.0
24;27;30	638.4	7.208	3.28	6.1
20;21;22;23;24;26;27;28;29;30	621.5	7.172	0.46	6.7
		3-factor		
27	514.5	5.908	4.83	5.3
24;30	507.3	5.907	3.23	5.7
24;27;30	504.6	5.913	2.34	6.7
20;21;22;23;24;26;27;28;29;30	491.6	5.880	0.40	6.8

5-year investment horizon is quite notable. The risk profiles are very attractive for all models, with very high Sharpe ratios except for the extremely diversified portfolio (all maturities) which is not to be recommended. Again, the two-factor model gives us the best portfolio profiles. As suggested by Korn and Koizol (2006) and also by Puhle (2008), a portfolio with risky zero-coupon bonds roughly equal in number to the number of state factors in the model, appears to gives us better risk profiles. Though our results do not confirm that the numbers should be exactly equal, we clearly recommend against using a much higher number of bonds compared to the number of factors.

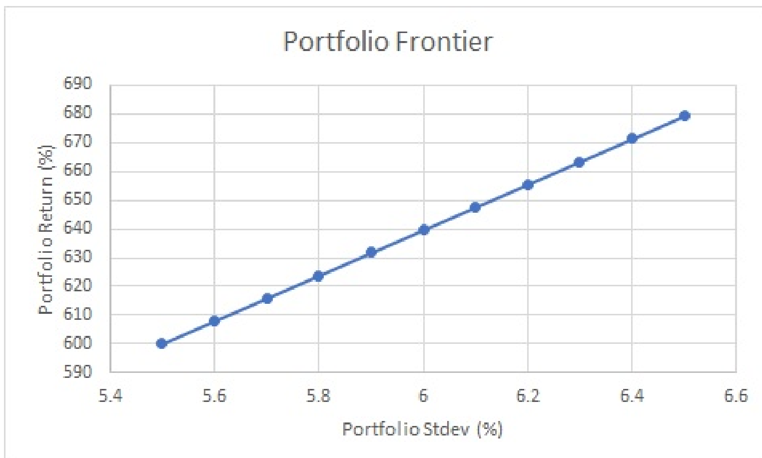

FIGURE 1. Portfolio frontier of two risky zero-coupon bonds; 24-year and 30-year under two-factor Vasicek model invested over 25-year investment period

Figure 6 presents the portfolio frontier of two risky zero coupon bonds invested over 25 years under the two-factor Vasicek model. The straight-line aspect of this frontier is due to the portfolio containing two risky assets only, similarly to what we presented in the one-year horizon case.

3.4. **Short-term investment of one year.** Spot interest rates of different maturities are only functions of the short rate in this one-year maturity case. In the Vasicek model, this function is affine. Risky zero coupon bonds are known for being highly positively correlated. We present the correlation matrix ρ of risky zero bonds with maturities: $2, 3, 4, 5, 6, 7, 8, 9, 10$. The matrix shows that risky zero-coupon bonds with a short investment period of one year are highly positively correlated.

$$\rho = \begin{pmatrix} 1 & 0.998 & 0.994 & 0.989 & 0.983 & 0.976 & 0.970 & 0.964 & 0.957 \\ 0.998 & 1 & 0.999 & 0.996 & 0.992 & 0.987 & 0.982 & 0.977 & 0.972 \\ 0.994 & 0.999 & 1 & 0.999 & 0.997 & 0.993 & 0.990 & 0.986 & 0.982 \\ 0.989 & 0.996 & 0.999 & 1 & 0.999 & 0.997 & 0.995 & 0.992 & 0.989 \\ 0.983 & 0.992 & 0.997 & 0.999 & 1 & 0.999 & 0.998 & 0.996 & 0.994 \\ 0.976 & 0.987 & 0.993 & 0.997 & 0.999 & 1 & 0.999 & 0.998 & 0.997 \\ 0.970 & 0.982 & 0.990 & 0.995 & 0.998 & 0.999 & 1 & 0.999 & 0.999 \\ 0.964 & 0.977 & 0.986 & 0.992 & 0.996 & 0.998 & 0.999 & 1 & 0.999 \\ 0.957 & 0.972 & 0.982 & 0.989 & 0.994 & 0.997 & 0.999 & 0.999 & 1 \end{pmatrix}$$

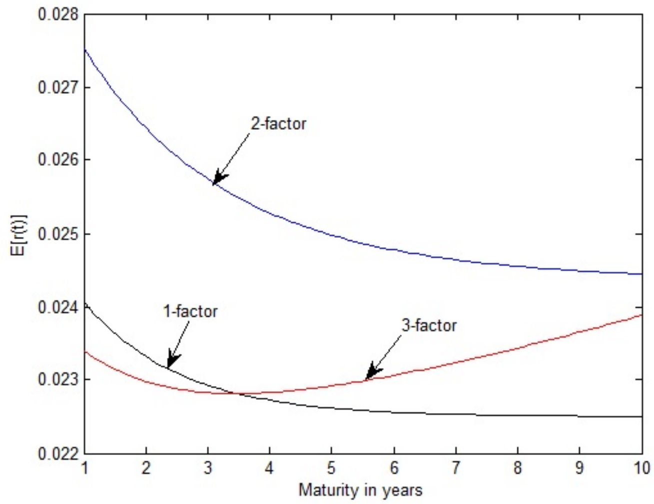

FIGURE 2. Vasicek: Expectation structure of r(T) over one-year investment horizon

We present our Vasicek short rate model in state space model where the short $r(t)$ is represented by J economic state factors $X_j(t)$. The distribution of these state variables is the same as that of $r(t)$. Since $r(t)$ is an affine function of $X(t)$, and X is Gaussian process, then so is r, and we find, using (31), that: $\mathbb{E}_0[r(1)] = 0.0242$, $\mathbb{E}_0[r(1)] = 0.0275$ and $\mathbb{E}_0[r(1)] = 0.0242$ for one-factor, two-factor and three-factor model respectively. Furthermore, the standard deviations of $r(1)$ are 0.019, 0.0194 and 0.017 for one-, two- and three-factor model respectively. Figure 1 shows the graph of expected values of $r(T)$. It indicates that the 2-factor model produce higher expected values of the short rate $r(t)$, followed by the 1-factor model. The expected values decrease as the discount bond maturity increases. This situation is expected since expected values of individual state factors decrease as the number of factors increases, i.e., $\mathbb{E}_0[X_1(T)] > \mathbb{E}_0[X_2(T)] > \mathbb{E}_0[X_3(T)]$. The value or contribution of the economic factors appears to decay exponentially with the increasing number of factors. In the 3-factor model, we see the graph increases after a few maturities. This upward movement is caused by the third factor in the model which turns to be negative in our estimation, thus increasing the total value of the expectation (see Equation (24)). Figure 2 shows the volatility of the short rate $r(t)$ measured in the one-, two-, and three-factor models on different bond maturities. At one-year maturity ($T = 1$), $\sqrt{\text{Var}}[r(1)] = 0.0192$ for 1-factor model, $\sqrt{\text{Var}}[r(1)] = 0.0273$ for a 2-factor and $\sqrt{\text{Var}}[r(1)] = 0.0305$ for the 3-factor model. The volatility of $r(T)$ increases with an increase in the numbers of state factors in the model. The longer the maturity of a discount bond, the higher the volatility associated with it, though this moderates significantly for longer maturities.

Figure 3 graphs the zero-coupon bond prices over one year investment planning horizon with respect to different Vasicek factor models. The zero-coupon bond prices decrease rapidly as the bond maturity increases. The one-factor and three-factor models have higher bond prices.

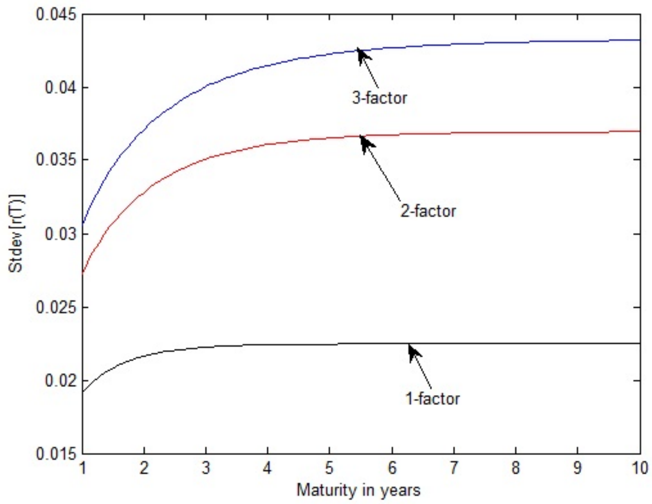

FIGURE 3. Vasicek: Volatility structure of r(T) over one-year investment horizon

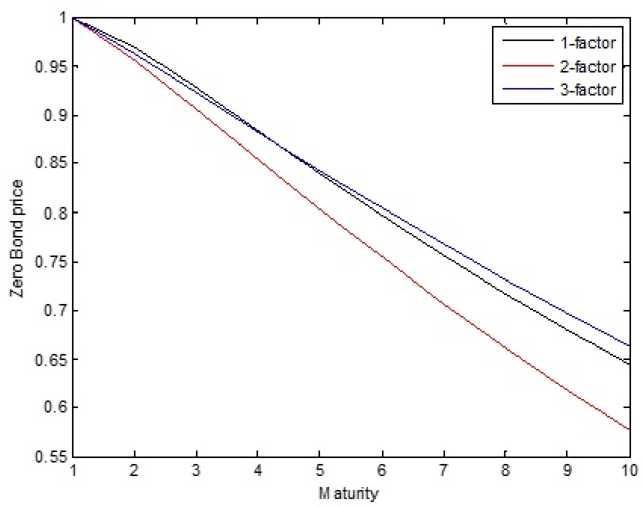

FIGURE 4. Vasicek: Zero coupon bond price over one year investment period

Table 3 shows the same information (expected bond prices) and includes standard deviations over one year investment horizon using the Vasicek multi-factor model, where we note a seven-to-ten-fold difference in size between the expected bond prices and their standard deviations, with lower number of factors generally resulting in lower standard deviations.

We can also compute the current spot prices $P(0,T_i)$ and use them to calculate the continuously compounded expected returns on each zero bond over the next period, via the formula below in Equation (48) for expected holding bond returns, with results collected in Table 4. The two-factor model results in significantly higher returns than the other two models, the one-factor model yielding the lowest returns.

$$r_i = \frac{\mathbb{E}_0^{\mathbb{P}}\left(P(T, T_i)\right)}{P(0, T_i)} - 1 \tag{39}$$

We also draw attention to the mean-variance efficient portfolios. We let the initial wealth $\hbar_0 = 1$ and calculate the tangency portfolio for three model variants.

$$\mathcal{N}_{tan} = \underbrace{\begin{pmatrix} -3.7614 \\ 23.1263 \\ 1.4057 \\ 19.8599 \\ -41.7431 \\ 30.0286 \\ -46.7030 \\ -25.5748 \\ 41.2788 \end{pmatrix}}_{\text{1-factor}}, \text{ or } \underbrace{\begin{pmatrix} -0.1692 \\ 3.8492 \\ 1.5905 \\ -0.3968 \\ -0.1588 \\ -3.3247 \\ -3.8359 \\ -3.1119 \\ 6.2418 \end{pmatrix}}_{\text{2-factor}}, \text{ or } \underbrace{\begin{pmatrix} 0.4856 \\ 4.8729 \\ -2.9893 \\ -4.4380 \\ 5.2781 \\ -11.7924 \\ 12.8108 \\ -0.9761 \\ -2.1595 \end{pmatrix}}_{\text{3-factor}}$$

The tangency portfolios of zero bonds contain long and short positions primarily due to high correlations of bonds, especially immediately preceding and following the bond's maturity (Puhle, 2008). High correlations means that zero bonds are highly inter-dependent resulting in instability in the optimization scheme. The scheme tries to give way to arbitrage opportunities by purchasing one kind of zero bond and shorting the other. For instance, in the one-factor model, the investor is to buy 23.12 units of the 3-year zero-coupon bonds and short 2-year zero-coupon bonds by 3.7 units. The one-factor model exhibits higher long and short positions than the other models. This corresponds to higher values in the Tobin fund. The long positions in all models vary from 0.07 to 41.28 while the short positions range from -0.19 to -46.7, with the two-factor and three-factor model positions not exceeding 13 in absolute value. We view these ranges as reasonable given the high correlations of bonds, meaning that they represent levels of leverage which are not unrealistic, especially for the two- and three-factor models. This is an indication that one should presumably not attempt to reduce the bond market's financial economic indicators to a single factor, even at a one-year horizon, when forming a portfolio of such bonds.

3.5. **Efficient portfolios with risky returns.** In this section, we still consider the investment planning horizon of one year. We construct efficient portfolios for this short investment horizon, shown in Table 5, for 10 different zero-coupon bonds with maturities ranging from 1-year to 10-years. Following Korn and Koziol (2006), we grouped these bonds in different portfolios. We thus have at our disposal 9 risky zero-coupon bonds and one risk-free zero-coupon bond with time to maturity of one year. The first combination reported in Table 5 consists of two bonds, one risk-less and a 7-year zero-coupon bond. The second portfolio consists of one risk-free and two risky zero-coupon bonds: a 4-year bond and a 10-year bond. Table 5 can be consulted for four other portfolio we consider, and various statistics under our usual three models with one, two, and three factors. Following the argument in Korn and Kozoil (2006), to compare bond combinations directly, the average maturity of risky bonds all have to be equal, thus equal to the time to maturity of one risky bond in the first portfolio which is 7. To calculate each portfolio's expected terminal wealth $E[h_1]$ and expected return, we allocated equal portfolio weights to each risky zero-coupon bond, for instance, two bonds contribute each 0.5 weight.

We report also in Table 5, the Sharpe ratio and the risk or standard deviation in each efficient portfolio. According to Sharpe (1966), a Sharpe ratio is the measure of risk adjusted return of a financial portfolio (excess return per unit volatility). Results indicate higher Sharpe ratios in one-factor model than the other models. The Sharpe ratios range from 0.09 to 0.78 considering all model variants. The Sharpe ratios in the one-factor and two-factor models are favourable as they are more than the traditional 0.3 minimum threshold (Sharpe, 1966), which is appreciable considering that risky zero-coupon bonds used in these combinations are highly correlated, which would typically imply higher risk (less benefit of diversification) than less correlated assets. As we can see from the table, higher risks are associated with lower Sharpe ratios. The three-factor model produces higher risks and very low Sharpe ratios on each portfolio, which do not meet the minimum requirements of the aforementioned threshold. The investor sets the initial wealth to 1 in her portfolio allocation problem, the expected terminal wealth at the end of the one-year investment period ranges from 1.039 to 1.060. The two-factor model produces higher expected terminal wealth and expected portfolio returns with moderate portfolio risks. The Sharpe ratio improves in the one-factor. In addition, the portfolio expected terminal wealth and the Sharpe ratio improve as one increases the number of risky bonds in the portfolio. On the other hand, portfolio risk increases with the increase of bonds in the combinations. The question is whether it is worth combining more risky bonds to increase portfolio performance, and the answer appears to be that on the contrary, low expected portfolio returns and expected terminal wealth come as a result of having more risky bonds, i.e. a riskier portfolio.

Figure 4 shows the portfolio frontier of two risky zero coupon bonds; 4-year and 10-year maturities invested over the one-year planning horizon, under a one-factor Vasicek (OU) model. The blue dotted line represents the inefficient portfolio frontier and the orange line indicates the efficient portfolio frontier. On the bottom end of the frontier, the portfolio risk is very much minimised but the portfolio return is low. On the other hand, at the top end of the frontier, the investor receives higher returns but accepts higher portfolio risks.

3.6. **Medium-term investment of 5 years.** The correlations of risky bonds measured over the 5-year investment vary from essentially zero (e.g. -0.001) to 0.99, as can be seen in the correlation matrix ρ given here for risky bonds with maturities.$1, 2, 3, 4, 6, 7, 8, 9$ and 10 years:

$$\rho = \begin{pmatrix} 1 & 0.329 & -0.214 & -0.207 & 0.131 & 0.130 & 0.131 & 0.130 & 0.128 \\ 0.329 & 1 & -0.112 & -0.103 & 0.084 & 0.086 & 0.089 & 0.089 & 0.091 \\ -0.214 & -0.112 & 1 & 0.023 & -0.007 & -0.001 & 0.004 & 0.010 & 0.014 \\ -0.207 & -0.103 & 0.023 & 1 & 0.121 & 0.126 & 0.132 & 0.137 & 0.141 \\ 0.131 & 0.084 & -0.007 & 0.121 & 1 & 0.99 & 0.99 & 0.99 & 0.99 \\ 0.130 & 0.086 & -0.001 & 0.126 & 0.99 & 1 & 0.99 & 0.99 & 0.99 \\ 0.131 & 0.089 & 0.004 & 0.132 & 0.99 & 0.99 & 1 & 0.99 & 0.99 \\ 0.130 & 0.089 & 0.010 & 0.137 & 0.99 & 0.99 & 0.99 & 1 & 0.99 \\ 0.128 & 0.091 & 0.014 & 0.141 & 0.99 & 0.99 & 0.99 & 0.99 & 1 \end{pmatrix}$$

TABLE 7. Vasicek: Expected zero coupon bond prices and standard deviations over one year investment period

1-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(1,T_i)]$	1	0.9694	0.9280	0.8840	0.8400	0.7971	0.7559	0.7165	0.6791	0.6436
$\text{std}[P(1,T_i)]$	0.0000	0.0755	0.0721	0.0686	0.0656	0.0616	0.0592	0.0556	0.0529	0.0500
2-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(1,T_i)]$	1	0.9563	0.9066	0.8555	0.8046	0.7550	0.7073	0.6619	0.6188	0.5783
$\text{std}[P(1,T_i)]$	0.0000	0.0693	0.0787	0.0848	0.0883	0.0894	0.0883	0.0860	0.0831	0.0800
3-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(1,T_i)]$	1	0.9626	0.9231	0.8838	0.8449	0.8068	0.7696	0.7336	0.6988	0.6653
$\text{std}[P(1,T_i)]$	0.0000	0.0748	0.0927	0.1063	0.1140	0.1183	0.1191	0.1140	0.1170	0.1127

TABLE 8. Vasicek: Expected zero coupon bond holding period returns over one - year investment horizon

1-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	3.1660	4.4612	4.9773	5.2381	5.3820	5.4505	5.4990	5.5072	5.5158	5.5428
2-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	4.5697	5.4820	5.9731	6.3261	6.5695	6.7440	6.8590	6.9651	7.0032	7.0728
3-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	3.8853	4.2791	4.4467	4.6041	4.7224	4.8337	4.9073	4.9800	5.0353	5.0861

Note that highly correlated zero-coupon bonds are exactly those pairs with both maturities greater than 5 years, while other pairs are more correlated when one bond has shorter maturity.

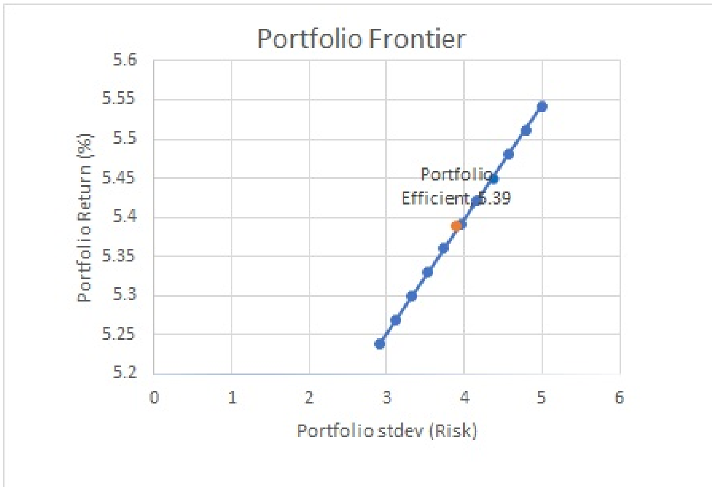

FIGURE 5. Portfolio frontier of two risky zero coupon bonds; 4-year and 10-year, under one-factor Vasicek model invested over one-year period

Table 6 shows the expected zero-coupon bond prices over a 5-year investment period, for the one-, two-, and three-factor models, and their standard deviations. Bonds with maturities less than the planning investment horizon have higher bond prices and higher standard deviations. That is short maturity bonds with maturities less than the investment period are exposed to more risk than bond with maturities higher than the investment horizon. The expected bond holding returns over the 5-year investment period are given in Table 7. Here the expected bond returns have significantly increased as compared to the one-year investments discussed in the previous subsections. The least zero bond holding return is 22.6%, the highest is 40.0%. The two-factor model gives us highest bond returns among the three models. Longer-maturity bonds have higher expected holding returns but have low expected zero-coupon bond spot and continuously compounded prices.

Letting the initial wealth $\hbar_0 = 1$, we calculate the tangency portfolio on all three different Vasicek models:

$$\mathcal{N}_{tan} = \underbrace{\begin{pmatrix} -4.8281 \\ -55.7762 \\ 19.2126 \\ 80.5810 \\ 41.8448 \\ -63.3895 \\ -33.2324 \\ -22.8307 \\ 39.3304 \end{pmatrix}}_{\textbf{1-factor}}, \underbrace{\begin{pmatrix} -371.5681 \\ 25.9509 \\ 720.1300 \\ -383.5450 \\ 148.3111 \\ 446.8819 \\ -307.6525 \\ -784.3152 \\ 498.0267 \end{pmatrix}}_{\textbf{2-factor}}, \underbrace{\begin{pmatrix} -31.6031 \\ 256.0999 \\ -290.2897 \\ 71.6459 \\ -241.1885 \\ 240.1848 \\ 65.0307 \\ -75.5351 \\ 8.3501 \end{pmatrix}}_{\textbf{3-factor}}$$

Table 9. Expected returns, Expected terminal wealth and Sharpe Ratios of risky zero-coupon bond portfolios over one-year investment period

Risky bonds(yrs)	**Exp Returns (%)**	$E[\hbar_1]$	**Sharpe Ratio**	**Risk (%)**
		1-factor		
7	5.4990	1.039	0.78	3.0
4;10	5.3900	1.042	0.57	3.9
4;7;10	5.3720	1.043	0.50	4.4
3;7;8;10	5.3820	1.043	0.47	4.7
4;6;7;8;10	5.4475	1.045	0.46	5.0
2;3;...;10	5.1885	1.047	0.47	6.2
		2-factor		
7	6.8590	1.054	0.37	6.2
4;10	6.6990	1.055	0.39	5.5
4;7;10	6.6851	1.057	0.34	6.3
3;7;8;10	6.7270	1.057	0.32	6.7
4;6;7;8;10	6.7934	1.059	0.30	7.3
2;3;...;10	6.4894	1.060	0.24	8.2
		3-factor		
7	4.9073	1.043	0.09	11.9
4;10	4.8451	1.044	0.08	7.3
4;7;10	4.8172	1.045	0.11	8.5
3;7;8;10	4.8550	1.045	0.11	8.9
4;6;7;8;10	4.8822	1.046	0.11	9.8
2;3;...;10	4.7184	1.046	0.09	10.7

Results from the tangency portfolios suggest that, using one-factor model, the investor should short sell significant units of 1-year, 2-year, 7-year, 8-year and 9-year zero-coupon bonds, whle buying similarly significant units of the other four bonds. Similar results hold for the two-factor model, though the amounts of bonds are greater by more than an order of magnitude, a highly leveraged portfolio. The three-factor model suggest a similar level of investments as the two-factor model, slightly less leveraged than with two factors. Which bonds are shorted is not consistent across models.

Table 8 shows the expected zero bond holding returns, the expected terminal wealth of the risky bond portfolio, the Sharpe ratio, and the portfolio risk over the 5-year investment period, for a $1 investment. In each risky bond portfolio, there is a risk-less 5-year zero-coupon bond.

Expected holding zero-coupon bond returns have been at least multiplied five for the 5-year investment period compared to the one-year investment period. Considering all portfolios and model variants, expected returns range from 24.89% to 37.0%, with the two-factor model recording the highest bond return percentages, and higher Sharpe ratios, while risk and expected terminal wealth are largely consistent across all models and all portfolios. Also, the expected terminal wealth have significantly increased when the investment period shifts to 5 years. When one risky zero-coupon bond is combined with a single risk-free zero-coupon bond, we get the highest Sharpe ratio (or nearly highest, for 3-factor model). Judging by our Sharpe ratios our portfolios seem to be less attractive when the number of risky bonds is close to or equal to the number of factors in the model. This modeling inconsistency was identified to some extent previously in the literature: Puhle, 2008; Korn and

Kozoil, 2006; Wilhelm, 1992. The greater the number of risky zero-coupon bonds are included in a portfolio, the more the risk is potentially included. Therefore it is imperative to reduce this number, while not not compromising on diversification. For instance, when the investor includes all risky zero-coupon bonds available at her disposal in one portfolio, the expected percentage return shrinks and the Sharpe ratio reduces, due to high portfolio risk (standard deviation). Overall, the two-factor model seem to produce more attractive portfolio risk profiles, indicating that trying to identify a higher number of financial economic factors may produce more risk in a systematic portfolio allocation procedure.

Figure 5 shows the portfolio efficient frontier for five risky zero-coupon bonds (4, 6, 7, 8, and 10-year maturity zero-coupon bonds). The frontier is slightly curved because the portfolio contains only securities with similar characteristics and expected holding returns. The orange dot indicates the portfolio on the efficient frontier where risk is minimized. The portfolio frontier becomes less efficient as it shifts away from this orange point.

4. **Conclusions.** In this paper, we analyse a portfolio selection problem of default-free zero-coupon bonds with times to maturity varying from one year to thirty years. The portfolio is constructed based on the Markowitz's mean-variance approach of portfolio selection. The paper proposes the use of dynamic term structure models of the Vasicek (Ornstein-Uhlenbeck) type because of their flexibility, their analytic (albeit complex) tractability (closed-form expressions for zero-coupon bond prices), and they being accommodating to model reconstruction techniques like the Kalman filter. In particular, the paper proposes various multi-factor Vasicek models for analysis. Emphasis is on long-term maturity zero-coupon bonds in terms of our initial motivation. However short-term and medium-term maturity bonds are also considered in this research. Since the investor's main focus is on expected terminal portfolio wealth and variance of the expected portfolio return, her main objective is to minimize the variance of this expected terminal portfolio wealth given the expected terminal wealth and the budget (self-financing) constraint.

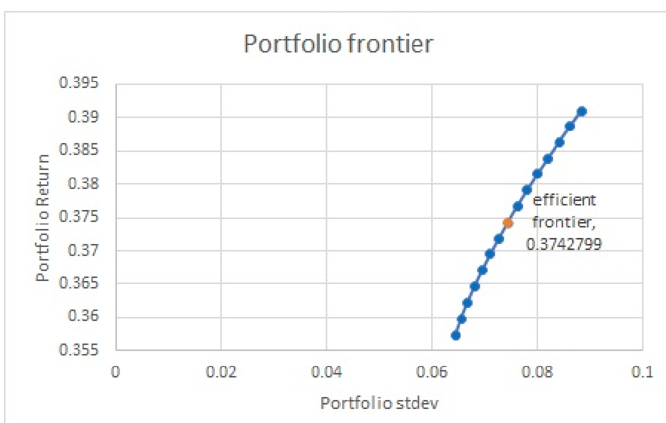

FIGURE 6. Portfolio frontier of five risky zero-coupon bonds; 4-year, 6-year, 7-year, 8-year and 10-year under two-factor Vasicek model invested over 5-year investment period

TABLE 10. Vasicek: Expected zero-coupon bond prices and standard deviations over 5-year investment period

1-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(5,T_i)]$	1.1976	1.1380	1.0841	1.0378	1.000	0.9694	0.9280	0.8840	0.8400	0.7971
$\text{std}[P(5,T_i)]$	0.0066	0.0062	0.0059	0.0057	0.000	0.0053	0.0051	0.0049	0.0046	0.0044
2-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(5,T_i)]$	1.2500	1.1751	1.1083	1.0502	1.000	0.9563	0.9066	0.8555	0.8046	0.7550
$\text{std}[P(5,T_i)]$	0.0052	0.0041	0.0030	0.0017	0.000	0.0016	0.0025	0.0031	0.0033	0.0035
3-factor										
T_i	1	2	3	4	5	6	7	8	9	10
$\mathbb{E}_0[P(5,T_i)]$	1.1926	1.1390	1.0893	1.0435	1.000	0.9626	0.9231	0.8838	0.8449	0.8068
$\text{std}[P(5,T_i)]$	0.0089	0.0077	0.0064	0.0052	0.000	0.0048	0.0055	0.0060	0.0063	0.0065

TABLE 11. Vasicek: Expected zero coupon bond holding period returns over five - year investment horizon

1-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	23.5	22.6	22.6	23.5	25.5	28.2	29.5	30.2	30.5	30.7
2-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	30.7	29.6	29.5	30.5	32.4	35.2	37.0	38.3	39.1	40.0
3-factor										
T_i	1	2	3	4	5	6	7	8	9	10
ExpReturn(%)	23.8	23.4	23.3	23.5	23.9	25.1	25.8	26.5	27.0	27.9

Table 12. Expected returns, Expected terminal wealth and Sharpe Ratios over 5-year investment period

Risky bonds(yrs)	**Exp Returns (%)**	$E[\hbar_5]$	**Sharpe Ratio**	**Risk (%)**
		1-factor		
7	29.5	1.271	0.58	6.9
4;10	27.1	1.263	0.23	7.1
4;7;10	27.6	1.268	0.30	7.0
3;7;8;10	28.25	1.271	0.38	7.2
4;6;7;8;10	28.42	1.273	0.41	7.1
1;2;3;...;10	26.54	1.261	0.14	75
		2-factor		
7	37.0	1.339	0.62	7.7
4;10	35.25	1.337	0.51	5.6
4;7;10	35.48	1.343	0.50	6.2
3;7;8;10	36.20	1.345	0.56	6.6
4;6;7;8;10	36.20	1.347	0.56	6.8
1;2;3;...;10	34.09	1.333	0.26	6.2
		3-factor		
7	25.8	1.246	0.26	7.4
4;10	25.70	1.250	0.31	5.9
4;7;10	25.48	1.251	0.22	7.1
3;7;8;10	25.88	1.252	0.26	7.5
4;6;7;8;10	25.76	1.252	0.24	7.9
1;2;3;...;10	24.89	1.247	0.16	6.2

Theoretical calculations, based on the affine term structure for bond price models with arbitrary numbers of Vasicek factors, yield explicit (closed-form) formulas for expected returns, variance-covariance matrices, expected terminal wealth, tangency portfolios as well as Tobin funds.

We calibrate these models by assuming some level of modeling ambiguity for the pricing formulas, in the form of statistical errors at the level of the factors; this leads naturally to a Kalman filtering method for estimating latent factor model parameters statistically via maximum likelihood. This is based on German Government bond data. We use these estimated models to test various portfolio strategies on one-year, five-year and twenty-five investment periods. The investor considers rebalancing strategies where the face value of zero-coupon bonds with maturities less than the investment period is reinvested at times less than the investment period at the current spot interest rate until the investment horizon. To understand the effect of varying the number of economic variables (state factors in the model), we consider models with one, two, and three latent Vasicek factors. We analyze the impact of different planning investment horizons on different portfolio strategies of both short and long-term maturity zero bonds. We report on the impact of different investment planning horizons on expected terminal portfolio wealth, Sharpe ratio (excess return per unit volatility) and other risk and reward indicators.

We report that risky zero-coupon bonds with a very short investment period (one year say) are highly positively correlated, while pairs of bonds with medium to long investment periods have correlations which can be much lower, often indistinguishable from zero. We find that bond portfolio risk profiles are attractive (e.g. from the standpoint of Sharpe ratios) under different investment planning horizons. However, the three-factor model's predictions are rejected (based on a minimum

Sharpe-ratio threshold of 0.3) in 11 out of 16 cases in all investment strategies. The Sharpe ratios however improve as the planning horizon increases to 25 years when long-term zero-coupon bonds are invested. Results also suggests that the investor should consider significant short-selling positions especially when investing in long-term maturity bonds over a long period of time. Short selling is encouraged in this paper's reported portfolios even though its negative effects are known when the dynamics on interest rates are volatile. The one-factor model produces attractive risk portfolio profiles. The two-factor model consistently ranks as the best of all three models in this respect. The two-factor model's predictions are rejected, via the 0.3 threshold on Sharpe ratio, in only 2 out of 16 cases. The one-factor model's predictions are rejected in 3 out of 16 cases. These numbers of rejected cases are reasonable, leading us to accept these two models for in- and out-of-sample predictions. This indicates that risky investments in the bond market can benefit from multi-factor risk modeling to some extent, but that considering more than two financial econometric factors in modeling is not recommended, particularly when one admits to some degree of modeling ambiguity. We note that under longer investments, the Sharpe ratios, the expected terminal portfolio wealth, and the expected holding portfolio returns, all increase significantly (e.g. when the investment horizon increases from one year to five years, the expected terminal portfolio wealth roughly increases by 30%.) This confirms that long-term risky investments in bond markets are attractive in terms of their high reward-to-risk profiles despite their inherent higher risk. On the modeling side, our study is largely consistent with reports in the literature that a portfolio strategy with a number of risky zero-coupon bonds in excess of the number of state model factors, is less attractive than those portfolios with a number of risky bonds roughly equal to the number of state variables.

Finally, some comments on future directions which lie outside the scope of this paper. Our portfolio selection models and methodology are restricted to zero-coupon bonds. In practice, it will be helpful also to handle coupon bonds. There are several short-rate term structure models which can be used in portfolio selection strategies (e.g. the Hull White, Cox, Ingersoll, Ross (CIR), and others). The Vasicek models used herein allow the convenience of tractable closed-form solutions, but some of these other models, which may require numerical, non-closed form, resolution, might produce better prediction performance, and/or lower uncertainty levels in the statistical estimation.

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