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Optimal reinsurance pricing with ambiguity aversion and relative performance concerns in the principal-agent model

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ABSTRACT

This paper first studies the optimal reinsurance problems for two competitive insurers and then studies the optimal reinsurance premium pricing problem for their common reinsurer by using the dynamic programming technique. The two insurers are subject to common insurance systematic risk. Each purchases proportional or excess-of-loss reinsurance for risk control. They aim to maximize the expected utilities of their relative terminal wealth. With the insurers' optimal reinsurance strategies, the reinsurer decides the reinsurance premiums for each insurer, also aiming to maximize the expected utility of her terminal wealth. Thus, the optimal reinsurance pricing problem is formulated as a Stackelberg game between two competitive insurers and a reinsurer, where the reinsurer is the leader, and the insurers are followers. Besides, all three players take model ambiguity into account. We characterize the optimal strategies for the insurers and the reinsurer and provide some numerical examples to show the impact of competition and model ambiguity on the pricing of reinsurance contracts.

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Reinsurance; dynamic programming; compound Poisson risk model; principal-agent model; model ambiguity

1. Introduction

With the development of the insurance business, in recent years, many companies enter the insurance market. In an increasingly complex environment, these companies face new sources of risk. On the one hand, these insurance companies call for efficient measures to reduce insurance risk and adapt their optimal portfolio strategy for capital investment to stay in business. On the other hand, they have to take efficient measures to fight for a market share against their competitors. This has attracted interest from researchers in actuarial mathematics for some years. The primary tool for insurance companies to control insurance risk remains reinsurance to reduce the shock of uncertain claims. The insurers also invest their surplus into financial markets to improve profits compared to holding pure cash positions. Consequently, for several decades, reinsurance and investment problems have been studied in the literature using various optimization criteria.[1]

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[1] We refer to Browne (1995), Bai & Guo (2010), Xu et al. (2008) and Gu et al. (2012), Zhao et al. (2013), Xu et al. (2017), Chen & Shen (2018), and Brachetta & Ceci (2019) for maximizing the expected utility of terminal surplus, to Chen et al. (2010) and Zhang

The above-cited works, and many others, only consider a single insurer's decision and ignore the interaction among insurers in the insurance and reinsurance markets. In practice, however, insurers tend to evaluate their performances relative to their peers, as established by some economic and sociological research, emphasizing the importance of relative concerns (see, for example, Demarzo et al. 2008).

Considering that different insurers have different risk-tolerance levels, many scholars have investigated insurers' relative performance concerns. Most of the problems are solved under the framework of non-zero-sum stochastic differential games. Bensoussan et al. (2014) study the non-zero-sum reinsurance and investment game between two insurers; Espinosa & Touzi (2015) discuss the optimal investment strategy and prove the existence of the non-stationary Nash equilibrium in an N-person game. Siu et al. (2017) study a class of non-zero-sum excess-of-loss reinsurance-investment games between two competitive insurers subject to systematic risk described by a general compound Poisson risk model. Deng et al. (2018) study the reinsurance and investment strategy with defaultable corporation bonds in a non-zero-sum stochastic differential game. Zhu et al. (2019) consider the optimal time-consistent investment and proportional reinsurance strategies for two mean-variance insurers subject to relative performance concerns. These works illustrate the importance of taking competition into account in making reinsurance and investment decisions. Nevertheless, these works all neglect to consider the reinsurer's point of view in discussing the optimal reinsurance problem. In fact, since the reinsurer typically has a monopoly position in the reinsurance market, one must presume that the reinsurer plays a critical role in designing any reinsurance contract. Typically, the reinsurer will decide the contract price of reinsurance, and at the very least, this reinsurance premium will be the subject of negotiation with each insurer. Each insurer's decision is then to determine what level of risk transfer they can afford, in the view of their own constraints, risk aversion levels, and concern about what their competition might be doing. Therefore, in this paper, we investigate the reinsurer's premium pricing optimization problem as well as the insurers' reinsurance optimization problem.

On the question of the joint interests of insurers and reinsurers, many scholars study this as an optimization problem, but only between one insurer and one reinsurer. For example, Hu et al. (2018a, 2018b) and Hu & Wang (2019) put one insurer and one reinsurer into the principal-agent model and discuss the optimal proportional reinsurance problem for the insurer and the optimal reinsurance premium pricing problem for the reinsurer. Gu et al. (2019) studied a similar optimization problem, which focuses on the optimal excess-of-loss reinsurance and investment problem in the principal-agent model. In a stochastic Stackelberg differential game setting, Chen & Shen (2018, 2019) depict the leader–follower relationship between reinsurer and insurer in the insurance market and analyze an optimal reinsurance strategy. Recently, Bai et al. (2022) investigate a hybrid stochastic differential reinsurance and investment game between one reinsurer and two insurers, including a stochastic Stackelberg differential subgame and a non-zero-sum stochastic differential subgame. This work is closest to our motivation. However, Bai et al. (2022) neglect modeling uncertainty (i.e. ambiguity), a form of uncertainty that cannot be measured accurately or hedged, but which is on the minds of all insurance professionals who wonder about the robustness of the risk models which they use to make their business decisions.

Ambiguity was introduced as a form of unmeasured uncertainty by Knight (1921). It has been adopted and developed as a way of addressing modeling uncertainty in portfolio allocation. Some scholars investigate modeling uncertainty on yields and other drift parameters in stochastic models for risky assets. For example, Maenhout (2004, 2006) assume ambiguity about the stock's expected return rate and applies the general robust control framework of Anderson et al. (2003) to the dynamic portfolio choice problem in finance, deriving closed-form expressions for optimal strategies for an inter-temporal consumption problem. Some scholars study model uncertainty on interest rates (e.g.

et al. (2016) for minimizing the probability of ruin, and to Bi & Guo (2013), Zeng et al. (2012, 2016), Zhang et al. (2017), and Chen & Shen (2019) for mean-variance criteria.

Flor & Larsen 2013), and some discuss model uncertainty on the inflation rates. We refer to Munk & Rubtsov (2014), who solve a stock-bond-cash portfolio problem by assuming that the investor is averse to ambiguity about the inflation model. Branger & Larsen (2013) and Jin et al. (2013) investigate the optimal investment problem with ambiguity induced by the uncertainty over the intensity of jumps in a market with event risk. It is worth noting that these works mainly focus on the drift of assets in financial markets. In recent years, the idea of model ambiguity was also applied to actuarial research. Some works consider the uncertainty of a diffusion risk model (see Yi et al. 2013, Hu et al. 2018b); some consider the uncertainty over the claim intensity of a classic risk model (the compound Poisson model), see Zeng et al. (2016) and Gu et al. (2017).

Motivated by the papers mentioned above and by the joint interest of two insurers and one reinsurer, we study the optimization problems for all three parties. First, we consider the reinsurance problem for two competing insurers who are subject to common shock (or systematic risk) from the insurance business. They purchase reinsurance for risk control and invest their surpluses at a fixed income rate. The aims are to maximize the expected utilities of their relative performances against their competitors' at a terminal time. Besides, we assume that the two insurers are averse to ambiguity (model uncertainty) about the intensity of common shock. We formulate the insurers' problem as a robust non-zero-sum game for the proportional reinsurance case and the excess-of-loss reinsurance case. The robustness is built against the model uncertainty mentioned above by adding an appropriate penalty term in the objective function and by considering the optimization problem within a worst-case scenario. The penalty compactifies the optimization on the space of models, ensuring a well-posed problem where the worst-case model can be computed. We characterize the insurers' strategies theoretically and numerically.

Second, given the insurers' optimal demand for risk control, we study the question of optimizing reinsurance premiums (i.e. the prices of reinsurance contracts) for the reinsurer. Like the insurers, the reinsurer is assumed to be averse to ambiguity about the intensity of common impact. She determines the reinsurance premiums dynamically for both insurers, and she invests the premium income into a risk-free asset. The aim is to maximize the expected utility of her terminal wealth. In line with Hu et al. (2018a, 2018b), Hu & Wang (2019), Gu et al. (2019), and Bai et al. (2022), we determine the optimal reinsurance premium under the principal-agent framework and, by applying the dynamic programming principle, we write down the Hamilton-Jacobi-Bellman (HJB) equation for the problem. Equilibrium strategies are considered for the cases of proportional and excess-of-loss reinsurance contracts. Again, robustness is achieved in the optimization space by using a penalty term to build a well-posed worst-case model. When the claims against the insurers are exponentially distributed, we obtain the optimal reinsurance premium numerically. Further numerical analyses are provided to illustrate the impact of competition, model uncertainty on the reinsurer's decision.

To our knowledge, we are the first to consider the robust optimal reinsurance and optimal reinsurance premium for two insurers and one reinsurer. Recently, several papers have also considered the optimal pricing of reinsurance contracts, see, e.g. Hu et al. (2018a, 2018b), Hu & Wang (2019), and Gu et al. (2019). However, these papers only consider the game between one insurer and one reinsurer. They cannot effectively reflect how competition in the insurance business impacts the pricing of reinsurance contracts. We think that this aspect is particularly important in view of the fact that reinsurers are often in a monopoly position, where the marketplace constraint on pricing cannot come merely from supply and demand. Our numerics illustrate clearly how reinsurance pricing responds to insurers' competitive behavior; see for instance the explanations surrounding Figure 6 in Section 3.

The outline of this paper is as follows. We introduce optimization problems for two insurers in Section 2. This section assumes that the two competing insurers put their surpluses into the bank and purchase reinsurance for risk control. Optimal reinsurance strategies are determined in the cases of proportional reinsurance and excess-of-loss reinsurance. In Section 3, we study the optimization problem for the reinsurer to derive the optimal reinsurance premium rate based on the insurers' strategies determined in Section 2. Numerical analyses are given in Sections 2 and 3 to illustrate our theoretical results. Section 4 concludes the paper.

2. Optimization problems for two insurers

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space indexed by a finite time horizon $[0, T]$, where $\mathcal{F}_t$ is right-continuous, $\mathbb{P}$ complete filtration. In this section, we consider the optimal reinsurance problem for two competing insurers.

2.1. Model setup

We start by describing the surpluses of two competing insurers, namely Insurer 1 and 2, who are subject to common insurance shock. The surplus of Insurer $k = 1, 2$ is described by Cramér-Lundberg (C-L) model, i.e.

$$\mathrm{d}W_k(t) = (\lambda_k + \lambda)C_k^0 \,\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N_k(t)+N(t)} Z_{k,i}, \quad W_k(0) = w_k,$$

where $w_k > 0$ is the initial surplus, $\{Z_{k,i}\}_{i\in\mathbb{N}}$ are the claims against Insurer k, which are independent and identically distributed (i.i.d) positive random variables taking values in $[0, D_k]$, where $D_k(\leq +\infty)$ is the maximum claim amount for Insurer k. $\sum_{i=1}^{N_k(t)+N(t)} Z_{k,i}$ represents the total claims that Insurer k pays up to time t and $N_k(t) + N(t)$ represents the cumulative number of claims. We assume that $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ and $\{N(t)\}_{t\geq 0}$ are three mutually independent Poisson processes with intensity $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda > 0$ respectively. $N(t)$ represents a common systematic insurance risk that affects both Insurers 1 and 2,[2] and $N_k(t)$ represents Insurer k's idiosyncratic insurance risk. C_k^0 is the premium rate of Insurer k determined according to the *expected value principle*, i.e. $C_k^0 = (1 + \eta_k)\mathbb{E}[Z_{k,i}]$, where $\eta_k > 1$ is the *insurance safety loading* of Insurer k.

Assumption 2.1: *We assume that $\{Z_{k,i}\}$ have distribution function $F_k(z)$ and*

$$\mathbb{E}[Z_{k,i}^2 \, e^{c\, e^{rT} Z_{k,i}}] < +\infty, \quad k = 1, 2,$$

where $c := \max(\gamma_1, \gamma_2, \gamma)$, γ_1 and γ_2 are the risk preference parameters of Insurer 1 and 2, and γ is the risk preference parameter of the reinsurer.

In this paper, both insurers are allowed to purchase proportional reinsurance or excess-of-loss reinsurance for risk control, and to put their surpluses into the bank with risk-free interest rate $r > 0$. At any time $t \in [0, T]$, let $a_k(t)$ be the self-retention level of Insurer k and

$$\mathbf{Z}_{k,i} := \mathbf{Z}_{k,i}(a_k(t))(< Z_{k,i})$$

be the part of claims paid by Insurer k. Specially, $\mathbf{Z}_{k,i} = a_k(t)Z_{k,i}$ with $0 \leq a_k(t) \leq 1$ if the insurer adopts a proportional reinsurance contract; and $\mathbf{Z}_{k,i} = \min(a_k(t), Z_{k,i})$ if the insurer adopts an excess-of-loss reinsurance contract. Then the remainder $\bar{\mathbf{Z}}_{k,i} := Z_{k,i} - \mathbf{Z}_{k,i}$ is covered by the reinsurer. We refer to $\{a_k(t)\}_{0\leq t\leq T}$ as the reinsurance strategy of Insurer k. The surplus process with reinsurance

[2] This setting is also adopted by Bai et al. (2013) who use a similar model to describe an insurer who has two lines of insurance business that are subject to common shock in the industry, and Siu et al. (2017) who use $N(t)$ to describe the common systematic insurance risk, see also Liang & Yuen (2016), Bi et al. (2016), and so on.

protection, denoted by $\{W_k^{a_k}(t)\}_{t\geq 0}$, for Insurer $k(k=1,2)$ then becomes

$$\mathrm{d}W_k^{a_k}(t) = \left[rW_k^{a_k}(t) + (\lambda + \lambda_k)C_k(t)\right]\mathrm{d}t - \mathrm{d}\left(\sum_{i=1}^{N_k(t)+N(t)} \mathbf{Z}_{k,i}\right), \quad W_k(0) = w_k, \tag{1}$$

where $C_k(t)$ is the premium rate defined by

$$\begin{aligned} C_k(t) &:= C_k^0 - (1+\theta_k)\mathbb{E}[Z_{k,i} - \mathbf{Z}_{k,i}] \\ &= C_k^0 - (1+\theta_k)\bar{\mu}_k + (1+\theta_k)\mathbb{E}[\mathbf{Z}_{k,i}], \end{aligned}$$

$\theta_k(\geq \eta_k)$ is the *reinsurance safety loading* for Insurer k, and $\bar{\mu}_k := \mathbb{E}[Z_{k,i}]$. Since θ_k uniquely defines Insurer k's payment for his reinsurance protection, it can be seen as the price of the reinsurance contract offered to Insurer k. In this section, we assume that $\theta_k,\ k=1,2$, are exogenous constants. The topic of how the reinsurer adjusts the prices dynamically is the subject of Section 3.

Definition 2.1: A process $\{a_k(t)\}_{t\in[0,T]}$ is an admissible strategy if $a_k(t)$ is a $\mathcal{F}_t$-adapted process; and the SDE (1) admits a unique strong solution. We denote by Π_k the set of all admissible reinsurance strategies for Insurer $k(k=1,2)$.

The two insurers are competing with each other. Each not only cares about his performance but also puts an eye on that of his competitor. The optimization problem of each insurer is to choose an optimal reinsurance strategy to maximize the expected utility of his relative performance over his competitor at the terminal time T:

$$\begin{aligned} &\max_{a_k\in\Pi_k} \mathbb{E}\left[U_k\left((1-\kappa_k)W_k^{a_k}(T) + \kappa_k(W_k^{a_k}(T) - W_j^{a_j}(T))\right)\right] \\ &= \max_{a_k\in\Pi_k} \mathbb{E}\left[U_k\left(W_k^{a_k}(T) - \kappa_k W_j^{a_j}(T)\right)\right], \end{aligned} \tag{2}$$

where $\kappa_k \in [0,1]$ is the *sensitivity parameter* of Insurer k towards Insurer j's performance, with $j \neq k$. A larger κ_k indicates that Insurer k cares more about his competitor's performance. $W_k^{a_k}(T) - \kappa_k W_j^{a_j}(T)$ represents Insurer k's relative performance at the terminal time T. $U_k(\cdot)$ is the utility function for Insurer k, which measures his risk-return preference. In line with Yi et al. (2013), Li et al. (2014), Siu et al. (2017) and Deng et al. (2018), we assume that $U_k(\cdot)$ is a constant absolute risk-aversion (CARA) utility function, i.e.

$$U_k(w) = -\frac{1}{\gamma_k}e^{-\gamma_k w}, \quad k=1,2,$$

in which γ_k is Insurer k's absolute risk aversion coefficient representing Insurer k's preference for risk.

Problem (2) is a typical non-zero-sum game between the two insurers that is rigorously formulated as below.

Problem 2.1: *Insurer k's objective is to seek the optimal reinsurance strategy a_k^* to maximize his objective given Insurer $j(\neq k)$ adopts the optimal strategy a_j^* :*

$$\begin{aligned} \mathbb{E}\left[U_1\left(W_1^{a_1}(T) - \kappa_1 W_2^{a_2^*}(T)\right)\right] &\leq \mathbb{E}\left[U_1\left(W_1^{a_1^*}(T) - \kappa_1 W_2^{a_2^*}(T)\right)\right], \\ \mathbb{E}\left[U_2\left(W_2^{a_2}(T) - \kappa_2 W_1^{a_1^*}(T)\right)\right] &\leq \mathbb{E}\left[U_2\left(W_2^{a_2^*}(T) - \kappa_2 W_1^{a_1^*}(T)\right)\right]. \end{aligned}$$

Thus, (a_1^*, a_2^*) is the Nash equilibrium of a stochastic differential game. To characterize a Nash equilibrium, define the relative surplus of Insurer k to Insurer j as

$$\hat{W}_k(t) := W_k^{a_k}(t) - \kappa_k W_j^{a_j}(t),$$

for $k, j = 1, 2$ and $k \neq j$. Then, it follows that

$$\begin{aligned} \mathrm{d}\hat{W}_k(t) &= \left[r\hat{W}_k(t) + (\lambda_k + \lambda)C_k(t) - \kappa_k(\lambda_j + \lambda)C_j(t)\right]\mathrm{d}t \\ &\quad - \mathrm{d}\left(\sum_{i=1}^{N_k(t)+N(t)} Z_{k,i} - \kappa_k \sum_{i=1}^{N_j(t)+N(t)} Z_{j,i}\right), \quad \hat{W}_k(0) = \hat{w}_k := w_k - \kappa_k w_j. \end{aligned}$$

We now model robustness due to concern about model misspecification or model ambiguity. Assume that both insurers are averse to modeling ambiguity about the arrival of claims, and they wish to make robust reinsurance decisions to guard against the possible model misspecification. We interpret this by defending against a worst-case scenario while allowing each model to be penalized by how far it deviates from the estimated model. Especially, since the systematic risk that impacts both insurers is challenging to measure, and the intensity λ is challenging to assess, in this paper, we only consider that both insurers have ambiguity-averse attitudes towards the claim intensity λ and the associated probability measure $\mathbb{P}$.[3] On the one hand, both insurers regard $\mathbb{P}$ as the reference probability measure; on the other hand, each insurer would like to consider other alternative probability measures $\{\mathbb{Q}_k\}$, $k = 1, 2$, which preferably do not deviate very far from the reference model. We handle this by assuming that $\mathbb{Q}_k$ is absolutely continuous and is equivalent to $\mathbb{P}$, because it is in this sense that we can measure distance between models in a tractable way. We denote this class of probability measures equivalent to $\mathbb{P}$ by $\mathcal{Q}_k$:

$$\mathcal{Q}_k = \{\mathbb{Q}_k \,|\, \mathbb{Q}_k \sim \mathbb{P}\}, \quad k = 1, 2.$$

According to Girsanov's theorem, for each $\mathbb{Q}_k \in \mathcal{Q}_k$ there exists a progressively measurable process $\{\phi_k(t)\}_{t\in[0,T]}$ such that

$$\left.\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \Lambda^{\phi_k}(t),$$

where

$$\Lambda^{\phi_k}(t) = \exp\left\{\int_0^t \ln\phi_k(s)\,\mathrm{d}N(s) + \int_0^t (1 - \phi_k(s))\lambda\,\mathrm{d}s\right\}$$

is a $\mathbb{P}$-martingale and ϕ_k satisfies

$$\exp\left\{\int_t^T \int_0^{D_k} (\phi_k(s)\ln\phi_k(s) - \phi_k(s) + 1)\lambda\,\mathrm{d}F_k(z)\,\mathrm{d}s\right\} < \infty.$$

Under the new probability measure $\mathbb{Q}_k$, the claim intensity λ becomes $\lambda\phi_k(t)$. Accordingly, the relative surplus $\hat{W}_k(t)$ evolves according to

$$\begin{aligned} \mathrm{d}\hat{W}_k(t) &= \left\{r\hat{W}_k(t) + (\lambda_k + \lambda\phi_k(t))C_k(t) - \kappa_k(\lambda_j + \lambda\phi_k(t))C_j(t)\right\}\mathrm{d}t \\ &\quad - \mathrm{d}\left(\sum_{i=1}^{N_k(t)+N^{\phi_k}(t)} Z_{k,i} - \kappa_k \sum_{i=1}^{N_j(t)+N^{\phi_k}(t)} Z_{j,i}\right), \end{aligned} \tag{3}$$

[3] While uncertainty over $\lambda_k, k = 1, 2$, can also be incorporated in our model, such a modeling strategy deviates from our primary concern and will not be considered in this paper, for the sake of presenting a tighter modeling framework.

where the superscript on N emphasizes that the intensity λ has changed to $\lambda\phi_k$.

Although both insurers are suspicious about the intensity λ and the reference probability measure $\mathbb{P}$, they don't want the alternative probability measure to deviate from $\mathbb{P}$ too far. According to their model robustness preferences, each insurer puts a penalization term into his objective function and considers his optimal problem under a robustness framework. Inspired by Maenhout (2004, 2006), Hu et al. (2018a, 2018b) and Bai et al. (2022), we consider the distance between the reference measure $\mathbb{P}$ and the alternative measure $\mathbb{Q}_k$, $k = 1, 2$, as the penalty term. Both insurers aim to maximize the expected utilities of their relative terminal wealth under a worst-case scenario. As such, each insurer first takes infimum in the objective function to determine the worst-case probability measure $\mathbb{Q}_k^* \in \mathcal{Q}_k$, and then determines his optimal strategy a_k^* based on the worst-case probability measure.

Definition 2.2: A strategy $\{a_k(t)\}_{0\le t\le T}$ is called an admissible reinsurance strategy with ambiguity aversion if the following conditions are satisfied

(i) $a_k(t)$ is adapted to $\{\mathcal{F}_t\}$;
(ii) For any probability measure $\mathbb{Q}_k \in \mathcal{Q}_k$, Equation (3) has a unique solution $\{\hat{W}_k(t)\}_{0\le t\le T}$ such that

$$\mathbb{E}^{\mathbb{Q}_k}\left[|U_k(\hat{W}_k(T))| + \left|\int_t^T \lambda \frac{\phi_k(s)\ln\phi_k(s) - \phi_k(s) + 1}{\varphi_k(s)}\,\mathrm{d}s\right|\right] < \infty,$$

for any $t \in [0, T)$, where $\mathbb{E}^{\mathbb{Q}_k}[\cdot]$ denotes the conditional expectation operator under probability measure $\mathbb{Q}_k$ and $\int_t^T \lambda \frac{\phi_k(s)\ln\phi_k(s)-\phi_k(s)+1}{\varphi_k(s)}\mathrm{d}s$ is the penalty term.

To avoid confusion, we also denote the set of all admissible reinsurance strategies with ambiguity aversion for Insurer k by Π_k. Thus, given the optimal strategy of his competitor $a_j^* \in \Pi_j$, Insurer k's value function is defined as

$$J^k(t, \hat{w}_k) = \max_{a_k\in\Pi_k}\min_{\mathbb{Q}_k\in\mathcal{Q}_k} \mathcal{J}^k(t, \hat{w}_k; a_k, a_j^*, \phi_k), \quad k = 1, 2 \text{ and } k \neq j, \tag{4}$$

where $\mathcal{J}^k$ is defined by

$$\mathcal{J}^k(t, \hat{w}_k; a_k, a_j^*, \phi_k) := \mathbb{E}^{\mathbb{Q}_k}\left[U_k\left(\hat{W}_k(T)\right) + \int_t^T \lambda \frac{\phi_k(s)\ln\phi_k(s) - \phi_k(s) + 1}{\varphi_k(s)}\mathrm{d}s\right]. \tag{5}$$

The last term of (5) represents the deviation of $\mathbb{Q}_k$ from $\mathbb{P}$, in which $\varphi_k(s)$, representing the intensity of the ambiguity aversion, can be tailored to aid in tractability.

Problem 2.2 (Nash equilibrium under ambiguity aversion): *With ambiguity aversion, the two competing insurers aim to find a Nash equilibrium (a_1^*, a_2^*) such that*

$$\begin{aligned}
\inf_{\mathbb{Q}_1\in\mathcal{Q}_1} \mathcal{J}^1(t, \hat{w}_1, a_1, a_2^*, \phi_1) &\le \inf_{\mathbb{Q}_1\in\mathcal{Q}_1} \mathcal{J}^1(t, \hat{w}_1, a_1^*, a_2^*, \phi_1),\\
\inf_{\mathbb{Q}_2\in\mathcal{Q}_2} \mathcal{J}^2(t, \hat{w}_2, a_2, a_1^*, \phi_2) &\le \inf_{\mathbb{Q}_2\in\mathcal{Q}_2} \mathcal{J}^2(t, \hat{w}_2, a_2^*, a_1^*, \phi_2),
\end{aligned}$$

for any $(a_1, a_2) \in \Pi_1 \times \Pi_2$.

For analytical tractability, we follow Maenhout (2004, 2006) and Hu et al. (2018a, 2018b) and assume that

$$\varphi_k(t) := -\frac{\alpha_k}{\gamma_k J^k(t,\hat{w}_k)}, \tag{6}$$

where $\alpha_k \geq 0$ is the ambiguity aversion coefficient of Insurer k, representing the extent of Insurer k's concern over his ambiguity to the claim intensity λ. A larger α_k means that Insurer k is more concerned about the model robustness. When $\alpha_k = 0$, Insurer k is ambiguous neutral, and his problem becomes a standard optimization problem.

In the following, we proceed to find the optimal strategies for Problems 2. To start with, we write down the HJB equation for each insurer according to the dynamic programming approach:

$$\begin{aligned}
&J_t^k(t,\hat{w}_k) + \max_{a_k}\min_{\phi_k}\Big\{J^k_{\hat{w}_k}(t,\hat{w}_k)\left[r\hat{w}_k + (\lambda_k + \lambda\phi_k)C_k - \kappa_k(\lambda_j + \lambda\phi_k)C_j\right] \\
&\quad + \lambda_k\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k - \mathrm{Z}_k) - J^k(t,\hat{w}_k)\right] + \lambda_j\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k + \kappa_k\mathrm{Z}_j) - J^k(t,\hat{w}_k)\right] \\
&\quad + \lambda\phi_k\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k - \mathrm{Z}_k + \kappa_k\mathrm{Z}_j) - J^k(t,\hat{w}_k)\right] \\
&\quad - \gamma_k J^k(t,\hat{w}_k)\frac{\lambda}{\alpha_k}\left(\phi_k\ln\phi_k - \phi_k + 1\right)\Big\} = 0
\end{aligned} \tag{7}$$

with terminal condition $J^k(T,\hat{w}_k) = -\frac{1}{\gamma_k}e^{-\gamma_k\hat{w}_k}$, where

$$C_k = C_k(a_k) = (1+\theta_k)\mathbb{E}[\mathrm{Z}_k] + C_k^0 - (1+\theta_k)\bar{\mu}_k$$

and $\mathrm{Z}_k := \mathrm{Z}_k(a_k)$. The subscripts on J^k denote partial derivatives resulting from time and diffusion dynamics, and the stochastic jump terms result in finite differences where the spatial step sizes are determined by claim sizes and reinsurance contract choices.

Theorem 2.1: *Let $\tilde{\gamma}_k := \gamma_k\, e^{r(T-t)}$. With given reinsurance safety loadings θ_1 and θ_2, the value function of Insurer k is given by $J^k(t,\hat{w}_k) = -\frac{1}{\gamma_k}e^{-\tilde{\gamma}_k\hat{w}_k + g_k(t)}$, where the function $g_k(t)$ is time-dependent and is given by* (A3) *in the* A.1. *The equilibrium reinsurance strategies are determined by*

$$a_k^* = \arg\min_{a_k}\left\{\lambda_k\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k\mathrm{Z}_k} - (1+\theta_k)\tilde{\gamma}_k\mathrm{Z}_k\right] + \gamma_k\frac{\lambda}{\alpha_k}\phi_k^*\right\}, \tag{8}$$

for $k = 1,2$, where $\phi_k^* := e^{\frac{\alpha_k}{\gamma_k}f_k}$ and

$$f_k = \mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k(\mathrm{Z}_k - \kappa_k\mathrm{Z}_j)}\right] - \tilde{\gamma}_k(C_k(a_k) - \kappa_k C_j(a_j^*)) - 1, \quad k = 1,2, \;\; j \neq k. \tag{9}$$

Proof: See A.1. ■

Remark 2.1: (i) Without systematic risk (i.e. $\lambda = 0$),

$$a_k^* = \arg\min_{a_k}\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k\mathrm{Z}_k} - (1+\theta_k)\tilde{\gamma}_k\mathrm{Z}_k\right]. \tag{10}$$

That is, the two insurers' decisions are independent of each other. The result is consistent with our common sense: Competition exists only if the two insurers are related to each other.

(ii) When $\kappa_k = 0$, i.e. when Insurer k ignores competition,

$$f_k = \mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k \mathbf{Z}_k} - (1+\theta_k)\tilde{\gamma}_k \mathbf{Z}_k\right] + \text{constants}.$$

Thus we also have Equation (10), i.e. Insurer k only cares about the risk control of his own business. This result is straightforward. However, so long as $\kappa_j \neq 0$, Insurer j's decision is still affected by Insurer k.

When the insurers have full confidence on the claim intensity λ, by performing a similar procedure as in the proof of Theorem 2.1, we have the following results.

Corollary 2.2: *When $\alpha_k = 0$, the equilibrium reinsurance strategies are determined by*

$$a_k^* = \arg\min_{a_k}\left\{\lambda_k \mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k \mathbf{Z}_k} - (1+\theta_k)\tilde{\gamma}_k \mathbf{Z}_k\right] + \lambda \mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k(\mathbf{Z}_k - \kappa_k \mathbf{Z}_j)}\right]\right\}. \tag{11}$$

Next, we propose detailed analyses over the optimal strategies when the insurers adopt proportional or excess-of-loss reinsurance protection.

2.2. The proportional reinsurance case

In this subsection, we assume that both insurers purchase proportional reinsurance protection to manage their insurance business risk. Then, $\mathbf{Z}_k(a_k) = a_k Z_k$. We report our results in the following proposition.

Proposition 2.3: *When both insurers adopt proportional reinsurance protection, for any $t \in [0, T]$, let $(\hat{a}_1, \hat{a}_2)$ be the pair of non-negative solutions (the solutions' non-negativity is to be proved below in Proposition 2.5) to the following system:*

$$\begin{cases} \lambda_1\left(\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1 a_1 Z_1}] - (1+\theta_1)\bar{\mu}_1\right) + \lambda\phi_1^*(a_1, a_2)\left(\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1(a_1 Z_1 - \kappa_1 a_2 Z_2)}] - (1+\theta_1)\bar{\mu}_1\right) = 0, \\ \lambda_2\left(\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2 a_2 Z_2}] - (1+\theta_2)\bar{\mu}_2\right) + \lambda\phi_2^*(a_1, a_2)\left(\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2(a_2 Z_2 - \kappa_2 a_1 Z_1)}] - (1+\theta_2)\bar{\mu}_2\right) = 0, \end{cases} \tag{12}$$

where ϕ_1^* and ϕ_2^* are defined in Theorem 2.1. Then, the equilibrium reinsurance strategies (a_1^*, a_2^*) admit one of the following four cases:

(1) *If $(\hat{a}_1, \hat{a}_2) \in [0,1] \times [0,1]$, then $(a_1^*, a_2^*) = (\hat{a}_1, \hat{a}_2)$;*
(2) *If $\hat{a}_1 > 1$ and $\hat{a}_2 \in [0,1]$, then $a_1^* = 1$ and a_2^* is determined by*

$$\lambda_2\left(\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2 a_2 Z_2}] - (1+\theta_2)\bar{\mu}_2\right) + \lambda\phi_2^*(1, a_2)\left(\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2(a_2 Z_2 - \kappa_2 Z_1)}] - (1+\theta_2)\bar{\mu}_2\right) = 0; \tag{13}$$

(3) *If $\hat{a}_2 > 1$ and $\hat{a}_1 \in [0,1]$, then $a_2^* = 1$ and a_1^* is determined by*

$$\lambda_1\left(\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1 a_1 Z_1}] - (1+\theta_1)\bar{\mu}_1\right) + \lambda\phi_1^*(a_1, 1)\left(\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1(a_1 Z_1 - \kappa_1 Z_2)}] - (1+\theta_1)\bar{\mu}_1\right) = 0; \tag{14}$$

(4) *Otherwise, $(a_1^*, a_2^*) = (1, 1)$.*

The worst case measures are given by $\phi_k^*(a_1^*, a_2^*)$, $k = 1, 2$.

Proof: See A.2. ■

Note that we do not specify the distribution of claim sizes here. Thus the equations in (12), (13) and (14) are very nonlinear in $\hat{a}_1$ and $\hat{a}_2$, so that analytic solutions in closed form are impossible. However, we will prove the existence of solutions $(\hat{a}_1, \hat{a}_2)$, and will present some economically intuitive properties.

The following result is consistent with literatures considering a single insurer, see e.g. Hu et al. (2018a).

Corollary 2.4: *When* $\lambda = 0$ *or* $\kappa_k = 0$, *the equilibrium reinsurance strategy* $a_k^* = \underline{a}_k^p \wedge 1$, *where* $\underline{a}_k^p$ *is determined by*

$$\mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\tilde{\gamma}_k a_k Z_k}] = (1 + \theta_k)\bar{\mu}_k, \quad a_k > 0, \tag{15}$$

for $k = 1, 2$.

Proof: When $\lambda = 0$ or $\kappa_k = 0$, from Equation (12) we have Equation (15). It is clear that $\mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\tilde{\gamma}_k a_k Z_k}]$ is strictly increasing in a_k with $\mathbb{E}^{\mathbb{Q}_k}[Z_k] = \bar{\mu}_k < (1 + \theta_k)\bar{\mu}_k$ and $\lim_{a_k \to \infty} \mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\tilde{\gamma}_k a_k Z_k}] = +\infty$. Thus $\underline{a}_k^p$ is uniquely determined by Equation (15), and $a_k^* = \underline{a}_k^p \wedge 1$ is the optimal reinsurance strategy. ■

Corollary 2.4 shows that, in the absence of systematic risk or without competition intention, Insurer k's decision is only affected by time-adjusted risk preference $\tilde{\gamma}_k$, the expected value of each claim $\bar{\mu}_k$, and the cost of reinsurance protection θ_k.

However, when $\kappa_k \neq 0$ and $\lambda \neq 0$, Insurer k's decision is further affected by the competition level κ_k and the prior estimation of systematic risk intensity λ. Specially, when $a_k^* \in [0, 1]$, Insurer k determines his reinsurance strategy by making a tradeoff between the controls of his own heterogeneity risk and the systematic risk:

$$\underbrace{\lambda_k \left(\mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\tilde{\gamma}_k a_k^* Z_k}] - (1 + \theta_k)\bar{\mu}_k \right)}_{\text{control on heterogeneity risk}} + \underbrace{\lambda \phi_k^* \left(\mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\tilde{\gamma}_k (a_k^* Z_k - \kappa_k a_j^* Z_j)}] - (1 + \theta_k)\bar{\mu}_k \right)}_{\textbf{control on systematic risk}} = 0, \tag{16}$$

in which the first and the second items on the left-hand side (LHS) decide the risk control for the heterogeneity risk and the systematic risk, respectively. Since Insurer k is ambiguous about the systematic risk, i.e. he is uncertain about λ, the intensity of the systematic risk is adjusted from λ to $\lambda \phi_k^*$. When Insurer k overestimates the intensity of $N(t)$, he focuses more on the control of systematic risk and a_k^* approaches the systematic risk control. However, due to competition, Insurer k does not always overestimate systematic risk, but will underestimate it in certain cases. We will numerically illustrate this counter intuitive phenomena later.

Insurer k's decision is also affected by the factor $\mathbb{E}^{\mathbb{Q}_k}[e^{\tilde{\gamma}_k(-\kappa_k a_j^* Z_j)}]$, which represents the effect of competition and affects the risk control on systematic risk. When Insurer k is more concerned about his competitor, he has a larger risk control on the systematic risk (i.e. $\bar{a}_k^p$, which is defined below in Equation (17), is larger) and thus his equilibrium strategy a_k^* increases.

The following proposition shows the existence of solutions to Equation (12). It also illustrates the influences of the prior estimation of the systematic risk and the prices of the reinsurance contracts on the optimal reinsurance strategies.

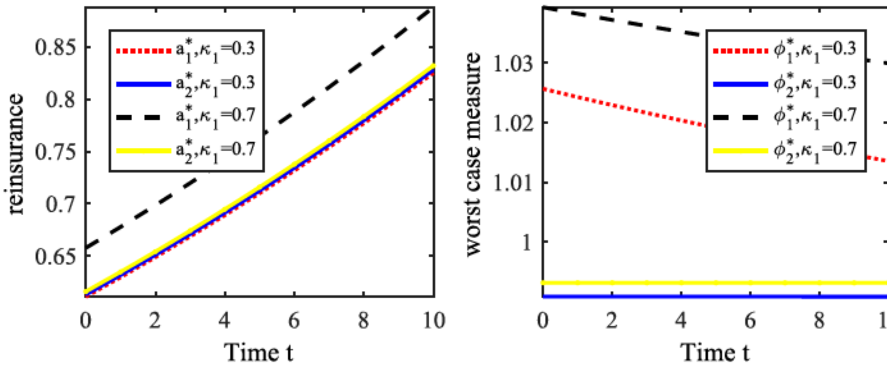

Figure 1. Proportional reinsurance strategies and worst case measures for fixed reinsurance safety loading. The model parameters are $r = 0.03, \lambda_1 = 1, \lambda_2 = 2, \lambda = 1, \kappa_2 = 0.5, \gamma_1 = 0.3, \gamma_2 = 0.3, \eta_1 = 0.2, \eta_2 = 0.3, T = 10, \theta_1 = 0.4, \theta_2 = 0.4, \alpha_1 = 0.3, \alpha_2 = 0.3, F_1(z) = F_2(z) = 1 - e^{-1.5z}$.

Proposition 2.5: *For any $\lambda > 0$, Equation (12) has a pair of solutions $(\hat{a}_1, \hat{a}_2)$ with $\underline{a}_k^p < \hat{a}_k < \bar{a}_k^p$, $k = 1, 2$, where $\bar{a}_k^p$ is the risk control for the systematic risk and is determined by*

$$\mathbb{E}^{\mathbb{Q}_k}[Z_k \, e^{\bar{\gamma}_k(a_k Z_k - \kappa_k \hat{a}_j Z_j)}] = (1 + \theta_k)\bar{\mu}_k, \quad a_k > 0. \tag{17}$$

Moreover, $\hat{a}_k$ is strictly increasing with λ and θ_k.

Proof: See A.3. ∎

Proposition 2.5 shows that, in a competitive market, with larger systematic risk intensity both insurers counter-intuitively increase their risk exposure to have a better performance against their competitor. Similar results have been reported by Siu et al. (2017), Chen et al. (2018), whereas these papers mainly focus on models with diffusion risk processes and do not consider the uncertainty on systematic risk. It is straightforward that a larger reinsurance price θ_k also leads to less reinsurance demand for reinsurance protection, and thus a larger self-retention level a_k^*. Due to competition and systematic risk, the increase of a_k^* also leads to an increase of the self-retention level of his competitor a_j^*.

We now turn to numerical solutions to verify our theoretical results above. By numerically solving the system of equations (12), we may determine the insurers' optimal controls (a_1^*, a_2^*) and the worst case measures (ϕ_1^*, ϕ_2^*) as illustrated in Figure 1. As expected, a_1^* and a_2^* increase as t approaches T. That is, both insurers take more risk for themselves to have better performances at time T. Similar results have been reported by Chen et al. (2018). However, ϕ_1^* and ϕ_2^* behave differently with time t. In fact, when t approaches T, on the one hand both insurers have larger risk exposures, leading to higher estimation over claim intensity $\lambda\phi^*$. On the other hand, as the expiration time $T-t$ decreases, the impact of the uncertainty over λ decreases, leading to underestimation over systematic risk. As for Insurer 1, the latter dominates the former, thus ϕ_1^* decreases with t. However, as for Insurer 2, both factors seem to be equal, thus ϕ_2^* does not change with t.

Table 1 shows the influences of reinsurance prices, competition, and model ambiguity on the insurers' self-retention levels and the worst-case measures. Since the influences are similar for any time $t \in [0, T]$, we just need to show the results for $t = 0$. As has been proved in Proposition 2.5, when λ and θ_1 increase, both insurers tend to keep more risk for themselves. Especially, θ_1 has a more remarkable impact on Insurer 1's self-retention level while it has a very limited indirect impact on that of Insurer 2. When θ_1 increases from 0.4 to 0.5, Insurer 1 increases his self-retention level by 0.0632 when $\lambda = 1$ and 0.0628 when $\lambda = 1.5$. In contrast, Insurer 2 only increases his self-retention level by 0.0033 when $\lambda = 1$ and 0.0043 when $\lambda = 1.5$. However, since his relative risk exposure over Insurer 1,

Table 1. The impact of model parameters on reinsurance strategies and worst case measures at $t = 0$ when both insurers adopt proportional reinsurance protection.

	$\lambda = 1.0$				$\lambda = 1.5$			
	a_1^*	a_2^*	ϕ_1^*	ϕ_2^*	a_1^*	a_2^*	ϕ_1^*	ϕ_2^*
$\theta_1 = 0.4$	0.4297	0.6042	1.0166	1.0283	0.4465	0.6124	1.0172	1.0285
$\theta_1 = 0.5$	0.4929	0.6075	1.0248	1.0254	0.5093	0.6167	1.0254	1.0258
$\kappa_1 = 0.0$	0.3521	0.5999	1.0207	1.0267	0.3521	0.6058	1.0207	1.0266
$\kappa_1 = 0.5$	0.4097	0.6031	1.0157	1.0278	0.4221	0.6108	1.0161	1.0280
$\kappa_1 = 1.0$	0.4571	0.6056	1.0207	1.0289	0.4801	0.6148	1.0216	1.0292
$\alpha_1 = 0.2$	0.6574	0.6162	1.0261	0.9931	0.6762	0.6300	1.0266	0.9938
$\alpha_1 = 0.4$	0.6585	0.6162	1.0528	0.9932	0.6773	0.6301	1.0539	0.9939

Note: The model parameters are $r = 0.03, \lambda_1 = 1, \lambda_2 = 2, \kappa_1 = 0.7, \kappa_2 = 0.5, \gamma_1 = \gamma_2 = 0.3, \eta_1 = 0.2, \eta_2 = 0.3, T = 10, \theta_1 = \theta_2 = 0.4, \alpha_1 = \alpha_2 = 0.3, F_1(z) = F_2(z) = 1 - e^{-1.5z}$.

$a_2^* - a_1^*$, decreases, Insurer 2 will instead decrease his estimation over the systematic risk from 1.0283 down to 1.0254.

Table 1 also shows that, as κ_1 increases, a_1^* increases dramatically and a_2^* increases slightly. That is, Insurer 1 takes more risks by himself when he becomes more concerned about the relative performance of his competitor. As a response, Insurer 2 also increases his self-retention level. We also observe that the impact of κ_1 becomes more prominent when λ becomes larger.[4] Thus, higher systematic risk intensity and more intensive competition jointly lead to less demand for reinsurance protection.

Finally, we observe that α_1 has negligible impact on both insurers' decision making. This result is somewhat confusing. In fact, when the insurers are less certain about the systematic risk, they tend to overestimate it. However, due to competition concern, both insurers choose not to seek more reinsurance protection. This result indicates that, in a competitive insurance market, insurers may do nothing over systematic risk, no matter how serious it might be. It emphasizes the importance of risk supervision on insurance companies, and possibly increasing this supervision on companies engaging in more competitive insurance markets.

Remark 2.2: We consider the special case where both insurers are certain about the systematic risk. From Equation (11) and the first-order condition, we see that in this case $a_k^*(\in (0,1))$ satisfies

$$\begin{cases} \lambda_1 \mathbb{E}[Z_1 e^{\tilde{\gamma}_1 a_1^* Z_1}] + \lambda \mathbb{E}[Z_1 e^{\tilde{\gamma}_1 (a_1^* Z_1 - \kappa a_2^* Z_2)}] = (\lambda_1 + \lambda)(1 + \theta_1)\bar{\mu}_1, \\ \lambda_2 \mathbb{E}[Z_2 e^{\tilde{\gamma}_2 a_2^* Z_2}] + \lambda \mathbb{E}[Z_2 e^{\tilde{\gamma}_2 (a_2^* Z_2 - \kappa a_1^* Z_1)}] = (\lambda_2 + \lambda)(1 + \theta_2)\bar{\mu}_2. \end{cases} \tag{18}$$

According to (18), the optimal reinsurance strategies $\{a_k^*(t)\}$, $k = 1, 2$ are the functions of reinsurance premiums θ_1 and θ_2. Conversely, for each $t \in [0, T]$, we can also consider θ_k as the function of optimal reinsurance strategies a_1^* and a_2^*. In other words, the reinsurer can, given the insurers' optimal reinsurance strategies, determine the reinsurance premium (i.e. θ_k) and try to adjust it for maximizing her profit and attracting more reinsurance business. We will discuss this point in the next section.

2.3. The excess-of-loss reinsurance case

In this subsection, we assume that both insurers adopt excess-of-loss reinsurance for risk control, i.e. $Z_k = Z_k \wedge a_k$.

Proposition 2.6: *When both insurers adopt excess-of-loss reinsurance protection, for any $t \in [0, T]$, let $(\tilde{a}_1, \tilde{a}_2)$ be the pair of non-negative solutions (the solutions' non-negativity is to be proved below in*

[4] This result can also be inferred from (16).

Proposition 2.8) to the following system:

$$\begin{cases} \lambda_1[e^{\tilde{\gamma}_1 a_1} - (1+\theta_1)] + \lambda\phi_1^*\left[e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - (1+\theta_1)\right] = 0, \\ \lambda_2[e^{\tilde{\gamma}_2 a_2} - (1+\theta_2)] + \lambda\phi_2^*\left[e^{\tilde{\gamma}_2 a_2}\mathbb{E}^{\mathbb{Q}_2}[e^{-\kappa_2\tilde{\gamma}_2(Z_1\wedge a_1)}] - (1+\theta_2)\right] = 0, \end{cases} \tag{19}$$

where ϕ_1^* and ϕ_2^* are defined in Theorem 2.1,

$$\mathbb{E}^{\mathbb{Q}_k}[e^{-\kappa_k\tilde{\gamma}_k(Z_j\wedge a_j)}] = -\int_0^{a_j}\bar{F}_j(z)\kappa_k\tilde{\gamma}_k\,e^{-\kappa_k\tilde{\gamma}_k z}\,dz + 1, \quad k=1,2, \ j\neq k.$$

Then, the equilibrium reinsurance strategies (a_1^*, a_2^*) admit one of the following four cases:

(1) *If $(\tilde{a}_1, \tilde{a}_2) \in [0, D_1] \times [0, D_2]$, then $(a_1^*, a_2^*) = (\tilde{a}_1, \tilde{a}_2)$;*

(2) *If $\tilde{a}_1 > D_1$ and $\tilde{a}_2 \in [0, D_2]$, then $a_1^* = D_1$ and a_2^* is determined by*

$$\lambda_2[e^{\tilde{\gamma}_2 a_2} - (1+\theta_2)] + \lambda\phi_2^*\left[e^{\tilde{\gamma}_2 a_2}\mathbb{E}^{\mathbb{Q}_2}[e^{-\kappa_2\tilde{\gamma}_2 Z_1}] - (1+\theta_2)\right] = 0;$$

(3) *If $\tilde{a}_2 > D_2$ and $\tilde{a}_1 \in [0, D_1]$, then $a_2^* = D_2$ and a_1^* is determined by*

$$\lambda_1[e^{\tilde{\gamma}_1 a_1} - (1+\theta_1)] + \lambda\phi_1^*\left[e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1 Z_2}] - (1+\theta_1)\right] = 0;$$

(4) *Otherwise, $(a_1^*, a_2^*) = (D_1, D_2)$.*

The worst case measures are given by $(\phi_1^*(a_1^*, a_2^*), \phi_2^*(a_1^*, a_2^*))$.

Proof: See A.4. ■

The system of equations (19) is almost identical to Equation (12) in the previous subsection for the proportional reinsurance contract.

Without systematic risk or competition, by setting $\lambda = 0$ or $\kappa_k = 0$ in Equation (19) we have the following results directly.[5]

Corollary 2.7: *When $\lambda = 0$ or $\kappa_k = 0$, the optimal reinsurance strategy is $a_k^* = \underline{a}_k^e \wedge D_k$, where $\underline{a}_k^e := \frac{1}{\tilde{\gamma}_k}\ln(1+\theta_k)$.*

In both cases, each insurer decides his risk exposure (or self-retention level) according to his time-adjusted risk preference and the price of reinsurance contract.

With systematic risk and competition, when $a_k^* \in (0, D_k)$, Insurer k determines his reinsurance strategy by making a tradeoff between his controls over heterogeneity risk and systematic risk:

$$\underbrace{\lambda_k[e^{\tilde{\gamma}_k a_k^*} - (1+\theta_k)]}_{\text{control on heterogeneity risk}} + \underbrace{\lambda\phi_k^*[e^{\tilde{\gamma}_k a_k^*}\mathbb{E}^{\mathbb{Q}_k}[e^{-\kappa_k\tilde{\gamma}_k(Z_j\wedge a_j^*)}] - (1+\theta_k)]}_{\text{control on systematic risk}} = 0, \tag{20}$$

in which we can use the first and second items on the LHS of the above to define the risk control determined by the heterogeneity risk, $\underline{a}_k^e$, and the risk control determined by the systematic risk, $\bar{a}_k^e$, by setting each aforementioned item to zero. See for instance $\bar{a}_k^e$ defined below in Equation (21). This is consistent with our previous definition of $\underline{a}_k^p$ from Corollary 2.4. The intensity of the systematic risk is adjusted from λ to $\lambda\phi_k^*$ due to model uncertainty. The multiplier $\mathbb{E}^{\mathbb{Q}_k}[e^{-\kappa_k\tilde{\gamma}_k(Z_j\wedge a_j^*)}]$ in the second item reflects the impact of Insurer k's competitive preference. When Insurer k becomes more competitive or the reinsurance becomes more costly, Insurer k has a larger risk exposure to the systematic risk $\bar{a}_k^e$; thus his self-retention level a_k^* increases.

[5] This result is consistent with literatures considering a single insurer, see e.g. Hu et al. (2018a, 2018b) and Hu & Wang (2019).

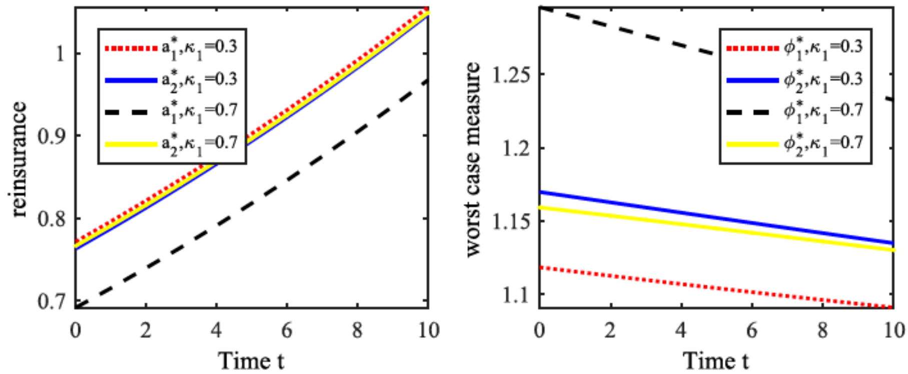

Figure 2. Equilibrium excess-of-loss reinsurance strategies with fixed reinsurance safety loading. The model parameters are $r=0.03, \lambda_1=1, \lambda_2=2, \lambda=1, \kappa_2=0.5, \gamma_1=0.3, \gamma_2=0.3, \eta_1=0.2, \eta_2=0.3, T=10, \theta_1=\theta_2=0.4, \alpha_1=\alpha_2=0.3, F_1(z)=F_2(z)=1-e^{-2z}$.

Table 2. The impact of model parameters on reinsurance strategies and worst-case measures at $t=0$ when both insurers adopt excess-of-loss reinsurance protection.

	$\lambda=1.0$				$\lambda=1.5$			
	a_1^*	a_2^*	ϕ_1^*	ϕ_2^*	a_1^*	a_2^*	ϕ_1^*	ϕ_2^*
$\theta_1=0.4$	0.8071	0.7184	1.0229	0.9773	0.8340	0.7301	1.0231	0.9775
$\theta_1=0.5$	0.9469	0.7205	1.0263	0.9774	0.9740	0.7327	1.0265	0.9776
$\kappa_1=0.0$	0.6803	0.7158	1.0026	0.9761	0.6803	0.7261	1.0026	0.9760
$\kappa_1=0.5$	0.7720	0.7177	1.0146	0.9770	0.7915	0.7291	1.0147	0.9771
$\kappa_1=1.0$	0.8582	0.7192	1.0391	0.9778	0.8962	0.7314	1.0395	0.9780
$\alpha_1=0.2$	0.8066	0.7184	1.0152	0.9773	0.8336	0.7301	1.0154	0.9775
$\alpha_1=0.4$	0.8076	0.7184	1.0307	0.9773	0.8345	0.7301	1.0310	0.9775

Note: The model parameters are $r=0.05, \lambda_1=1, \lambda_2=2, \kappa_1=0.7, \kappa_2=0.3, \gamma_1=\gamma_2=0.3, \eta_1=0.2, \eta_2=0.3, T=10, \theta_1=\theta_2=0.4, \alpha_1=\alpha_2=0.3, F_1(z)=F_2(z)=1-e^{-2z}$.

Similar to the previous subsection, we present the following results to characterize the equilibrium reinsurance strategies.

Proposition 2.8: *For any $\lambda>0$, the system of equations (19) has a solution $(\tilde{a}_1,\tilde{a}_2)$ with $\underline{a}_k^e<\tilde{a}_k<\bar{a}_k^e$, $k=1,2$, where*

$$\bar{a}_k^e=\frac{1}{\tilde{\gamma}_k}\ln\frac{1+\theta_k}{\mathbb{E}^{\mathbb{Q}_k}[e^{-\kappa_k\tilde{\gamma}_k(Z_j\wedge\tilde{a}_j)}]}, \tag{21}$$

and where $\underline{a}_k^e$ is defined in Corollary 2.7. Moreover, $\tilde{a}_k$ is strictly increasing with λ and θ_k.

Proof: See A.5. ■

By solving Equation (19), we show the equilibrium excess-of-loss reinsurance strategies and the worst-case measures as functions of t in Figure 2, and the impact of model parameters in Table 2. The results are similar to the proportional reinsurance case, and thus the discussions are omitted.

Remark 2.3: When both insurers are certain about the systematic risk, similar to Remark 2.2, we see that in this case $a_k^*(\in[0,D_k])$ satisfies

$$\begin{cases} e^{\tilde{\gamma}_1 a_1^*}[\lambda_1+\lambda\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2^*)}]]=(\lambda_1+\lambda)(1+\theta_1), \\ e^{\tilde{\gamma}_2 a_2^*}[\lambda_2+\lambda\mathbb{E}^{\mathbb{Q}_2}[e^{-\kappa_2\tilde{\gamma}_2(Z_1\wedge a_1^*)}]]=(\lambda_2+\lambda)(1+\theta_2). \end{cases} \tag{22}$$

3. Optimization problem for reinsurer

In the previous section, we have shown that reinsurance prices significantly impact the demand for reinsurance protection. Thus, it is vital for the reinsurer to determine the reinsurance prices according to her risk-return preferences. Recently, Hu et al. (2018a, 2018b), Hu & Wang (2019), Gu et al. (2019), and Bai et al. (2022) have considered the dynamic reinsurance pricing problem in a continuous-time principal-agent framework, under which the reinsurer observes an insurer's optimal decision and then prices the reinsurance contract dynamically. In line with these works, in this section, we dynamically determine the reinsurer's optimal safety loadings $\theta_k,\ k = 1,2$ for the two competing insurers to investigate the impact of their competition and the common shock (systematic risk) on the reinsurance prices. As the reinsurer has less information about the systematic risk than the insurers, we assume that both insurers are certain about the intensity of $\{N(t)\}$, whereas the reinsurer is ambiguity-averse about it.[6]

With possible misspecification in λ, the reinsurer would make conservative pricing strategies to ensure robustness of the reinsurance business across λ and its nearby values. That is, she considers a possible rate, $\lambda^{\mathbb{Q}}$, relative to a new probability measures $\mathbb{Q}$ belonging to the set of probabilities measures $\mathcal{Q}$ that are absolutely continuous and equivalent to $\mathbb{P}$, i.e. $\mathcal{Q} = \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}$. According to Girsanov's theorem, there exists a progressively measurable process, $\upsilon(t)$, such that

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \upsilon(t),$$

where

$$\upsilon(t) = \exp\left\{\int_0^t \ln\phi(s)\,\mathrm{d}N(s) + \lambda\int_0^t (1-\phi(s))\,\mathrm{d}s\right\}$$

is a $\mathbb{P}$-martingale and $\{\phi(s)\}_{0\le t\le T}$ satisfies

$$\exp\left\{\int_t^T \int_0^{D_k} (\phi(s)\ln\phi(s) - \phi(s) + 1)\lambda\,\mathrm{d}F_k(z)\,\mathrm{d}s\right\} < \infty, \quad k = 1,2.$$

According to Branger & Larsen (2013), under the probability measure $\mathbb{Q}$, $N(t)$ has jump intensity $\lambda\phi(t)$, i.e. $\lambda^{\mathbb{Q}} = \lambda\phi(t)$.

Based on the insurers' strategies $\{a_k^*(t)\}$ given in the previous section, the reinsurer decides the reinsurance prices $\{\theta_k(t)\}$ dynamically. Besides, the reinsurer invests her wealth in the bank with the same return rate r. Then, under probability measure $\mathbb{Q}$, the reinsurer's wealth process X has the following dynamics:

$$\begin{aligned}\mathrm{d}X(t) &= \left(rX(t) + \sum_{k=1,2}(1+\theta_k(t))(\lambda_k + \lambda\phi(t))\mathbb{E}[\bar{Z}_{k,i}]\right)\mathrm{d}t\\ &\quad - \mathrm{d}\left(\sum_{k=1,2}\sum_{i=1}^{N_k(t)+N^{\phi}(t)}\bar{Z}_{k,i}\right), \quad X(0) = x, \qquad (23)\end{aligned}$$

where $\bar{Z}_{k,i} := Z_{k,i} - Z_{k,i}(a_k^*(t))$ is the reinsurer's payment for the ith claim of insurer k. A strategy $\{\theta_1(t),\theta_2(t)\}_{t\in[0,T]}$ is said to be admissible if $\theta_k(t)$ is $\mathcal{F}_t$-measurable and that (23) has a unique solution. The reinsurer's objective is to maximize the exponential utility of her wealth at terminal time

[6] In this model, reinsurance price is determined based on demand for reinsurance protection. The insurers' model ambiguity impact the reinsurer's decision indirectly through a_k^*; however, our numerical experiments in Section 2 show that model ambiguity has very limited impact on both insurers' decisions. Thus, our assumption that both insurers are certain about the intensity of systematic risk is rational. Besides, this assumption greatly simplifies our notations and calculations without affecting our main result.

T under the worst case scenario, with a penalty term added in the objective function to penalize any probability measure $\mathbb{Q}$ that is far from the priory probability measure $\mathbb{P}$:

$$V(t,x) = \max_{\{\theta_1,\theta_2\}} \min_{\phi} \mathbb{E}^{\mathbb{Q}}\left[-\frac{1}{\gamma}\mathrm{e}^{-\gamma X(T)} + \int_t^T \lambda \frac{\phi(s)\ln\phi(s) - \phi(s) + 1}{\varphi(s,V)}\,\mathrm{d}s\right], \tag{24}$$

where $\gamma > 0$ is an absolute risk aversion coefficient representing the reinsurer's risk preference, $\varphi(s,V) > 0$ captures the degree of concern over model robustness. A larger φ means that the reinsurer is more uncertain about the intensity of systematic risk and is more concerned about the model robustness. As in the previous section, we set

$$\varphi(t,V) = \frac{-\alpha}{\gamma V(t,x)},$$

where $\alpha > 0$ is the reinsurer's ambiguity aversion parameter describing her attitude toward model uncertainty. When $\alpha = 0$, the reinsurer is certain about λ, and Problem (24) becomes the standard dynamic optimization problem.

Based on the insurers' optimal equilibrium reinsurance strategies (a_1^*, a_2^*) and the reinsurer's surplus process (23), we are able to write down the corresponding HJB equation of the optimization problem (24):

$$\begin{aligned} V_t(t,x) + \max_{(\theta_1,\theta_2)} \min_{\phi} \Bigg\{ & V_x(t,x)\left[rx + \sum_{k=1,2}(1+\theta_k)(\lambda_k + \phi\lambda)\mathbb{E}[\bar{\mathrm{Z}}_k]\right] \\ & + \sum_{k=1,2}\lambda_k \mathbb{E}^{\mathbb{Q}}\left[V(t,x-\bar{\mathrm{Z}}_k) - V(t,x)\right] \\ & + \lambda\phi\mathbb{E}^{\mathbb{Q}}\left[V(t,x-\bar{\mathrm{Z}}_1-\bar{\mathrm{Z}}_2) - V(t,x)\right] - \gamma V(t,x)\frac{\lambda}{\alpha}(\phi\ln\phi - \phi + 1)\Bigg\} = 0, \end{aligned} \tag{25}$$

with boundary condition $V(T,x) = \frac{-1}{\gamma}\mathrm{e}^{-\gamma x}$.

Theorem 3.1: *Let $\tilde{\gamma} = \gamma\,\mathrm{e}^{r(T-t)}$. With given insurers' self-retention levels (a_1^*, a_2^*) in Section 2, the value function for the reinsurer is given by*

$$V(t,x) = -\frac{1}{\gamma}\mathrm{e}^{-\tilde{\gamma}x + h(t)},$$

where the function h depends only on t. In the reinsurance contract, the optimal safety loadings of reinsurer are given by

$$(\theta_1^\star, \theta_2^\star) := \arg\min_{\{\theta_1,\theta_2\}} \left\{\sum_{k=1,2}\lambda_k\mathbb{E}^{\mathbb{Q}}[-\tilde{\gamma}\bar{\mathrm{Z}}_k(1+\theta_k) + \mathrm{e}^{\tilde{\gamma}\bar{\mathrm{Z}}_k}] + \lambda\frac{\gamma}{\alpha}\phi^\star\right\}, \tag{26}$$

where $\bar{\mathrm{Z}}_k = Z_k - \mathrm{Z}_k(a_k^*)$, $\phi^\star = \mathrm{e}^{\frac{\alpha}{\tilde{\gamma}}f}$, and

$$f = f(a_1^*, a_2^*) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\tilde{\gamma}(\bar{\mathrm{Z}}_1+\bar{\mathrm{Z}}_2)} - \tilde{\gamma}\sum_{k=1,2}(1+\theta_k)\bar{\mathrm{Z}}_k\right] - 1.$$

Remark 3.1: When $\lambda = 0$, a_k^* is the function of θ_k, yet is independent of θ_j, $j \neq k$ (see Remark 2.1). Thus, we have

$$\theta_k^\star = \arg\min_{\theta_k} \mathbb{E}^{\mathbb{Q}}[-\tilde{\gamma}\bar{Z}_k(1+\theta_k) + e^{\tilde{\gamma}\bar{Z}_k}].$$

That is, the reinsurer only needs to specify the prices for each insurer respectively. The prices are independent of the claim intensities.

Next, we consider the case where the insurers adopt proportional reinsurance or excess-of-loss reinsurance respectively. According to Eqs. (18) and (22), θ_k can be seen as the function of a_1^* and a_2^*, i.e. $\theta_k = \theta_k(a_1^*, a_2^*)$. Thus, instead of seeking the optimal reinsurance prices $\{\theta_k^\star\}$ directly, in the sequel we first seek the optimal reinsurance strategies (denoted as $a_k^\star$) to minimize the RHS of (26) among all the optimal reinsurance strategies determined in the previous section; then, we determine the optimal reinsurance prices by $\theta_k^\star = \theta_k(a_1^\star, a_2^\star)$. The reinsurer's problem can be rewritten as

$$\begin{cases} (a_1^\star, a_2^\star) := \arg\min_{(a_1^*, a_2^*)} \Gamma(a_1^*, a_2^*), \\ \text{s.t.} \quad (18) \text{ or } (22), \end{cases} \tag{27}$$

where

$$\begin{cases} \Gamma(a_1^*, a_2^*) = \sum_{k=1,2} \lambda_k \mathbb{E}^{\mathbb{Q}}\left[-\tilde{\gamma}(1+\theta_k)\bar{Z}_k + e^{\tilde{\gamma}\bar{Z}_k}\right] + \lambda \frac{\gamma}{\alpha} e^{\frac{\alpha}{\gamma} f}, \\ f = \mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{\gamma}\bar{Z}_1 + \tilde{\gamma}\bar{Z}_2} - \sum_{k=1,2}(1+\theta_k)\tilde{\gamma}\bar{Z}_k\right] - 1. \end{cases}$$

After solving Problem (27), we may obtain the safety loading $\theta_k^\star = \theta_k^\star(a_1^\star, a_2^\star)$, $k = 1, 2$, for the reinsurer.

To simplify our analysis, we further assume that the claim sizes $\{Z_{k,i}\}_{i\in\mathbb{N}}$ are exponentially distributed with density functions $\frac{dF_k(z)}{dz} = \xi_k e^{-\xi_k z}$, $k = 1, 2$, with $\xi_k > \tilde{\gamma}_k$.

Example 3.1 (Proportional reinsurance case): Direct calculation shows

$$\mathbb{E}^{\mathbb{Q}}[e^{\tilde{\gamma}\bar{Z}_k}] = \mathbb{E}^{\mathbb{Q}}[e^{\tilde{\gamma}(1-a_k^*)Z_k}] = \frac{\xi_k}{\xi_k - \tilde{\gamma}(1-a_k^*)},$$

$$\mathbb{E}^{\mathbb{Q}}[\bar{Z}_k] = \mathbb{E}^{\mathbb{Q}}[(1-a_k^*)Z_k] = (1-a_k^*)\bar{\mu}_k,$$

and

$$\begin{aligned}(1+\theta_k)[(\lambda_k+\lambda)\bar{\mu}_k] &= \mathbb{E}[Z_k e^{\tilde{\gamma}_k a_k^* Z_k}](\lambda_k + \lambda\mathbb{E}[e^{-\tilde{\gamma}_k\kappa_k a_j^* Z_j}]) \\ &= \frac{\xi_k}{(\xi_k - \tilde{\gamma}_k a_k^*)^2}\left(\lambda_k + \lambda\frac{\xi_j}{\xi_j + \kappa_k\tilde{\gamma}_k a_j^*}\right).\end{aligned}$$

Since Γ is highly nonlinear in a_1^* and a_2^*, we cannot prove the existence of a maximum point and have to resort to numerical solutions. In fact, we determine the optimizer $(a_1^\star, a_2^\star)$ by solving the following system of equations

$$\frac{\partial\Gamma}{\partial a_1^*} = \frac{\partial\Gamma}{\partial a_2^*} = 0$$

numerically. Then, the Insurer k's optimal reinsurance price $\theta_k^\star(t)$ and the worst-case measure $\phi^\star(t)$ are determined. Next, we present some numerical results that provide more insights into the nature of the solution.

Figure 3 shows the dependence of the reinsurance prices, worst-case measures, and the insurers' self-retention levels on the competition levels. From Panel (b) we see that, as Insurer 1 becomes

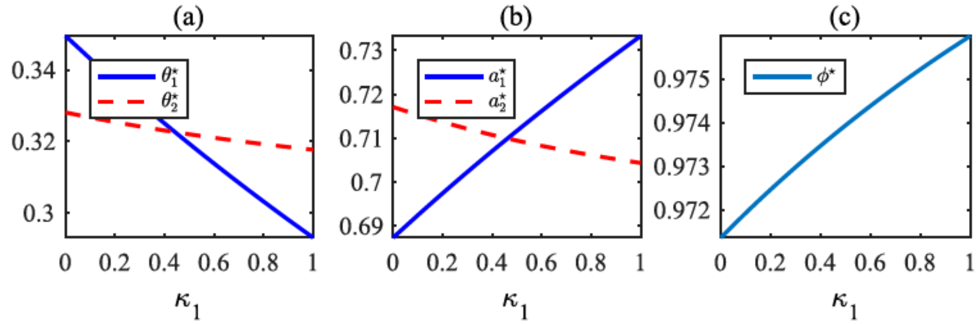

Figure 3. The impact of competition. The model parameters are: $r = 0.03, t = 0, T = 10, \lambda_1 = 1, \lambda_2 = 2, \lambda = 1, \kappa_2 = 0.7, \gamma_1 = \gamma_2 = \gamma = 0.3, \eta_1 = \eta_2 = 0.2, \xi_1 = \xi_2 = 2, \alpha = 0.3$.

more competitive against Insurer 2, he increases his risk exposure by keeping more risk $a_1^\star$ for himself. This result is similar to the case with fixed reinsurance prices (see Table 1). Accordingly, the reinsurer will appropriately decrease the reinsurance price $\theta_1^\star$ to drag Insurer 1 back to business (see Panel (a)). For example, when Insurer 1 is totally indifferent about his competitor's business, he keeps 68.73% of each claim payment for himself, and receives a reinsurance contract with $\theta_1^\star = 0.3495$; however, as Insurer 1 becomes highly competitive with $\kappa_1 = 1.0$, he increases his self-retention level to 73.33% and receives a contract with a lower price of $\theta_1^\star = 0.2931$. The progression between these two extremes is nearly linear. Neither of these two effects on $a_1^\star$ and $\theta_1^\star$ is highly pronounced. Still, they are non-negligible and much more pronounced than the very slight differences we observe in Figure 6 for Insurer 2's risk retention and his reinsurance price. As κ_1 increases, the reinsurer only slightly decreases the reinsurance safety loading for Insurer 2, leading to a decrease of Insurer 2's risk exposure from 71.71% to 70.43% over the entire range of Insurer 1's competitive behavior.

Panel (c) in Figure 6 shows that $\phi^\star$ is slightly increasing and smaller than 1. This phenomenon is non-trivial and is challenging to explain. A possible explanation is that when the reinsurer is allowed to adjust the reinsurance prices based on an attempt at optimizing her utility while worrying about model misspecification, her strategy makes the insurers bear too much risk, leaving insufficient risk for herself, leading to a small $\phi^\star$. This could be a sign that the robust optimization strategy as a reinsurance pricing mechanism could be further improved to avoid leaving business on the table. As for the result that $\phi^\star$ increases slightly with κ_1, the following interpretation is possible. There are two competing effects. As κ_1 increases, Insurer 1 increases his self-retention level, and Insurer 2 keeps his self-retention level almost unchanged, as we observed, thus the reinsurer's risk level decreases. However, as κ_1 increases, the reinsurance price $\theta_1^\star$ also decreases, leading to a higher risk level since the reinsurance premium shrinks. We observe here that the latter effect appears to be the dominant one, causing $\phi^\star$ to increase. However, the rate of increase is quite moderate, and in practice, one may think of the two effects as being equally strong.

Figure 4 shows the impact of model ambiguity, specifically the reinsurer's overall concern about the uncertainty over λ, on the two insurers' risk-retention, the two reinsurance prices, and her estimation over systematic risk: $a_k^\star, \theta_k^\star$ and $\phi^\star$. For the latter, the effect is more pronounced than in Figure 6. In other words, the reinsurer's prudent reaction on risk estimation is mostly influenced by her modeling ambiguity level, and comparatively, is hardly sensitive at all to her clients' competitive attitudes. Interestingly, the very stable values for $\theta_k^\star$ in Panel (a) show that the reinsurer's pricing scheme does not vary much at all with her modeling ambiguity. She doesn't reduce her prices at all as her ambiguity aversion rises significantly. The two insurers' self retentions are similarly insensitive (see Panel (b)), if slightly more responsive: Insurer 1, who has a lower competition level, reduces his risk exposure by about 0.04% in response to the very small reduction in reinsurance price; Insurer 2, who is much more competitive, chooses to remain largely at a constant risk-retention level.

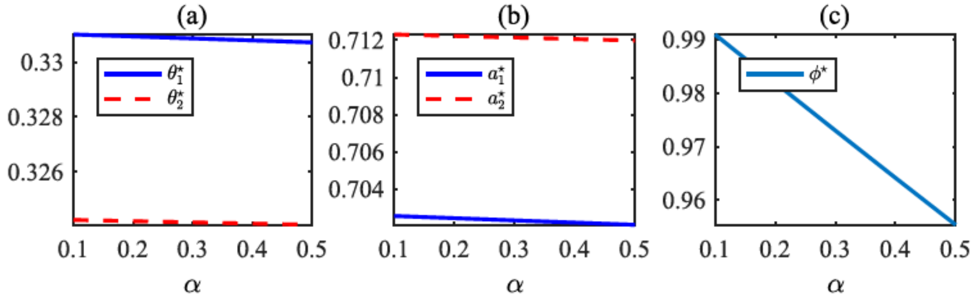

Figure 4. The impact of model ambiguity. The model parameters are: $t = 0, , T = 10, r = 0.03, \lambda_1 = 1, \lambda_2 = 2, \lambda = 1, \kappa_1 = 0.3, \kappa_2 = 0.7, \gamma_1 = \gamma_2 = \gamma = 0.3, \eta_1 = \eta_2 = 0.2, \xi_2 = \xi_2 = 2$.

Figure 5 shows the impact of attitudes towards risk, or the reinsurer and insurer's aversion to financial risk, considered separately (parameters γ and γ_1) on $a_k^\star, \theta_k^\star$ and $\phi^\star$. The effect of the reinsurer's risk aversion on the insurers' risk retention is significant and increasing (Panel (a)), which is corroborated by the strongly increasing dependence of reinsurance prices on the reinsurer's risk aversion (Panel (b)). As the reinsurer worries about financial risk, she responds by increasing prices and then sees her clients pull back correspondingly. The story is much murkier when looking at an insurer's increase in risk aversion. There, Insurer 1's response is convex and non-monotone, with a minimal risk retention level when the risk aversion coefficient γ_1 is around 0.45 (Panel (d)). The range of this effect is smaller than when the reinsurer's risk aversion changes, but an intuitive explanation for this non-monotone response is harder to provide. Intuitively, for low and increasing risk aversion, risk retention would decrease, and we think the change in monotonicity is due to the following: the insurer's demand for reinsurance eventually runs against the reinsurer's willingness to sell additional risk protection at a price that allows a profit. Panel (e) shows that the sensitivities of prices to risk aversion are all increasing, and very significantly so in response to the client's risk aversion: the insurers will be willing to pay much higher prices when increasing their own risk aversion. Note that they do not care about their competitor's risk aversion, however. Finally, the reinsurer's prudent reaction to model ambiguity on her risk estimation, in this worst-case scenario framework, is influenced by both her risk aversion and her client's (see Panels (c) and (f)). This effect is not strong but is still appreciable. We note a 0.0164 increase under the reinsurer's own increasing risk aversion, which causes her to be more prudent in modeling. We see a 0.0451 decrease under her client's increasing risk aversion; we think this is strongly related to the client's willingness to pay a much higher price for reinsurance, making it possible for the reinsurer to be less concerned about modeling uncertainty.

Example 3.2 (Excess-of-loss reinsurance): In this case, after simple calculation, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\bar{Z}_k] &= \mathbb{E}^{\mathbb{Q}}[Z_k - Z_k \wedge a_k^*] = \int_{a_k^*}^{\infty} (z - a_k^*)\xi_k \,\mathrm{e}^{-\xi_k z}\,\mathrm{d}z \\
&= \mathrm{e}^{-\xi_k a_k^*} \int_0^{\infty} z\xi_k \,\mathrm{e}^{-\xi_k z}\,\mathrm{d}z = \frac{\mathrm{e}^{-\xi_k a_k^*}}{\xi_k}, \\
\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\tilde{\gamma}\bar{Z}_k}] &= \int_0^{a_k^*} \mathrm{e}^{\tilde{\gamma}(z-z)}\,\mathrm{d}F_k(z) + \int_{a_k^*}^{\infty} \mathrm{e}^{\tilde{\gamma}(z-a_k^*)}\,\mathrm{d}F_k(z) \\
&= F_k(a_k^*) - F_k(0) + \mathrm{e}^{\tilde{\gamma}(z-a_k^*)}F_k(z)\,|_{a_k^*}^{\infty} - \int_{a_k^*}^{\infty} \tilde{\gamma}F_k(z)\,\mathrm{e}^{\tilde{\gamma}(z-a_k^*)}\,\mathrm{d}z \\
&= \int_{a_k^*}^{\infty} (1 - F_k(z))\tilde{\gamma}\,\mathrm{e}^{\tilde{\gamma}(z-a_k^*)}\,\mathrm{d}z + 1 = \frac{\xi_k}{\xi_k - \tilde{\gamma}}\mathrm{e}^{-\xi_k a_k^*},
\end{aligned}$$

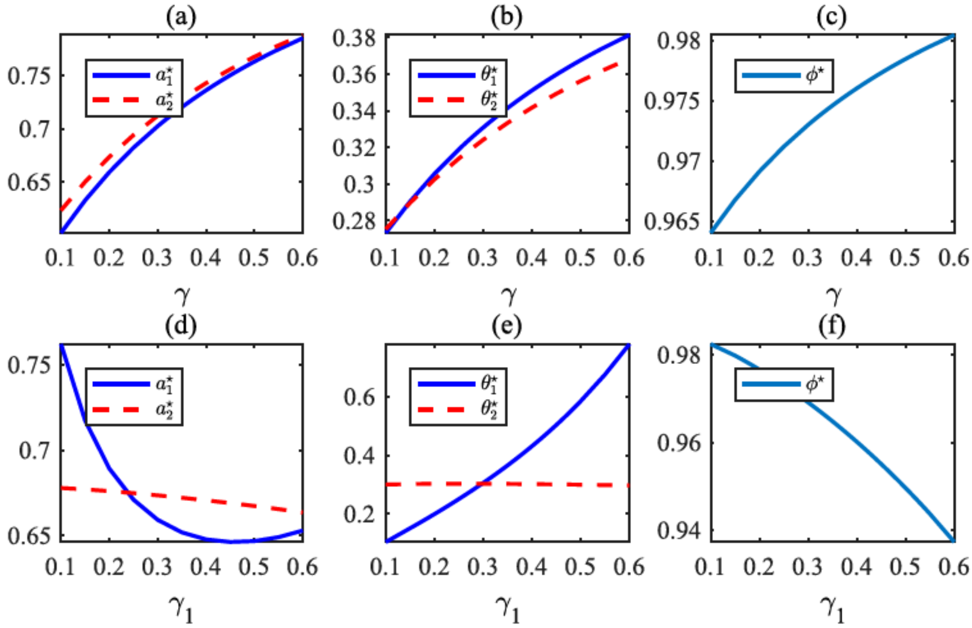

Figure 5. The impact of risk preferences. The model parameters are: $t=0, T=10, r=0.03, \lambda_1=1, \lambda_2=2, \lambda=1, \kappa_1=0.3, \kappa_2=0.7, \gamma_1=\gamma_2=\gamma=0.3, \eta_1=\eta_2=0.2, \xi_2=\xi_2=2, \alpha=0.3$.

and

$$\begin{aligned}(\lambda_k+\lambda)(1+\theta_k) &= \mathrm{e}^{\tilde{\gamma}_k a_k^*}\left[\lambda_k+\lambda\left(\int_0^{a_j}\bar{F}_j(z)\kappa_k\tilde{\gamma}_k\,\mathrm{e}^{-\kappa_k\tilde{\gamma}_k z}\,\mathrm{d}z-1\right)\right]\\ &= \mathrm{e}^{\tilde{\gamma}_k a_k^*}\left[\lambda_k-\frac{\lambda}{\xi_j+\kappa_k\tilde{\gamma}_k}\left(\xi_j+\kappa_k\tilde{\gamma}_k\,\mathrm{e}^{-(\xi_j+\kappa_k\tilde{\gamma}_k)a_j}\right)\right].\end{aligned}$$

Again, substituting these expressions into (27) we can solve the optimization problem numerically. The results are similar to the proportional reinsurance case (see Figure 6), so we omit the discussions here.

4. Conclusion

In this paper, we investigate the optimal reinsurance problems for two competitive insurers subject to common impacted claims (or systematic risk) and for a reinsurer who handles both contracts, where all three actors are averse to ambiguity in the model specifications. First, given reinsurance prices or safety loadings, we formulate the insurers' problem as a stochastic differential game and derive the Nash equilibrium reinsurance strategy for them. Second, based on the insurers' optimal reinsurance strategies, from the viewpoint of the reinsurer, we derive the optimal safety loadings for the reinsurance contracts by maximizing the expected utility of her terminal wealth when she is ambiguity averse regarding the intensity of the systematic risk. We obtain several interesting results: (i) With competition and systematic risk, the insurers always make a trade-off between their heterogeneous risk and the systematic risk to determine the optimal reinsurance strategy; the more intense the competition is, the more aggressive the insurers are. Moreover, ambiguity aversion has an essential impact on their estimation over the systematic risk while it has little effect on the reinsurance strategies. (ii) The optimal reinsurance premium is sensitive to the risk attitudes of reinsurer and insurers, but it is insensitive to the competition between the insurers.

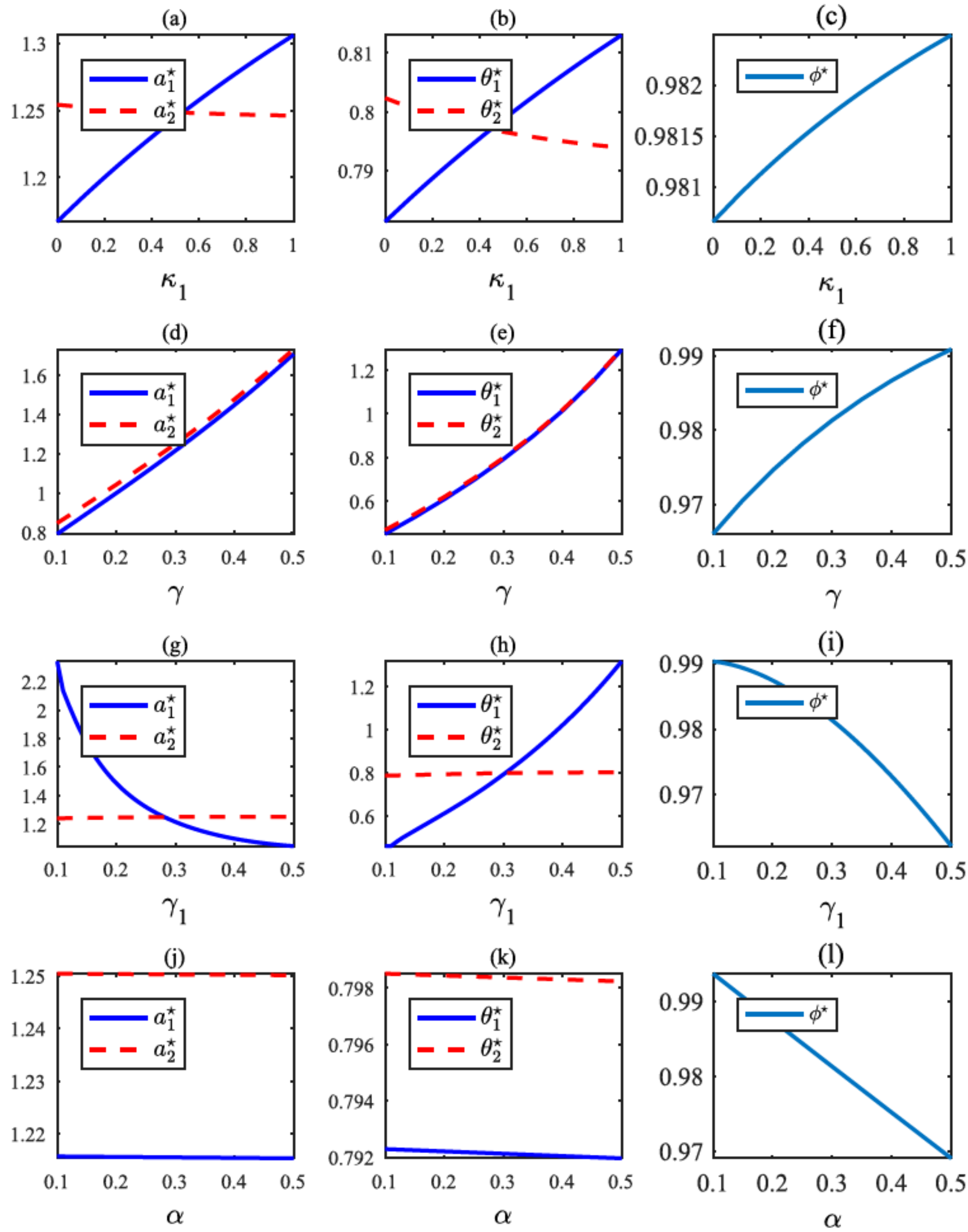

Figure 6. The impact of competition, risk preferences, and model ambiguity on the insurers' self-retention levels and the reinsurance prices. The model parameters are: $t = 0, r = 0.05, \lambda_1 = 1, \lambda_2 = 2, \lambda = 1, \kappa_1 = 0.3, \kappa_2 = 0.9, \gamma_1 = \gamma_2 = \gamma = 0.3, \xi_1 = \xi_2 = 1, \alpha = 0.3$.

Possible extensions of this work could include considering the problem for multiple insurers, leading to more complex competition and collision scenarios. We may consider other utility or objective functions, such as ruin probabilities, for the insurers and the reinsurer.

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Appendix. Proof

A.1 Proof of Theorem 2.1

Proof: We conjecture $J(t,\hat{w}_k) = -\frac{1}{\gamma_k}\mathrm{e}^{-\tilde{\gamma}_k\hat{w}_k + g_k(t)}$. Then

$$J^k_t(t,\hat{w}_k) = \left(r\tilde{\gamma}_k\hat{w}_k + g_k'(t)\right)J^k(t,\hat{w}_k), \quad J^k_{\hat{w}_k}(t,\hat{w}_k) = -\tilde{\gamma}_k J^k(t,\hat{w}_k),$$
$$\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k - \mathbf{Z}_k) - J^k(t,\hat{w}_k)\right] = J^k(t,\hat{w}_k)\left(\mathbb{E}^{\mathbb{Q}_k}[\mathrm{e}^{\tilde{\gamma}_k\mathbf{Z}_k}] - 1\right),$$
$$\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k + \kappa_k\mathbf{Z}_j) - J^k(t,\hat{w}_k)\right] = J^k(t,\hat{w}_k)\left(\mathbb{E}^{\mathbb{Q}_k}[\mathrm{e}^{-\kappa_k\tilde{\gamma}_k\mathbf{Z}_j}] - 1\right),$$

and

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k - \mathbf{Z}_k + \kappa_k\mathbf{Z}_j) - J^k(t,\hat{w}_k)\right]\\ &= J^k(t,\hat{w}_k)\left(\mathbb{E}^{\mathbb{Q}_k}[\mathrm{e}^{\tilde{\gamma}_k(\mathbf{Z}_k - \kappa_k\mathbf{Z}_j)}] - 1\right).\end{aligned} \tag{A1}$$

Applying the first-order condition on HJB equation (7), we find the minimizer ϕ_k^* for the inner minimization problem by solving

$$J^k_{\hat{w}_k}(C_k - \kappa_k C_j) + \mathbb{E}^{\mathbb{Q}_k}\left[J^k(t,\hat{w}_k - \mathbf{Z}_k + \kappa_k\mathbf{Z}_j) - J^k\right] - \frac{\gamma_k}{\alpha_k}J^k\ln\phi_k^* = 0, \tag{A2}$$

where $C_k = C_k^0 - (1+\theta_k)\bar{\mu}_k + (1+\theta_k)\mathbb{E}[\mathbf{Z}_k]$. Then, for $k=1,2, j\neq k$, $\phi_k^* = \mathrm{e}^{\frac{\alpha_k}{\gamma_k}f_k}$ and Equation (7) becomes

$$\begin{aligned}&g_k'(t) + \min_{a_k}\left\{-\tilde{\gamma}_k[\lambda_k C_k - \kappa_k\lambda_j C_j] + \lambda_k\left[\mathbb{E}^{\mathbb{Q}_k}[\mathrm{e}^{\tilde{\gamma}_k\mathbf{Z}_k}] - 1\right] + \lambda_j\left[\mathbb{E}^{\mathbb{Q}_k}[\mathrm{e}^{-\tilde{\gamma}_k\kappa_k\mathbf{Z}_j}] - 1\right]\right.\\ &\left.+\gamma_k\frac{\lambda}{\alpha_k}(\phi_k^* - 1)\right\} = 0, \quad g_k(T) = 0.\end{aligned} \tag{A3}$$

Hence,

$$a_k^* = \arg\min_{a_k}\left\{\lambda_k\mathbb{E}^{\mathbb{Q}_k}\left(\mathrm{e}^{\tilde{\gamma}_k\mathbf{Z}_k} - (1+\theta_k)\tilde{\gamma}_k\mathbf{Z}_k\right) + \gamma_k\frac{\lambda}{\alpha_k}\phi_k^*\right\}.$$

Finally, the worst case measure ϕ_k^* is obtained by substituting $a_k = a_k^*$ into Equation (A2). J^k is obtained by solving Equation (A3). A standard verification procedure shows that J^k is indeed the value function. The proof is completed. ■

A.2 Proof of Proposition 2.3

Proof: In this case,

$$f_k = \mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k(a_k Z_k - \kappa_k a_j Z_j)}\right] - \tilde{\gamma}_k\left(C_k - \kappa_k C_j\right) - 1, \tag{A4}$$

where $C_k = \bar{\mu}_k[\eta_k - \theta_k + (1+\theta_k)a_k]$. Using first-order condition on the RHS of Equation (8) yields the following equation for $\hat{a}_k$:

$$\begin{aligned}
&\lambda_k\tilde{\gamma}_k\left[\mathbb{E}^{\mathbb{Q}_k}[Z_k\, e^{\tilde{\gamma}_k a_k Z_k}] - (1+\theta_k)\bar{\mu}_k\right] + \lambda\phi_k^*\frac{\partial f_k}{\partial a_k} = 0\\
&\Rightarrow \lambda_k\left[\mathbb{E}^{\mathbb{Q}_k}[Z_k\, e^{\tilde{\gamma}_k a_k Z_k}] - (1+\theta_k)\bar{\mu}_k\right]\\
&\quad + \lambda\phi_k^*\left[\mathbb{E}^{\mathbb{Q}_k}[Z_k\, e^{\tilde{\gamma}_k(a_k Z_k - \kappa_k a_j Z_j)}] - (1+\theta_k)\bar{\mu}_k\right] = 0, \quad k = 1,2,\ j \neq k.
\end{aligned}$$

That is,

$$\lambda_k\mathbb{E}^{\mathbb{Q}_k}[Z_k\, e^{\tilde{\gamma}_k a_k Z_k}] + \lambda\phi_k^*\mathbb{E}^{\mathbb{Q}_k}[Z_k\, e^{\tilde{\gamma}_k(a_k Z_k - \kappa_k a_j Z_j)}] = (\lambda_k + \lambda\phi_k^*)(1+\theta_k)\bar{\mu}_k. \tag{A5}$$

Since the RHS of (8) is convex in a_k, the candidate equilibrium reinsurance strategies $\hat{a}_k$ is determined by (A5). The existence of $(\hat{a}_1, \hat{a}_2)$ are equivalent to the existence of solutions to the system of nonlinear equation in Equation (12), which will be proved in Proposition 2.5. Thus, the equilibrium reinsurance strategy $a_k^* = \hat{a}_k \wedge 1$ is determined and the worst-case measure ϕ_k^* follows. ■

A.3 Proof of Proposition 2.5

Proof: (i) Denote

$$\begin{cases}
H_1(a_1,a_2) := \lambda_1\left[\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1 a_1 Z_1}] - (1+\theta_1)\bar{\mu}_1\right] + \dfrac{\lambda}{\tilde{\gamma}_1}\phi_1^*\dfrac{\partial f_1}{\partial a_1},\\[2ex]
H_2(a_1,a_2) := \lambda_2\left[\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2 a_2 Z_2}] - (1+\theta_2)\bar{\mu}_2\right] + \dfrac{\lambda}{\tilde{\gamma}_2}\phi_2^*\dfrac{\partial f_2}{\partial a_2},
\end{cases}$$

where $\phi_k^* = e^{\frac{\alpha_k}{\gamma_k}f_k}$ and

$$\frac{\partial f_k}{\partial a_k} = \mathbb{E}^{\mathbb{Q}_k}[\tilde{\gamma}_k Z_k\, e^{\tilde{\gamma}_k(a_k Z_k - \kappa_k a_j Z_j)}] - (1+\theta_k)\tilde{\gamma}_k\bar{\mu}_k.$$

Thus, we just need to show that the system of non-linear equations $H_1(a_1,a_2) = H_2(a_1,a_2) = 0$ admit a solution.

Step 1, we fix $a_2 > 0$ and determine a solution to the equation $H_1(a_1,a_2) = 0$. Firstly, using Equation (15), it is clear that

$$\begin{aligned}
H_1(\underline{a}_1^p, a_2) &= \lambda\phi_1^*\left[\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1(\underline{a}_1^p Z_1 - \kappa_1 a_2 Z_2)}] - (1+\theta_1)\bar{\mu}_1\right]\\
&< \lambda\phi_1^*\left[\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1(\underline{a}_1^p Z_1)}] - (1+\theta_1)\bar{\mu}_k\right] = 0.
\end{aligned}$$

Secondly, for each $a_2 > \underline{a}_2^p$, it is easy to see that the non-linear equation

$$\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1(a_1 Z_1 - \kappa_1 a_2 Z_2)}] = (1+\theta_1)\bar{\mu}_1, \quad a_1 \in (0,+\infty), \tag{A6}$$

admits a unique solution $\bar{a}_1^p$. It is clear that $\bar{a}_1^p > \underline{a}_1^p$ and

$$H_1(\bar{a}_1^p, a_2) = \lambda_1\left[\mathbb{E}^{\mathbb{Q}_1}[Z_1\, e^{\tilde{\gamma}_1\bar{a}_1^p Z_1}] - (1+\theta_1)\bar{\mu}_1\right] > 0.$$

Thus, $H_1(a_1,a_2) = 0$ admits a solution $\hat{a}_1 = \hat{a}_1(a_2) \in (\underline{a}_1^p, \bar{a}_1^p)$. Finally,

$$\begin{aligned}
\frac{\partial H_1}{\partial a_1} &= \lambda_1\mathbb{E}^{\mathbb{Q}_1}[\tilde{\gamma}_1 Z_1^2\, e^{\tilde{\gamma}_1 a_1 Z_1}] + \lambda\phi_1^*\mathbb{E}^{\mathbb{Q}_1}[Z_1^2\tilde{\gamma}_1\, e^{\tilde{\gamma}_1(a_1 Z_1 - \kappa_1 a_2 Z_2)}]\\
&\quad + \lambda\frac{\phi_1^*}{\tilde{\gamma}_1}\frac{\alpha_1}{\gamma_1}\left(\frac{\partial f_1}{\partial a_1}\right)^2 > 0,
\end{aligned} \tag{A7}$$

i.e. H_1 is strictly increasing with a_1, indicating that $\hat{a}_1(a_2) \in (\underline{a}_1^p, \bar{a}_2^p)$ is uniquely determined. It is obvious that H_1 is the decreasing function of a_2. Thus, $\hat{a}_1$ strictly increases with a_2.

Step 2, we determine $\hat{a}_2$ by solving $H_2(\hat{a}_1(a_2), a_2) = 0, a_2 \in [\underline{a}_2^p, \bar{a}_2^p]$, where $\bar{a}_2^p$ is determined by $\frac{\partial f_2}{\partial a_2} = 0$ such that $\bar{a}_2^p > \underline{a}_2^p$. In fact, it is clear that

$$H_2(\hat{a}_1(\underline{a}_2^p), \underline{a}_2^p) = \lambda \frac{\phi_2^*}{\tilde{\gamma}_2} \frac{\partial f_2}{\partial a_2}\bigg|_{a_2=\underline{a}_2^p} < \lambda \frac{\phi_2^*}{\tilde{\gamma}_2} \frac{\partial f_2}{\partial a_2}\bigg|_{a_2=\bar{a}_2^p} = 0.$$

Since $\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2 \bar{a}_2^p Z_2}] > (1+\theta_2)\bar{\mu}_2$, we also have

$$H_2(\hat{a}_1(\bar{a}_2^p), \bar{a}_2^p) = \lambda_2 \left[\mathbb{E}^{\mathbb{Q}_2}[Z_2\, e^{\tilde{\gamma}_2 \bar{a}_2^p Z_2}] - (1+\theta_2)\bar{\mu}_2\right] > 0.$$

Thus, there exists $\hat{a}_2 \in (\underline{a}_2^p, \bar{a}_2^p)$ such that $H_2(\hat{a}_1(\hat{a}_2), \hat{a}_2) = 0$. The pair $(\hat{a}_1, \hat{a}_2)$ is the solution to Equation (12).

(ii) Since $\frac{\partial f_1}{\partial a_1} < \frac{\partial f_1}{\partial a_1}\,|_{a_1=\bar{a}_1^p} = 0$ for $a_1 < \hat{a}_1^p$, $H_1(a_1, a_2)$ is strictly decreasing with λ. Thus, using the fact that H_1 is the increasing function of a_1, we can see that $\hat{a}_1$ strictly increases with λ. Similarly, we are able to show that $\hat{a}_2$ strictly increases with λ.

Finally, since $H_k(a_1, a_2)$ is strictly decreasing with θ_k, the argument that $\hat{a}_k$ strictly increases with θ_k for $k = 1, 2$ can be similarly proved. ∎

A.4 Proof of Proposition 2.6

Proof: In this case, direct calculation shows $\mathbb{E}^{\mathbb{Q}_k}[Z_k] = \int_0^{a_k} \bar{F}_k(z)\,dz$ and

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k Z_k}\right] &= \int_0^{a_k} e^{\tilde{\gamma}_k z}\, dF_k(z) + \int_{a_k}^{D_k} e^{\tilde{\gamma}_k a_k}\, dF_k(z) \\
&= e^{\tilde{\gamma}_k z} F_k(z)|_0^{a_k} - \int_0^{a_k} F_k(z)\tilde{\gamma}_k\, e^{\tilde{\gamma}_k z}\, dz + e^{\tilde{\gamma}_k a_k}\,(F_k(D_k) - F_k(a_k)) \\
&= e^{\tilde{\gamma}_k a_k} - \int_0^{a_k} F_k(z)\tilde{\gamma}_k\, e^{\tilde{\gamma}_k z}\, dz, \\
\frac{\partial}{\partial a_k}\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k Z_k}\right] &= \tilde{\gamma}_k\, e^{\tilde{\gamma}_k a_k} \bar{F}_k(a_k).
\end{aligned}$$

By first-order condition, for $k = 1, 2$, the minimizer $\tilde{a}_k$ of the RHS of (8) satisfies

$$\begin{aligned}
&\lambda_k \left[\frac{\partial}{\partial a_k}\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k Z_k}\right] - (1+\theta_k)\tilde{\gamma}_k \frac{\partial}{\partial a_k}\mathbb{E}^{\mathbb{Q}_k}[Z_k]\right] \\
&\quad + \lambda\phi_k^* \left[\frac{\partial}{\partial a_k}\mathbb{E}^{\mathbb{Q}_k}\left[e^{\tilde{\gamma}_k Z_k}\right]\mathbb{E}^{\mathbb{Q}_k}\left[e^{-\kappa_k \tilde{\gamma}_k Z_j}\right] - (1+\theta_k)\tilde{\gamma}_k \frac{\partial}{\partial a_k}\mathbb{E}^{\mathbb{Q}_k}[Z_k]\right] = 0 \\
&\Rightarrow \lambda_k[e^{\tilde{\gamma}_k a_k} - (1+\theta_k)] + \lambda\phi_k^*[e^{\tilde{\gamma}_k a_k}\mathbb{E}^{\mathbb{Q}_k}\left[e^{-\kappa_k \tilde{\gamma}_k Z_j}\right] - (1+\theta_k)] = 0.
\end{aligned}$$

The existence of solutions $(\tilde{a}_1, \tilde{a}_2)$ will be proved in Proposition 2.8. According to this result, the desired equilibrium reinsurance strategies (a_1^*, a_2^*) are characterized. ∎

A.5 Proof of Proposition 2.8

Proof: (i) Let us define

$$\begin{cases} G_1(a_1, a_2) := \lambda_1[e^{\tilde{\gamma}_1 a_1} - (1+\theta_1)] + \lambda\phi_1^*[e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - (1+\theta_1)], \\ G_2(a_1, a_2) := \lambda_2[e^{\tilde{\gamma}_2 a_2} - (1+\theta_2)] + \lambda\phi_2^*[e^{\tilde{\gamma}_2 a_2}\mathbb{E}^{\mathbb{Q}_2}[e^{-\kappa_2\tilde{\gamma}_2(Z_1\wedge a_1)}] - (1+\theta_1)]. \end{cases}$$

First, for given $a_2 > 0$, we determine $\tilde{a}_1 = \tilde{a}_1(a_2)$ by solving $G_1(a_1, a_2) = 0, a_1 > 0$. It is clear that

$$G_1(\underline{a}_1^e, a_2) = \lambda\phi_1^*[e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - (1+\theta_1)] < 0, \tag{A8}$$

where $\underline{a}_1^e$ is defined in Corollary 2.7. Note that $\bar{a}_1^e$ satisfies

$$e^{\tilde{\gamma}_1 \bar{a}_1^e}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - (1+\theta_1) = 0. \tag{A9}$$

Then,

$$G_1(\bar{a}_1^e, a_2) = \lambda_1[e^{\tilde{\gamma}_1 \bar{a}_1^e} - (1+\theta_1)] > 0.$$

Also note that $G_1 = \lambda_1[e^{\tilde{\gamma}_1 a_1} - (1+\theta_1)] + \lambda \frac{\phi_1^*}{\tilde{\gamma}_1}\frac{\partial f_1}{\partial a_1}$ with

$$\frac{\partial f_1}{\partial a_1} = \tilde{\gamma}_1 e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - \tilde{\gamma}_1(1+\theta_1),$$

then

$$\frac{\partial G_1}{\partial a_1} = \lambda_1\tilde{\gamma}_1 e^{\tilde{\gamma}_1 a_1} + \lambda\frac{\phi_1^*}{\tilde{\gamma}_1}\frac{\alpha_1}{\gamma_1}\left(\frac{\partial f_1}{\partial a_1}\right)^2 + \lambda\frac{\phi_1^*}{\tilde{\gamma}_1}\frac{\partial^2 f_1}{\partial a_1^2} > 0.$$

Thus, $\tilde{a}_1 = \tilde{a}_1(a_2) \in (\underline{a}_1^e, \bar{a}_1^e)$ is uniquely determined.

Second, we determine $\tilde{a}_2$ by solving the non-linear equation $G_2(\tilde{a}_1(a_2), a_2) = 0$. Similar to (A8), it is clear that $G_2(\tilde{a}_1(\underline{a}_2^e), \underline{a}_2^e) < 0$. Note that $\bar{a}_2^e > \underline{a}_2^e$ and that $\bar{a}_2^e$ satisfies

$$e^{\tilde{\gamma}_2\bar{a}_2^e}\mathbb{E}^{\mathbb{Q}_2}\left[e^{-\kappa_2\tilde{\gamma}_2(Z_1\wedge\tilde{a}_1)}\right] - (1+\theta_1) = 0.$$

Then,

$$G_2(\tilde{a}_1(\bar{a}_2^e), \bar{a}_2^e) = \lambda_2[e^{\tilde{\gamma}_2\bar{a}_2^e} - (1+\theta_2)] > 0$$

and there exists $\tilde{a}_2 \in (\underline{a}_2^e, \bar{a}_2^e)$ such that $G_2(\tilde{a}_1(\tilde{a}_2), \tilde{a}_2) = 0$. That is, $(\tilde{a}_1, \tilde{a}_2)$ are the solutions to (19).

(ii) Since

$$e^{\tilde{\gamma}_1 a_1}\mathbb{E}^{\mathbb{Q}_1}[e^{-\kappa_1\tilde{\gamma}_1(Z_2\wedge a_2)}] - (1+\theta_1) < 0, \quad a_1 < \bar{a}_1^e,$$

G_1 strictly decreases with λ; thus $\tilde{a}_1$ is strictly increasing with λ. It is clear that G_2 is also a decreasing function of λ, thus $\tilde{a}_2$ is strictly increasing with λ either.

Finally, since G_k is strictly decreasing with θ_k, the argument that $\tilde{a}_k$ strictly increase with θ_k for $k = 1, 2$ can be similarly proved. ■

A.6 Proof of Theorem 3.1

Proof: Inspired by the results in the previous section, we postulate that $V(t,x) = -\frac{1}{\gamma}e^{-\tilde{\gamma}x+h(t)}$. Then, direct calculation shows

$$\mathbb{E}^{\mathbb{Q}}\left[V(t,x-\bar{Z}_k) - V(t,x)\right] = V(t,x)\mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{\gamma}\bar{Z}_k} - 1\right]$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[V(t,x-\bar{Z}_1-\bar{Z}_2) - V(t,x)\right] = V(t,x)\mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{\gamma}(\bar{Z}_1+\bar{Z}_2)} - 1\right].$$

By first-order condition, we have

$$V_x(t,x)\sum_{k=1,2}(1+\theta_k)\mathbb{E}^{\mathbb{Q}}[\bar{Z}_k] + \mathbb{E}^{\mathbb{Q}}\left[V(t,x-\bar{Z}_1-\bar{Z}_2) - V(t,x)\right] - V(t,x)\frac{\gamma}{\alpha}\ln\phi^\star = 0. \tag{A10}$$

That is, $\phi^\star = e^{\frac{\alpha}{\gamma}f}$ with

$$f = \mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{\gamma}(\bar{Z}_1+\bar{Z}_2)} - \sum_{k=1,2}(1+\theta_k)\tilde{\gamma}\bar{Z}_k\right] - 1.$$

Thus, the HJB equation (25) can be rewritten as

$$h_t + \min_{(\theta_1,\theta_2)}\left\{\sum_{k=1,2}\lambda_k\left[(1+\theta_k)\mathbb{E}^{\mathbb{Q}}[-\tilde{\gamma}\bar{Z}^k]\right]\right.$$
$$\left. + \sum_{k=1,2}\lambda_k\mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{\gamma}\bar{Z}^k} - 1\right] + \gamma\frac{\lambda}{\alpha}(\phi^\star - 1)\right\} = 0. \tag{A11}$$

The optimal reinsurance safety loadings are given by

$$(\theta_1^\star, \theta_2^\star) := \arg\min\left\{\sum_{k=1,2}\lambda_k\mathbb{E}^{\mathbb{Q}}[e^{\tilde{\gamma}\bar{Z}^k} - \tilde{\gamma}\bar{Z}^k(1+\theta_k)] + \lambda\frac{\gamma}{\alpha}\phi^\star\right\}.$$

Once the optimal reinsurance safety loadings $(\theta_1^\star, \theta_2^\star)$ are determined, we may obtain $\phi^\star$ and V. By a standard verification theorem, we are able to show that V is indeed the value function and $(\theta_1^\star, \theta_2^\star)$ and $\phi^\star$ are the optimal pricing strategies and the worst-case factor. ■