

BERTINI THEOREMS FOR F -SIGNATURE AND HILBERT–KUNZ MULTIPLICITY

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ABSTRACT. We show that Bertini theorems hold for F -signature and Hilbert–Kunz multiplicity. In particular, if $X \subseteq \mathbb{P}^n$ is quasi-projective with F -signature greater than λ (respectively the Hilbert–Kunz multiplicity is less than λ) at all points $x \in X$, then for a general hyperplane $H \subseteq \mathbb{P}^n$ the F -signature (respectively Hilbert–Kunz multiplicity) of $X \cap H$ is greater than λ (respectively less than λ) at all points $x \in X \cap H$.

1. INTRODUCTION

A common tool for studying a quasi-projective algebraic variety $X \subseteq \mathbb{P}_k^n$, $k = \bar{k}$, is to perform induction on dimension by intersecting with a general hyperplane H . When doing this, we want the resulting intersection $X \cap H$ to have similar properties to the original variety X . Bertini’s theorem accomplishes exactly this: the classical result asserts that if X is smooth then so is $X \cap H$ for a general choice of H [Har77, II, Theorem 8.18], [Kle98]. Many classes of singularities also satisfy this property. For example, in characteristic zero, if X is log terminal (respectively log canonical), then so is $X \cap H$ [KM98, Lemma 5.17]. Even more generally the multiplier ideal of a divisor pair restricts to the multiplier ideal of the intersection

$$\mathcal{J}(X, \Delta)|_{X \cap H} = \mathcal{J}(X \cap H, \Delta|_{X \cap H}),$$

see [Laz04, Example 9.5.9]. In characteristic zero, Bertini theorems can be generalized to the case where H is a general member of a base point free linear system.

In characteristic $p > 0$, the situation is more complicated. It is essential that H is a general member of a very ample linear system (or something close to that) if you expect Bertini-type results to hold. Since strongly F -regular and F -pure singularities are analogous to log terminal and log canonical singularities respectively [HW02], it is natural to expect that the corresponding Bertini-results hold. In [SZ13], this is exactly what was shown.

Theorem ([SZ13]). *If (X, Δ) is a strongly F -regular (resp. sharply F -pure) pair such that $X \subseteq \mathbb{P}_k^n$ is quasi-projective and $k = \bar{k}$ is of characteristic $p > 0$, then $(X \cap H, \Delta|_{X \cap H})$ is also strongly F -regular (resp. sharply F -pure) for a general choice of hyperplane $H \subseteq \mathbb{P}_k^n$.*

However, the corresponding result for test ideals is false:

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Theorem ([Byd18]). *For any $p > 0$ and $n \geq 3$, there exists a \mathbb{Q} -divisor Δ on $X = \mathbb{A}_k^n$, where $k = \bar{k}$ is of characteristic $p > 0$, such that*

$$\tau(X, \Delta)|_H \neq \tau(X \cap H, \Delta|_{X \cap H})$$

for a general hyperplane $H \subseteq \mathbb{A}^n$.

It is then natural to ask about other types of F -singularities in characteristic $p > 0$. For example, the behavior of F -rational singularities under restriction to general hyperplanes is still unknown. In this paper, we show that the above sort of Bertini-theorem holds for F -signature $s(\mathcal{O}_{X,x})$ and Hilbert–Kunz multiplicity $e_{\text{HK}}(\mathcal{O}_{X,x})$ in the following sense.

Main Theorem (Theorem 5.4, Theorem 5.5). *Suppose that $X \subseteq \mathbb{P}_k^n$ is a normal quasi-projective variety, $k = \bar{k}$ is of characteristic $p > 0$, and $\Delta \geq 0$ is a \mathbb{Q} -divisor. Suppose that $\lambda \geq 0$ is a number such that the F -signature is bigger than λ ,*

$$s(\mathcal{O}_{X,x}, \Delta) > \lambda,$$

for all $x \in X$. Then for a general hyperplane $H \subseteq \mathbb{P}_k^n$,

$$s(\mathcal{O}_{X \cap H,x}, \Delta|_{X \cap H}) > \lambda$$

for all $x \in X \cap H$.

Similarly suppose that $\lambda \geq 1$ is a number such that the Hilbert–Kunz multiplicity is less than λ ,

$$e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda,$$

for all $x \in X$. Then for a general hyperplane $H \subseteq \mathbb{P}_k^n$,

$$e_{\text{HK}}(\mathcal{O}_{X \cap H,x}) < \lambda$$

for all $x \in X \cap H$.

We actually prove a slightly stronger result by weakening the hypothesis that $X \subseteq \mathbb{P}_k^n$ and we also make statements about the locus U where $s(\mathcal{O}_{X,x}, \Delta) > \lambda$ for all $x \in U$ or likewise with the locus where $e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda$.

Recall that F -signature measures how strongly F -regular a variety or pair is. Explicitly, if R is finite type over $k = \bar{k}$, then R is regular if and only if R^{1/p^e} is a locally free R -module by [Kun69]. The F -signature refines this. By definition, $s(R)$ is a number that indicates what percentage of R^{1/p^e} is locally free asymptotically as e goes to ∞ . Thus $1 \geq s(R) \geq 0$ and

- $s(R) = 1$ if and only if R is regular [HL02] (cf. [Yao05]) and
- $s(R) > 0$ if and only if R is strongly F -regular [AL03].

The F -signature should be thought of some sort of local volume of the singularity.

On the other hand, Hilbert–Kunz measures how close a ring is to being regular. If $(R, \mathfrak{m}, k = k^p)$ is a local ring of dimension d , then $e_{\text{HK}}(R)$ is the asymptotic value of the ratio between the number of generators of R^{1/p^e} as an R -module with the number of generators expected for a regular ring (p^{ed}) . One has that $e_{\text{HK}}(R) \geq 1$ and

- $e_{\text{HK}}(R) = 1$ if and only if R is regular [WY00, Theorem 1.5].

Hilbert–Kunz multiplicity is another sort of volume of a singularity.

We prove our main result by relying on the axiomatic Bertini framework as introduced in [CGM86]. In particular, to show the type of result in our Main Theorem, it suffices to show the following two properties for a property of singularities \mathcal{P} (such as $s(\mathcal{O}_{X,x}) > \lambda$):

- (A1) If $\phi : Y \rightarrow Z$ is a flat morphism with regular fibers and Z is \mathcal{P} , then Y is \mathcal{P} too.
- (A2) Let $\phi : Y \rightarrow S$ be a morphism of finite type where Y is excellent and S is integral with generic point η . If Y_η is geometrically \mathcal{P} , then there exists an open neighborhood U of η in S such that the fibers Y_s are geometrically \mathcal{P} for each $s \in U$. (In fact, it suffices to check this for $S = (\mathbb{P}_k^n)^*$, the space of hyperplanes).

Property (A1) was already proven for F -signature in [Yao06]. In Section 3, we generalize this result to the context of pairs and give a new proof in the classical non-pair setting. In Section 4, we show that property (A2) holds for F -signature.

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2. PRELIMINARIES

2.1. Hilbert–Kunz multiplicity and F -signature. Throughout this article, we shall assume all schemes X are Noetherian, separated, and have prime characteristic $p > 0$. If $x \in X$, we let $k(x)$ denote the residue field of the local ring $\mathcal{O}_{X,x}$. We let $F^e : X \rightarrow X$ denote the e -iterated Frobenius endomorphism or p^e -th power map. We say X is F -finite if F^e is a finite morphism, in which case X is automatically excellent and has a dualizing complex [Kun76, Gab04].

When $X = \text{Spec}(A)$ is affine, we often conflate scheme-theoretic and ring-theoretic notation. In particular, $F^e : A \rightarrow A$ denotes the e -iterated Frobenius, and for an A -module M we write $F_*^e M$ for $\Gamma(\text{Spec}(A), F_*^e \widetilde{M})$ where \widetilde{M} is the associated quasi-coherent sheaf on $\text{Spec}(A)$. In other words, $F_*^e M$ is the A -module arising from M via restriction of scalars for F^e . In case A is reduced, we also identify F^e with the inclusion $A \subseteq A^{1/p^e}$, and shall at times use M^{1/p^e} to denote $F_*^e M$ accordingly.

If $J \subseteq A$ is an ideal, then the e -th Frobenius power of J is the expansion of J under the e -iterated Frobenius and denoted $J^{[p^e]} = \langle F^e(J) \rangle = \langle j^{[p^e]} \mid j \in J \rangle$. It follows $J(F_*^e M) = F_*^e(J^{[p^e]} M)$ or $J(M^{1/p^e}) = (J^{[p^e]} M)^{1/p^e}$ for any A -module M . In the local setting, the Frobenius powers give rise to the following well-studied variant on the Hilbert–Samuel multiplicity.

Definition 2.1. If (A, \mathfrak{m}) is a local ring of dimension d , the Hilbert–Kunz multiplicity of A is

$$e_{\text{HK}}(A) = \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_A(A/\mathfrak{m}^{[p^e]}),$$

where we write $\ell_A(_)$ for the length of an A -module

Theorem 2.2. Suppose (A, \mathfrak{m}) is a local ring of dimension d .

- (a) [Mon83] The limit defining the Hilbert–Kunz multiplicity $e_{\text{HK}}(A)$ exists, and moreover

$$\ell_A(A/\mathfrak{m}^{[p^e]}) = e_{\text{HK}}(A) \cdot p^{ed} + O(p^{e(d-1)}).$$

- (b) [WY00] The Hilbert–Kunz multiplicity $e_{\text{HK}}(A) \geq 1$, and if A is equidimensional then $e_{\text{HK}}(A) = 1$ if and only if A is regular.

The F -signature, like the Hilbert–Kunz multiplicity, is another important numerical invariant of a local ring in positive characteristic defined in terms of the iterates of Frobenius. For any positive characteristic ring A , recall that an A -module inclusion $M_1 \rightarrow M_2$

is said to be pure if $M_1 \otimes_A N \rightarrow M_2 \otimes_A N$ remains injective for any A -module N . An inclusion $A \rightarrow M$, where M is a finitely generated A -module, is pure if and only if it is split, *i.e.* admits an A -module section; see [HR76, Corollary 5.2]. If (A, \mathfrak{m}) is local, $A \rightarrow M$ is pure if and only if $E_A(k) \rightarrow M \otimes_A E_A(k)$ is injective, where $E_A(k)$ is an injective hull of the residue field $k = A/\mathfrak{m}$; this follows from Matlis duality [HH95, Lemma 2.1 (e)]. We write $\ell_A(_)$ for the length of an A -module, omitting the subscript at times to simplify notation.

Definition 2.3. If (A, \mathfrak{m}) is an excellent local ring of dimension d , the e -th Frobenius degeneracy ideal

$$I_e(A) = \langle a \in A \mid A \xrightarrow{1 \mapsto F_*^e a} F_*^e A \text{ is not a pure } A\text{-module inclusion} \rangle$$

is an ideal of A , and the F -signature is

$$s(A) = \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_A(A/I_e(A)).$$

Recall the following results on F -signature.

Theorem 2.4. *Suppose (A, \mathfrak{m}) is an excellent local ring of dimension d .*

(a) [Tuc12] *The limit defining the F -signature $s(A)$ exists, and moreover*

$$\ell_A(A/I_e(A)) = s(A) \cdot p^{ed} + O(p^{e(d-1)}).$$

(b) [HL02] *The F -signature $s(A) \leq 1$, and $s(A) = 1$ if and only if A is regular.*

(c) [AL03, Yao05] *The F -signature $s(A) \geq 0$, and $s(A) > 0$ if and only if A is strongly F -regular. In this case, A is necessarily a Cohen–Macaulay normal domain.*

The Hilbert–Kunz multiplicity and F -signature are also known to satisfy additional properties in the F -finite setting, such as semi-continuity.

Theorem 2.5. [Smi16, Pol18, PT18] *Consider an F -finite domain A .¹*

(a) *The Hilbert–Kunz multiplicity determines an upper semi-continuous function*

$$Q \in \text{Spec}(A) \mapsto e_{\text{HK}}(A_Q)$$

on $\text{Spec}(A)$.

(b) *The F -signature determines a lower semi-continuous function*

$$Q \in \text{Spec}(A) \mapsto s(A_Q)$$

on $\text{Spec}(A)$.

Moreover, if (A, \mathfrak{m}) is an F -finite local ring of dimension d , note that one can alternately describe the degeneracy ideals as

$$\begin{aligned} I_e(A) &= \langle a \in A \mid A \xrightarrow{1 \mapsto F_*^e a} F_*^e A \text{ is not a split } A\text{-module inclusion} \rangle \\ &= \langle a \in A \mid \phi(F_*^e a) \in \mathfrak{m} \text{ for all } \phi \in \text{Hom}_A(F_*^e A, A) \rangle, \end{aligned}$$

¹Note that upper semi-continuity of the Hilbert–Kunz multiplicity is also known to hold for a ring which is essentially of finite type over an excellent local ring.

and the F -signature can be viewed as giving an asymptotic measure of the number of splittings of the e -iterated Frobenius. In particular, if (A, \mathfrak{m}) is an F -finite local domain, we have

$$e_{\text{HK}}(A) = \lim_{e \rightarrow \infty} \frac{\mu_A(A^{1/p^e})}{\text{rank}_A(A^{1/p^e})} \quad \text{and} \quad s(A) = \lim_{e \rightarrow \infty} \frac{\text{frk}_A(A^{1/p^e})}{\text{rank}_A(A^{1/p^e})}.$$

where $\mu_A(_)$ denotes the minimal number of generators and $\text{frk}_A(_)$ denotes free rank; see [PT18] for details. Recall that, for arbitrary (and not necessarily local) A , the free rank of an A -module M is the maximal rank $\text{frk}_A(M)$ of a free A -module quotient of M . *For us going forward, we will use $a_e(R)$ (resp. $b_e(R)$) to denote the free rank (resp. minimal number of generators) of R^{1/p^e} , and may write simply a_e (resp. b_e) if the context is clear*

One can generalize the interpretation of Hilbert–Kunz multiplicity and F -signature for F -finite rings beyond the local setting as well. To make this more precise, recall first the following result of Kunz.

Lemma 2.6. [Kun76] *If A is a reduced equidimensional F -finite ring, the function*

$$Q \in \text{Spec}(A) \mapsto [k(Q)^{1/p^e} : k(Q)] \cdot p^{e \text{ht } Q}$$

is constant on $\text{Spec}(A)$. In particular, if A is a domain,

$$\text{rank}_A(A^{1/p^e}) = [k(Q)^{1/p^e} : k(Q)] \cdot p^{e \text{ht } Q}$$

for any $e \geq 0$ and $Q \in \text{Spec}(A)$.

We recall a recent result globalizing Hilbert–Kunz multiplicity and F -signature.

Theorem 2.7. [DSPY19] *If A is a reduced equidimensional F -finite ring, and $\gamma \in \mathbb{Z}_{\geq 0}$ with $p^\gamma = [k(Q)^{1/p} : k(Q)] \cdot p^{\text{ht } Q}$ for all $Q \in \text{Spec}(A)$, then the limit*

$$e_{\text{HK}}(A) = \lim_{e \rightarrow \infty} \frac{\mu_A(A^{1/p^e})}{p^{e\gamma}}$$

exists and equals $\max\{e_{\text{HK}}(A_Q) \mid Q \in \text{Spec}(A)\} = \max\{e_{\text{HK}}(A_{\mathfrak{m}}) \mid \mathfrak{m} \in \max \text{Spec}(A)\}$. Similarly, the limit

$$s(A) = \lim_{e \rightarrow \infty} \frac{\text{frk}_A(A^{1/p^e})}{p^{e\gamma}}$$

exists and equals $\min\{s(A_Q) \mid Q \in \text{Spec}(A)\} = \min\{s(A_{\mathfrak{m}}) \mid \mathfrak{m} \in \max \text{Spec}(A)\}$.

2.2. Divisors. In this subsection, we review the definitions and properties of the F -signature of divisor pairs.

Definition 2.8. If (A, \mathfrak{m}) is a normal excellent local domain of dimension d and D is an effective Weil divisor on $\text{Spec}(A)$, the e -th Frobenius degeneracy ideal along D is

$$I_e(A, D) = \langle a \in A \mid A \xrightarrow{1 \mapsto F_*^e a} F_*^e(A(D)) \text{ is not a pure } A\text{-module inclusion} \rangle.$$

If Δ is an effective \mathbb{Q} -divisor on $\text{Spec}(A)$, the F -signature of (A, Δ) is

$$s(A, \Delta) = \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_A\left(A/I_e(A, \lceil (p^e - 1)\Delta \rceil)\right).$$

Lemma 2.9. *Suppose (A, \mathfrak{m}) is a normal excellent local domain of dimension d and Δ is an effective \mathbb{Q} -divisor on $\text{Spec}(A)$. Let $\{D_e\}_{e>0}$ be a sequence of Weil divisors on $\text{Spec}(A)$ with bounded difference from $\{\lceil (p^e - 1)\Delta \rceil\}_{e>0}$ independent of $e > 0$. In other words, there exists an effective Cartier divisor C such that*

$$-C \leq D_e - \lceil (p^e - 1)\Delta \rceil \leq C$$

for all $e > 0$. Then

$$s(A, \Delta) = \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_A(A/I_e(A, D_e)).$$

Proof. This is essentially the same argument as [BST12, Lemma 4.17] and [PT18, Theorem 4.13], and so we omit it. \square

Theorem 2.10. [BST12, PT18] *Suppose (A, \mathfrak{m}) is a normal excellent local domain of dimension d and Δ is an effective \mathbb{Q} -divisor on $\text{Spec}(A)$.*

(a) *The limit defining the F -signature $s(A, \Delta)$ exists, and moreover*

$$\ell_A\left(A/I_e(A, \lceil (p^e - 1)\Delta \rceil)\right) = s(A, \Delta) \cdot p^{ed} + O(p^{e(d-1)}).$$

(b) *The F -signature $s(A, \Delta) \geq 0$, and $s(A, \Delta) > 0$ if and only if (A, Δ) is strongly F -regular.*

If A is an F -finite normal excellent domain of dimension d and D is an effective Weil divisor on $\text{Spec}(A)$, one can define the *free rank* of A^{1/p^e} along D

$$(2.10.1) \quad a_e^D(A) = \text{frk}_A^D(A^{1/p^e})$$

to be the maximal rank $a_e(D)$ of a simultaneous free A -module quotient of A^{1/p^e} and $(A(D))^{1/p^e}$. In other words, $a_e(D)$ is the largest non-negative integer such that there is a commuting diagram

$$\begin{array}{ccc} & (A(D))^{1/p^e} & \\ \subseteq \nearrow & & \searrow \\ A^{1/p^e} & \xrightarrow{\quad} & A^{\oplus a_e(D)}. \end{array}$$

In case (A, \mathfrak{m}) is local, we have that $\text{frk}_A^D(A^{1/p^e}) = [k(\mathfrak{m})^{1/p^e} : k(\mathfrak{m})] \cdot \ell_A(A/I_e(A, D))$ (see [BST12, Proposition 3.5]), and once more this leads to a recent global interpretation of the F -signature along a divisor.

Theorem 2.11. [DSPY19] *Let A be an F -finite normal excellent domain of dimension d , and $\gamma \in \mathbb{Z}_{\geq 0}$ with $p^\gamma = [k(Q)^{1/p} : k(Q)] \cdot p^{\text{ht} Q}$ for all $Q \in \text{Spec}(A)$. Suppose Δ is an effective \mathbb{Q} -divisor on $\text{Spec}(A)$. The F -signature along Δ determines a lower semi-continuous function*

$$Q \in \text{Spec}(A) \mapsto s(A_Q, \Delta)$$

on $\text{Spec}(A)$. Moreover, the limit

$$s(A, \Delta) = \lim_{e \rightarrow \infty} \frac{\text{frk}_A^{\lceil p^e \Delta \rceil}(A^{1/p^e})}{p^{e\gamma}}$$

exists and equals $\min\{s(A_Q, \Delta) \mid Q \in \text{Spec}(A)\} = \min\{s(A_{\mathfrak{m}}, \Delta) \mid \mathfrak{m} \in \max \text{Spec}(A)\}$.

In light of Theorem 2.11, and following [DSPY19], we also make the following global definition.

Definition 2.12. For a normal F -finite scheme X and effective \mathbb{Q} -divisor Δ we set

$$s(X, \Delta) = \min\{s(\mathcal{O}_{X,x}, \Delta) \mid x \in X\} = \min\{s(\mathcal{O}_{X,x}, \Delta) \mid x \in X \text{ a closed point}\}.$$

When $X = \text{Spec } A$ is affine, we write $s(A, \Delta)$ for $s(X, \Delta)$.

2.3. Divisors and families. Finally, we discuss the correspondence between \mathbb{Q} -divisors and p^{-e} -linear maps in the relative setting of $A \subseteq R$ (or in other words, for families). What follows is contained in [PSZ18] although we work in a less general setting.

Setting 2.13. Suppose that A is an F -finite regular domain and suppose we have $A \subseteq R$ a flat finite type extension of rings with geometrically² normal fibers. Additionally assume that for some choice of ω_A ,

$$(\dagger) \quad F^! \omega_A \cong \omega_A.$$

This always holds for rings essentially of finite type over a Gorenstein semi-local ring.

For any A -algebra B , we write $R_B = R \otimes_A B$. Frequent values of B include A^{1/p^e} , the fraction field $K := K(A)$ and $k(Q)$, the residue field of a point $Q \in \text{Spec } A$.

We make some quick observations.

Lemma 2.14. *In the setting of Setting 2.13, each $R_{A^{1/p^e}}$ is a normal integral domain, as are $R_{K^{1/p^e}}$ and R_{K^∞} as well.*

Proof. $A^{1/p^e} \rightarrow R_{A^{1/p^e}}$ is flat with normal fibers over a regular base, and hence $R_{A^{1/p^e}}$ is normal by [Mat89, Theorem 23.9]. Since $R \rightarrow R_{A^{1/p^e}}$ is purely inseparable and $R_{A^{1/p^e}}$ is reduced, it follows that $R_{A^{1/p^e}}$ is a domain. Localizing, we have that $K \rightarrow R_K$ also has geometrically normal fibers, and the same argument gives that $R_{K^{1/p^e}}$ and R_{K^∞} are normal domains as well. \square

Lemma 2.15. *In the setting of Setting 2.13, for each $Q \in \text{Spec } A$ and $x \in \text{Spec } R_{K(Q)} \subseteq \text{Spec } R$ a point of codimension 1 on the fiber, we have that R_x is regular and thus Δ is \mathbb{Q} -Cartier at x . In particular, we can restrict $\Delta|_{\text{Spec } R_{k(Q)}}$ to any fiber.*

Proof. Choose a codimension 1 point $x \in \text{Spec } R_{K(Q)}$, in other words a codimension one point of a fiber over $Q \in \text{Spec}(A)$. In particular, $(R_{K(Q)})_x$ is normal and hence regular. It follows that R_x is also regular since $R_{K(Q)}$ is obtained from R by killing a regular sequence and localizing. \square

We now discuss the correspondence between divisors and maps in Setting 4.1.

Lemma 2.16. [PSZ18, 2.8–2.11] *Suppose that A is an F -finite regular domain and suppose we have $A \subseteq R$ a flat finite type extension of rings with geometrically normal fibers. Then for every $R_{A^{1/p^e}}$ -linear map*

$$\phi : R^{1/p^e} \rightarrow R_{A^{1/p^e}}$$

which generates $\text{Hom}_{R_{A^{1/p^e}}}(R^{1/p^e}, R_{A^{1/p^e}})$ at the generic point of every fiber, there exists a corresponding $\mathbb{Z}_{(p)}$ -divisor³ on $\text{Spec } R$

$$\Delta_\phi \sim_{\mathbb{Q}} -K_{R/A}$$

²Here we mean that the fibers are normal after any base change, including inseparable ones.

³A \mathbb{Q} -divisor in which no denominators contain p .

which does not contain any fiber in its support.

Conversely, given an effective $\mathbb{Z}_{(p)}$ -divisor $\Delta \sim_{\mathbb{Q}} -K_{R/A}$ on $\text{Spec } R$ whose support does not contain any fiber, we can construct a map $\phi : R^{1/p^e} \rightarrow R_{A^{1/p^e}}$ such that $\Delta_{\phi} = \Delta$.

Finally, we recall the interaction between divisors and maps behaves under base change. While not crucial for the following statement, in this paper we restrict ourselves to base changes which are either flat or restriction to a fiber followed by a flat base change, which is easier to work with than the generality of [PSZ18].

Lemma 2.17. [PSZ18, Lemma 2.21] *In the setting of Setting 2.13 assume that $\Delta = \Delta_{\phi}$ is constructed as in Lemma 2.16. For any regular A -algebra B satisfying (\dagger) , let $\pi : \text{Spec } R_B \rightarrow \text{Spec } R$ denote the canonical map. Set $\phi_B := \phi \otimes_{A^{1/p^e}} B^{1/p^e}$ to be the base changed map*

$$\phi_B : (R_B)^{1/p^e} = R^{1/p^e} \otimes_{A^{1/p^e}} B^{1/p^e} \rightarrow R_{A^{1/p^e}} \otimes_{A^{1/p^e}} B^{1/p^e} = R_{B^{1/p^e}}.$$

In this case,

$$\Delta_{\phi_B} = \pi^* \Delta = \pi^* \Delta_{\phi}.$$

Remark 2.18. Frequently $B = A^{1/p^d}$ in which case the based changed map ϕ_B in Lemma 2.17 is simply

$$\phi_{A^{1/p^d}} : (R_{A^{1/p^d}})^{1/p^e} \rightarrow R_{A^{1/p^{e+d}}}.$$

3. F -SIGNATURE TRANSFORMATION FOR REGULAR FIBERS

In this section, we will be concerned with the behavior of the F -signature under flat local extensions, building on the following result of Y. Yao.

Theorem 3.1 ([Yao06]). *Suppose that $(A, \mathfrak{m}) \subseteq (R, \mathfrak{n})$ is a flat local extension of excellent local rings of characteristic $p > 0$. Then if $R/\mathfrak{m}R$ is regular, we have*

$$s(A) = s(R).$$

Our goal is to generalize the above result to the context of divisor pairs (R, Δ) , for which we will first need to give a variation on the proof of the original statement. We begin with some preliminary lemmas.

Lemma 3.2. *Suppose that $(A, \mathfrak{m}) \subseteq (R, \mathfrak{n})$ is a flat local extension of local rings. If $x_1, \dots, x_{\delta} \in R$ is a regular sequence on $R/\mathfrak{m}R$, then $R/\langle x_1, \dots, x_{\delta} \rangle$ is a flat A -algebra. Moreover, $x_1, \dots, x_{\delta} \in R$ are a regular sequence on $M \otimes_A R$ for any finitely generated A -module M , and lastly for any $t \geq 0$ the R -module inclusion*

$$R/\langle x_1^t, \dots, x_{\delta}^t \rangle \xrightarrow{1 \mapsto \left[\frac{1}{x_1^t \cdots x_{\delta}^t} \right]} H_{\langle x_1, \dots, x_{\delta} \rangle}^{\delta}(R)$$

is pure as an inclusion of A -modules.

Proof. See [Mat80, Corollary 20.F, page 151] or [HH94, Lemma 7.10]. For the final statement, note that it suffices to check purity after tensoring with finitely generated A -modules, where injectivity follows from the previous regular sequence assertion. \square

The following was used in Hochster and Huneke's original study of F -regularity and base change.

Lemma 3.3. [HH94, Lemma 7.10] *Let $(A, \mathfrak{m}) \subseteq (R, \mathfrak{n})$ be a flat local extension of local rings. Suppose $R/\mathfrak{m}R$ is regular and $x_1, \dots, x_\delta \in R$ give a regular system of parameters of $R/\mathfrak{m}R$. If E_A is an injective hull of A/\mathfrak{m} over A with socle generated by u , then $E_R = H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R) \otimes_A E_A$ is an injective hull of R/\mathfrak{n} over R with socle generated by $\left[\frac{1}{x_1 \cdots x_\delta} \right] \otimes u$.*

Now, we give a new proof of Yao’s result.

Proof of Theorem 3.1. If $x_1, \dots, x_\delta \in R$ give a regular system of parameters of $R/\mathfrak{m}R$, then by Lemma 3.3 we have $E_R = H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R) \otimes_A E_A$ with socle generated by $v = \left[\frac{1}{x_1 \cdots x_\delta} \right] \otimes u$.

Consider now $R^{1/p^e} \otimes_R E_R$, so that $I_e(R)^{1/p^e} = \text{Ann}_{R^{1/p^e}}(1 \otimes v)$. With this observation in place, we claim the following equality of ideals of R^{1/p^e} :

Claim 3.4. $I_e(R)^{1/p^e} = (I_e(A)R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle)^{1/p^e}$

Proof of claim. We may identify $R^{1/p^e} \otimes_R H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R) = (H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R))^{1/p^e}$ with $1 \otimes \left[\frac{1}{x_1 \cdots x_\delta} \right] \leftrightarrow \left[\frac{1}{x_1^{p^e} \cdots x_\delta^{p^e}} \right]^{1/p^e}$. Using that

$$\left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e} \xrightarrow{1 \mapsto \left[\frac{1}{x_1^{p^e} \cdots x_\delta^{p^e}} \right]^{1/p^e}} (H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R))^{1/p^e}$$

is pure as an inclusion of A^{1/p^e} -modules, this gives further identifications

$$\begin{aligned} R^{1/p^e} \otimes_R E_R &= (H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R))^{1/p^e} \otimes_{A^{1/p^e}} (A^{1/p^e} \otimes_A E_A) \\ 1 \otimes v &\leftrightarrow \left[\frac{1}{x_1^{p^e} \cdots x_\delta^{p^e}} \right]^{1/p^e} \otimes (1 \otimes u) \\ &\supseteq \left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e} \otimes_{A^{1/p^e}} (A^{1/p^e} \otimes_A E_A) \\ &\leftrightarrow 1 \otimes (1 \otimes u). \end{aligned}$$

In particular, $\text{Ann}_{R^{1/p^e}}(1 \otimes v) = \text{Ann}_{R^{1/p^e}}(1 \otimes (1 \otimes u))$. It is worth noting that this ideal contains the kernel of $R^{1/p^e} \rightarrow \left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$ and its image along this quotient homomorphism is the same as the annihilator of $1 \otimes (1 \otimes u)$ over $\left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$.

On the other hand, since $A^{1/p^e} \rightarrow \left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$ is flat, it follows that the annihilator of $1 \otimes (1 \otimes u)$ over $\left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$ is the extension of

$$\text{Ann}_{A^{1/p^e}}(1 \otimes u \in A^{1/p^e} \otimes_A E_A) = I_e(A)^{1/p^e}.$$

Putting everything together proves the claim. \square

Thus, using the flatness of $A \rightarrow R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle$ once again, it follows

$$\begin{aligned} \ell_R \left(\frac{R}{I_e(R)} \right) &= \ell_R \left(\frac{R}{I_e(A)R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle} \right) = \ell_A \left(\frac{A}{I_e(A)} \right) \ell_R \left(\frac{R}{\mathfrak{m}R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle} \right) \\ &= p^{e\delta} \ell_A \left(\frac{A}{I_e(A)} \right) \ell_R \left(\frac{R}{\mathfrak{m}R + \langle x_1, \dots, x_\delta \rangle} \right) = p^{e\delta} \ell_A \left(\frac{A}{I_e(A)} \right). \end{aligned}$$

Since $\dim R = \dim A + \delta$, the desired equality now follows after dividing by $p^{e \dim R}$ and taking limits. \square

We now generalize the above proof to the context of pairs. We break off the main technical step into a lemma.

Lemma 3.5. *Suppose that $(A, \mathfrak{m}) \subseteq (R, \mathfrak{n})$ is a flat local extension of normal local rings of characteristic $p > 0$ and write $f : \text{Spec } R \rightarrow \text{Spec } A$ for the induced map. For any effective Weil D on $\text{Spec } A$ and $e > 0$, define*

$$\begin{aligned} I_e(A, D) &= \langle a \in A \mid A \rightarrow A(D)^{1/p^e} \text{ with } 1 \mapsto a^{1/p^e} \text{ is not } A\text{-pure} \rangle \\ I_e(R, f^*D) &= \langle r \in R \mid R \rightarrow R(f^*D)^{1/p^e} \text{ with } 1 \mapsto r^{1/p^e} \text{ is not } R\text{-pure} \rangle. \end{aligned}$$

Then if $R/\mathfrak{m}R$ is regular,

$$\ell_R \left(\frac{R}{I_e(R, f^*D)} \right) = p^{e(\dim R - \dim A)} \cdot \ell_A \left(\frac{A}{I_e(A, D)} \right).$$

Proof. If $x_1, \dots, x_\delta \in R$ give a regular system of parameters of $R/\mathfrak{m}R$, we have that $E_R = H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R) \otimes_A E_A$ with socle generated by $v = \left[\frac{1}{x_1 \cdots x_\delta} \right] \otimes u$. Consider now $R(f^*D)^{1/p^e} \otimes_R E_R$, so that $I_e(R, f^*D)^{1/p^e} = \text{Ann}_{R^{1/p^e}}(1 \otimes v)$. Using that $R(f^*D) = R \otimes_A A(D)$ and the same identifications made in the proof above, we see that

$$\begin{aligned} R(f^*D)^{1/p^e} \otimes_R E_R &= \left(H_{\langle x_1, \dots, x_\delta \rangle}^\delta(R) \right)^{1/p^e} \otimes_{A^{1/p^e}} (A(D)^{1/p^e} \otimes_A E_A) \\ 1 \otimes v &\leftrightarrow \left[\frac{1}{x_1^{p^e} \cdots x_\delta^{p^e}} \right]^{1/p^e} \otimes (1 \otimes u) \\ &\supseteq \left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e} \otimes_{A^{1/p^e}} (A(D)^{1/p^e} \otimes_A E_A) \\ &\leftrightarrow 1 \otimes (1 \otimes u). \end{aligned}$$

But since $A^{1/p^e} \rightarrow \left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$ is flat, it follows that the annihilator of $1 \otimes (1 \otimes u)$ over $\left(R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle \right)^{1/p^e}$ is the expansion of $\text{Ann}_{A^{1/p^e}}(1 \otimes u \in A(D)^{1/p^e} \otimes_A E_A) = I_e(A, D)^{1/p^e}$. In other words, we have shown $I_e(R, f^*D)^{1/p^e} = (I_e(A, D)R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle)^{1/p^e}$.

Thus, using the flatness of $A \rightarrow R/\langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle$ once again, it follows

$$\begin{aligned} \ell_R \left(\frac{R}{I_e(R, f^*D)} \right) &= \ell_R \left(\frac{R}{I_e(A, D)R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle} \right) \\ &= \ell_A \left(\frac{A}{I_e(A, D)} \right) \ell_R \left(\frac{R}{\mathfrak{m}R + \langle x_1^{p^e}, \dots, x_\delta^{p^e} \rangle} \right) \\ &= p^{e\delta} \ell_A \left(\frac{A}{I_e(A, D)} \right) \ell_R \left(\frac{R}{\mathfrak{m}R + \langle x_1, \dots, x_\delta \rangle} \right) = p^{e\delta} \ell_A \left(\frac{A}{I_e(A, D)} \right) \end{aligned}$$

as desired. \square

We now can prove the main result of the section.

Theorem 3.6. *Suppose that $(A, \mathfrak{m}) \subseteq (R, \mathfrak{n})$ is a flat local extension of normal local rings of characteristic $p > 0$ and write $f : \text{Spec } R \rightarrow \text{Spec } A$ the induced map. Suppose further that $\Delta \geq 0$ is a \mathbb{Q} -divisor on $\text{Spec } A$. Then if $R/\mathfrak{m}R$ is regular, we have*

$$s(A, \Delta, \mathfrak{m}) = s(R, f^*\Delta, \mathfrak{n}).$$

Proof. We will first apply Lemma 3.5 to $D = \lfloor p^e \Delta \rfloor$. We see that $f^*D = f^*\lfloor p^e \Delta \rfloor \leq \lfloor p^e f^* \Delta \rfloor$. Hence, recalling that $d = \dim R$ and applying both Lemma 2.9 and Lemma 3.5,

$$\begin{aligned} s(R, f^*\Delta) &= \lim_{e \rightarrow \infty} \frac{1}{p^{de}} \ell_R \left(R/I_e(R, \lceil (p^e - 1)f^*\Delta \rceil) \right) = \lim_{e \rightarrow \infty} \frac{1}{p^{de}} \ell_R \left(R/I_e(R, \lfloor p^e f^* \Delta \rfloor) \right) \\ &\leq \lim_{e \rightarrow \infty} \frac{1}{p^{de}} \cdot \ell_R \left(R/I_e(R, f^*\lfloor p^e \Delta \rfloor) \right) = \lim_{e \rightarrow \infty} \frac{1}{p^{(d-\delta)e}} \ell_A \left(A/I_e(A, \lfloor p^e \Delta \rfloor) \right) \\ &= s(A, \Delta). \end{aligned}$$

On the other hand, if we choose $D = \lceil p^e \Delta \rceil$, then $f^*D = f^*\lceil p^e \Delta \rceil \geq \lceil p^e f^* \Delta \rceil$ and arguing as above gives $s(R, f^*\Delta) \geq s(A, \Delta)$. This completes the proof. \square

We also address Hilbert–Kunz multiplicity under flat extensions (with regular fibers). There is little work to do here since Kunz proved the result (even before a limit is taken).

Theorem 3.7. ([Kun76, Proposition 3.9b]) *Let $(A, \mathfrak{m}) \hookrightarrow (R, \mathfrak{n})$ be a flat local extension of rings of positive characteristic. Further suppose that $R/\mathfrak{m}R$ is regular. Then*

$$e_{\text{HK}}(A) = e_{\text{HK}}(R).$$

4. F -SIGNATURE AND HILBERT–KUNZ MULTIPLICITY OF GENERAL FIBERS

Before proving Bertini-type theorems, we need one more result. We need to show that if $A \subseteq R$ is a finite type extension of rings such that the perfectified generic fiber has F -signature greater than λ , then so do *most* of the closed fibers.

Setting 4.1. We assume that $A \subseteq R$ is a flat finite type morphism of Noetherian F -finite integral domains with fraction fields $K = \text{Frac}(A) \subseteq L = \text{Frac}(R)$. Suppose further that A is regular and that $A \subseteq R$ has geometrically normal fibers. Further assume that $\Delta \geq 0$ is a \mathbb{Q} -divisor on $\text{Spec } R$ whose support does not contain any fiber.

We will not universally assume this setting in this section, but we will always be able to reduce to it. In order to motivate the main result of this section, we first give an easy proof of a weaker statement.

Proposition 4.2. *In the setting of Setting 4.1, further suppose that A is finite type over an uncountable algebraically closed field of characteristic $p > 0$. If*

$$s(R_{K^\infty, x}) \geq \lambda$$

for all $x \in \text{Spec } R_{K^\infty}$, then for a very general⁴ closed point $Q \in \text{Spec } A$ with residue field $k(Q)$,

$$s(R_{k(Q), x}) \geq \lambda$$

for all $x \in \text{Spec } R_{k(Q)}$.

Proof. By [DSPY19, Theorem 4.13], for each $e > 0$, and by Lemma 4.8 below, we can spread out our splitting and obtain some a_e, d_e and $0 \neq g_e \in A$ so that there is a surjection

$$(4.2.1) \quad R_{A[1/g_e]^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{A[1/g_e]^{1/p^{e+d_e}}}^{\oplus a_e}$$

and so that

$$\lambda \leq \min_{x \in \text{Spec } R_{K^\infty}} \{s(R_{K^\infty, x})\} = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e \dim R}}.$$

Since our Q is very general, $Q \notin V(g_e)$ for any e . Hence we have surjections $A[1/g_e] \rightarrow k(Q)$ for all e . We now apply

$$_ \otimes_{A[1/g_e]^{1/p^{e+d_e}}} k(Q)^{1/p^{e+d_e}}$$

to (4.2.1) which yields a surjective map

$$R_{k(Q)^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{k(Q)^{1/p^{e+d_e}}}^{\oplus a_e}.$$

But $k(Q)$ is perfect and so this can be identified with a surjective map

$$(R_{k(Q)})^{1/p^e} \rightarrow R_{k(Q)}^{\oplus a_e}.$$

The result follows. □

Remark 4.3. We do not expect this result to hold for simply general fibers; see [Mon98] for an example where the analogous Hilbert–Kunz statement for general fibers does not hold.

We now need the following result of Pérez, the third author, and Yao.

Theorem 4.4. [PTY17] *For every Noetherian ring A of characteristic $p > 0$, and every finitely generated A -algebra R , and every finitely generated R -module M , there exists a positive constant C with the following property: for all primes $Q \in \text{Spec}(A)$, all regular $k(Q)$ -algebras Γ , and all $P \in \text{Spec}(R_\Gamma := R \otimes_A \Gamma)$, and all $e \geq 1$, we have that $\ell_{R_\Gamma}(M_\Gamma/P^{[p^e]}(M_\Gamma)) \leq Cp^{e \dim(M_\Gamma)}$ where $M_\Gamma := M \otimes_A \Gamma$.*

The next result is the technical heart of the section. We state and prove it first in the non-pairs setting and then explain how to generalize it to pairs in a proposition which follows it.

Notation 4.5. Recall again that if S is a local ring, $a_e(S)$ denotes the free rank of S^{1/p^e} whereas $b_e(S)$ denotes its minimal number of generators. Moreover, if Δ is a \mathbb{Q} -divisor on $\text{Spec } S$ we shall denote by $a_e^\Delta(S)$ the number $a_e^{[(p^e-1)\Delta]}(S)$ defined in (2.10.1); which is a slight abuse of notation.

⁴Meaning outside a countable union of proper closed subsets of $\text{Spec } A$

Proposition 4.6. *Suppose we are in the setting of Setting 4.1. There exists a positive constant C and $0 \neq g \in A$ with the following property: for all $Q \in \text{Spec}(B := A[g^{-1}])$, all $d > 0$, all $x \in \text{Spec}(R_{k(Q)^{1/p^d}})$, and all $e > 0$, we have*

$$\left| s(R_{k(Q)^{1/p^d},x}) - \frac{a_e(R_{k(Q)^{1/p^d},x})}{\text{rank}_{R_{k(Q)^{1/p^d},x}}(R_{k(Q)^{1/p^d},x})^{1/p^e}} \right| \leq \frac{C}{p^e}.$$

Similarly we have

$$\left| e_{\text{HK}}(R_{k(Q)^{1/p^d},x}) - \frac{b_e(R_{k(Q)^{1/p^d},x})}{\text{rank}_{R_{k(Q)^{1/p^d},x}}(R_{k(Q)^{1/p^d},x})^{1/p^e}} \right| \leq \frac{C}{p^e}.$$

Proof. Let $\delta = \dim R_K = \dim R_{K^\infty}$, so that $\text{rank}_{R_{K^\infty}}(R_{K^\infty})^{1/p^e} = p^{e\delta}$. Since $R_{A^{1/p^{e+d}}} \rightarrow R_{A^{1/p^{e+d}}}^{1/p^e}$ base changes to $R_{K^\infty} \rightarrow R_{K^\infty}^{1/p^e}$ for any $e, d > 0$, we see that

$$\text{rank}_{R_{A^{1/p^{e+d}}}} R_{A^{1/p^{e+d}}}^{1/p^e} = p^{e\delta}$$

as well. Note that $A^{1/p^d} \subseteq R_{A^{1/p^d}}$ is also flat, and for any $Q \in \text{Spec}(A)$ and $x \in \text{Spec}(R_{k(Q)^{1/p^d}})$, we have that $\text{ht } x - \text{ht } Q = \dim R_{k(Q)^{1/p^d},x}$. Using that $\text{Frac}(R_{A^{1/p^{e+d}}}) = L_{K^{1/p^{e+d}}}$ as $R_{A^{1/p^{e+d}}}$ is a domain, we also compute

(4.6.1)

$$\begin{aligned} p^{e\delta} &= \text{rank}_{R_{A^{1/p^{e+d}}}} R_{A^{1/p^{e+d}}}^{1/p^e} = \left[L_{K^{1/p^{e+d}}}^{1/p^e} : L_{K^{1/p^{e+d}}} \right] = \frac{\left[L_{K^{1/p^{e+d}}}^{1/p^e} : L_{K^{1/p^d}} \right]}{\left[L_{K^{1/p^{e+d}}} : L_{K^{1/p^d}} \right]} \\ &= \frac{\left[L_{K^{1/p^{e+d}}}^{1/p^e} : L_{K^{1/p^d}} \right]}{\left[K^{1/p^e} : K \right]} \\ &= \frac{\left[k(x)^{1/p^e} : k(x) \right]}{\left[k(Q)^{1/p^e} : k(Q) \right]} \cdot p^{e(\text{ht } x - \text{ht } Q)} \\ &= \frac{\left[k(x)^{1/p^e} : k(x) \right]}{\left[k(Q)^{1/p^e} : k(Q) \right]} \cdot p^{e \dim R_{k(Q)^{1/p^d},x}} \end{aligned}$$

whence

$$\left[k(Q)^{1/p^e} : k(Q) \right] \cdot p^{e\delta} = \text{rank}_{R_{k(Q)^{1/p^d},x}}(R_{k(Q)^{1/p^d},x})^{1/p^e}.$$

Form exact sequences

$$(4.6.2) \quad (R_{A^{1/p}})^{\oplus p^\delta} \xrightarrow{\alpha_1} R^{1/p} \rightarrow M_1 \rightarrow 0$$

$$(4.6.3) \quad R^{1/p} \xrightarrow{\alpha_2} (R_{A^{1/p}})^{\oplus p^\delta} \rightarrow M_2 \rightarrow 0$$

of $R_{A^{1/p}}$ -modules so that both M_1, M_2 are torsion. Take $0 \neq c \in R_{A^{1/p}}$ that kills both; replacing c with c^p if necessary, we may further assume $0 \neq c \in R$. The image of $U = \text{Spec } R[1/c] \subseteq \text{Spec } R$ in $\text{Spec } A$ is open [The20, Tag 01UA] and contains the image of the generic point. Thus, after inverting an element of A , we may assume c does not

vanish along any fiber. In other words, for any $Q \in \text{Spec } A$ and $x \in \text{Spec } R_{k(Q)^{1/p^d}}$, the image of c in $R_{k(Q)}$ is non-zero, and hence also in $R_{k(Q)^{1/p^d},x}$.

Applying $\underline{} \otimes_{A^{1/p}} k(Q)^{1/p^{d+1}}$ to the sequences above gives that

$$(R_{k(Q)^{1/p^{d+1}}})^{\oplus p^\delta} \rightarrow R_{k(Q)^{1/p^{d+1}}}^{1/p} \rightarrow M_1 \otimes_{A^{1/p}} k(Q)^{1/p^{d+1}} \rightarrow 0$$

$$R_{k(Q)^{1/p^{d+1}}}^{1/p} \rightarrow (R_{k(Q)^{1/p^{d+1}}})^{\oplus p^\delta} \rightarrow M_2 \otimes_{A^{1/p}} k(Q)^{1/p^{d+1}} \rightarrow 0$$

are exact sequences of $R_{k(Q)^{1/p^{d+1}}}$ -modules. But we have that $R_{k(Q)^{1/p^{d+1}}}$ is a free $R_{k(Q)^{1/p^d}}$ -module of rank $[k(Q)^{1/p} : k(Q)]$, so we may view these as sequences of $R_{k(Q)^{1/p^d}}$ -modules and localize at $x \in \text{Spec } R_{k(Q)^{1/p^d}}$ to give the exact sequences of $R_{k(Q)^{1/p^d},x}$ -modules

$$(R_{k(Q)^{1/p^d},x})^{\oplus p^\delta[k(Q)^{1/p}:k(Q)]} \xrightarrow{\psi_1} (R_{k(Q)^{1/p^d},x})^{1/p} \rightarrow \left(M_1 \otimes_{A^{1/p}} k(Q)^{1/p^d} \right)_x^{\oplus [k(Q)^{1/p}:k(Q)]} \rightarrow 0$$

$$(R_{k(Q)^{1/p^d},x})^{1/p} \xrightarrow{\psi_2} (R_{k(Q)^{1/p^d},x})^{\oplus p^\delta[k(Q)^{1/p}:k(Q)]} \rightarrow \left(M_2 \otimes_{A^{1/p}} k(Q)^{1/p^d} \right)_x^{\oplus [k(Q)^{1/p}:k(Q)]} \rightarrow 0$$

so that the summands of the quotients $(M_i \otimes_{A^{1/p}} k(Q)^{1/p^d})_x$ for $i = 1, 2$ are killed by the image of c in $R_{k(Q)^{1/p^d},x}$. If P is the maximal ideal of $R_{k(Q)^{1/p^d},x}$ and $\ell(\underline{})$ denotes length over $R_{k(Q)^{1/p^d},x}$, applying Theorem 4.4 (for $A^{1/p} \rightarrow R_{A^{1/p}}$ with the $R_{A^{1/p}}$ -modules M_1, M_2), we have that there is a positive constant C' so that

$$\ell \left(\frac{(M_i \otimes_{A^{1/p}} k(Q)^{1/p^d})_x}{P^{[p^e]}(M_i \otimes_{A^{1/p}} k(Q)^{1/p^d})_x} \right) \leq \frac{C'}{p^e} p^{e \cdot \dim R_{k(Q)^{1/p^d},x}} = \frac{C'}{p^e} \frac{[k(x)^{1/p} : k(x)] p^{(e+1) \dim R_{k(Q)^{1/p^d},x}}}{[k(Q)^{1/p} : k(Q)] p^\delta}$$

for $i = 1, 2$ and all $e > 0$, and where the last equality follows from setting $e = 1$ in the computation (4.6.1).

Let J_e denote either the e -th Frobenius power $P^{[p^e]}$ of the maximal ideal P used to define the Hilbert–Kunz multiplicity $e_{\text{HK}}(R_{k(Q)^{1/p^d},x})$, or the e -th Frobenius degeneracy ideal $I_e(R_{k(Q)^{1/p^d},x})$ used to define the F -signature. Using the well-known properties $(J_e)^{[p]} \subseteq J_{e+1}$ and $\phi((J_{e+1})^{1/p}) \subseteq J_e$ for all $\phi \in \text{Hom}_{R_{k(Q)^{1/p^d},x}} \left((R_{k(Q)^{1/p^d},x})^{1/p}, R_{k(Q)^{1/p^d},x} \right)$,⁵ the maps ψ_1, ψ_2 induce

$$\begin{aligned} (R_{k(Q)^{1/p^d},x}/J_e)^{\oplus p^\delta[k(Q)^{1/p}:k(Q)]} &\xrightarrow{\psi_{1,e}} (R_{k(Q)^{1/p^d},x}/J_{e+1})^{1/p} \\ (R_{k(Q)^{1/p^d},x}/J_{e+1})^{1/p} &\xrightarrow{\psi_{2,e}} (R_{k(Q)^{1/p^d},x}/J_e)^{\oplus p^\delta[k(Q)^{1/p}:k(Q)]} \end{aligned}$$

with $\text{coker } \psi_{i,e}$ a quotient of $\text{coker } \psi_i$ killed by $P^{[p^e]}$ for $i = 1, 2$. Taking lengths and dividing by

$$[k(x)^{1/p} : k(x)] p^{(e+1) \dim R_{k(Q)^{1/p^d},x}}$$

⁵See for instance [PT18] (and references therein) where these properties are used systematically in the study of Hilbert–Kunz multiplicities and F -signatures.

gives

$$\left| \frac{\ell\left(\frac{R_{k(Q)^{1/p^d},x}}{J_e}\right)}{p^{e \dim R_{k(Q)^{1/p^d},x}}} - \frac{\ell\left(\frac{R_{k(Q)^{1/p^d},x}}{J_{e+1}}\right)}{p^{(e+1) \dim R_{k(Q)^{1/p^d},x}}} \right| \leq \frac{C'}{p^e} \cdot \frac{1}{[k(Q)^{1/p} : k(Q)]p^\delta}$$

so that the proposition follows from [PT18, Lemma 3.5] with $C = 2C'/[k(Q)^{1/p} : k(Q)]p^\delta$. \square

As mentioned above, we need to generalize the above to the context of pairs.

Proposition 4.7. *Suppose we are in the setting of Setting 4.1. There exists a positive constant C and $0 \neq g \in A$ with the following property: for all $Q \in \text{Spec}(B := A[g^{-1}])$, all $d > 0$, all $x \in \text{Spec}(R_{k(Q)^{1/p^d}})$, and all $e > 0$, we have*

$$\left| s(R_{k(Q)^{1/p^d},x}, \Delta_{Q,d}) - \frac{a_e^{\Delta_{Q,d}}(R_{k(Q)^{1/p^d},x})}{\text{rank}_{R_{k(Q)^{1/p^d},x}}(R_{k(Q)^{1/p^d},x})^{1/p^e}} \right| \leq \frac{C}{p^e}$$

where $\Delta_{Q,d} = \Delta|_{\text{Spec}(R_{k(Q)^{1/p^d}})}$.

Proof. The desired result follows the argument in Proposition 4.6, with modifications we now describe to account for the addition of Δ . Choose $0 \neq c' \in R$ so that $\text{div}_R(c') \geq p\Delta$. After inverting an element of A , we may assume c' does not vanish along any fiber and thus $\text{div}_R(c')|_{\text{Spec } R_{k(Q)}} \geq p\Delta|_{\text{Spec } R_{k(Q)}}$ on fibers as well. In particular, for any $\phi \in \text{Hom}_{R_{k(Q)^{1/p^d},x}}\left((R_{k(Q)^{1/p^d},x})^{1/p}, R_{k(Q)^{1/p^d},x}\right)$ and $\psi(_) = \phi((c')^{1/p} \cdot _)$, we have that $\Delta_\psi \geq \Delta_{Q,d}$ and $\text{div}_R(c')|_{\text{Spec } R_{k(Q)^{1/p^d}}} \geq p\Delta_{Q,d}$ where $\Delta_{Q,d} = \Delta|_{\text{Spec } R_{k(Q)^{1/p^d}}}$.

Replace α_1, α_2 in the right exact sequences 4.6.2 and 4.6.3 with their premultiples

$$\begin{aligned} (R_{A^{1/p}})^{\oplus p^\delta} &\xrightarrow{\cdot c'} (R_{A^{1/p}})^{\oplus p^\delta} \xrightarrow{\alpha_1} R^{1/p} \\ R^{1/p} &\xrightarrow{\cdot (c')^{1/p}} R^{1/p} \xrightarrow{\alpha_2} (R_{A^{1/p}})^{\oplus p^\delta}, \end{aligned}$$

respectively. In [PT18, proof of Theorem 4.12], the properties

$$\begin{aligned} c'\left(I_e(R_{k(Q)^{1/p^d},x}, \lceil (p^e - 1)\Delta_{Q,d} \rceil)^{[p]}\right) &\subseteq I_{e+1}(R_{k(Q)^{1/p^d},x}, \lceil (p^{e+1} - 1)\Delta_{Q,d} \rceil) \\ \phi\left((c'I_{e+1}(R_{k(Q)^{1/p^d},x}, \lceil (p^{e+1} - 1)\Delta_{Q,d} \rceil))^{1/p}\right) &\subseteq I_e(R_{k(Q)^{1/p^d},x}, \lceil (p^e - 1)\Delta_{Q,d} \rceil) \end{aligned}$$

are shown to hold. The proof of Proposition 4.6 can now be traced through without further modification. The corresponding maps ψ_i satisfy the analogs of the above properties with respect to the ideals $I_e(R_{k(Q)^{1/p^d},x}, \lceil (p^e - 1)\Delta_{Q,d} \rceil)$ and pass to maps $\psi_{i,e}$ on the quotients, with $\text{coker } \psi_{i,e}$ a quotient of $\text{coker } \psi_i$ killed by $P^{[p^e]}$ for $i = 1, 2$. In particular, the constant

C' derived in the proof of Proposition 4.6 from Theorem 4.4 once more gives

$$\left| \frac{\ell \left(\frac{R_{k(Q)^{1/p^d}, x}}{I_e(R_{k(Q)^{1/p^d}, x}, \lceil (p^e - 1)\Delta_{Q,d} \rceil)} \right)}{p^{e \cdot \dim R_{k(Q)^{1/p^d}, x}}} - \frac{\ell \left(\frac{R_{k(Q)^{1/p^d}, x}}{I_{e+1}(R_{k(Q)^{1/p^d}, x}, \lceil (p^{e+1} - 1)\Delta_{Q,d} \rceil)} \right)}{p^{(e+1) \dim R_{k(Q)^{1/p^d}, x}}} \right| \\ \leq \frac{C'}{p^e} \cdot \frac{1}{[k(Q)^{1/p} : k(Q)]p^\delta}$$

so that once more the proposition follows from [PT18, Lemma 3.5] with $C = 2C'/[k(Q)^{1/p} : k(Q)]p^\delta$. \square

Lemma 4.8. *In the setting of Setting 4.1, suppose that there is a surjective R_{K^∞} -linear map*

$$(4.8.1) \quad (R_{K^\infty})^{1/p^e} \rightarrow R_{K^\infty}^{\oplus a_e}$$

for some $a_e > 0$. Then for some $d_e > 0$ and $0 \neq g \in A$, setting $B = A[1/g]$, there is a surjective $R_{B^{1/p^e+d_e}}$ -linear map

$$R^{1/p^e} \otimes_B B^{1/p^{e+d_e}} = R_{B^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{B^{1/p^{e+d_e}}}^{\oplus a_e}$$

which tensors with $\otimes_{B^{1/p^{e+d_e}}} K^\infty$ to recover (4.8.1).

Furthermore, suppose there exists a Weil divisor Δ on $\text{Spec } R$ (still in Setting 4.1) such that each component projection $\rho : R_{K^\infty}^{1/p^e} \rightarrow R_{K^\infty}$ corresponds to a \mathbb{Q} -divisor $\Delta_\rho \geq \xi^* \Delta$ (where $\xi : \text{Spec } R_{K^\infty} \rightarrow \text{Spec } R$ is the canonical map). In this case we can choose our g such that the map

$$R_{B^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{B^{1/p^{e+d_e}}}^{\oplus a_e}$$

also has the property that each component projection $\gamma : R_{B^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{B^{1/p^{e+d_e}}}$ corresponds to a \mathbb{Q} -divisor Δ_γ on $\text{Spec } R_{B^{1/p^{d_e}}}$ such that $\Delta_\gamma \geq \eta^* \Delta$ (where $\eta : \text{Spec } R_{B^{1/p^{d_e}}} \rightarrow \text{Spec } R$ is the canonical map).

Proof. First notice since we are planning to invert an element of A , we may assume that $\omega_A \cong A$. Furthermore, any future B satisfies the same property. Note also that R_{K^∞} is a normal domain by Lemma 2.14. We have $(R_{K^\infty})^{1/p^e} \cong R^{1/p^e} \otimes_{K^{1/p^e}} K^\infty$ and so we can view our initial map as an R_{K^∞} -linear map, and in particular a K^∞ -linear map

$$\phi : (R^{1/p^e})_{K^\infty} \rightarrow R_{K^\infty}^{\oplus a_e}.$$

In other words, we are simply identifying relative and absolute Frobenius over a perfect field. Fix x_1, \dots, x_t a generating set for R^{1/p^e} over $R_{A^{1/p^e}}$. By base change, the images of those elements are also a generating set for $R_{K^\infty}^{1/p^e}$ over R_{K^∞} or for any intermediate base change. We may assume that all of the $\phi(x_i)$ land inside $R_{K^{1/p^{e+d_e}}}^{\oplus a_e} \hookrightarrow R_{K^\infty}^{\oplus a_e}$ for some $d_e > 0$. Note the $\phi(x_i)$ generate $\phi(R_{K^{1/p^{e+d_e}}}^{1/p^e})$ as a $R_{K^{1/p^{e+d_e}}}$ -module. This implies that

$$\phi(R_{K^{1/p^{e+d_e}}}^{1/p^e}) \subseteq R_{K^{1/p^{e+d_e}}}^{\oplus a_e}$$

and hence we have a map (which we also call ϕ)

$$\phi : R_{K^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{K^{1/p^{e+d_e}}}^{\oplus a_e}.$$

Since this map becomes surjective after the faithfully flat base change to K^∞ , it is surjective.

By the same argument as above, we may find a denominator g' so that

$$\phi \left(R_{A^{1/p^{e+d_e}}[1/g']}^{1/p^e} \right) \subseteq R_{A^{1/p^{e+d_e}}[1/g']}^{\oplus a_e}$$

which produces a map

$$\phi : R_{A^{1/p^{e+d_e}}[1/g']}^{1/p^e} \rightarrow R_{A^{1/p^{e+d_e}}[1/g']}^{\oplus a_e}.$$

We do not know that this map is surjective but the cokernel is zero if we tensor with $R_{K^{1/p^{e+d_e}}}$. Inverting another element g'' , setting $g = g'g''$ and $B = A[g^{-1}]$, we may assume that

$$\phi : R_{B^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{B^{1/p^{e+d_e}}}^{\oplus a_e}$$

is surjective as desired.

Now we move on to the statement involving Δ . We begin in exactly the same way and produce a surjective map

$$\phi : R_{B^{1/p^{e+d_e}}}^{1/p^e} \rightarrow R_{B^{1/p^{e+d_e}}}^{\oplus a_e}$$

for some $d_e > 0$ where $B = A[1/g]$. We need to show that the component projection maps γ coming from ϕ produce divisors Δ_γ on $\text{Spec } R_{B^{1/p^{d_e}}}$ via Section 2.3 such that $\Delta_\gamma \geq \eta^* \Delta$ where $\eta : \text{Spec } R_{B^{1/p^{d_e}}} \rightarrow \text{Spec } R$ is the canonical map. Consider the following diagram where all of these maps are labeled.

$$\begin{array}{ccc} \text{Spec } R_{K^\infty} & \xrightarrow{\zeta} & \text{Spec } R_{B^{1/p^{d_e}}} & \xrightarrow{\eta} & \text{Spec } R \\ & \searrow \xi & & & \end{array}$$

We also know that $\zeta^* \Delta_\gamma = \Delta_\rho$ by Lemma 2.17 since γ base changes to a projection ρ . Since $\Delta_\rho \geq \xi^* \Delta = \zeta^* \eta^* \Delta$, we see that $\zeta^* \Delta_\gamma \geq \zeta^* \eta^* \Delta$. Since Δ has no vertical components neither does $\eta^* \Delta$. Therefore because $\Delta_\gamma \geq 0$, we conclude that $\Delta_\gamma \geq \eta^* \Delta$ as desired. \square

Lemma 4.9. *In the setting of Setting 4.1, suppose that there is a surjective R_{K^∞} -linear map*

$$(4.9.1) \quad R_{K^\infty}^{\oplus b_e} \rightarrow (R_{K^\infty})^{1/p^e}$$

for some $b_e > 0$. Then for some $d_e > 0$ and $0 \neq g \in A$, setting $B = A[1/g]$, there is a surjective $R_{B^{1/p^{e+d_e}}}$ -linear map

$$R_{B^{1/p^{e+d_e}}}^{\oplus b_e} \rightarrow R_{B^{1/p^{e+d_e}}}^{1/p^e} \otimes_B B^{1/p^{e+d_e}} = R_{B^{1/p^{e+d_e}}}^{1/p^e}$$

which tensors with $\otimes_{B^{1/p^{e+d_e}}} K^\infty$ to recover (4.9.1).

Proof. The proof strategy is the same as before in Lemma 4.8. Note that R_{K^∞} is a normal domain by Lemma 2.14. We have $(R_{K^\infty})^{1/p^e} \cong R^{1/p^e} \otimes_{K^{1/p^e}} K^\infty$ and so we can view our initial map as an R_{K^∞} -linear map, and in particular a K^∞ -linear map

$$\psi : R_{K^\infty}^{\oplus b_e} \rightarrow (R^{1/p^e})_{K^\infty}.$$

In other words, we are simply identifying relative and absolute Frobenius over a perfect field.

The images of the standard basis $e_i \in R_{K^\infty}^{\oplus b_e}$ form a generating set for $(R^{1/p^e})_{K^\infty}$ by hypothesis. We may assume that $\psi(e_i) \in (R^{1/p^e})_{K^{1/p^{e+d_e}}}$ for some $d_e > 0$. Now, the $\psi(e_i)$ generate $\psi(R_{K^{1/p^{e+d_e}}}^{\oplus b_e})$ as a $R_{K^{1/p^{e+d_e}}}$ -module and so we have a map which we also call ψ

$$\psi : R_{K^{1/p^{e+d_e}}}^{\oplus b_e} \rightarrow (R^{1/p^e})_{K^{1/p^{e+d_e}}}.$$

Since the faithfully flat base change of this map with K^∞ is the other map called ψ , this ψ is also surjective. Likewise, we also can find a denominator g' and so induce a map

$$\psi : R_{A^{1/p^{e+d_e}}[1/g']}^{\oplus b_e} \rightarrow (R^{1/p^e})_{A^{1/p^{e+d_e}}[1/g']}.$$

Inverting yet another element if necessary, let us assume that this map is also surjective as desired. \square

Theorem 4.10. *In the setting of Setting 4.1, further suppose that A is finite type over a perfect field of characteristic $p > 0$. If*

$$s(R_{K^\infty}, \Delta_{K^\infty}) > \lambda$$

then there exists an open dense $U \subseteq \text{Spec } A$ such that for any closed point $Q \in U$,

$$s(R_{k(Q)}, \Delta_{k(Q)}) > \lambda.$$

Proof. Inverting an element of A if necessary, we may choose a positive constant C as in Proposition 4.7. By [DSPY19], fix $0 < \epsilon \ll 1$ such that $s(R_{K^\infty, x}, \Delta_{K^\infty}) > \lambda + 2\epsilon$ for all $x \in \text{Spec } R_{K^\infty}$. Pick $e \gg 0$ so that $C/p^e < \epsilon$, so that we have

$$a_e^{\Delta_{K^\infty}}(R_{K^\infty, x}) / \text{rank}_{R_{K^\infty, x}}(R_{K^\infty, x}^{1/p^e}) > \lambda + \epsilon.$$

By [DSPY19, Theorem 4.22] and by Lemma 4.8, after inverting an element of A we may assume there is a $d \geq 0$ and a surjective $R_{A^{1/p^{e+d}}}$ -linear map

$$R_{A^{1/p^{e+d}}}^{1/p^e} \rightarrow R_{A^{1/p^{e+d}}}^{\oplus a_e}, \text{ where } a_e := a_e^{\Delta_{K^\infty}}(R_{K^\infty})$$

satisfying the divisorial condition on projections from Lemma 4.8. Applying $_ \otimes_{A^{1/p^{e+d}}}$ $k(Q)^{1/p^{e+d}}$ for maximal $Q \in \text{Spec}(A)$ gives a surjection

$$R_{k(Q)^{1/p^{e+d}}}^{1/p^e} \rightarrow R_{k(Q)^{1/p^{e+d}}}^{\oplus a_e}.$$

of $R_{k(Q)^{1/p^{e+d}}}$ -modules where still $a_e = a_e^{\Delta_{K^\infty}}(R_{K^\infty})$. Note the projections corresponding to this map also have the property that their corresponding divisors are $\geq \Delta_Q := \Delta|_{R_{k(Q)^{1/p^d}}}$ by Lemma 2.17.

Since A is finite type over a perfect field and Q is maximal, $k(Q)$ is also perfect and so $k(Q)^{1/p^{e+d}} = k(Q)^{1/p^e} = k(Q)$. It also follows that

$$\text{rank}_{R_{K^\infty}}(R_{K^\infty}^{1/p^e}) = \text{rank}_{R_{k(Q)}}(R_{k(Q)}^{1/p^e})$$

since $A \subseteq R$ is flat and of finite type and A is F -finite.

Therefore we have a surjection

$$(R_{k(Q), x})^{1/p^e} \rightarrow R_{k(Q), x}^{\oplus a_e}$$

showing that

$$\frac{a_e^{\Delta_Q}(R_{k(Q), x})}{\text{rank}_{R_{k(Q), x}}(R_{k(Q), x})^{1/p^e}} > \lambda + \epsilon.$$

Thus, it follows once again from Proposition 4.7 that

$$s(R_{k(Q),x}, \Delta_Q) > \lambda$$

for all $x \in \text{Spec } R_{k(Q)}$ as desired. \square

Theorem 4.11. *In the setting of Setting 4.1, further suppose that A is finite type over a perfect field of characteristic $p > 0$. If*

$$e_{\text{HK}}(R_{K^\infty}) < \lambda$$

then there exists an open dense $U \subseteq \text{Spec } A$ such that for any closed point $Q \in U$,

$$e_{\text{HK}}(R_{k(Q)}) < \lambda.$$

Proof. Inverting an element of A if necessary, we may choose a positive constant C as in Proposition 4.6. By [DSPY19], fix $0 < \epsilon \ll 1$ such that $e_{\text{HK}}(R_{K^\infty,x}) < \lambda + 2\epsilon$ for all $x \in \text{Spec } R_{K^\infty}$. Pick $e \gg 0$ so that $C/p^e < \epsilon$, so that we have

$$b_e(R_{K^\infty,x}) / \text{rank}_{R_{K^\infty,x}}(R_{K^\infty,x}^{1/p^e}) < \lambda + \epsilon.$$

By Lemma 4.9, after inverting an element of A we may assume there is a $d \geq 0$ and a surjective $R_{A^{1/p^e+d}}$ -linear map

$$R_{A^{1/p^e+d}}^{\oplus b_e} \rightarrow R_{A^{1/p^e+d}}^{1/p^e}, \text{ where } b_e := b_e(R_{K^\infty}).$$

Applying $_ \otimes_{A^{1/p^e+d}} k(Q)^{1/p^{e+d}}$ for maximal $Q \in \text{Spec}(A)$ gives a surjection

$$R_{k(Q)^{1/p^e+d}}^{\oplus b_e} \rightarrow R_{k(Q)^{1/p^e+d}}^{1/p^e}$$

of $R_{k(Q)^{1/p^e+d}}$ -modules.

Since A is of finite type over a perfect field and Q is maximal, $k(Q)$ is also perfect and so $k(Q)^{1/p^{e+d}} = k(Q)^{1/p^e} = k(Q)$. It also follows that

$$\text{rank}_{R_{K^\infty}}(R_{K^\infty}^{1/p^e}) = \text{rank}_{R_{k(Q)}}(R_{k(Q)}^{1/p^e})$$

since $A \subseteq R$ is flat and of finite type and A is F -finite. Therefore we have a surjection

$$R_{k(Q),x}^{\oplus b_e} \rightarrow (R_{k(Q),x})^{1/p^e}$$

showing that

$$b_e(R_{k(Q),x}) / \text{rank}_{R_{k(Q),x}}(R_{k(Q),x}^{1/p^e}) < \lambda + \epsilon.$$

Thus, it follows once again from Proposition 4.6 that

$$e_{\text{HK}}(R_{k(Q),x}) < \lambda$$

for all $x \in \text{Spec } R_{k(Q)}$ as desired. \square

5. BERTINI THEOREMS FOR F -SIGNATURE AND HILBERT–KUNZ MULTIPLICITY

In this section, we conclude by proving our Bertini theorems for F -signature. We first recall the main result of [CGM86] and the very slight generalization to the context of pairs of [SZ13].

Suppose \mathcal{P} is a local property for locally Noetherian schemes (respectively pairs $(X, \Delta \geq 0)$).

- (A1) Whenever $\phi : Y \rightarrow Z$ is a flat morphism with regular fibers and Z (resp. (Z, Δ)) is \mathcal{P} , then Y (resp. $(Y, \phi^* \Delta)$) is \mathcal{P} too.
- (A2) Let $\phi : Y \rightarrow S$ be a morphism of finite type where Y is excellent and S is integral with generic point η . If Y_η (resp. $(Y_\eta, \Delta|_{Y_\eta})$) is geometrically \mathcal{P} , then there exists an open neighborhood U of η in S such that Y_s (resp. $(Y_s, \Delta|_{Y_s})$) is geometrically \mathcal{P} for each $s \in U$.
- (A3) \mathcal{P} is open on schemes X (resp. pairs (X, Δ)) of finite type over a field.

Theorem 5.1. [CGM86, Theorem 1] *Let X be a scheme of finite type over an algebraically closed field k , let $\phi : X \rightarrow \mathbb{P}_k^n$ be a morphism with separable generated residue field extensions. Suppose X (resp. (X, Δ)) has a property \mathcal{P} satisfying conditions (A1) and (A2). Then there exists a nonempty open subscheme U of $(\mathbb{P}_k^n)^*$ such that $\phi^{-1}(H)$ has property \mathcal{P} for each hyperplane $H \in U$.*

Remark 5.2. In the proof of Theorem 5.1, when using (A2), S is (an open subset) of $(\mathbb{P}_k^n)^*$ and $\phi^{-1}(s) = Y_s$ are fibers that are exactly equal to the hyperplane sections. In particular, one may additionally assume that S is of finite type over an algebraically closed field and we only need to verify (A2) for the closed fibers.

Suppose that $k = \bar{k}$ is uncountable and consider the following weakening of (A2):

- (B2) Let $\phi : Y \rightarrow S$ be a morphism of finite type where S is integral of finite type over k , with generic point η . If Y_η (resp. $(Y_\eta, \Delta|_{Y_\eta})$) is geometrically \mathcal{P} , then for a very general closed point $s \in S$ we have that Y_s (resp. $(Y_s, \Delta|_{Y_s})$) is geometrically \mathcal{P} for each $s \in U$.

If (A1) and (B2) hold for \mathcal{P} , then it immediately follows that the weakening of Theorem 5.1 holds for *very general* hyperplane sections.

Corollary 5.3. [CGM86, Corollary 2] *Let $k = \bar{k}$, $V \subseteq \mathbb{P}_k^n$ be a closed subscheme (resp. and let Δ be a \mathbb{Q} -divisor on V) and let \mathcal{P} be a local property satisfying (A1).*

- (a) *If \mathcal{P} satisfies (A2), and V (resp. (V, Δ)) is \mathcal{P} , then the general hyperplane section of V (resp. (V, Δ)) satisfies \mathcal{P} .*
- (b) *If k is uncountable, \mathcal{P} satisfies (B2), and V (resp. (V, Δ)) is \mathcal{P} , then the very general hyperplane section of V (resp. (V, Δ)) satisfies \mathcal{P} .*
- (c) *Suppose \mathcal{P} satisfies (A1), (A2) and (A3), and set $\mathcal{P}(V)$ to be the \mathcal{P} locus of V then*

$$\mathcal{P}(V \cap H) \supseteq \mathcal{P}(V) \cap H$$

for a general hyperplane H .

Combining this machinery with our work of the previous sections, we immediately obtain the main results of the paper. We first state the result for F -signature.

Theorem 5.4. *Suppose that $\psi : X \rightarrow \mathbb{P}_k^n$ is a map of varieties over $k = \bar{k}$ with separably generated residue field extensions (for example, a closed embedding) and that $\Delta \geq 0$ is a \mathbb{Q} -divisor on X . Suppose that $\lambda \geq 0$.*

(a) *Suppose $s(\mathcal{O}_{X,x}, \Delta_x) > \lambda$ for all closed points $x \in X$. Choose a general hyperplane $H \subseteq \mathbb{P}_k^n$, and set $Y = \psi^{-1}(H)$. Then*

$$s(\mathcal{O}_{Y,y}, \Delta_y|_Y) > \lambda$$

for all closed points $y \in Y$.

(b) *Suppose $\psi : X \subseteq \mathbb{P}_k^n$ is a closed embedding. Let $U_X \subseteq X$ be the subset of points $x \in X$ such that $s(\mathcal{O}_{X,x}, \Delta_x) > \lambda$. For any hyperplane $H \subseteq \mathbb{P}_k^n$ let $U_{H \cap X}$ denote the set of points $x \in X \cap H$ such that $s(\mathcal{O}_{H,x}, \Delta_x|_H) > \lambda$. Then for H a general hyperplane in \mathbb{P}_k^n*

$$U_{H \cap X} \supseteq U_X \cap H.$$

(c) *Suppose additionally that k is uncountable, and that $s(\mathcal{O}_{X,x}, \Delta_x) \geq \lambda$ for all closed points $x \in X$. Choose a very general hyperplane $H \subseteq \mathbb{P}_k^n$, and set $Y = \psi^{-1}(H)$. Then*

$$s(\mathcal{O}_{Y,y}, \Delta_y|_Y) \geq \lambda$$

for all closed points $y \in Y$.

Proof. For part (a), we consider the condition \mathcal{P} that $s(\mathcal{O}_{X,x}, \Delta) > \lambda$. We apply Theorem 5.1 using the fact that properties (A1) and (A2) are satisfied by Theorem 3.6 and Theorem 4.10 respectively (see also Remark 5.2).

For part (b), we simply use Corollary 5.3 and use the fact that $s(\mathcal{O}_{X,x}, \Delta) > \lambda$ is an open condition by semicontinuity [Pol18, PT18] so that (A3) is satisfied.

Part (c) either follows from (a) by considering a sequence of $\lambda_i = \lambda - 1/i$ or alternately can be directly proven via Remark 5.2 by replacing property (A2) with (B2), which was verified in Proposition 4.2. \square

Theorem 5.5. *Suppose that $\psi : X \rightarrow \mathbb{P}_k^n$ is a map of normal varieties over $k = \bar{k}$ with separably generated residue field extensions (for example, a closed embedding). Suppose that $\lambda \geq 1$.*

(a) *Suppose $e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda$ for all closed points $x \in X$. Choose a general hyperplane $H \subseteq \mathbb{P}_k^n$, and set $Y = \psi^{-1}(H)$. Then*

$$e_{\text{HK}}(\mathcal{O}_{Y,y}) < \lambda$$

for all closed points $y \in Y$.

(b) *Suppose $\psi : X \subseteq \mathbb{P}_k^n$ is a closed embedding. Let $U_X \subseteq X$ be the subset of points $x \in X$ such that $e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda$. For any hyperplane $H \subseteq \mathbb{P}_k^n$ let $U_{H \cap X}$ denote the set of points $x \in X \cap H$ such that $e_{\text{HK}}(\mathcal{O}_{H,x}) < \lambda$. Then for H a general hyperplane in \mathbb{P}_k^n*

$$U_{H \cap X} \supseteq U_X \cap H.$$

(c) *Suppose additionally that k is uncountable, and that $e_{\text{HK}}(\mathcal{O}_{X,x}) \leq \lambda$ for all closed points $x \in X$. Choose a very general hyperplane $H \subseteq \mathbb{P}_k^n$, and set $Y = \psi^{-1}(H)$. Then*

$$e_{\text{HK}}(\mathcal{O}_{Y,y}) \leq \lambda$$

for all closed points $y \in Y$.

Proof. For part (a), we consider the condition \mathcal{P} that $e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda$. We apply Theorem 5.1 using the fact that properties (A1) and (A2) are satisfied by Theorem 3.7 and Theorem 4.11 respectively.

For part (b), we simply use Corollary 5.3 and use the fact that $e_{\text{HK}}(\mathcal{O}_{X,x}) < \lambda$ is an open condition by semicontinuity [Smi16, PT18] so that (A3) is satisfied.

Part (c) follows from (a) by considering a sequence of $\lambda_i = \lambda - 1/i$. \square

Remark 5.6. We expect that the normality hypothesis on X is not needed in the statement of Theorem 5.5. However, this would require us at least to generalize our base setup to incorporate normal varieties (to work with divisors and F -signature) and also handle general quasi-projective schemes (to deal the Hilbert–Kunz multiplicity). We feel that trying to work this out would add substantial complication to the paper. Thus we hope that someone will determine if Bertini-type theorems hold for Hilbert–Kunz multiplicity more generally.

Remark 5.7. We expect that the very general hypothesis in (c) above cannot be removed. Indeed, see [Mon98].

REFERENCES

- [AL03] I. M. ABERBACH AND G. J. LEUSCHKE: *The F -signature and strong F -regularity*, Math. Res. Lett. **10** (2003), no. 1, 51–56. MR1960123 (2004b:13003)
- [BST12] M. BLICKLE, K. SCHWEDE, AND K. TUCKER: *F -signature of pairs and the asymptotic behavior of Frobenius splittings*, Adv. Math. **231** (2012), no. 6, 3232–3258. 2980498
- [Byd18] A. BYDLON: *Counterexamples to Bertini theorems for test ideals*, J. Algebra **501** (2018), 150–165. 3768130
- [CGM86] C. CUMINO, S. GRECO, AND M. MANARESI: *An axiomatic approach to the second theorem of Bertini*, J. Algebra **98** (1986), no. 1, 171–182. 825140 (87k:14008)
- [DSPY19] A. DE STEFANI, T. POLSTRA, AND Y. YAO: *Globalizing F -invariants*, Adv. Math. **350** (2019), 359–395. 3947648
- [Gab04] O. GABBER: *Notes on some t -structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734.
- [HW02] N. HARA AND K.-I. WATANABE: *F -regular and F -pure rings vs. log terminal and log canonical singularities*, J. Algebraic Geom. **11** (2002), no. 2, 363–392. MR1874118 (2002k:13009)
- [Har77] R. HARTSHORNE: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)
- [HH94] M. HOCHSTER AND C. HUNEKE: *F -regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), no. 1, 1–62. MR1273534 (95d:13007)
- [HH95] M. HOCHSTER AND C. HUNEKE: *Applications of the existence of big Cohen–Macaulay algebras*, Adv. Math. **113** (1995), no. 1, 45–117. 1332808
- [HR76] M. HOCHSTER AND J. L. ROBERTS: *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), no. 2, 117–172. MR0417172 (54 #5230)
- [HL02] C. HUNEKE AND G. J. LEUSCHKE: *Two theorems about maximal Cohen–Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391–404. MR1933863 (2003j:13011)
- [Kle98] S. L. KLEIMAN: *Bertini and his two fundamental theorems*, Rend. Circ. Mat. Palermo (2) Suppl. (1998), no. 55, 9–37, Studies in the history of modern mathematics, III. 1661859 (99m:14001)
- [KM98] J. KOLLÁR AND S. MORI: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 (2000b:14018)
- [Kun69] E. KUNZ: *Characterizations of regular local rings for characteristic p* , Amer. J. Math. **91** (1969), 772–784. MR0252389 (40 #5609)

- [Kun76] E. KUNZ: *On Noetherian rings of characteristic p* , Amer. J. Math. **98** (1976), no. 4, 999–1013. MR0432625 (55 #5612)
- [Laz04] R. LAZARSFELD: *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR2095472 (2005k:14001b)
- [Mat80] H. MATSUMURA: *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR575344 (82i:13003)
- [Mat89] H. MATSUMURA: *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR1011461 (90i:13001)
- [Mon83] P. MONSKY: *The Hilbert–Kunz function*, Math. Ann. **263** (1983), no. 1, 43–49. MR697329 (84k:13012)
- [Mon98] P. MONSKY: *Hilbert–Kunz functions in a family: point- S_4 quartics*, J. Algebra **208** (1998), no. 1, 343–358. 1644019
- [PSZ18] Z. PATAKFALVI, K. SCHWEDE, AND W. ZHANG: *F -singularities in families*, Algebr. Geom. **5** (2018), no. 3, 264–327. 3800355
- [PTY17] F. PÉREZ, K. TUCKER, AND Y. YAO: *Uniformity in reduction to characteristic p* , in preparation.
- [Pol18] T. POLSTRA: *Uniform bounds in F -finite rings and lower semi-continuity of the F -signature*, Trans. Amer. Math. Soc. **370** (2018), no. 5, 3147–3169. 3766845
- [PT18] T. POLSTRA AND K. TUCKER: *F -signature and Hilbert–Kunz multiplicity: a combined approach and comparison*, Algebra Number Theory **12** (2018), no. 1, 61–97. 3781433
- [SZ13] K. SCHWEDE AND W. ZHANG: *Bertini theorems for F -singularities*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 4, 851–874. 3108833
- [Smi16] I. SMIRNOV: *Upper semi-continuity of the Hilbert–Kunz multiplicity*, Compos. Math. **152** (2016), no. 3, 477–488. 3477638
- [The20] THE STACKS PROJECT AUTHORS: *stacks project*, 2020.
- [Tuc12] K. TUCKER: *F -signature exists*, Invent. Math. **190** (2012), no. 3, 743–765. 2995185
- [WY00] K.-I. WATANABE AND K.-I. YOSHIDA: *Hilbert–Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra **230** (2000), no. 1, 295–317. MR1774769 (2001h:13032)
- [Yao05] Y. YAO: *Modules with finite F -representation type*, J. London Math. Soc. (2) **72** (2005), no. 1, 53–72. 2145728 (2006b:13012)
- [Yao06] Y. YAO: *Observations on the F -signature of local rings of characteristic p* , J. Algebra **299** (2006), no. 1, 198–218. MR2225772 (2007k:13007)

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