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Characterizing quantum physics students' conceptual and procedural knowledge of the characteristic equation

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ABSTRACT

Research on student understanding of eigentheory in linear algebra has expanded recently, yet few studies address student understanding of the Characteristic Equation. In this study, we explore quantum physics students' conceptual and procedural knowledge of deriving and using the Characteristic Equation. We developed the Conceptual and Procedural Knowledge framework for classifying the quality of students' conceptual and procedural knowledge of both deriving and using the Characteristic Equation along a continuum. Most students exhibited deeper conceptual and procedural knowledge of using the Characteristic Equation than of deriving the Characteristic Equation. Furthermore, most students demonstrated deeper procedural knowledge than conceptual knowledge of deriving the Characteristic Equation. Most students demonstrated conceptual knowledge that was as deep or deeper than their procedural knowledge of using the Characteristic Equation. Examples of student work are provided, including descriptions of student work exhibiting rich knowledge of the characteristic equation. Implications for instruction and future research are discussed.

1. Introduction

Considering recent demands for enhanced student understanding of concepts in science, technology, engineering, and mathematics (STEM) fields, education researchers are tasked with exploring how students make sense of mathematical concepts in interdisciplinary settings. Indeed, the National Research Council (2012) charged that the U.S. must improve undergraduate STEM education, specifically recommending investigations into student learning of cross-cutting concepts in STEM courses. It further stated that these interdisciplinary studies "could help to increase the coherence of students' learning experience across disciplines … and could facilitate an understanding of how to promote the transfer of knowledge from one setting to another" (p. 202). Our research contributes towards this need for basic research by investigating students' reasoning about and use of mathematics within upper-division physics, particularly focusing on quantum physics students' understanding of concepts related to eigentheory from linear algebra.

At a broad level, linear algebra is a key course in the undergraduate education of students across the STEM-related majors. According to the *2015 CBMS Survey of Undergraduate Programs* (Blair, Kirkman, & Maxwell, 2018), approximately 55,000 students were enrolled in introductory linear algebra in the Fall 2015 term, up from 46,000 in 2010; in addition, the majority of these students

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took the course in the mathematics department of a PhD-granting institution. Although this enrollment total is fifth highest for introductory-level mainstream mathematics courses (behind calculus courses and Differential Equations), it is more than triple that of advanced level courses such as Introduction to Proof (pp. 213–215). This implies that linear algebra is a central course in undergraduate STEM students' required mathematical education and that it is one of the last mathematics courses that non-mathematics STEM majors encounter in their undergraduate education. As such, it provides a fertile area for investigation whose results regarding the teaching and learning of linear algebra have the potential to impact not only the field of mathematics education but various discipline-based educational research fields as well.

We use the term *eigentheory* to encompass topics related to eigenvalues, eigenvectors, and eigenspaces. We choose to focus on eigentheory because it is a conceptually complex idea that builds from and relies upon student understanding of multiple key ideas in mathematics, and its application is widespread in mathematics and beyond. Students encounter the topic in a wide spectrum of mathematics courses—such as introductory and advanced linear algebra, differential equations, probability, graph theory, cryptography—as well as courses in chemistry, physics, economics, and engineering. For instance, within mathematics, the applications of eigentheory include stochastic processes, predator-prey models, and connectivity of graphs and digraphs. One example in quantum physics contexts is the energy eigenvalue equation $H|E_n\rangle = E_n|E_n\rangle$. In this equation, the eigenvalues E_n of the Hamiltonian H are the allowed energies of the quantum system, and the eigenstates $|E_n\rangle$ of H are the energy eigenstates of that system (McIntyre, Manogue, & Tate, 2012, p. 68). We believe it is essential for researchers to examine student understanding of eigentheory due to its interdisciplinary nature.

The ubiquity of eigentheory across different fields situates our work, which is part of a larger research project that investigates students' understanding, symbolization, and interpretation of eigentheory and related key ideas from linear algebra in quantum physics (NSF- DUE-1452889). In this paper, we specifically focus on quantum physics students' conceptual and procedural knowledge of deriving and using the characteristic equation (CE), which is used to calculate eigenvalues of a matrix (see Section 2). We believe this work is relevant to the mathematics education research field because it provides insight into how university students reason about content that they not only learn in a mathematics course but also are expected to use in a physics course after they have completed their mathematics coursework. Understanding more about students' procedural and conceptual knowledge of the CE can inform and improve instruction in linear algebra classrooms.

Students are often comfortable with using the CE to find eigenvalues, yet some researchers posit that conceptually understanding why the CE is valid or how it connects to related concepts can be complicated for some students (e.g., Bouhjar, Andrews-Larson, Haider, & Zandieh, 2018; Thomas & Stewart, 2011). Bouhjar et al. (2018) found:

> There is a disconnect between students' understanding of standard procedures for finding eigenvalues and the formal definition of an eigenvector and eigenvalue, and… students are more able to execute the standard procedure than draw on conceptual understandings aligned with the formal definition. (p. 213)
>
> This documented disconnect between students' understanding of the procedural use of the CE and the conceptual derivation of the CE from the eigenvector definition motivated us to explore students' conceptual and procedural knowledge of the CE. We address the following research question: How do quantum physics students reason with and about the CE?[1]

Through our exploration of students' understanding of the CE, we developed what we call the Conceptual and Procedural Knowledge framework for characterizing the quality and type of students' knowledge of the CE. Through the Conceptual and Procedural Knowledge framework, we offer a theoretical contribution by providing nuanced characterizations of different qualities of conceptual and procedural knowledge of the CE. This framework also offers the distinction of students' knowledge of deriving the CE and using the CE. These classifications of the quality of students' conceptual and procedural knowledge of deriving and using the CE provide insight into how students reason about these different aspects of the CE.

2. Mathematics background

We review some linear algebra content in this section to provide the reader with some context of the mathematical terms used throughout this paper. An *eigenvector* of an $n \times n$ matrix A is defined as a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ, called an *eigenvalue*. A central tool often used to calculate the eigenvalues of an $n \times n$ matrix A is the *characteristic equation* of A, defined as $det(A - \lambda I) = 0$, for an $n \times n$ identity matrix I and scalar λ. The determinant of $A - \lambda I$ gives the characteristic polynomial of A, a degree-n polynomial in terms of λ. The roots of this polynomial are the eigenvalues of A. In addition to symbolically representing a procedure, the CE is conceptually related to what Lay (2003) refers to as the Invertible Matrix Theorem (IMT). The IMT relates a multitude of concepts in linear algebra through the notion of equivalence (see Fig. 1 for one version; even more statements could be included): if one statement is true (or not true) for an $n \times n$ matrix, then all the other statements are also true (or not true). One can derive the CE from the eigenequation $A\vec{x} = \lambda\vec{x}$ by subtracting $\lambda\vec{x}$ from both sides ($A\vec{x} - \lambda\vec{x} = \vec{0}$), introducing the identity matrix to get the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$, and reasoning about the equivalence of the statements in the IMT. For example, one can recognize that for the equation $(A - \lambda I)\vec{x} = \vec{0}$ to yield more than just the trivial solution for $\vec{x}$ (as eigenvectors cannot be the zero vector), the matrix $A - \lambda I$ must not be invertible. This implies that the determinant of $A - \lambda I$ must be zero.

[1] This paper builds from and is an extension of a conference presentation given at the 2019 Conference on Research in Undergraduate Mathematics Education (Serbin et al., 2019).

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\vec{x} = \vec{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.
g. The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.
m. Col $A = \mathbb{R}^n$.
n. Nul $A = \{\vec{0}\}$.
o. Det $A \neq 0$.
p. The number 0 is not an eigenvalue of A.
q. A is nonsingular.

Fig. 1. The Invertible Matrix Theorem (constructed from Lay, 2003).

3. Literature review

Various research studies (e.g., Bouhjar et al., 2018; Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Plaxco, Zandieh, & Wawro, 2018; Salgado & Trigueros, 2015; Thomas & Stewart, 2011) have explored student understanding of eigenvalues, eigenvectors, and related concepts, yet we have not found any that specifically focus on characterizing students' understanding and use of the CE. Thomas and Stewart (2011) focused on how students interpreted $A\vec{x} = \lambda\vec{x}$, finding many students were comfortable with the procedural algebraic manipulations of matrices and vectors, as in Tall's (2004) symbolic world, but the students did not hold embodied ideas regarding eigenvalues and eigenvectors. They asserted that students' fluency in symbolic manipulations should be paired with understanding what the symbols represent. In particular, they argued the importance of understanding the result of the product on both sides of the equation $A\vec{x} = \lambda\vec{x}$ is the same vector, although both sides of the equation represent different processes (matrix multiplication and scalar multiplication). Thomas and Stewart (2011) also highlighted the importance of understanding why the identity matrix is used in transitioning from $A\vec{x} = \lambda\vec{x}$ to $(A - \lambda I)\vec{x} = \vec{0}$, which many students in their study did not clearly articulate.

Other studies demonstrated students' rich understanding of connections between concepts related to eigenvalues and eigenvectors (e.g., Larson, Rasmussen, & Zandieh, 2008; Salgado & Trigueros, 2015). Larson et al. (2008) highlighted one student's ability to make connections between the linearly dependent column vectors and the zero determinant of a matrix by reasoning about the determinant graphically as the area of the parallelogram formed by two column vectors. This student reasoned that because the determinant of a matrix with linearly dependent column vectors must be zero, the area of the parallelogram formed by the column vectors of the matrix is zero, implying that linearly dependent vectors lie along the same line. These concepts are related to the CE of a matrix A because the column vectors of the matrix $A - \lambda I$ are assumed to be linearly dependent and the determinant of $A - \lambda I$ is assumed to be zero in order to determine the eigenvalues λ of A. More directly related to student understanding of the CE, Salgado and Trigueros (2015) described the reasoning of a group of three students who derived the CE without prior instruction by reasoning about the equivalence of statements in the IMT. In particular, this group of students determined that their model equation $(A - \lambda I)\vec{x} = \vec{0}$ had multiple solutions, so the matrix $A - \lambda I$ had no inverse matrix, implying the determinant of $A - \lambda I$ should be zero. These students demonstrated the conceptual understanding needed to reinvent the CE on their own. The IOLA curriculum (Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013) utilizes the notion of guided reinvention (Freudenthal, 1991) in the classroom to facilitate students' interpretation of "stretch factors and stretch directions" of linear transformations in terms of eigenvectors and eigenvalues and to develop ways to determine them. Similar to Salgado and Trigueros (2015), this derivation relies on students making connections to other linear algebra concepts. Plaxco et al. (2018) analyzed student reasoning about the eigenequation as the class engaged in the IOLA curriculum. Of particular relevance here, the authors showed examples of student work and suggested associated instructor moves that could be leveraged in the classroom towards the guided reinvention (Freudenthal, 1991) of the CE.

Other studies have focused on individual and class wide student understanding of the IMT (Payton, 2019; Wawro, 2014; Wawro, 2015), which can be used to derive the CE. Wawro (2014) described the progression of a linear algebra class's reasoning about the IMT over time. The class reasoned about connections between statements of the IMT by discussing the equivalence of the statements by definition or making if-then deductions to prove the equivalence of the statements. Wawro (2015) exemplified a student who made logical implication connections between statements in the IMT by reasoning about solutions to matrix equations. The student used the equivalence of the existence of a solution $\vec{x}$ to the matrix equation $A\vec{x} = \vec{b}$ for every vector $\vec{b}$ in $\mathbb{R}^n$ and the uniqueness of the trivial solution to the equation $A\vec{x} = \vec{0}$ to connect the IMT statements of the columns of A spanning $\mathbb{R}^n$ and the columns of A forming a linearly independent set. This student also used reasoning about the aforementioned matrix equations to explain the equivalence of the IMT statements of the matrix A having a trivial null space and A not having the eigenvalue zero. Payton (2019) described how some students reasoned about the IMT by using their understanding of free and basic variables to make chains of logical implications between statements in the IMT.

Most relevant to our current study, Bouhjar et al. (2018) characterized students' responses to an open-ended written question that asked if 2 was an eigenvalue of a given 2×2 matrix. The authors claimed students who reasoned about the determinant to solve the problem used a more procedural approach, and students who reasoned about the matrix $A - \lambda I$ without computing the determinant to solve the problem used a more conceptual approach (Hiebert & Lefevre, 1986). However, the authors described their difficulty in determining whether students' written work demonstrated any associated conceptual knowledge of the CE:

> It was often unclear from the responses of students who used the standard procedure [seeing if $det(A - 2I) = 0$] whether they understood links among the equation used in defining eigenvectors, the solution set of $(A - \lambda I)\vec{x} = \vec{0}$, and the equivalencies in the invertible matrix theorem that lead to use of the determinant as a tool for determining when the solution is non-trivial. (Bouhjar et al., 2018, p. 212)

Furthermore, because many students simply used the CE to calculate the eigenvalues of the given matrix A directly with no additional explanation, the authors were unable to explore those students' conceptual understanding of the derivation of the CE. Bouhjar et al. (2018) claimed more work is needed to distinguish whether a student using the CE to find eigenvalues just uses the procedure by rote or actually has deep conceptual understanding of why the CE works. Our analysis of students' interview responses about the derivation and use of the CE contributes toward this need by characterizing, along a continuum, students' conceptual and procedural knowledge in this context.

4. Theoretical background

Conceptual Knowledge (hereafter CK) and *Procedural Knowledge* (hereafter PK) are qualitative constructs commonly used by mathematics education researchers to classify students' mathematical knowledge. These constructs were originally introduced by Hiebert and Lefevre in 1986. They defined conceptual knowledge as "knowledge that is rich in relationships…a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3–4). They described PK as "familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (p. 7), which consist of "rules or procedures for solving mathematical problems" (p. 7). Star (2005) argued that Hiebert and Lefevre (1986) original definitions of CK as richly connected knowledge and PK as knowledge of conventions or rules for manipulating symbols conflate students' *type* of knowledge with *quality* of knowledge, as if PK could never be as rich in connections as CK. Star further argued that holding CK is not necessarily better than PK; instead, both types of knowledge are essential for proficient understanding of mathematics.

To resolve this potential conflation between type and quality of knowledge, Star (2005) proposed classifying knowledge according to both quality (either deep or superficial) and type (either procedural or conceptual). He defined *deep PK* as "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (p. 408) and *superficial PK* is knowledge of procedures that is not richly connected. According to Baroody, Feil and Johnson (2007), a student demonstrates deep PK by demonstrating flexibility or explaining how steps are interrelated to achieve a goal, and superficial PK is associated to a routine. Star (2005) characterized *deep CK* as richly connected knowledge of concepts and *superficial CK* as knowledge of concepts that is not richly connected. Deep PK and CK are characterized by a high degree of organization, abstractness, and accuracy (Baroody et al., 2007); finally, CK is related to PK because it helps students evaluate the best procedure in a specific situation, increases flexibility in solving problems, allows for the generalization of a procedure in novel problems, and enables one to check if the solution of a problem is reasonable (Crooks & Alibali, 2014).

Furthermore, Crooks and Alibali (2014) stipulated two overarching facets of CK in mathematics: general principle knowledge and knowledge of principles underlying procedures. *General principle knowledge* involves understanding general principles as fundamental laws or regularities in a particular domain. This type of CK corresponds to rules, definitions, and aspects of domain structure, in general, which are not associated with specific procedures; it also incorporates knowledge of symbols. *Knowledge of principles underlying procedures* captures understanding the "why" of solving problems; it includes understanding why specific procedures work, as well as understanding the purpose of each step in a procedure. This type of knowledge is related to knowledge of connections because it involves understanding the "connections among the steps in a procedure and between individual steps and their conceptual underpinnings" (p. 367).

We sought to operationalize Star (2005) characterizations of superficial and deep CK and PK as a way to classify students' knowledge quality. Classifying students' knowledge quality as deep or superficial can seem quite extreme, however, given that not all students exhibit strictly deep or superficial CK and PK. Therefore, in this paper, we include a moderate knowledge quality as a way to classify the knowledge of students who demonstrate deeper knowledge than students with superficial CK and PK, yet less deep knowledge than those with deep CK or PK. We based our interpretation of moderate CK and PK on Baroody et al. (2007) hypothesized model in which students' knowledge quality transitions from superficial CK and PK to deep CK and PK through intermediate levels of relatively shallow and relatively deep knowledge. Although we are not modeling a progression of the quality of CK and PK as Baroody et al. (2007) proposed, their intermediate levels of relatively shallow and relatively deep knowledge informed our characterization of moderate CK and PK. We included only one intermediate level between deep and superficial because it gave us the desired level of nuance in the categories to distinguish knowledge quality.

Informed and inspired by Star's (2005) conceptualizations of deep and superficial CK and PK, Baroody's et al. (2007) intermediate levels of knowledge, and Crooks and Alibali's (2014) notions of general principle knowledge and knowledge of principles underlying procedures, we developed a framework for characterizing aspects of superficial, moderate, and deep CK and PK of the characteristic equation, as described in the Methods section.

5. Methods

5.1. Data collection

The data for this study consist of video, transcript, and written work from individual semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 17 students enrolled in a quantum mechanics course. Nine of the students were from a junior-level course at a large public research university in the northwest United States (school A), and the other eight were in a senior-level course at a medium public research university in the northeast United States (school B). All students are pseudonymed with "A#" or "B#," with the numeric identifier assigned by the lead researcher to identify the interview participants from the roster of all students enrolled in the courses. The interviews occurred during the first week of the course. These interviews were part of a larger research program, in which we endeavor to investigate how students reason about and symbolize concepts related to eigentheory in quantum physics. Interview questions aimed to elicit student understanding of several linear algebra concepts (e.g., normalization, basis, inner products and orthogonality, eigentheory), which the students would use throughout their quantum mechanics course.

For this paper, we focus on students' attempts to recall, derive, and/or use the CE within their response to one particular interview question. Students were first asked, "Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?" After follow-ups inquiring whether they had a geometric or graphical way to think about the equation and how they would think about the equal sign in this context, students were asked how they would think about the equation if $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. The participants were then asked to determine the values of x and y that would make the equation true. Finally, students were asked to find the eigenvalues and eigenvectors of, if they had not already done so. Note that the interview question was designed so the terms "eigenvector" and "eigenvalue" were not used until the end; however, many students immediately recognized the first matrix equation as an eigen-equation and often brought up eigentheory terminology on their own. As students derived the CE from the eigenequation, they sometimes brought up concepts from the IMT. Given the semi-structured nature of the interview, some students were asked follow-up questions pertaining to why the determinant of $A - \lambda I$ should be zero. The protocol never explicitly asked students about the connections between the CE and linear algebra concepts in the IMT but rather asked students to explain their various responses throughout their work on the problem.

5.2. Development of the conceptual and procedural knowledge framework

As we considered the student work exhibited in the interview data, we recognized most students used the CE successfully to find eigenvalues, yet most students struggled with explaining the derivation of the CE. Student understanding of deriving the CE and using the CE seemed to be distinct units of analysis. This led us to examine the quality of students' conceptual and procedural knowledge across both deriving the CE and using the CE. Thus, we have four categories of analysis regarding student knowledge of the CE that comprise our framework (see Fig. 2).

Our first step in developing the framework was creating a conceptual analysis (Thompson, 2008) of the ways in which students might understand the derivation and use of the CE. We elaborated on our conceptual analysis by using evidence from students' work in the interview data. We considered the Star (2005) definitions of deep and superficial conceptual and procedural knowledge, and

	N/A	*Superficial*	*Moderate*	*Deep*
Procedural Knowledge of Deriving the CE	Does not attempt to write the CE	Incorrectly writes the CE (e.g., $A - \lambda I = 0$) and does not attempt to explain the symbolic derivation of the CE	Attempts to write CE and make connections between $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = 0$, and $det(A - \lambda I) = 0$, but uses symbols incorrectly	Accurately manipulates symbols among $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = 0$, and $det(A - \lambda I) = 0$ to derive the CE, and writes the CE correctly
Conceptual Knowledge of Deriving the CE	Does not attempt to explain how the CE is derived	States they do not know where the CE comes from or gives irrelevant explanation of how the CE is derived	Gives explanation of how CE is derived from $(A - \lambda I)\vec{x} = 0$ that is relevant to the IMT, yet incorrect	Accurately explains how CE is derived from $(A - \lambda I)\vec{x} = 0$, while referencing connections to the IMT
Procedural Knowledge of Using the CE	Does not use the CE procedure to find eigenvalues	Correctly uses the CE procedure to find eigenvalues, without exhibiting fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting either fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting both fluency in algebraic manipulations and rigor in the calculations
Conceptual Knowledge of Using the CE	Does not recognize eigenvalues are the results of using the CE and does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE but does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE and makes only one connection between the eigenvalues resulting from the CE and other eigentheory concepts; OR makes two or more connections with at least one being incorrect	Recognizes eigenvalues are the results of using the CE and correctly makes two or more connections between the eigenvalues resulting from the CE and other eigentheory concepts.

Fig. 2. Conceptual and Procedural Knowledge (CPK) framework for the CE.

we discussed how those knowledge qualities were demonstrated by the students in the context of the eigenvalue interview task. Through our discussion of students' work on the interview task, we refined our conceptual analysis and developed lists of characteristics of student work demonstrating superficial, moderate, and deep PK and CK for both deriving and using the CE. We also included "N/A" for instances in which the student work did not involve the CE.

After creating these preliminary characterizations of each knowledge type and quality, we worked as a team to operationalize them as a framework for coding student data. This involved an iterative process of using the knowledge characterizations to classify the level of knowledge demonstrated in student work and revising the knowledge characterizations. We used these lists of characteristics of knowledge types to collaboratively classify students' work. In doing so, we used the constant comparison method (Strauss & Corbin, 1998) to compare students' work and determine if responses that were classified the same way were actually indicative of the same knowledge type. We found instances in which the categories were too vague to distinguish students' type of knowledge demonstrated, so we refined the lists to make the descriptions of each knowledge type more detailed for the sake of practicality in using the framework to code student work. We then organized the lists into general characterizations of knowledge that could be demonstrated on the interview task with respect to deriving or using the CE. The result is what we refer to as the *Conceptual and Procedural Knowledge* (hereafter CPK) *Framework,* presented in Figure 2. In the remainder of this section, we describe each row of the framework and provide characteristics of student responses demonstrating each of the qualities of PK of deriving the CE, CK of deriving the CE, PK of using the CE, and CK of using the CE.

5.2.1. *Procedural knowledge of deriving the characteristic equation*

In the CPK framework, PK of deriving the CE relates to demonstrating an understanding of how the CE is derived symbolically by moving from the eigenequation $A\vec{x} = \lambda\vec{x}$ to the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ to $det(A - \lambda I) = 0$. Deep PK is demonstrated by accurately performing symbolic steps of a procedure and showing how they are interrelated; thus, deep PK of deriving the CE is characterized by the student accurately manipulating symbols, such as $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = \vec{0}$, and $det(A - \lambda I) = 0$, to derive the CE and write it correctly. Moderate PK of deriving the CE is characterized by an attempt to make connections between $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = \vec{0}$, and $det(A - \lambda I) = 0$, but with some incorrect symbolic manipulations. This action demonstrates knowledge that is not as rich in connections as deep PK. Superficial PK of deriving the CE is demonstrated by incorrectly writing the CE and not attempting to explain the symbolic derivation of the CE. Students who do not attempt to write the CE have their responses coded as N/A because their PK of deriving the CE is unobservable.

5.2.2. *Conceptual knowledge of deriving the characteristic equation*

Exhibiting CK of deriving the CE involves explaining how the CE is derived conceptually. Deep CK is rich in connections, so in the CPK framework, deep CK of deriving the CE is characterized by accurately explaining how the CE is derived from the equation $(A - \lambda I)\vec{x} = \vec{0}$ while referencing connections to concepts from the IMT to explain why the determinant of $A - \lambda I$ must be zero. These connections might include the singularity of the matrix $- \lambda I$, the linear dependence of the column vectors of $A - \lambda I$, or the infinite number of solutions to the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Note that we did not expect students to explicitly reference the IMT by name; rather, we use the IMT as a way to collectively reference all ideas students may bring up as related to the CE in some way. Students exhibiting moderate CK give explanations of how the CE is derived conceptually that are relevant to the IMT, yet incorrect. This attempt to make a connection to concepts in the IMT demonstrates knowledge that is deeper than superficial CK yet less connected than deep CK. Superficial CK is not rich in connections, so in the CPK framework, superficial CK of deriving the CE is characterized by the students acknowledging they have no idea where the CE comes from or giving incorrect or irrelevant explanations. Students who do not attempt to give an explanation have their responses coded as N/A because their CK of deriving the CE is unobservable.

5.2.3. *Procedural knowledge of using the characteristic equation*

Demonstrating PK of using the CE is associated with understanding that the CE is an appropriate procedure to use to find eigenvalues, as well as demonstrating fluency and rigor in employing the CE. Star (2005) claimed deep PK is associated with "comprehension, flexibility, and critical judgment" (p. 408). Flexibility and critical judgment are concerned with knowing various procedures one could use to approach a problem and choosing the most efficient one to employ. These characteristics of deep PK are unobservable in the context of using the CE, since there is only one way to use the CE to find eigenvalues. Therefore, rather than focusing on flexibility and critical judgment as indicators of deep PK, we focus on students' comprehension of the procedure, as evidenced by their fluency and rigor demonstrated while performing the procedure. Demonstrating fluency while using the CE to find eigenvalues involves efficiently and accurately performing the algebraic manipulations of the procedure. Exhibiting rigor in performing this procedure involves accurate manipulation of symbols through using precise notation. Thus, the CPK framework associates deep PK of using the CE with correctly using the determinant procedure to find the eigenvalues while exhibiting fluency in algebraic manipulations and rigor in using proper notation. Students with moderate PK of using the CE exhibit less fluency and rigor than students with deep PK by correctly finding eigenvalues but exhibiting either fluency or rigor in their written work. Superficial PK of using the CE is characterized by correctly finding the eigenvalues, yet doing so without demonstrating rigor in notation and fluency in symbolic manipulations. Students who do not use the CE procedure have their responses coded as N/A because their PK of using the CE is unobservable.

5.2.4. Conceptual knowledge of using the characteristic equation

Exhibiting CK of using the CE entails demonstrating an understanding that the solutions of the CE are eigenvalues and making connections to other aspects of eigentheory. Deep, moderate, and superficial CK of using the CE are all characterized by recognizing that eigenvalues are the results of using the CE. Deep CK is richly connected, so deep CK of using the CE also entails correctly making two or more connections between the eigenvalues resulting from the CE and other eigentheory concepts.[2] Examples of such connections include using eigenvalues (2 and 5 in our given question) in either of the equations $A\vec{x} = \lambda\vec{x}$ or $(A - \lambda I)\vec{x} = \vec{0}$ to find eigenvectors, giving a graphical interpretation of scaling an eigenvector by the corresponding eigenvalue of 2 or 5, or verifying that 2 is the same eigenvalue as the one originally given in the problem statement $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. Students with moderate conceptual understanding of using the CE make only one connection (correct or incorrect) between the eigenvalues resulting from using the CE and other eigentheory concepts, or they make two or more connections with at least one being incorrect. An example of an incorrect connection is a student using the eigenvalue 2 in an incorrect equation, such as $(A - \lambda I)\vec{x} = \lambda\vec{x}$, to find eigenvectors. The correctness of the connection depends on the correctness of the first statement made while connecting the concepts. For instance, using the resulting eigenvalues in $A\vec{x} = \lambda\vec{x}$ to find eigenvectors would be considered a correct connection, regardless of whether the student correctly found the eigenvectors, since the first statement of $A\vec{x} = \lambda\vec{x}$ is correct. Students with superficial CK do not use or discuss the resulting eigenvalues in the context of other eigentheory concepts. Students who do not recognize the results of the CE procedure as eigenvalues and do attempt to use the eigenvalues in related contexts have their responses coded as N/A because their CK of using the CE is unobservable.

5.3. Data analysis

To begin our analysis, we wrote detailed descriptions for each student of their work on the aforementioned interview task, focusing on their reasoning about the CE. These descriptions contained evidence from the transcripts and images of students' written work. We considered the students' written work and transcripts of their verbal explanations as evidence of their understanding of the CE, so we analyzed these as we classified the students' knowledge quality demonstrated on the task. After we finalized the CPK framework, we used the CPK framework to code each student's response to the eigenvalue task as demonstrating either N/A, superficial, moderate, or deep conceptual and procedural knowledge of deriving and using the CE. Each member of the research team, including the five authors and a mathematics graduate student, coded each participant's response. We met to compare our resulting characterizations and resolve any differing views regarding our codes of each student. The research team demonstrated high inter-rater reliability in our coding, given that we had over 80 % agreement on 90 % of our codes. We resolved differing views by having each author explain their rationale for their code choice. We discussed whether the evidence in the data supported their characterization, and we voted on which code best characterized the student's knowledge as evident in the data. After reaching consensus and finalizing our characterizations of the students' responses, we looked for trends by comparing the depth of students' CK within the dimensions of deriving and using the CE and by comparing the depth of their PK within the dimensions of deriving and using the CE. We then compared the depth of students' CK and PK of deriving the CE and the depth of their CK and PK of using the CE. We note that the latter comparisons for relative depth were observed according to two different categorical measures, one for CK and one for PK. It is possible that the differences in levels of students' CK and PK we found would not appear in a different coding scheme of the same knowledge.

6. Results

Responses of three of the 17 participating students were coded as N/A in all four categories, and one was coded as N/A in all but one category; thus, we focus our Results section on analyzing the remaining 13 students. Our four-part theoretical framework allowed us to unpack different aspects of students' understanding of the CE. Our analysis is summarized in Table 1, which conveys how we characterized each student's knowledge that was demonstrated during the task. The total number of these students exhibiting N/A, superficial, moderate, and deep knowledge in each of the four categories is also provided in Table 1. In this section, we first share descriptions of student work exhibiting superficial, moderate, and deep knowledge qualities within each category of the CPK framework. We then highlight three students who demonstrated high sophistication in all categories. Finally, we share notable results from our analyses in the remainder of this section; overall, based on our rubrics, we found that PK of deriving the CE appears to be generally stronger than CK of deriving the CE among the students, their CK of using the CE seems stronger than their PK of using the CE, and students' PK and CK of using the CE tends to be stronger than their PK and CK of deriving the CE.

[2] We chose two or more connections made by a student as an indicator of deep CK based on natural thresholds in the data. Students were not explicitly asked to make connections to other eigentheory concepts, so very few of the students in our study exhibited more than two connections. We decided that a student who correctly made at least two connections between eigentheory concepts exhibited richly connected knowledge of eigentheory. We chose one connection correctly made by a student as an indicator of moderate CK because this demonstrated less richly connected knowledge than a student exhibiting deep CK. We acknowledge that these thresholds of one and two connections might differ in an interview setting where students are explicitly prompted to make connections.

Table 1
Characterizations of students' knowledge demonstrated during the task.

Student	Dimension	N/A	Superficial	Moderate	Deep
A8	PK of Deriving CE				X
	CK of Deriving CE				X
	PK of Using CE			X	
	CK of Using CE				X
A11	PK of Deriving CE			X	
	CK of Deriving CE			X	
	PK of Using CE				X
	CK of Using CE				X
A13	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE			X	
	CK of Using CE			X	
A24	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE				X
	CK of Using CE				X
A30	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE			X	
	CK of Using CE			X	
A32	PK of Deriving CE		X		
	CK of Deriving CE		X		
	PK of Using CE	X			
	CK of Using CE	X			
B2	PK of Deriving CE		X		
	CK of Deriving CE		X		
	PK of Using CE		X		
	CK of Using CE				X
B3	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE				X
	CK of Using CE				X
B4	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE				X
	CK of Using CE			X	
B5	PK of Deriving CE				X
	CK of Deriving CE			X	
	PK of Using CE				X
	CK of Using CE				X
B6	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE	X			
	CK of Using CE				X
B8	PK of Deriving CE		X		
	CK of Deriving CE		X		
	PK of Using CE			X	
	CK of Using CE				X
B11	PK of Deriving CE			X	
	CK of Deriving CE		X		
	PK of Using CE			X	
	CK of Using CE		X		
Total	**PK of Deriving CE**	**0**	**3**	**8**	**2**
	CK of Deriving CE	**0**	**10**	**2**	**1**
	PK of Using CE	**2**	**1**	**5**	**5**
	CK of Using CE	**1**	**1**	**3**	**8**

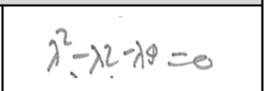
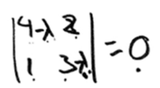
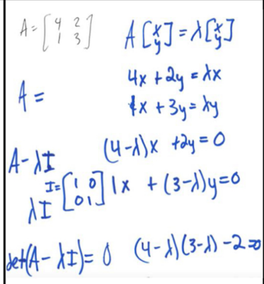
Knowledge Quality	Student	Transcripts and Images of Student Work Demonstrating Students' Procedural Derivation of the CE	
Superficial	A32	"There's another equation that I'm missing that I remember… where it's actually a lot easier, where I'll have, I'll be subtracting, uh, actual λ values… I'll like have a form where it's like λ—like, it'll be a, a quadratic equation, where it's like λ^2 minus, like, λ, you know, 2, or, like you know, minus λ4."	
Moderate	A30	"Um, well, when you take the determinant of *A* … I know I can just subtract this – I essentially subtracted this portion [pointing to the right side of the equation $A\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$ and took, uh, and took the determinant of *A*... 4 minus the eigenvector [pointing to λ], 2, 1, and then 3 minus the eigenvector, and I set the determinant equal to 0, and then you can find the determinant, and you get, like, a quadratic equation with the eigenvector... [pointing to λ]. And you can find the eigenvalue."	
Deep	B5	"So, $4x$ plus… $2y$ is equal to some λx and then we have- Let's see now. $1x$ plus $3y$ is equal to λy. Then again, I move the lambdas over and I get new equations. So, like 4 minus λx and these 2 equals 0 and then I get $1x$ plus, oh excuse me, um 3 minus 1, 3 minus λ, I mean, y equals 0… and now I'll plug that in for matrix *A* like this new matrix, I think it's *A* minus λI or something like that. And so, λI, where *I* is equal to the identity matrix… now I have *A* minus λI equals 0 … the determinant of that has to equal 0."	

Fig. 3. Evidence of Students' Procedural Knowledge of Deriving the CE.

6.1. Descriptions of student work exemplifying each category of the CPK framework

Below we give examples of student work that were characterized according to each category of the CPK Framework. We exemplify students' work demonstrating superficial, moderate, and deep CK and PK of deriving and using the CE. The examples are representative of the other students' work in that they are similar to the work of students classified with the same category of the CPK framework. The examples were chosen based on how well they illustrated each category of the CPK framework. Our intentionality in demonstrating a wide variety of students in our examples also influenced our example choice.

6.1.1. Procedural knowledge of deriving the characteristic equation

The quality of PK for deriving the CE is characterized by the ability to correctly write the CE and make different connections between the eigenequation $A\vec{x} = \lambda\vec{x}$, homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$, and the CE, which reveals an understanding of how the CE is derived symbolically. In Fig. 3, we illustrate each level of quality from this knowledge category with examples of student work from our data. Three students' responses were coded as demonstrating superficial PK of deriving the CE. For example, we coded A32′s response as superficial because A32, when asked to find the eigenvectors and eigenvalues of matrix *A*, first recalled the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ but got a bit stuck on how to proceed. The interviewer attempted to assist him by telling him one of the lambda values was 2; A32 then attempted to write the characteristic polynomial, but he never did so correctly. When asked by the interviewer, "Okay, and you're not quite sure where that comes from?" A32 responded, "I forgot the, I forgot the equation, yeah." A majority of the students in our study (8 of 13) demonstrated moderate PK of deriving the CE. In particular, A30 exhibited moderate PK because he wrote the CE correctly, but he made no attempt to make connections with the homogeneous equation. He seemed to attempt to explain where the CE comes from when he explained that he (mentally) subtracted the right side of the equation, pointing to $A\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$ (see Fig. 3). However, A30 did not explicitly manipulate symbols in the eigenequation and the homogeneous equation to derive the CE. Only two students showed deep PK of deriving the CE. For instance, B5 demonstrated deep PK because he both wrote the CE correctly and he established a connection between the system of equations and the CE while manipulating the symbols accurately (see Fig. 3).

Knowledge Quality	Student	Transcripts Demonstrating Students' Conceptual Derivation of the CE
Superficial	B4	"I guess it would have to do with, it [$det(A-\lambda I)$] has to be equal to 0… for the polynomial set up to be able to factor."
	B6	"Because of something in linear algebra that says it needs to be this way that I wish I could remember."
	B11	"I have no idea why, or I do, I just don't remember… I, like I, I remember learning why that is the thing that I do, but like when I- if I ever encounter a problem where I need eigenvalues, like, this [points to CE] is the first thing that comes to mind and not like where that comes from."
Moderate	A11	"I know an invertible matrix is one that's - that has a non-zero determinant, but if you want this to have a zero determinant, then [pause]."
	B5	"So, if we want the matrix to be invertible, um, then we want the determinant to equal 0, so we can solve for λ that way."
Deep	A8	"If this resulting matrix is, um, nonsingular, then it has a unique solution, then its only – the only $\vec{x}$ that satisfies this [points to $(A-\lambda I)\vec{x} = \vec{0}$] is zero. So, we're looking for weird cases where this is not true. So, you're looking for cases where the determinant of A minus that is equal to 0. [Writes out $det(A-\lambda I) = 0$]. So, that means a singular matrix, so it's gonna give you non – it's gonna give you an infinite number of solutions for this case."

Fig. 4. Evidence of students' CK of deriving the CE.

6.1.2. Conceptual knowledge of deriving the characteristic equation

The students in our study gave various explanations of how the CE is derived conceptually (see Fig. 4 for examples). Most (10 out of 13) students in our study exhibited superficial CK of deriving the CE. Several students claimed they did not know where the CE comes from or why the determinant of $A-\lambda I$ should be zero (e.g., B11). Some students claimed the CE is true because of something they learned in class or "something in linear algebra that says it needs to be this way" (B6). Another student, B4, posited that the determinant of $A-\lambda I$ should be zero for the characteristic polynomial to factor nicely. These explanations seemed irrelevant, so we coded these students as exhibiting superficial CK of deriving the CE. Two students, A11 and B5, demonstrated moderate CK of deriving the CE because both referenced invertibility and its relationship to the determinant. These students' explanations of the derivation of the CE were relevant to the equivalent statements in the IMT (see Fig. 1), yet they were either incomplete or incorrect. For instance, A11 mentioned invertibility and that $A-\lambda I$ should have a zero determinant, but he did not explain why the matrix $A-\lambda I$ should not be invertible. B5 also mentioned invertibility, but his explanation incorrectly implied that $A-\lambda I$ should be invertible. Only one student, A8, exhibited deep CK of deriving the CE by giving an accurate explanation of why the determinant of $A-\lambda I$ should be zero. A8 made connections between the equivalent IMT statements of $A-\lambda I$ being nonsingular and the homogeneous equation $(A-\lambda I)\vec{x} = \vec{0}$ having a unique trivial solution. He reasoned that $(A-\lambda I)\vec{x} = \vec{0}$ should not have a unique solution, implying that $A-\lambda I$ is singular, and the determinant of $A-\lambda I$ is zero.[3] Here, A8 demonstrated deep CK of the CE by reasoning about connections between statements in the IMT and relating them to the homogeneous equation and the CE.

6.1.3. Procedural knowledge of using the characteristic equation

In documenting students' PK of using the CE, we were looking for evidence of students writing the CE, or part of it, and performing some algebraic operations to find the eigenvalues. We categorized the quality of this knowledge by determining their fluency and rigor demonstrated along with the different steps of the procedure, starting with computing the determinant of the matrix of $A-\lambda I$, setting the resulting characteristic polynomial equal to 0 to write the CE, factoring the characteristic polynomial, and finding the roots of the equation as the eigenvalues. Two students did not use the CE to find eigenvalues on the interview task, so they were coded as N/A. Of the remaining eleven students, only one student's response demonstrated superficial PK of using the CE. As shown in Fig. 5, B2 did not set the polynomial equal to zero, and he required the help of the interviewer for finding $\lambda - 2$ and $\lambda - 5$ as the factors of the characteristic polynomial. B2's response showed both a lack of fluency and rigor. Five students demonstrated moderate PK of using the CE on the interview task. For instance, A13 first made a mistake in an algebraic calculation, which demonstrated a lack of fluency. Some students seemed to perform algebraic calculations in their mind, but they made mistakes with neglecting to set some factors equal to zero, which influenced the quality of their demonstrated procedural knowledge. Five students' responses exhibited deep PK of using the CE. In particular, B5 navigated fluently along with the steps of the procedure, while being careful about setting the polynomial equal to zero at each step.

[3] These equivalent statements are respectively the negations of the statements d., q., and o. of the IMT in Fig. 1 for the matrix A-λI instead of the matrix A.

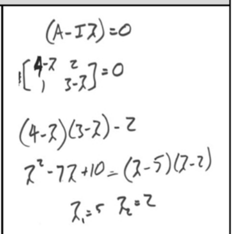
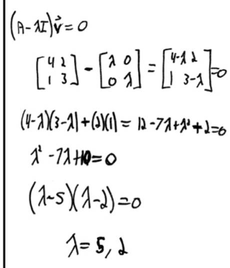
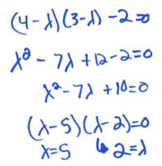
Knowledge Quality	Student	Transcripts and Images of Work Demonstrating Students' Procedural Use of the CE	
Superficial	B2	Interviewer: "I can help you factor, if you want. Is that what you're trying to work on? (B2: yeah.) Try 5 and 2." B2: "I was thinking 5 and 2. And I don't know why. Yeah. Sorry, I was making it way harder in my head… I knew it was 5 and 2 and then I was like 'How did I do that one?' I forgot it was just the 2 multiplying it. I was thinking maybe I was going to need to throw in some imaginaries."	
Moderate	A13	"So, that's 12 minus 7λ plus λ^2 plus 2 equals zero. [Writes $\lambda^2 + 7\lambda + 14 = 0$ while he talks] So, I'd have λ^2 minus 7λ plus 14 equals 0. And I'd want to solve this for λ. Let's see if I can factor. I'm horrible at factoring. Just saying right off the bat…" Interviewer 2: "I think it's a minus 2 at the end there. That might make it easier to factor." A13: "Oh! Yeah, I think you're right. [Tries to draw a minus sign over the +2 he had written in the determinant calculation and in other places].	
Deep	B5	"So, okay, so I go through this and I'll do it out. λ^2 minus 7λ plus 12 minus 2 so you have λ^2 minus 7λ plus 10. 5 times 2 is 10, 5 plus 2 is 7 or negative 5 minus 2 is 7. So, I'll say λ minus 5 and this is all - this all has to be equal to 0 based on that equation. So, yes in fact 2 does solve this equation."	

Fig. 5. Evidence of students' PK of using the CE.

6.1.4. Conceptual knowledge of using the characteristic equation

We classified students' CK of using the CE by considering the number and validity of connections they made between the eigenvalues resulting from the CE and other eigentheory concepts. Examples of making these connections include referencing the equations $A\vec{x} = \lambda\vec{x}$ and $(A - \lambda I)\vec{x} = \vec{0}$, explaining a graphical interpretation of the eigenvalue, or recognizing that an eigenvalue resulting from the CE was the same eigenvalue in the eigenequation from the task prompt. One student did not recognize that eigenvalues resulted from the CE and did not make connections from the CE to other eigentheory concepts, so he was coded as N/A. Only one of the remaining 12 students, B11, demonstrated superficial CK of using the CE in his response to the eigenvalue task. Superficial CK is characterized by making no connections, as we see with B11's response. He used the CE to find eigenvalues, but then admitted a lack of knowledge (see Fig. 6) and did not go further.

Three students exhibited moderate CK of using the CE. Moderate CK is characterized by making either a single valid connection with no others attempted, or attempting multiple connections with at least one that is not valid. The second fits both of our examples of students' responses demonstrating moderate CK of using the CE (see Fig. 6). B4 made a connection back to the eigenvalue in the eigenequation in the task prompt, but then he worked with the equation $\lambda I - A = \vec{0}$ in order to find eigenvectors, rather than $(\lambda I - A)\vec{x} = \vec{0}$. He later admitted that he was not sure of the source of this and did not go further. In this case, B4 incorrectly made a connection between the result of solving the CE and the homogeneous equation by trying to use an incorrect equation $\lambda I - A = \vec{0}$ to find eigenvectors. In A30's response, again a connection was made back to the eigenvalues in the eigenequation from the previous part of the problem. Moving forward from this, A30 was asked how he would find the eigenvectors, and started by considering the matrix $(A - 2I)$ and attempting row reduction. However, after doing the row reduction, A30 did not produce any vector or other interpretation of the result. A30 incorrectly made a connection between the result of solving the CE and the eigenvector associated with the eigenvalue 2.

A majority of the students (8 of 12) exhibited deep CK of using the CE. Deep CK in this category is characterized by making multiple valid connections and no incorrect ones. A particular example of a student demonstrating deep CK is B6, who made two connections to eigentheory concepts in her response to the interview task (see Fig. 6). B6 made a connection to the eigenequation in previous part of the problem by recognizing that the eigenvalue she found was the same as the eigenvalue 2 in the eigenequation from the task prompt. She also made a connection to the definition of eigenvector. Making these connections was indicative of richly connected CK of using the CE.

6.2. Students demonstrating high sophistication across knowledge types

In this subsection, we highlight the mathematical reasoning of students who demonstrated relatively high sophistication in

Knowledge Quality	Student	Transcripts Demonstrating Students' Conceptual Use of the CE
Superficial	B11	"And I did that because [4-second pause] I have no idea why, or I do, I just don't remember."
Moderate	B4	"So, I have 2 eigenvalues, one of 2 and one of 5. Um...So we have verified that I do have an eigenvalue of positive 2 [referencing the previous part of the task]." "Well it's for an eigen system it has to be equal to 0… at least, if I remember my, how that, how that's, how that method works."
	A30	"I got one of them right because it equals 2, and that's what eigenvalue's here [points back to original matrix equation in question prompt] in the original question." "I kind of, I kind of forgot this step... If I multiply by [writes a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the right of the $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ matrix] it's supposed to equal 2 times $\begin{bmatrix} x \\ y \end{bmatrix}$ Is that what it was? I'm not really sure after this."
Deep	B6	"Yep. 2 is definitely one of them." "So, eigenvectors are vectors that when operated on by a matrix A, … That you get the same vector back multiplied by a scalar quantity."

Fig. 6. Evidence of students' CK of using the CE.

deriving and using the CE. This sophistication was determined by our classification of the quality of these students' demonstrated knowledge as deep or moderate across all four categories of the CPK Framework. Our analysis of student work illuminated that three students (A8, A11, and B5) demonstrated sophisticated conceptual and procedural knowledge of the CE. In particular, two students (A8 and B5) demonstrated deep knowledge in three of the four categories and moderate in a fourth, and another student (A11) showed deep knowledge in two categories and moderate in the other two (see Table 1). This exemplifies our finding that some students do indeed exhibit strong conceptual and procedural knowledge of the CE. The mathematical knowledge demonstrated by these three students serves as an example of the powerful ways of reasoning that are possible for students as they make sense of the CE and related mathematical concepts.

A8 and B5 both demonstrated deep PK of deriving the CE, and A11 showed moderate PK of deriving the CE. A8 manipulated $A\vec{x} = \lambda\vec{x}$ cleanly into $(A - \lambda I)\vec{x} = \vec{0}$, and wrote the CE correctly (see Fig. 7). A11 also manipulated the symbols in the eigenequation and the homogeneous equation, but he made a notational error in writing the vector v on the wrong side of the matrix $A - \lambda I$ in the equation $\vec{v}(A - \lambda I) = \vec{0}$ (see Fig. 7). Thus, A11's response demonstrated moderate PK of deriving the CE. Rather than manipulating the eigenequation and homogeneous equation in symbolic form, B5 reasoned about those equations by translating between the matrix equation $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$ and a system of equations with $4x + 2y = \lambda x$ and $1x + 3y = \lambda y$ (see Fig. 7). B5 subtracted λx from both sides of the first equation, subtracted λy from both sides of the second equation, and combined like terms to write the equations $(4 - \lambda)x + 2y = 0$ and $1x + (3 - \lambda)y = 0$. B5 seemed to recognize the coefficients of x and y in these equations as the entries of the matrix $A - \lambda I$. He first wrote $A - \lambda I = 0$, but he quickly corrected his work and wrote $det(A - \lambda I) = 0$. Here, B5 demonstrated deep PK of deriving the CE by making connections between different representations of the eigenequation, the homogeneous equation, and the CE. Overall, these three students demonstrated sophisticated responses in procedurally deriving the CE.

These three students also demonstrated strong CK in their explanations of the derivation of the CE. A8 demonstrated deep CK of deriving the CE, and A11 and B5 demonstrated moderate CK of deriving the CE. While A8 transitioned between writing the homogeneous equation and writing the CE, he correctly stated there is a nonzero solution for x when the matrix $A - \lambda I$ is singular and the determinant of $A - \lambda I$ is zero, connecting the CE to the IMT (see Fig. 7). As B5 first wrote the CE, he claimed, "That's just what I was taught." However, when the interviewer later asked B5 about what he meant by this, he explained that the determinant of the matrix $A - \lambda I$ had to be zero for it to be invertible. Even though B5's explanation is incorrect, his response still seemed relevant to the IMT, so his response demonstrated moderate CK of deriving the CE. Now, as A11 wrote the CE, he first claimed, "I don't know why that is." Later in the interview, the interviewer asked A11 why the determinant of $A - \lambda I$ should be zero, and A11 explained, "I know an invertible matrix is one that's – that has a non-zero determinant, but if you want this to have a zero determinant, then [pause]." Here, A11 referenced the connection between the invertibility and determinant of a matrix, but his explanation of how this related to the CE was incomplete. Thus, A11's response demonstrated moderate CK of deriving the CE. A8, A11, and B5 all demonstrated high sophistication in their CK of deriving the CE, given that they were the only students in our study who exhibited moderate or deep CK of deriving the CE in their responses to the interview task.

These students also demonstrated strong PK in using the CE to find eigenvalues of a matrix. A11 and B5 both demonstrated deep PK of using the CE, and A8 exhibited moderate PK of using the CE. A11 correctly and fluently calculated the determinant of the matrix $A - \lambda I$ and solved the CE for the eigenvalues 2 and 5 (see Fig. 7). B5 found the determinant of $A - \lambda I$ by using the coefficients of x and y from the system of equations $(4 - \lambda)x + 2y = 0$ and $1x + (3 - \lambda)y = 0$ as the entries of the matrix $A - \lambda I$. B5 correctly and fluently simplified the CE into $\lambda^2 - 7\lambda + 10 = 0$ and solved the CE to find the eigenvalues of 2 and 5. Both A11 and B5 demonstrated notational rigor in being careful to set the characteristic polynomial equal to 0 at each step of the calculation. When using the CE, A8

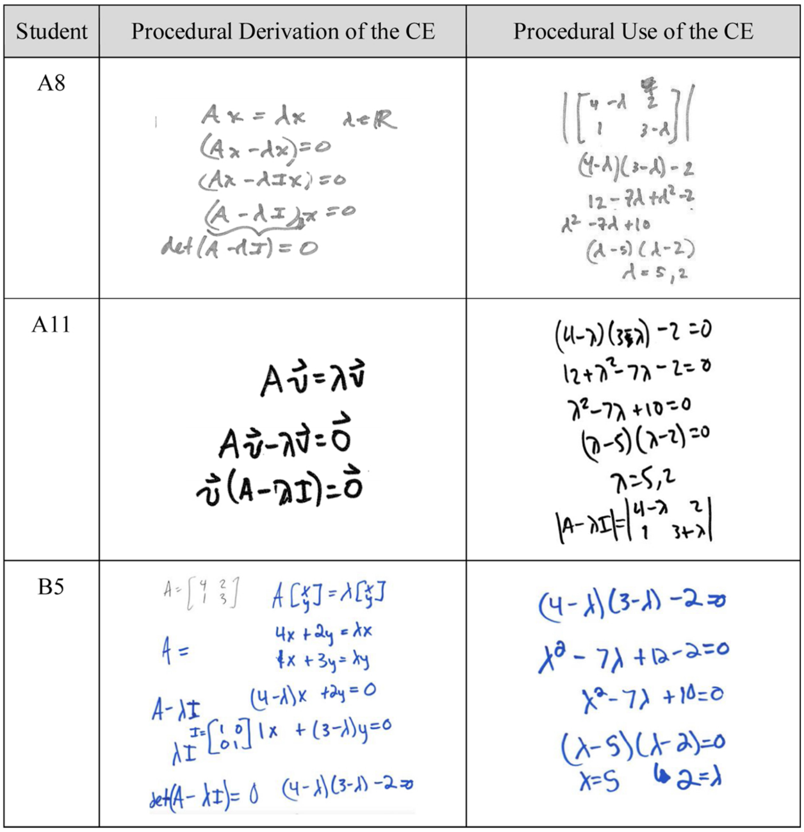


Fig. 7. Written work of students with high sophistication.

correctly calculated the eigenvalues with no apparent difficulty. However, his notation was somewhat improper, manipulating the polynomial in the CE by itself rather than as an equation (see Fig. 7). For this reason, we coded this as showing moderate PK, partially due to this omission being associated with other common student errors in factoring. These students' responses exhibited advanced PK of using the CE.

A8, A11, and B5 all demonstrated deep CK of using the CE by making connections to other concepts in eigentheory. These three students all seemed to understand the solutions of the CE are eigenvalues that can be used to find eigenvectors of A, and they all acknowledged that the eigenvalue 2, found by using the CE, was the same as the eigenvalue in the eigenequation $A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ from the previous part of the interview task. A8, A11, and B5 all used an eigenvalue they found using the CE to find a corresponding eigenvector, but they did so in slightly different ways. After finding the eigenvalues, A8 found an eigenvector corresponding to the eigenvalue 2 by using the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ and row reducing the associated augmented matrix. A11 used the eigenvalue 2 to find the values of x and y that satisfy the matrix representation of the eigenequation $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$. B5 used the eigenvalue 2 in one of the equations in his written system of equations, $(4 - \lambda)x + 2y = 0$, to find a corresponding eigenvector. Both A8 and A11 also gave a geometric or graphical interpretation of eigenvalues scaling or stretching eigenvectors that are all along the same line. A8, A11, and B5 all exhibited deep CK of using the CE by collectively making connections between the eigenvalues found by using the CE, the eigenequation, the homogeneous equation, eigenvectors, and a graphical interpretation of eigenvalues, eigenvectors, and eigenspaces.

We present these portraits of students demonstrating high sophistication as exemplars of richly connected knowledge of eigentheory concepts. We illustrated how these three students demonstrated moderate or deep CK of both deriving and using the CE. These students exhibited a strong understanding of how the CE is derived by reasoning or attempting to reason with the equivalent statements in the IMT that are used to justify why the determinant of $A - \lambda I$ is zero. The students further exhibited deep understanding of the connections between the CE and other concepts in eigentheory, such as the eigenequation or eigenvectors. These portraits of observed deep knowledge of the CE are notable given that conceptually understanding the connections between the CE and other concepts can be complicated for students (e.g., Bouhjar et al., 2018; Thomas & Stewart, 2011).

Students' deep knowledge is characterized by a high degree of organization and accuracy (Baroody et al., 2007). These three students demonstrated a high level of organization through their strong knowledge of connections between the symbolic forms of the eigenequation, homogeneous equation, and the CE. Their richly connected PK seems well-organized in the sense that the students made connections between the steps of the procedural derivation of the CE and referenced them in a logical order. This connectedness and logical coherence of their knowledge is indicative of a high degree of organization. These students also exhibited accuracy in their symbolic manipulations, as evidenced by their use of notational rigor in using the CE to find eigenvalues of a matrix. Overall, the strong knowledge of deriving and using the CE demonstrated by these three students serves as an example of the powerful ways of reasoning that are possible for students as they make sense of the CE and related mathematical concepts.

6.3. Comparisons of depth of students' knowledge types

In this section, we compare the depth of the different types of students' knowledge of the CE to illustrate various themes we found in the data. These comparisons provide insight into the differences between the nature of students' CK and PK of deriving and using the CE. We first draw a comparison between the depth of students' CK of deriving the CE and the depth of students' CK of using the CE. This is a comparison of the characterizations from the second and fourth row of the CPK framework in Fig. 2. Second, we compare the depth of students' PK of deriving the CE and the depth of students' PK of using the CE. This is a comparison of the characterizations from the first and third row of the CPK framework in Fig. 2. Third, we draw a comparison between the depth of students' CK of deriving the CE and the depth of students' PK of deriving the CE. This is a comparison of the characterizations from the first and second row of the CPK framework in Fig. 2. Finally, we draw a comparison between the depth of students' PK of using the CE and the depth of students' CK of using the CE. This is a comparison of the characterizations from the third and fourth row of the CPK framework in Fig. 2.

6.3.1. Comparison of students' conceptual knowledge of deriving versus using the characteristic equation

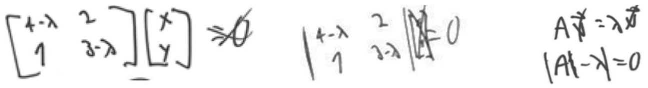
Our analysis suggested that the students in our study were better at using the CE than deriving the CE conceptually. Overall, according to our rubrics, students' CK of using the CE seemed stronger than their CK of deriving the CE, seen as 10 out of 13 students exhibited superficial CK of deriving the CE, but only 1 out of 13 students exhibited superficial CK of using the CE (see Table 1). Also, only 1 of 13 students exhibited deep CK of deriving the CE, but 8 out of 13 exhibited deep CK of using the CE. Furthermore, most students (10 of 13) individually exhibited stronger CK of using the CE than of deriving the CE (see Table 1). Each student individually had CK of using the CE that was as deep or deeper than their CK of deriving the CE. This shows a majority of the students made connections between the CE and another concept of eigentheory, but most students did not make connections to concepts in the IMT related to the derivation of the CE.

B3 exemplified this trend of exhibiting deeper CK of using the CE than of deriving the CE. In particular, when asked to find the eigenvalues and eigenvectors of A, B3 first wrote an appropriate homogeneous equation (Fig. 8a), crossed out the "= 0," and said it was the determinant of that which equaled zero (Fig. 8b). He then explained he could cross out the $\begin{bmatrix} x \\ y \end{bmatrix}$ "because you're dividing it out," claiming the vectors in the eigenequation $A\vec{v} = \lambda\vec{v}$ cancel (see Fig. 8c). In doing so, B3 demonstrated superficial CK of deriving the CE because his explanation was inaccurate and did not reference the IMT. However, once B3 found 2 and 5 as the eigenvalues of A, he mentioned "you could have given me 5," in reference to the original $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ equation, and he used $A\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$ to find other eigenvectors. Even though B3 did not seem to figure out a conceptual derivation of the CE, he recognized the CE solutions as eigenvalues and made connections between those and the eigenequation to find eigenvectors. This exemplar illustrates our result that our students connected the CE with eigentheory concepts, but most students did not seem to know why the CE is true.

Linking Crooks and Alibali (2014) facets of CK to the theoretical underpinnings of our CPK framework, general principle knowledge of the CE is characterized by recognizing the CE as the tool that is used to find eigenvalues, while knowledge underlying the procedure of the CE is characterized by understanding the derivation of the CE as a procedure that can be used to find eigenvalues. According to the CPK framework, demonstrating either superficial, moderate, or deep CK of using the CE involves recognizing that eigenvalues are the results of using the CE. Thus, general principle knowledge is related to CK of using the CE. A majority of the students (12 of 13) in our study demonstrated general principle knowledge in their responses, seen as only one student's response was coded as N/A for not demonstrating CK of using the CE (see Table 1). Furthermore, the knowledge of principles underlying the procedure of the CE is associated with understanding how the CE is derived from the homogeneous equation by making connections to the IMT. This knowledge underlying procedures is related to CK of deriving the CE. Most students (10 of 13) did not provide an explanation of the conceptual connections between the homogeneous equation and the CE, so they did not demonstrate strong knowledge of principles underlying the procedure of the CE.

6.3.2. Comparison of students' procedural knowledge of deriving versus using the characteristic equation

Students in our study were generally better at procedurally using the CE than deriving the CE. There were more students (5 of 13) who exhibited deep PK of using the CE than those (2 of 13) who exhibited deep PK of deriving the CE (see Table 1). Students in our study seemed to have stronger PK of using the CE than of deriving the CE, seen as 10 students each exhibited PK of using the CE that



(a) (b) (c)

Fig. 8. B3's written work for his derivation of the CE.

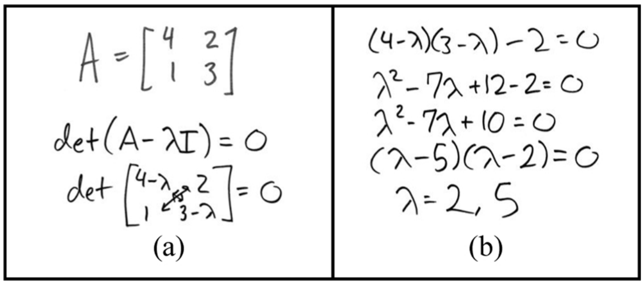


Fig. 9. A24′s written work for his derivation and use of the CE.

was as deep or deeper than their PK of deriving the CE (see Table 1). Most students wrote the CE correctly (or only made small mistakes writing it) but did not accurately make connections between equations such as $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = \vec{0}$, and $det(A - \lambda I) = 0$. However, nearly all students had little to no difficulty in using the CE to find eigenvalues. A24 exemplified this trend because he demonstrated deep PK of using the CE and moderate PK of deriving the CE. A24's symbolic manipulations (see Fig. 9a) revealed he wrote the CE correctly, but he did not make any attempt to connect to other equations for deriving the CE. Nevertheless, he fluently and rigorously used the CE to find eigenvalues (see Fig. 9b).

6.3.3. Comparison of students' procedural knowledge and conceptual knowledge of deriving the CE

All students demonstrated PK of deriving the CE that was as deep or deeper than their CK of deriving the CE (see Table 1). In some ways, this is not surprising as many students (10 of the 13) did not make any connection to the IMT in their explanation of deriving the CE, demonstrating superficial CK of deriving the CE, but most (10 of the 13) wrote the CE correctly and/or made connections to $A\vec{x} = \lambda\vec{x}$ or $(A - \lambda I)\vec{x} = \vec{0}$, demonstrating moderate or deep PK of deriving the CE (see Table 1). A13's response illustrates this trend by demonstrating deeper PK than CK of deriving the CE. A13 demonstrated moderate PK of deriving the CE by attempting to make connections between the CE and the homogeneous equation, but he did not write the CE correctly. A13 demonstrated superficial CK of deriving the CE because he did not give an explanation of the derivation of the CE that was relevant to the IMT. He claimed, "I remember this equation 'cause I remember it from the Math 305 class… Um, I don't really remember going through a proof of why this is the way it is. Why this equation works. Um, I think it's, I think that's what it is."

6.3.4. Comparison of students' procedural knowledge and conceptual knowledge of using the CE

A majority of students (11 of the 13) demonstrated CK of using the CE that was as deep or deeper than their PK of using the CE (see Table 1). This means the students demonstrated an ability to make connections to other eigentheory concepts that was somewhat stronger than their ability to fluently and rigorously use the CE to find eigenvalues. Looking back at A8 as a particular example, recall that he fluently used the CE to find eigenvalues and connected them back to both the homogeneous equation and the equation given in the initial problem statement, demonstrating deep CK for using the CE. However, he did not rigorously write " $= 0$" after each step in the calculations, demonstrating moderate PK for using the CE. Here, A8 exemplified the trend we found of students exhibiting deeper CK than PK of using the CE.

We recognize this trend between PK and CK in using the CE is largely a result of the choices we made on characterizing "deep knowledge" within each dimension. In particular, we note that categorizing students who do not rigorously write "= 0" as having moderate PK in using the CE (such as A8), and categorizing students who correctly connected the found eigenvalues to other eigentheory elements as having deep CK of using the CE, regardless of their abilities to find eigenvectors or explain what eigenvalues mean, are subjective decisions. However, we believe that our analysis highlights that many students do know how to find the values of λ that make $det(A - \lambda I) = 0$ true, despite work that appears non-rigorous; furthermore, the analysis suggests that students understand how this process produces the eigenvalues, which are essential to all other aspects of eigentheory.

7. Discussion

In this study, we investigated how quantum physics students reasoned with and about the characteristic equation. We developed the CPK framework to characterize the quality and type of students' knowledge of both using and deriving the CE. When performing the interview task, the students first procedurally derived the CE by algebraically manipulating symbols from the eigenequation $A\vec{x} = \lambda\vec{x}$, to the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$, to the CE $det(A - \lambda I) = 0$. Students exhibited CK of deriving the CE by explaining how the CE is derived from the homogeneous equation via making connections to the statements in the IMT. The students then calculated the eigenvalues of the matrix A by finding the roots of the CE, demonstrating PK of using the CE. The students demonstrated CK of using the CE by acknowledging that the solutions of the CE are eigenvalues and by making connections to other topics in eigentheory. Using the framework to classify students' knowledge demonstrated in their responses allowed us to gain insight into how students reasoned about the CE. Overall, we found that students' PK of deriving the CE seemed to be generally stronger than their CK of deriving the CE, their CK of using the CE seemed stronger than their PK of using the CE, and students' PK and CK of using the CE seemed to be stronger than their PK and CK of deriving the CE. Most students experienced little to no difficulty in using the CE to find eigenvalues and make connections to other eigentheory concepts. These students seem to exhibit strong general principle knowledge (Crooks & Alibali, 2014) of the CE by recognizing the CE as the procedure used to find eigenvalues. However, as students derived the CE from the eigenequation, most did not provide relevant conceptual explanations about how the CE is derived from the homogeneous equation. Only one student (A8) demonstrated a deep understanding of the conceptual derivation of the CE by reasoning about equivalent statements in the IMT. Several students claimed they did not know why the determinant of $A - \lambda I$ should be zero. These students' knowledge of principles underlying procedures (Crooks & Alibali, 2014) seemed to need further development to allow them to better understand how the CE is derived conceptually.

Perhaps a reason why students demonstrated deeper CK of using the CE than of deriving the CE is that the quantum physics students might not perceive knowledge of the derivation of the CE as particularly useful, or they might not have seen a need to discuss the conceptual derivation of the CE when they use it in the interview. Another reason could be that instructors might focus on the determinant procedure as a tool to find eigenvalues without focusing on the conceptual derivation of the CE, which might reduce the opportunity for the students to develop a deeper understanding of the derivation. Regardless, this result of quantum physics students demonstrating deeper CK of using the CE than of deriving the CE raises questions that warrant consideration by instructors and education researchers, such as if it is necessarily problematic for quantum physics students to not have deep CK of deriving the CE – to what extent is deep CK reasonable for STEM students in interdisciplinary fields versus those in mathematical fields? Physics education researchers (e.g., Christensen & Thompson, 2012; Wilcox, Caballero, Rehn, & Pollock, 2013) have suggested that students' difficulties with conceptually understanding mathematics can undermine their understanding of related physics concepts and their attempts to solve physics problems. Thus, we argue that it is useful for physics students to have deep conceptual understanding of the mathematical concepts that underlie the tools they use, and particularly, it is useful for physics students to have deep CK of the CE.

There are some limitations to this study's methods and findings. One such limitation is that the interview task only involved using the CE to find eigenvalues of a 2×2 matrix. For instance, if the matrix had a larger size, the students might have experienced more difficulty in using the CE to find eigenvalues. Thus, administering more challenging tasks to students during the interview might influence the quality of knowledge demonstrated in the students' responses. This could have influenced results regarding students' depth of PK of using the CE. As Crooks and Alibali (2014) argued, analyzing students' responses to additional tasks would provide more insight into students' CK and PK, possibly allowing for fuller and more nuanced characterizations. Furthermore, the participants in this study were all physics majors who had encountered eigentheory in a linear algebra, differential equations, or mathematical methods for physics course, so the findings from this study might not be generalizable to other student populations. Lastly, using empirical data from only quantum physics students' work to inform our development of the CPK framework for the CE might have had an impact on the knowledge characterizations we proposed within the framework. Using data from interviews with different STEM students might yield different characterizations of students' CK and PK of deriving and using the CE.

8. Conclusions and implications

This study offers a theoretical contribution through delineating the quality of CK and PK. In particular, the categories of N/A and Moderate provide additional nuance in classifying the quality of students' CK and PK. This elaboration of Star (2005) characterizations of deep and superficial CK and PK can be useful for exploring the nature of students' mathematical knowledge. We also offer a theoretical contribution in leveraging Crooks and Alibali (2014) two facets of CK (i.e., general principle knowledge and knowledge of principles underlying procedures) to facilitate our elaboration of Star's (2005) characterization of both CK and PK. We considered knowledge of principles underlying procedures to be related to both PK and CK of deriving the CE. While PK of the CE is characterized by an understanding of how the CE is derived symbolically through making connections between the eigenequation and the homogeneous equation, knowledge of principles underlying procedures supports students' understanding of why such connections make sense in the symbolic derivation. Thus, students' PK of deriving the CE could be strengthened by knowledge of principles underlying procedures. Although we did not focus our analysis on such a relationship between PK and knowledge of principles underlying procedures, it served as a starting point for elaborating both Star's (2005) characterization of and relationships between students' CK and PK. In the CPK framework for the CE, we also offer the distinction of student understanding of deriving and using the CE to provide more insight into how students reason about these different aspects of the CE. We believe that the general structure of the CPK framework would be a useful tool for researchers aiming to investigate and understand reasoning regarding topics other than the one presented here. It could be used to characterize students' understanding of how a concept or procedure is used and derived by utilizing the framework's general classifications of superficial, moderate, and deep conceptual and procedural knowledge. The CPK framework can be generalized for investigating student understanding of topics in linear algebra and other areas of mathematics, but we note that the characteristics of student work listed in each cell of the framework would change based on the mathematical content and the nature of the tasks that students perform.

Analyzing interview data allowed us to capture student understanding of deriving the CE, which was not apparent in the written task data in Bouhjar et al. (2018) study. Hence, we answered Bouhjar et al.'s call to distinguish whether a student employing the CE to find eigenvalues might only know how to use the procedure or also understands why the CE is an appropriate and valid procedural tool for that situation. The CPK framework seems most useful for analyzing student interview data because this setting allows interviewers to prompt students to both perform procedures and explain their thinking about concepts. This can yield data demonstrating students' CK and PK of deriving and using mathematical concepts and procedures. To use the CPK framework with written data, the written tasks would need to elicit evidence of students' reasoning about both the derivation and use of the mathematical topic.

Our study contributes to the research base on students' understanding of concepts and procedures related to eigentheory. This study focused on students' knowledge of the CE, which the students used to find eigenvalues of a 2 × 2 matrix. Future research can investigate the nature of students' knowledge of using the CE to finding eigenvalues of larger sized matrices or matrices with complex eigenvalues. This may present additional challenges for students, given the added complexity of finding the roots of a higher degree polynomial or one that cannot be factored over the real numbers. This could yield additional insight into how students' reason with and about the CE.

We conclude by offering some teaching suggestions that could support instructors in fostering students' deep CK and PK of the CE. To support students' development of deep CK of deriving the CE, instructors could intentionally aim to enhance students' understanding of the IMT by giving students opportunities in the context of small group work or class discussion to reason about the equivalence of the statements in the IMT through making logical implications (e.g., Payton, 2019; Wawro, 2014). Instructors could also give students opportunities to reinvent the CE themselves (see Plaxco et al., 2018; Salgado & Trigueros, 2015 for examples of class activities). Instructors could also help students make explicit connections between the IMT and the CE while explaining the derivation of the CE. This could involve explaining that the determinant of the matrix $A - \lambda I$ should be zero for there to be an infinite number of eigenvector solutions to the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Instructors should also use precision in demonstrating the symbolic derivation of the CE from the eigenequation $A\vec{x} = \lambda\vec{x}$ to help students learn how to accurately manipulate symbols associated with eigentheory concepts. Various research studies (e.g., Harel, 2000; Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010; Karakok, 2019; Thomas & Stewart, 2011) have indicated that keeping track of what the various objects and operations are (the number zero versus the zero vector, matrix-vector and scalar-vector multiplication both yielding vectors, the importance of the identity matrix in the homogeneous eigenequation, etc.) is central to developing a keen understanding and ease with linear algebra. Thus, integrating explicit conversations into class discussions and homework that promote thoughtful consideration of the objects, symbols, and operations of linear algebra would facilitate student learning. For instance, asking students about what each of the operations and symbols in the equations, $A\vec{x} = \lambda\vec{x}$, $(A - \lambda I)\vec{x} = \vec{0}$, and $det(A - \lambda I) = 0$, represent could support students' understanding of the steps in the procedure of deriving the CE. Implementing these suggestions for instruction might help students develop both deeper CK and PK of deriving the CE.

Declaration of Competing Interest

None

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