Students' Conceptual Understanding of Normalization of Vectors from $\mathbb{R}^2$ to $\mathbb{C}^2$

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Interdisciplinary studies illuminate ways mathematics is incorporated into core STEM courses. Vector normalization is a crosscutting idea that appears in several mathematics and physics courses. The research question pursued in this study is: how do quantum physics students reason about normalization of vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, before and after quantum mechanics instruction? The data are analyzed using the theory of coordination classes (diSessa & Sherin, 1998). Results focus on students' thinking as they normalize different types of vectors: (A) a real vector and (B) a complex vector before instruction; and (C) a complex vector after instruction. Analysis identifies the ideas students coordinate when problem solving, which problem aspects students attend to, and how students take up or disregard ideas while they problem solve.

Keywords: linear algebra, quantum mechanics, normalization, coordination class theory.

Interdisciplinary studies have a significant role within mathematics education research because they illuminate the ways mathematics is incorporated into core science, technology, engineering, and mathematics (STEM) courses. For example, many physics programs require students to take mathematics courses in integral and vector calculus, differential equations, and linear algebra. In these courses, problem solving and reasoning involve a coordination of discipline-specific and mathematical knowledge (e.g., Christensen & Thompson, 2012; Hu & Rebello, 2013; Uhden et al., 2012; Wagner et al., 2011; Wittman & Black, 2015).

Physics and mathematics education researchers have started investigating student thinking around how linear algebra concepts are used in quantum mechanics. These areas of research include eignetheory (Dreyfus et al. 2017; Her & Loverude, 2020; Wawro et al., 2019), notation (Gire & Price, 2015; Wawro et al., 2020), expectation values (Schermerhorn et al., 2019), basis and change of basis (Serbin, et al., 2021; Close et al., 2013), and boundary conditions (Ryan & Schermerhorn, 2020). One mathematical concept that is essential to quantum mechanics is vector normalization, a crosscutting idea that also appears in several other mathematics and physics courses. Despite students encountering normalization several times in their undergraduate studies, students' understanding of normalization has been relatively uninvestigated. In this study, we pursue the research question: *how do quantum physics students reason about normalization of vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, before and after quantum mechanics instruction?*

Literature Review

Research on students' understanding of norms and normalization at the undergraduate level is sparse. There is some education literature where these ideas are relevant, such as student understanding of unit vectors. Barniol and Zavala (2011) asked physics students at a Mexican university to draw a unit vector in the direction of a vector drawn from the origin to the point (2,2) on the Cartesian plane. 22% of students gave a correct answer, 25% drew a vector from the origin to the point (1,1), and 14% drew the component vectors $2\hat{\imath}$ and $2\hat{\jmath}$; these findings were confirmed in a subsequent study (Barniol & Zavala, 2014). Vega et al. (2016) investigated students' abilities to draw unit vectors representing the motion of a particle moving in a two-dimensional plane, where the unit vectors were in terms of polar unit vectors $\hat{r}$ and $\hat{\theta}$. Many students drew vectors that did not satisfy the definition of a unit vector. The authors found four fundamental ideas to correctly solve this problem, namely that unit vectors: are vectors, have a length or magnitude of one, point in the increasing direction of the corresponding coordinate, and are dimensionless. This shows that finding a unit vector (i.e., normalizing) is not trivial.

Theoretical perspective

We analyze the data using the theory of coordination classes (diSessa & Sherin, 1998). Derived from a knowledge-in-pieces perspective (diSessa, 1993), a coordination class (CC) is a model for student understanding of a concept based on a networked system (or coordination) of context-specific knowledge elements. The theory of CC was developed as a means to define conceptual understanding and conceptual change (diSessa & Sherin, 1998) for the analysis of thinking and learning. To account for how information is determined from observation and from inference, a CC is divided into two parts: readout strategies and a causal net.

The first component of a CC, Readout strategies, relates to how information is observed from the surrounding world, while the second component, a causal net, deals with inferences. Readout strategies are the processes by which the information is drawn out from the external source. diSessa and Sherin note that for many quantities in physics, the readout strategies mostly involve determining the value of the quantity in a given situation (1998). A readout strategy for a CC around the concept of "fastness" could mean attending to what is more (Parnafes, 2007).

The second structural component of the CC, the causal net, represents the network of knowledge elements that connect observation to targeted information. This network can include small elements such as phenomenological primitives (diSessa, 1993) or more complex resources (Buteler & Coleoni, 2016; Wittmann, 2002). The causal net is often intertwined with the readout strategies. How students read information out of a given task can determine which elements are invoked and/or the causal net might influence strategies for answering a question.

In addition to usefulness in describing conceptual understanding, CC theory was designed to map conceptual change as new information is integrated into understanding or previously held ideas are deemed inapplicable. Incorporation is the process of recruiting knowledge elements in a CC. New CCs can be constructed by an individual through reorganizing and extending existing readout strategies and causal nets from known CCs and other prior knowledge (diSessa & Sherin, 1998). Complimentary to incorporation, displacement is the acknowledgement that certain elements or ideas are not applicable or helpful in a particular context.

Methods

To analyze the impact of a quantum mechanics course on the ways students conceptualize normalization, interviews[1] were conducted at two universities identified as University A and University C. Both universities offered a spins-first approach following the textbook *Quantum Mechanics: A Paradigms Approach* (McIntyre et al., 2012) and incorporated student-centered activities which encouraged discussion in the classroom. University A is a large, public, research-intensive university in the northwestern United States. The course enrolled 35 students and met seven hours a week for three weeks after a week-long preface on matrix methods. Linear algebra was a prerequisite course. University C is a medium-sized, public, research university in the northeastern US. The course met three hours a week for 15 weeks and enrolled 17 students. Following the textbook, eigentheory was first introduced in the context of spins before learning about wave functions in a continuous positions space. The second round of interviews was

[1] All interviews were conducted in-person prior to the COVID-19 global pandemic.

conducted within a few weeks of the conclusion of the spins content. As the prerequisite, students could take differential equations and linear algebra as one combined or two separate courses, and students saw some linear algebra in a mathematical methods for physics course.

Both the pre-interview and post-interview were semi-structured (Bernard, 1988), containing targeted questions related to thinking about normalization. Interviews were conducted using a think-aloud protocol and students were encouraged to discuss their choices when problem solving. The main stem for the interview questions are given in Figure 1. In the first interview (pre-interview), students were given two different vectors. Each was written on a piece of paper with the statement "Normalize the following vector." The first vector did not include any complex terms. After students completed the normalization, they were asked what it means to normalize and why they chose that procedure to normalize v. Students were then presented with a second vector included complex terms in both components. This vector was changed at University C to align with a physics convention of having real-valued first components and complex-valued second components. Additionally, the vector components were chosen so that students who did not use the complex conjugate[2] would arrive at $\|w\| = 0$ and need to reconcile the connection of length with normalization. The second interview (post-interview) focused on a vector in $\mathbb{C}^2$. Students were asked out loud to "Normalize a vector whose components are 3 and 2i" so they could choose the representation of the vector. Students were again asked about their meaning for normalization and about their procedure.

At University A, interviews were conducted with nine and eight students during the preface and at the course's end, respectively; six students participated in both interviews. At University C, eight and nine students were interviewed in the first and eighth weeks, respectively; seven students participated in both interviews. All interviews were videotaped and transcribed. Written work was collected and scanned. Participants were assigned pseudonyms "A#" or "C#" to identify them from a roster of all students in the courses. Students were not asked for their pronouns, so we use the gender-neutral singular pronoun "they" throughout the paper.

Data analysis identified the knowledge elements (units of ideas) as students reasoned about normalization. CC theory is used to analyze (a) information students invoke when normalizing vectors and describing normalization, and (b) compare snapshots of students' CCs to determine how the CCs are impacted by learning quantum mechanics. The second focus places interest on how knowledge of complex terms is incorporated into a student's conceptual framework.

Interview transcriptions were analyzed synchronously with the video data to account for students' written work. Analysis specifically attended to ways students conceptualized or

	University A	University C
Pre-interview	Normalize the following vector: $v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$	
	Normalize the following vector: $w = \begin{bmatrix} 3+2i \\ 4-i \end{bmatrix}$	Normalize the following vector: $w = \begin{bmatrix} 3 \\ 3i \end{bmatrix}$
Post-interview	Please normalize a vector whose components are 3 and 2i.	

Figure 1: Overview of interview questions presented at University A and University C. The questions given during the first week of the course were written out on paper. The question given post-spins content was asked verbally.

[2]For $v = [v_1\ v_2] \epsilon \mathbb{R}^2$, $\|v\| = \sqrt{\langle v|v \rangle} = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2}$, but for vectors $w = [w_1\ w_2] \epsilon \mathbb{C}^2$, $\|w\| = \sqrt{\langle w|w \rangle} = \sqrt{w_1 \overline{w}_1 + w_2 \overline{w}_2}$. The conjugation in the inner product for $\mathbb{C}^2$ produces nonnegative magnitudes in $\mathbb{R}$ for $w \neq 0$.

characterized vectors, vector representation, complex quantities, mathematical norms, and normalization. We performed inductive open coding (Miles et al., 2013) to identify knowledge elements, which were student ideas (complete thoughts or utterances expressing an idea), representational choices, or procedural choices.

The initial coding of the knowledge elements incorporated the language used by students. For example, the two student utterances *"*that will give us the length, and we can divide that length to normalize the vector*"* and *"*So, I'm just going to divide *v* by the magnitude of *v"* resulted in two similar codes: "Dividing a vector by its length normalizes the vector" and "Normalization is vector divided by its magnitude," varying only by the concepts of magnitude and length. Codes based on representational choices or calculation were also framed around student statements. A student who said "I'm going to do this in Dirac Notation" and a student who rewrote the vector in Dirac notation[3] would both be coded as "A vector can be expressed using Dirac notation." To interpret students' readout strategies, the initial review of the transcripts involved identifying what elements or aspects students attended to when problem solving.

Results

We present students' coordination classes as they normalize different types of vectors: (A) a real vector and (B) a complex vector during pre-interviews; and (C) a complex vector during post-interviews. Using a CC perspective, we identify the ideas students coordinate (causal net), which aspects of the problem statement students attend to (readouts), and the ways in which students take up or disregard ideas while they problem solve (incorporation and displacement).

Pre-interviews for normalizing a real vector

Overall, students were successful in normalizing a real vector. All but one student read out the coefficients of the vector and coordinated that the coefficients were squared and added. Four of seventeen students explicitly identified the dot product. Others bypassed the dot product by directly correlating magnitude or length with the square root of the sum of squared components.

We first present the CC established by A11 to highlight the method of data analysis, then expand discussion to other students. A11 correctly approached the procedure for normalization:

> ***A11:*** Normalize the following vector. Uh, well, we just, we solve for the dot product, which is v-dot-v, which would be five-squared plus two-squared. And we want to square root that, and divide by it. So, you'd have v divided by the square root of v-dot-v.

A11 immediately connected normalization and the dot product. We identify a knowledge element "Normalization involves a dot product." They then coordinated this element with their reading out of the components, an element we label "Dot product adds the components squared." Lastly, they take the result of the calculation and divide it into the original vector, which is identified by the element "Normalization involves dividing by the square root of the dot product." The progression of ideas in the causal net is shown by (1)-(3) in Figure 2.

When asked to "explain why you chose that process," A11 introduced the idea of magnitude and unit vectors, and reiterates several of the earlier connections.

> ***A11:*** Oh, so, to find the magnitude of v, square root of that [v-dot-v] … To normalize it, I took v and I divided by the magnitude of v. It's taking the vector and dividing by its length. ... What's the symbol for a unit vector? … [the] hat kinda tells you it's normalized.

Figure 2 shows the causal net with the elements from the student's additional explanation. Since the student references the dot product they wrote in their initial calculation, we can draw a

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[3] Dirac notation uses bras, $\langle v|$, and kets, $|v\rangle$, to represent row and column vectors, respectively.

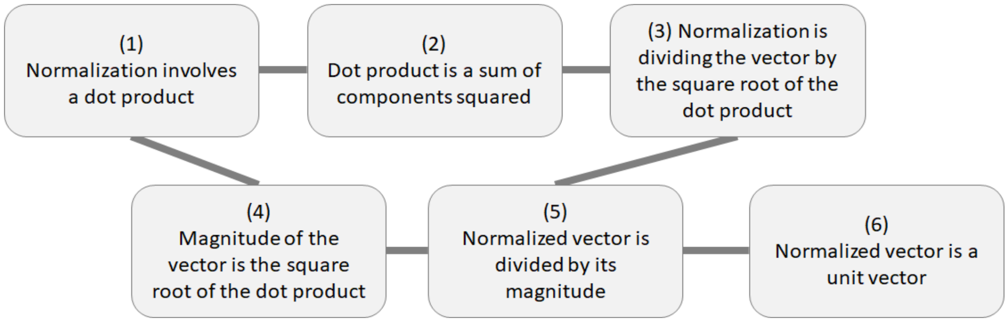

Figure 2. A depiction of A11 CC after they are asked to explain their process. Elements 1-3 were part of their original solution., while the remaining elements appear during further explanation.

connection between their first element and "Magnitude is the square root of the dot product." They coordinate this information with the element "Normalized vector is divided by its [magnitude/length]." The subsequent discussion of division, as the student finalizes their response, is why we connect this element with their initial element (3) in Figure 2.

Our analysis of A11's initial solution provides a look at the sequence of knowledge elements that led from the observed prompt to the determined output. As the student explained their process in response to the interviewer, other elements were identified ((4)-(6) in Figure 2).

Three students approached normalization geometrically, either through graphing on a set of Cartesian axes or by representing a triangle, as exemplified by C7 below.

> ***C7:*** 5 units along one basis vector and two units along the other basis vector which is orthogonal. [draws triangle] … the magnitude of this entire vector would be 5^2 plus 2^2.

C7 coordinated an earlier element "Normalization makes magnitude 1" with elements: "Vector forms a triangle," "Components are amounts along basis directions" and "Magnitude is sum of components squared." Reading out the components as a number of units, C7 identified the magnitude in a physicalized space, albeit incorrectly since they did not include a square root.

C7 was the only student in the pre-interviews to invoke the use of an arbitrary constant.

> ***C7 (continued)***: But we need to normalize that so if I say constant C times 5^2 plus 2^2 equals... C times 29 is one. … So, C is equal to 1 [over] 29, that way it is equal to one. [Adds 1/29 to expanded vector] But then how do I get// Oh, I could say that's 5 over the sqrt of 29 all of that squared plus 2 over the sqrt of 29. All that squared equals one.

A traditional physics approach is to include a *normalization constant* within a given vector and then solve $v \cdot v = 1$ to determine the constant. Inconsistent with the physics approach, C7 applied a constant after the dot product was calculated (Figure 3). We identify a coordination between "Normalize involves multiplying by a constant" and "Constant times the magnitude is one." These connections result in the determination of the constant and a normalized vector.

Three other students read out "normalize" from the question statement, and activate a CC for finding the normal vector. Two of these students end up displacing the incorrect elements by separating the ideas of finding a normal vector and normalization as involving magnitude.

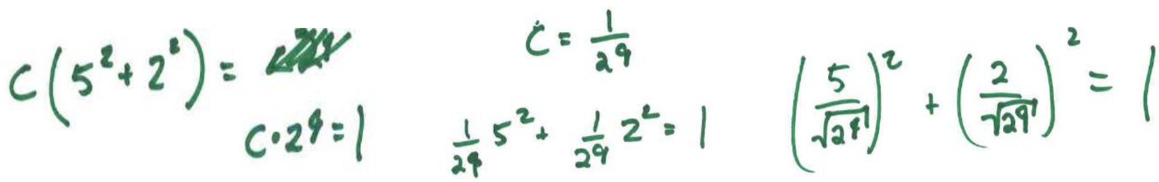

Figure 3. C7's written work where they multiply a constant by the result of their dot product. They then solve for the constant as part of finding the normalized vector.

Pre-interviews for normalizing a complex vector

The additional readout of an imaginary number requires a shift in thinking to include complex conjugates. Most commonly, 10 of 16 students[4] did not incorporate elements for complex vectors. The causal nets were similar to those for the first task. Students commonly invoked the summation of squared components and the division by the square root. As an example, A11 applies the same strategy as with the real vector.

A11: So, you do 3 + 2i squared, uh, plus 4 - i squared, square root of that. That's going to equal the w magnitude. [Calculates to $\sqrt{20+20i}$] … divide by that.

A11 begins the same process of summing the components squared as they did with the dot product for the initial vector. They take the square root and identify the result as the magnitude, consistent with the knowledge element "Magnitude of the vector is the square root of the dot product." Lastly, they divide the vector, w, by the magnitude. Figure 4 shows the relevant knowledge elements that were identified overlapping with the student's initial causal net.

Six students identify the imaginary component and change their strategies to incorporate the idea of a dot product of the vector with its complex conjugate, as shown by C7 below.

C7: I think I am going to be using a complex conjugate but I am not sure. So first of all, to get the magnitude of w, I am going to do w vector times w vector star. … that is 3+3i times 3-3i right. That is equal to ... 18 and then I have got to square root the whole thing.

After being presented with a complex vector, C7 identified the need for a complex conjugate, $w*$. They changed the sign of the i term and carried out a dot product between w and $w*$. They determined the magnitude as the result of the calculation, the square root of 18. The elements used by C7 up to this point are consistent with other students' causal nets for incorporating complex terms. However, C7 went on to use a constant and again invoked the elements "Normalized vector has a magnitude of one" and "Constant times multiplied vectors is one."

Post-Interviews with a complex vector

Following instruction in quantum mechanics, all students invoked the complex conjugate and only one of seventeen students incorrectly normalized the vector. The majority carry out normalization by independently finding the magnitude of the vector and then dividing by it.

The most common method was calculating the inner product of a vector with its complex conjugate, then dividing by the square root of the result. A11 now exemplifies this process.

A11: Ok, so normalize it. ... v dot v ... would be, uh, 3^2 plus negative $(2i)^2$ [laughs] because now, we know how to do that, and that's 9 plus 4 and that's going to be 13. So, you're going to have v normalized, is going to be 1 over root 13, … you take v, and you multiply by its complex conjugate, ... so this would be 3 times 3 plus 2i times negative 2i.

In addition to elements related to the dot product, A11 referenced the complex conjugate in their

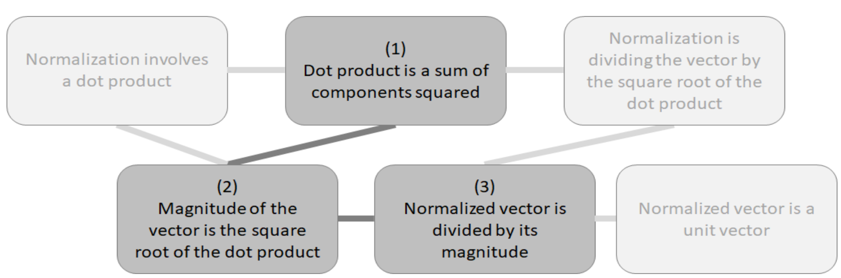

FIGURE 4. A11's causal net for normalizing a complex vector, which incorporates the same knowledge elements.

[4] One student did not get the question with the complex vector, since they did not complete the first task.

explanation. Here, we identify "Dot product is between vector and complex conjugate" and "Complex conjugate adds a negative sign to the imaginary term."

Three of the four students using Dirac notation were also among the four to apply a normalization constant. As part of the post-interviews, C7 accurately applied the constant to the vector (not the dot product), consistent with other students who applied this method.

> ***C7:*** Now, to normalize … there needs to be some scalar times these two [kets] such that when I square the components of them and add them together, I'll get a total of one. ... v bra times v ket is equal to one, alright? So complex conjugate times [writing]. All that times the normal thing [initial vector]. So, C 3 up ket plus C 2i minus ket. (Figure 5)

In C7's work, we identify the elements "Normalization involves a scalar" and "Multiplying by scalar means sum of the components squared is 1." C7 then sets the Dirac inner product equal to one and writes out both the bra and ket in terms of the basis vectors as shown in Figure 5. They then invoke elements "Bra is the complex conjugate," consistent with the application of Dirac notation in physics, and "Complex conjugate changes the sign of the imaginary terms."

Discussion

Vector normalization is a cross-cutting concept that spans both mathematics and physics instruction. This work informs on physics students' conceptualizations of normalization of both real and complex vectors. The use of interviews before and after relevant instruction explore the way the concept of normalization changes following instruction in quantum mechanics.

Pre-interviews establish a baseline for thinking about normalization, in which all but one student normalized the vector correctly. Students commonly coordinated elements related to dot products, the sum of squared components, dividing by a square root, and magnitude or length of a vector. The transition to the complex vector resulted in about two-thirds of students invoking the same knowledge elements, and thereby utilizing the same CC. Students that correctly normalized, incorporated additional elements related to complex vectors into their causal net, such as needing a dot product of the vector with its complex conjugate.

In the post-interviews, all but one student correctly normalized the complex vector. By the end of quantum mechanics, students successfully incorporated elements related to complex vectors into their causal nets. The identified CC became more consistent with the standard approaches for normalization within both physics and mathematics.

The results provide information to mathematics instructors about how mathematics is applied in a physics course. For example, the use of a constant to normalize a vector is a common practice in quantum mechanics that is taken up by students following quantum instruction. This research further reports common knowledge elements used by students to construct a process for normalization by way of coordination class theory from physics education research.

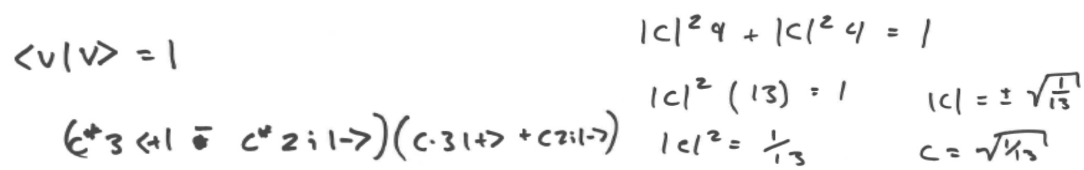

Figure 5. C7's work using a normalization constant. C7 applies C with the complex conjugate vector but later choose C to be real and positive, consistent with physics conventions.*

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References

Barniol, P., & Zavala, G. (2011). Students' difficulties with unit vectors and scalar multiplication of a vector. In N. S. Rebello, P. V. Engelhardt, & C. Singh (Eds.), *2011 PERC Proceedings* (pp. 115–118). Omaha, NE.

Barniol, P., & Zavala, G. (2014). Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics – Physics Education Research, 10*(010121), 1–14.

Bernard, R. H. (1988). *Research methods in cultural anthropology.* Sage Publications.

Buteler, L., & Coleoni, E., (2016). Solving problems to learn concepts, how does it happen? A case for buoyancy. *Physical Review Physics Education Research, 12*(2), 020144.

Close, H., Schiber, C., Close, E. W., & Donnelly, D. (2013, April). Students' dynamic geometric reasoning about quantum spin-1/2 states. In *Proceedings of the 2013 Physics Education Research Conference* (pp. 93-96), Portland, OR.

diSessa A. A. (1993). Toward an epistemology of physics, *Cognition and Instruction, 10*(2–3), 105–225.

diSessa A. A., & Sherin B. L. (1998). What changes in conceptual change? *International Journal of Science Education, 20*(10), 1155–1191.

Dreyfus, B. W., Elby, A., Gupta, A., & Sohr, E. R. (2017). Mathematical sense-making in quantum mechanics: An initial peek. *Physical Review Physics Education Research, 13*(2), 020141. doi:10.1103/PhysRevPhysEducRes.13.020141

Her, P., & Loverude, M. (2020, July). *Examining student understanding of matrix algebra and eigentheory*. Paper presented at Physics Education Research Conference 2020, Virtual Conference.

Gire, E., & Price, E. (2015). Structural features of algebraic quantum notations. *Physical Review Special Topics – Physics Education Research, 11*(2), 1–11.

Hu, D., & Rebello, N. S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. *Physical Review Special Topics - Physics Education Research, 9*(2), 020118. https://doi.org/10.1103/PhysRevSTPER.9.020118" https://doi.org/10.1103/PhysRevSTPER.9.020118

McIntyre, D., Manogue, C.A., & Tate, J. (2012). *Quantum Mechanics: A Paradigms Approach*. Addison-Wesley.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Parnafes O., (2007), What does ''fast'' mean? Understanding the physical world through computational representations, *Journal of the Learning Sciences, 16*(3), 415–450.

Rodriguez, J. G., Stricker, A. R., & Becker, N. M. (2020). Students' interpretation and use of graphical representations: Insights afforded by modeling the varied population schema as a coordination class, *Chemistry Education Research and Practice, 21*(2), 536–560.

Ryan, Q. X., & Schermerhorn, B. P. (2020). Students' use of symbolic forms when constructing equations of boundary conditions. *Physical Review Physics Education Research*, *16*(1), 010122.

Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G. (2012). Modelling Mathematical Reasoning in Physics Education. *Science and Education, 21*(4), 485–506.

Schermerhorn, B. P., Passante, G., Sadaghiani, H., & Pollock, S. J. (2019). Exploring student preferences when calculating expectation values using a computational features framework. *Physical Review Physics Education Research*, *15*(2), 020144.

Serbin, K.S., Wawro, M., & Storms, R. (2021). Characterizations of student, instructor, and textbook discourse related to basis and change of basis in quantum mechanics. *Physical Review Physics Education Research, 17,* 010140.
Vega, M., Christensen, W., Farlow, B., Passante, G., & Loverude, M. (2016). Student understanding of unit vectors and coordinate systems beyond cartesian coordinates in upper division physics courses. In D. L. Jones, L. Ding, & A. Traxler (Eds.), *2016 PERC Proceedings* (pp. 194-199), Sacramento, CA.
Wagner, J., Manogue, C., & Thompson, J. (2011, August). *Representation issues: Using mathematics in upper-division physics*. Paper presented at Physics Education Research Conference 2011, Omaha, Nebraska.
Wawro, M., Watson, K., & Christensen, W. (2019). Student reasoning about eigenvectors and eigenvalues from a Resources perspective. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 654-662), Oklahoma City, OK.
Wittmann M. C. (2002), The object coordination class applied to wave pulses: Analysing student reasoning in wave physics, *International Journal of Science Education, 24*(1), 97–118.
Wittmann, M. C., & Black, K. E. (2015). Mathematical actions as procedural resources: An example from the separation of variables. *Physical Review Special Topics – Physics Education Research*, *11*(2), 1–13.