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Transition fronts of Fisher–KPP equations in locally spatially inhomogeneous patchy environments

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ABSTRACT

This paper is devoted to the study of spatial propagation dynamics of species in locally spatially inhomogeneous patchy environments or media. For a lattice differential equation with monostable nonlinearity in a discrete homogeneous media, it is well-known that there exists a minimal wave speed such that a traveling front exists if and only if the wave speed is not slower than this minimal wave speed. We shall show that strongly localized spatial inhomogeneous patchy environments may prevent the existence of transition fronts (generalized traveling fronts). Transition fronts may exist in weakly localized spatial inhomogeneous patchy environments but only in a finite range of speeds, which implies that it is plausible to obtain a maximal wave speed of existence of transition fronts.

1. Introduction

Front propagation occurs in many applied fields such as population dispersals in biology, combustion in chemistry, neuronal waves in neuroscience, fluid dynamics in physics and more. Since the pioneering work of Fisher [1] and Kolmogorov–Petrovskii–Piskunov [2], front propagation dynamics of classical reaction–diffusion equation

$$u_t(t,x) = u_{xx} + f(x,u)u, x \in \mathbb{R} \tag{1.1}$$

and lattice differential equation

$$\dot{u}_j(t) = u_{j+1} - 2u_j + u_{j-1} + f_j(u_j)u_j, \quad j \in \mathbb{Z}. \tag{1.2}$$

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have been studied extensively. In biology (1.1) is used to model the spread of population in non-patchy environment with random internal interaction of the organisms and (1.2) is for species in patchy environment with nonlocal internal interaction of the organisms. Here we focus on (1.2). For nonlinearity term $f_j(u_j)$, we assume that

(H1) *$f_j \in C^2([0,\infty),\mathbb{R})$, $-L < \inf_{j\in\mathbb{Z},v\geq 0}\{f_j'(v)\} \leq \sup_{j\in\mathbb{Z},v\geq 0}\{f_j'(v)\} < 0$ for all $(j,v) \in \mathbb{Z}\times\mathbb{R}^+$ with some $L > 0$ and $f_j(v) < 0$ for all $(j,v) \in \mathbb{Z}\times\mathbb{R}^+$ with $v > L_0$ for some $L_0 > 0$.*

In the literature, (H1) is called Fisher–KPP type nonlinearity due to Fisher [1] and Kolmogorov–Petrovskii–Piskunov [2]. However, most existing works are concerned with the propagation dynamics in homogeneous or spatially periodic media. Fisher [1] and Kolmogorov–Petrovskii–Piskunov [2] considered a homogeneous case of (1.1), that is, $f(x,u) = f(u) = 1-u$. Fisher conjectured and Kolmogorov–Petrovskii–Piskunov proved that there exist traveling fronts of speeds not less than the minimal wave speed $c^* = 2$, which is a solution of (1.1) of form $u(t,x) = \phi(x-ct)$, $\phi(-\infty) = 1$ and $\phi(\infty) = 0$. Later, existence of periodic traveling waves of (1.1) or more general reaction–diffusion equations with Fisher–KPP nonlinearity has been studied by researchers including B. Zinner and his collaborators in 1995 [3], H.F. Weinberger in 2002 [4], and H. Berestycki et al. in 2005 [5]. For the case in non-periodic inhomogeneous media, we cannot expect wave profiles that take the form of constant or periodic front profiles. The notation of traveling waves has been extended to generalized traveling waves or transition fronts by several authors (e.g., [6,7]). In the past decade, transition fronts in non-periodic inhomogeneous media have attracted much attention (e.g., [6,8–11]). For instance, J. Nolen et al. considered in [10] the KPP equation of one dimension with random dispersal (classic reaction–diffusion equation) in compactly supported inhomogeneous media. More precisely, they considered (1.1) in the media which are localized perturbations of the homogeneous media. They showed that localized KPP inhomogeneity may prevent the existence of transition fronts and provided some examples.

The discrete system (1.2) has also been the subject of much research attention. The past two decades have seen vigorous research activities on applications to dynamics on lattice differential equations [12–19]. In numerical simulations, lattice differential equations have some advantages over classical reaction–diffusion equations in applications. For example, (1.2) can be viewed as the spatial discretization of (1.1). On the other hand, lattice differential equations are of interest as models in their own right. It is more reasonable to model some problems with spatial discrete structure such as population dispersal in a patchy environment by lattice differential equations. The main concerns include also the properties of spreading speed and propagation of waves such as traveling fronts, periodic(pulsating) traveling waves and transition fronts. For homogeneous or periodic discrete media with monostable or bistable nonlinearities, we refer the readers to [12–15,18,19]. The simplest case of transition fronts are traveling waves whose profiles are time-independent, that is, there exists some function ϕ such that

$$u_j(t) = \phi(j-ct), \phi(\infty) = 0 \text{ and } \phi(-\infty) = 1, \tag{1.3}$$

where c is the wave speed. For the homogeneous case with $f_j(u_j) = 1-u_j$, it is almost trivial that there exists a minimal wave speed c^* such that a traveling wave exists if and only if the wave speed $c \geq c^*$. Later, periodic traveling wave solutions have been investigated in [20,21] for the Fisher–KPP equation in periodically inhomogeneous media, where the periodic traveling wave solutions $u_j(t)$ to lattice differential equations such as (1.2) satisfy the following

$$u_j(t+p/c) = u_{j-p}(t), \lim_{j\to-\infty} u_j(t) = 1 \text{ and } \lim_{j\to\infty} u_j(t) = 0 \text{ locally in } t \in \mathbb{R}. \tag{1.4}$$

Work on entire solutions or transition fronts for bistable reaction–diffusion equations in discrete media includes [19,22]. However, less is known to the spreading dynamics to (1.2) with Fisher–KPP nonlinearity in non-periodic inhomogeneous media.

Kong and Shen [23] considered the KPP equations in higher space dimension with nonlocal, random or discrete dispersal in localized perturbations of the homogeneous media and investigate in [24] the KPP equations with nonlocal, random or discrete dispersal in localized perturbations of the periodic media. They showed that the localized spatial inhomogeneity of the medium preserve the spatial spreading in all the directions. The lower bound of mean wave speed of (1.2) can be obtained due to the spreading properties proved in [24] and in [23] for the particular case in localized perturbations of the homogeneous media. However, the existence and (general) non-existence of transition fronts have not yet been investigated for discrete dispersals.

We will focus on the study of existence and non-existence of transition fronts of (1.2) with Fisher–KPP type nonlinearity in localized perturbations of spatially homogeneous patchy environments or media. Hereafter, we assume the following:

(H2) *$f_j(0) > 0$ for all j and $f_j(0) = 1$ for any $|j| \geq N$ with some positive integer N.*

Throughout the paper, we assume (H1)–(H2). Let $\Lambda : \mathcal{D}(\Lambda) \subset X \to X$ be defined by

$$(\Lambda u)_j := u_{j+1} - 2u_j + u_{j-1} + f_j(0)u_j, \forall u \in X, \tag{1.5}$$

where $X = \{u||u_j| < L, \text{for someL¿0and allj} \in \mathbb{Z}\}$ with norm $\|u\|_X = \sup_{j\in\mathbb{Z}}\{|u_j|\}$.

Let $\lambda = \sup\{\mathbf{Re}\ \mu \mid \mu \in \sigma(\Lambda)\}$. Let $\{u_j^*\}_{j\in\mathbb{Z}}$ be the unique positive stationary solution of (1.2), where the existence of $\{u_j^*\}_{j\in\mathbb{Z}}$ was proved in Theorem 2.1 of [23] by Kong and Shen under the assumptions of (H1) and (H2). To study the propagation wave solutions in localized perturbations in patchy media, we will extend the traveling front of (1.3) in homogeneous media and the periodic traveling front of (1.4) in periodic media and define transition fronts of (1.2) and their mean speeds as follows:

Definition 1.1 (*Transition Front*)**.** A global-in-time solution $\{u_j(t)\}_{j\in\mathbb{Z}}$ of (1.2) is called a transition front if $0 \leq u_j(t) \leq u_j^*$ and there is a continuous function $X(t) : \mathbb{R} \to \mathbb{R}$ such that $\lim_{j-X(t)\to-\infty}(u_j(t) - u_j^*) = 0$ and $\lim_{j-X(t)\to\infty} u_j(t) = 0$ uniformly in $t \in \mathbb{R}$.

Definition 1.2 (*Mean Wave Speed*)**.** A transition front is said to admit a mean wave speed if the following limit exists: $c = \lim_{|t_j-t_k|\to\infty} \dfrac{j-k}{t_j - t_k}$, where t_i is the first time such that $u_i(t_i) = \dfrac{1}{2}\inf_j\{u_j^*\}$ for $i \in \mathbb{Z}$ and $u_l(t_i) < \dfrac{1}{2}\inf_j\{u_j^*\}$ for all $l > i$.

Remark 1.1.

(1) The analogous notion for continuous reaction–diffusion equation in [10] was referred to in [7,25];

(2) By Theorem 2.1 in [23], there exists a positive r such that $\inf_{|j|>r}\{u_j^*\} > 0$. Thus, $\inf_j\{u_j^*\} > 0$ in Definitions 1.1 and 1.2.

In the current study, our main result shows conditions for both existence and nonexistence of transition fronts of (1.2) for lattice differential KPP equation in patchy environment with a localized perturbation in media. There are several essential differences between classic reaction–diffusion equations and lattice differential equations. Among these are the use of fundamental PDE techniques including heat kernel estimates, Poincaré inequality, Harnack inequality and principal eigenvalue theory. We shall introduce discrete versions of these fundamental tools in later sections. Because of those significant differences, the approaches for classical reaction–diffusion equations in [10] cannot be applied directly to (1.2), that is a continuous-time discrete in space lattice differential equation. In this paper, we consider transition fronts in the localized perturbed homogeneous patchy media, and provide the variational formulas for both the upper bound and the lower bound of the wave speeds that transition fronts exist.

Principal eigenvalue theory plays a central and important role in studying transition fronts. Let $X_\mu = \{u \in X : \sup_j |e^{j\cdot\mu} u_j| < \infty\}$. Consider the following linear difference equation for $\psi \in X_\mu$,

$$\Lambda\psi_j = \gamma\psi_j, \tag{1.6}$$

where $j \in \mathbb{Z}$, Λ is as in (1.5).

Note that letting $\psi_j = e^{-\mu j}\phi_j$, we have the following equivalent problem for $\phi \in X$,

$$e^{-\mu}\phi_{j+1} - 2\phi_j + e^{\mu}\phi_{j-1} + a_j\phi_j = \gamma_\mu\phi_j, \tag{1.7}$$

where $j \in \mathbb{Z}$ with $a_j = f_j(0)$.

Let $\Lambda_\mu : \mathcal{D}(\Lambda_\mu) \subset X \to X$ be defined by $(\Lambda_\mu\phi)_j := e^{-\mu}\phi_{j+1} - 2\phi_j + e^{\mu}\phi_{j-1} + a_j\phi_j$. Λ_μ are of so called Jacobi operators in [26]. The positive principal eigenvectors to (1.7) play important roles in constructions of transition fronts to (1.2). We refer readers to [26] for spectral theory of Jacobi operators in a Hilbert space. For the particular case of periodic media, we refer readers to [20]. Due to lack of compactness, we apply an extension of Krein–Rutman theorem [27], Theorem 2.2 in [28] to prove that (1.5) has a principal eigenvalue λ (see Lemma 3.1). We investigate the positive solutions to (1.7)(see Lemma 3.3). If $a_j \equiv 1$ in (1.7), the principal eigenvalue is equal to $e^{\mu} - 1 + e^{-\mu}$ associated with constant eigenvector 1. We define one auxiliary function $\lambda(\mu) = e^{\mu} - 1 + e^{-\mu}$ for $\mu > 0$ and another auxiliary function for the wave speed, $c(\mu) = \frac{\lambda(\mu)}{\mu}$ for $\mu > 0$. Let (c^*, μ^*) be such that $c^* = \dfrac{\lambda(\mu^*)}{\mu^*} = \inf_{\mu>0} \dfrac{\lambda(\mu)}{\mu}$. It is well-known that if $a_j \equiv 1$, then c^* is so called spreading speed, that is the minimal speed such that a traveling front solution of (1.2) may exist. The existence of transition fronts of (1.2) relies on the constructions of super/sub-solutions with valid positive principal eigenvectors to (1.7). Let $\lambda^* = \lambda(\mu^*)$ and $(\hat{c}, \hat{\mu})$ be such that $\lambda(\hat{\mu}) = \lambda$ and $\hat{c} = c(\hat{\mu})$. We find that the parameter values of μ of valid positive principal eigenvectors to (1.7) must locate in $[\hat{\mu}, \mu^*)$ when $\lambda > 1$ and $\hat{\mu} < \mu^*$ (See Section 3). Due to the spreading properties of c^* proved in [23], the parameter values of μ of valid positive principal eigenvectors to (1.7) must be less than μ^*. We explore the minimal speed c^* in Section 4.1. On the other hand, the principal eigenvalue λ of (1.5) plays another important role that may prevent the existence of transition fronts. The $\hat{c}$ is corresponding to the maximal speed such that a traveling solution may exist (see Section 4.3). The following Fig. 1 shows the existence intervals of transition fronts to (1.2) with parameter values $(c, \mu) \in [c^*, \hat{c}] \times [\hat{\mu}, \mu^*]$.

We state the main theorem in the following.

Theorem 1.1 (*Existence and Non-Existence of Transition Fronts*). *Assume (H1)–(H2).*

(1) If $\lambda \in [1, \lambda^*)$ and $\hat{c} > c^*$, then transition front exists for any speed $c \in [c^*, \hat{c}]$. Moreover, if $c \in (c^*, \hat{c}]$, then for any $\epsilon > 0$, there exist $C_1,\ C_2,\ T > 0$ such that for $t > T$ and $j > ct$,

$$C_1 e^{-(\mu+\epsilon)(j-ct)} \le u_j(t) \le C_2 e^{-(\mu-\epsilon)(j-ct)}. \tag{1.8}$$

(2) No transition front with speed c exists for the following cases: (i) $\lambda > \lambda^*$; (ii)$c < c^*$ and (iii) $c > \hat{c}$.

Remark 1.2. Throughout the paper, we consider the existence and nonexistence of transition front with a mean wave speed. The uniqueness of a transition front remains an open question. It is unknown whether there is some transition front without a mean speed. However, for nonexistence, in general there are no transition fronts if $X(t)/t < c^*$ or $X(t)/t > \hat{c}$ as $t \to \infty$.

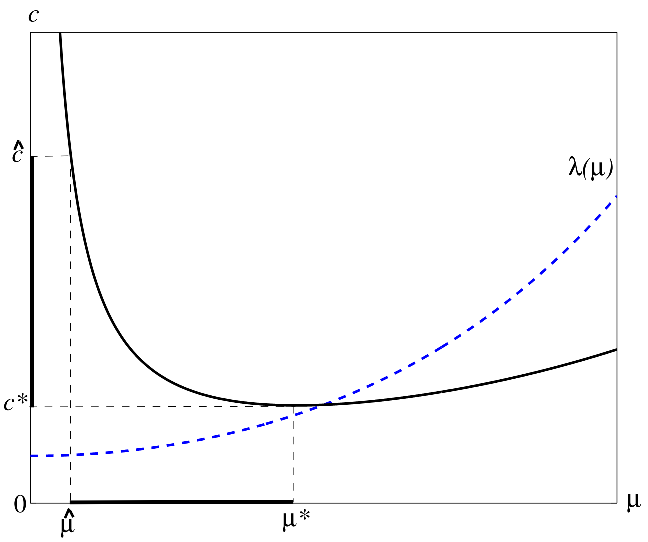

Fig. 1. The solid curve is for speed auxiliary function $c(\mu) = \frac{\lambda(\mu)}{\mu}$. The dashed curve is for principal eigenvalue $\lambda(\mu)$. The parameter values of existence region of transition fronts are located on $[c^*, \hat{c}] \times [\hat{\mu}, \mu^*]$, where $c^* = \inf_{\mu>0} c(\mu)$ and $\hat{c} = c(\hat{\mu})$ with $\hat{\mu}$ satisfying $\lambda(\hat{\mu}) = \lambda$.

This paper is organized as follows. In Section 2, we provide the discrete analogs of fundamental tools in classical reaction–diffusion equations, including semigroup theory, comparison principles, discrete heat kernel, discrete parabolic Harnack inequality and many others. In Section 3, we investigate the principal eigenvalue theory and construct the super/sub-solutions. Then we show the existence of transition fronts and also the asymptotic behaviors of transition fronts (1.8), that is, proof of Theorem 1.1 (1). In Section 4, we show nonexistence of transition fronts under $\lambda > \lambda^*$, the lower bound of wave speeds (minimal wave speed c^*), and the upper bound of wave speeds (maximal wave speed $\hat{c}$), that is proof of Theorem 1.1 (2). In Section 5, we provide a particular example with the simplest case: a perturbation at a single location. Finally, we provide some concluding remarks in Section 6.

2. Foundations of lattice differential equations

2.1. Initial value problem

Let $X^+ = \{u \in X | u_j \geq 0, \forall j \in \mathbb{Z}\}$. Let Λ be as in (1.5). It follows from the general semigroup approach (see [29]) that Λ generates a uniformly continuous semigroup $T(t)$ and (1.2) has a unique (local) solution $u(t; z)$ with $u(0) = \{z_j\}_{j\in\mathbb{Z}}$ for every $z \in X$, that is given by

$$u(t) = T(t)u(0) - \int_0^t T(t-s)g(s)ds, t > 0, \tag{2.1}$$

where $g_j(s) = (f_j(u_j) - f_j(0))u_j$ for $j \in \mathbb{Z}$, $g(s) = \{g_j(s)\}_{j\in\mathbb{Z}}$, and $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$.

2.2. Comparison principle

We introduce comparison principle in this subsection, which will play an important role in obtaining the existence of transition fronts of (1.2). We define super/sub-solutions and state the comparison principle as follows.

Definition 2.1 (*Super/Sub-Solution*)**.** For a given continuous-time and bounded function $u_j : [0,T) \to \mathbb{R}$, $\{u_j\}_{j\in\mathbb{Z}}$ is called a super-solution (sub-solution) of (1.2) on $[0,T)$ if for all j, $\dot{u}_j(t) \geq (\leq) u_{j+1} - 2u_j + u_{j-1} + f_j(u_j)u_j$.

Proposition 2.1 (*Comparison Principle*)**.**

(1) If $u(t)$ and $v(t)$ are sub-solution and super-solution of (1.2) on $[0,T)$, respectively, $u_j(0) \leq v_j(0)$, then

$$u_j(t) \leq v_j(t) \quad \text{for} \quad t \in [0,T).$$

Moreover, if $u_j(0) \neq v_j(0)$ for some j, then for all j,

$$u_j(t) < v_j(t) \quad \text{for} \quad t \in (0,T).$$

(2) If $z, w \in X$ and $z \leq w$, then $u_j(t;z) \leq u_j(t;w)$ for $t > 0$ at which both $u(t;z)$ and $u(t;w)$ exist. Moreover, if $z_j \neq w_j$ for some j, then for all j, $u_j(t;z) < u_j(t;w)$ for $t > 0$ at which both $u(t;z)$ and $u(t;w)$ exist.

Proof. The proof follows from arguments in Lemma 2.1 in [15]. □

With the comparison principle, we have that if $z \in X^+$, $u(t;z) \in X^+$.

In next two subsections, we introduce the discrete heat kernel and the discrete parabolic Harnack inequality, which play critical roles in studying the asymptotic behaviors and the bounds of wave speeds of transition fronts.

2.3. Discrete heat kernel

Discrete heat kernel is highly related to I-Bessel functions. The I-Bessel function $I_x(t)$ is defined as a solution to the differential equation

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (t^2 + x^2)y = 0.$$

In [30], the author derived an upper bound and lower bound for $I_x(t)$, for all $t > 0$ and $x \geq 0$,

$$e^{-\frac{1}{2\sqrt{t^2+x^2}}} \leq I_x(t)\sqrt{2\pi}(t^2 + x^2)^{\frac{1}{4}} e^{-\varsigma_0(t,x)} \leq e^{\frac{1}{2\sqrt{t^2+x^2}}},$$

with $\varsigma_0(x,t) = \sqrt{t^2 + x^2} + x ln(\frac{t}{x+\sqrt{t^2+x^2}})$.

By Proposition 3.1 in [31], the heat kernel on a 2-regular graph is given by

$$K(t,r) = e^{-2t} I_r(2t), \text{ for } (t,r) \in (0,\infty) \times \mathbb{Z}^+.$$

With the help of the above bounds of $I_r(t)$, we have the bounds of $K(t,r)$:

$$\frac{1}{\sqrt{2\pi}}(4t^2 + r^2)^{-\frac{1}{4}} e^{-2t - \frac{1}{2\sqrt{4t^2+r^2}} + \varsigma_0(2t,r)} \leq K(t,r) \leq \frac{1}{\sqrt{2\pi}}(4t^2 + r^2)^{-\frac{1}{4}} e^{-2t + \frac{1}{2\sqrt{4t^2+r^2}} + \varsigma_0(2t,r)}.$$

The authors in [31] showed that $\sqrt{t}e^{-t}I_x(t) \leq (1 + \frac{x}{t})^{-\frac{x}{2}}$, thus $K(t,r) \leq \frac{1}{\sqrt{2t}}(1 + \frac{r}{2t})^{-\frac{r}{2}}$.

By Theorem 2.3 in [32], $h_t^{\mathbb{Z}}(j) \asymp F(t,j)$, that is, there exist positive real constants $\epsilon > 0$ and $M_\epsilon > 0$ such that

$$(1 - \epsilon)F(t,j) \leq h_t^{\mathbb{Z}}(j) \leq (1 + \epsilon)F(t,j), \tag{2.2}$$

for $j^2+t^2 > M_\epsilon$, where $h_t^{\mathbb{Z}}(j)$ is the heat kernel associated with $\mathcal{L}f(j) = f(j) - \frac{f(j+1)+f(j-1)}{2}$ and $F(t,j)$ is given by if $j=0$,

$$F(t,j) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+t^2)^{\frac{1}{4}}},$$

else if $j \neq 0$,

$$F(t,j) = \frac{1}{\sqrt{2\pi}} \frac{exp[-t + |j|\varsigma(t/|j|)]}{(1+t^2+j^2)^{\frac{1}{4}}},$$

where $\varsigma(t/|j|) := \varsigma_0(1, t/|j|)$.

Recall the nonlinear equation (1.2),

$$\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + f_j(u_j)u_j, \quad j \in \mathbb{Z}.$$

Consider also the linearized equation

$$\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + a_j u_j, \quad j \in \mathbb{Z}, \tag{2.3}$$

where $a_j = f_j(0)$

Let $\Lambda_s : \mathcal{D}(\Lambda_s) \subset X \to X$ be defined by

$$(\Lambda_s u)_j := u_{j+1} - u_j + u_{j-1}, \forall u \in X. \tag{2.4}$$

Let $S(t)$ be the semigroup generated by Λ_s. Note that $(S(t)z)_j = e^t \sum_k h_{2t}^{\mathbb{Z}}(j-k)z_k$ for $z := \{z_j\}_{j\in\mathbb{Z}} \in X$. Then the solution of (1.2) is given by

$$u(t) = S(t-T)u(T) - \int_T^t S(t-s)g(s)ds, t > T,$$

where $g_j(t) = (1 - f_j(u_j))u_j(t)$. More precisely, we have the following, for $t > T$,

$$u_j(t) = e^{t-T} \sum_k h_{2(t-T)}^{\mathbb{Z}}(j-k)u_k(T) - \int_T^t e^{(t-s)} \sum_k h_{2(t-s)}^{\mathbb{Z}}(j-k)g_k(s)ds. \tag{2.5}$$

We should point out that the solution form with (2.5) is slightly different with that given by (2.1). With heat kernel $h_{2t}^{\mathbb{Z}}$ in (2.5), we can use the heat kernel estimate (2.2). Then there would be some advantages over (2.1) while exploring some estimates, such as the exponential tail estimates of transition fronts.

2.4. Discrete parabolic Harnack inequality

In this subsection, we shall introduce the discrete parabolic Harnack inequality for the solution to our main equation (1.2). Harnack inequalities have many significant applications in both elliptic and parabolic differential equations such as exploring boundary regularity, heat kernel estimate, and other solution estimates. Moser in [33] proved a parabolic Harnack inequality for classical parabolic PDEs. For discrete parabolic Harnack inequalities, we will adopt Definition 1.6 and apply Theorem 1.7 in [34] to prove that the discrete parabolic Harnack inequality holds on a 2-regular graph. Readers are referred to [34] for further information about parabolic Harnack inequality on graphs. For convenience, we recall necessary graph theory, and state the Definition 1.6 of [34] as the following Definition 2.2.

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx}$ a symmetric nonnegative weight on $\Gamma \times \Gamma$. We call x and y neighbors, denoted by $x \sim y$, when $\mu_{xy} \neq 0$. Vertices are measured by $m(x) = \sum_{x\sim y} \mu_{xy}$. The "volume" of subsets $E \subset \Gamma$ by $V(E) = \sum_{x\in E} m(x)$. We can further define $d(x,y)$ as the distance of x and y in Γ, that is, the shortest number of edges between x and y. Let $B_r(x)$ be the closed ball $\{y \in \Gamma | d(x,y) \le r\}$. We say that $u(t,x)$ satisfies continuous-time parabolic equation on (t,x) if

$$m(x)u_t(t,x) = \sum_y \mu_{xy}(u(t,y) - u(t,x)). \tag{2.6}$$

We remark that for a 2-regular graph, x has only two neighbors $y_- := x-1$ and $y_+ := x+1$. If we consider the same weight for $\mu_{xy_-} = \mu_{xy_+}$, then

$$\sum_y \mu_{xy}(u(t,y) - u(t,x)) = \mu_{xy_-}(u(t,x-1) - 2u(t,x) + u(t,x+1)),$$

that is the exactly same type equation as (1.2) we consider in the paper. In [34], Delmotte defines Harnack inequality of (2.6) on the graph as follows.

Definition 2.2 (*Harnack Inequality [34]*)**.** Set $\eta \in (0,1)$ and $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$. (Γ, μ) satisfies the continuous-time parabolic Harnack inequality $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C)$ if for all x_0, s, r and every nonnegative solution on $Q = [s, s+\theta_4 r^2] \times B_r(x_0)$ we have

$$\sup_{Q_-} u \le C \inf_{Q_+} u,$$

where $Q_- = [s+\theta_1 r^2, s+\theta_2 r^2] \times B_{\eta r}(x_0)$ and $Q_+ = [s+\theta_3 r^2, s+\theta_4 r^2] \times B_{\eta r}(x_0)$.

By Theorem 1.7 in [34], the discrete parabolic Harnack inequality holds if and only if the following three conditions are satisfied:

Definition 2.3 ($\Delta^*(\alpha)$ *Condition*)**.** Let $\alpha > 0$, the weighted graph satisfies $\Delta^*(\alpha)$ if

$$x \sim y \implies \mu_{xy} \ge \alpha m(x);$$

Definition 2.4 (*"Doubling Volume" Property*)**.** There exists a $C > 0$ such that

$$V(B_{2r}(x)) \le C V(B_r(x))$$

for any $x \in \Gamma$ and r;

and

Definition 2.5 (*Poincaré Inequality*)**.** There exists a $C_2 > 0$ such that for all $v \in \mathbb{R}^\Gamma$, all x_0, and $r > 0$,

$$\sum_{x \in B_r(x_0)} m(x)(v(x) - \bar{v})^2 \le C_2 r^2 \sum_{x,y \in B_{2r}(x_0)} \mu_{xy}(v(x) - v(y))^2,$$

where $\bar{v} = \frac{1}{V(B_r(x_0))} \sum_{x \in B_r(x_0)} m(x)v(x)$.

Now we claim that parabolic Harnack inequality holds on a 2-regular graph.

Theorem 2.1 (*Harnack Inequality on a 2-regular Graph*)**.** *The parabolic Harnack inequality $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C)$ holds on a 2-regular graph.*

Proof. It suffices to show a 2-regular graph satisfies the $\Delta^*(\alpha)$ condition, the "doubling volume" property and the Poincaré inequality. First, a 2-regular graph with $0 < \alpha \leq \frac{1}{2}$ satisfies the $\Delta^*(\alpha)$ condition. Second, for a 2-regular graph, $V(B_r(x)) = 2(2r+1)$ and $V(B_{2r}(x)) = 2(4r+1)$. Choose $C = 2$ and then the "doubling volume" property holds.

Finally, we prove the Poincaré inequality on a 2-regular graph. In fact, we have a strong Poincaré inequality, that is, $B_{2r}(x_0)$ can be reduced by $B_r(x_0)$. Without loss of generality, let $x_0 = 0$ and consider $v(x)$ for $-r \leq x \leq r$. Consider the same weights for all vertices, and let $\mu_{xy} = \mu_{xy} = 1$ if $|y - x| = 1$, otherwise 0. Then we have

$$\sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x) - v(y))^2 = \sum_{x\in B_r(x_0)} [(v(x) - v(x+1))^2 + (v(x) - v(x-1))^2]. \tag{2.7}$$

The sequence $v(x)$ oscillates around $\bar{v}$. In other words, if $v(x)$ moves from $-r$ to r, it must either hit the $\bar{v}$ at some point or cross $\bar{v}$ from one side to another. There exists at least one integer $\hat{x}$ such that either $v(\hat{x}) = \bar{v}$ or $(v(\hat{x}) - \bar{v})(v(\hat{x}+1) - \bar{v}) < 0$. In addition, there exists an $\hat{x} \in (-r, r)$ such that

$$\max\{|v(\hat{x}+1) - \bar{v}|, |v(\hat{x}) - \bar{v}|\} \leq |v(\hat{x}) - v(\hat{x}+1)|. \tag{2.8}$$

Thus, with (2.7), (2.8), Cauchy–Schwarz and triangle inequalities, we have that, for $x \leq \hat{x}$,

$$\begin{aligned}
|v(x) - \bar{v}| &= |\sum_{y=x}^{\hat{x}-1}(v(y) - v(y+1)) + (v(\hat{x}) - \bar{v})| \\
&\leq \sum_{y=x}^{\hat{x}-1} |(v(y) - v(y+1))| + |(v(\hat{x}) - \bar{v})| \\
&\leq \sum_{y=x}^{\hat{x}-1} |(v(y) - v(y+1))| + |(v(\hat{x}) - v(\hat{x}+1))| \\
&= \sum_{y=x}^{\hat{x}} |(v(y) - v(y+1))| \\
&\leq \sum_{y=-r}^{r-1} |(v(y) - v(y+1))| \\
&\leq [\sum_{y=-r}^{r-1} ((v(y) - v(y+1)))^2]^{\frac{1}{2}} [\sum_{y=-r}^{r-1} (1)^2]^{\frac{1}{2}} \\
&= [2r \sum_{y=-r}^{r-1} ((v(y) - v(y+1)))^2]^{\frac{1}{2}} \\
&\leq [2r \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x) - v(y))^2]^{\frac{1}{2}}.
\end{aligned}$$

If $x = \hat{x} + 1$, with (2.8), $|v(\hat{x}+1) - \bar{v}| \leq |v(\hat{x}) - v(\hat{x}+1)|$ and so we also have the above inequality. If $x > \hat{x} + 1$, then we can do backward arguments above and have

$$\begin{aligned}
|v(x) - \bar{v}| &= |\sum_{y=\hat{x}+2}^{x} (v(y) - v(y-1)) + (v(\hat{x}+1) - \bar{v})| \\
&\leq \sum_{y=\hat{x}+2}^{x} |(v(y) - v(y-1))| + |(v(\hat{x}+1) - \bar{v})|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{y=\hat{x}+2}^{x} |(v(y)-v(y-1))| + |(v(\hat{x})-v(\hat{x}+1))| \\
&= \sum_{y=\hat{x}+1}^{x} |(v(y)-v(y-1))| \\
&\le \sum_{y=-r+1}^{r} |(v(y)-v(y-1))| \\
&\le [\sum_{y=-r+1}^{r} ((v(y)-v(y+1)))^2]^{\frac{1}{2}} [\sum_{y=-r+1}^{r} (1)^2]^{\frac{1}{2}} \\
&= [2r \sum_{y=-r+1}^{r} ((v(y)-v(y+1)))^2]^{\frac{1}{2}} \\
&\le [2r \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2]^{\frac{1}{2}}.
\end{aligned}
$$

In summary, for all $x \in B_r(x_0)$, we have that

$$|v(x)-\bar{v}| \le [2r \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2]^{\frac{1}{2}}.$$

Take the square for both sides and thus

$$(v(x)-\bar{v})^2 \le 2r \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2.$$

Note that $m(x) = \sum_{x\sim y} \mu_{xy} = 2$. Then take the sum over $B_r(x_0)$ and we have

$$
\begin{aligned}
\sum_{x\in B_r(x_0)} m(x)(v(x)-\bar{v})^2 &\le (m(x)(2r+1)2r) \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2 \\
&= (8r^2+4r) \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2 \\
&\le (12r^2) \sum_{x,y\in B_r(x_0)} \mu_{xy}(v(x)-v(y))^2.
\end{aligned}
$$

Hence, Poincaré inequality holds for $C_2 = 12$. □

2.5. Some auxiliary functions

We recall some auxiliary functions. One is for the function $\varsigma(z)$ in the heat kernel estimate (2.2). Recall that $\varsigma(z) = \sqrt{1+z^2} + ln\frac{z}{1+\sqrt{1+z^2}}$ for $z \in \mathbb{R}^+$. Another is for the wave speed, $c(\mu) = \frac{\lambda(\mu)}{\mu}$ with $\lambda(\mu) = e^{\mu} - 1 + e^{-\mu}$ for $\mu > 0$. The properties of these auxiliary functions play important roles throughout later sections. We group them in the following lemma and their proofs are straightforward.

Lemma 2.1. *Let $g(z) = -1 + 2\frac{\varsigma(z)+\mu}{z}$ for $\mu > 0$ and $z > 0$.*

(1) $\varsigma(z)$ is strictly increasing in z on $(0,\infty)$ and then there exists a $l_0 > 0$ such that $\varsigma(l_0) = 0$.
(2) $g(z)$ is concave down and obtains an absolute maximum at $z_0 = csch(\mu) = \frac{2}{e^{\mu}-e^{-\mu}}$ for $z \in (0,\infty)$ and $g(z_0) = \lambda(\mu)$.
(3) For fixed $\mu > 0$, $c(\mu)$ is concave up and has a unique critical point at μ^*, that is, $c(\mu)$ strictly decreasing in $(0,\mu^*]$ and strictly increasing in (μ^*,∞).

(4) For $\mu \in (0, \mu^*)$, $c(\mu) > \frac{2}{z_0}$, for $\mu = \mu^*$, $c(\mu) = \frac{2}{z_0}$ and for $\mu > \mu^*$, $c(\mu) < \frac{2}{z_0}$, where $z_0 = csch(\mu)$.

(5) $\frac{\varsigma(z)}{z}$ is strictly increasing in z on $(0, \infty)$ and $\lim_{z \to \infty} \frac{\varsigma(z)}{z} = 1$.

Proof.

(1) By direct computation,

$$\varsigma'(z) = \frac{z}{1 + \sqrt{1+z^2}} + \frac{1}{z} = \frac{\sqrt{1+z^2}}{z} > 0.$$

Therefore $\varsigma(z)$ is strictly increasing on $(0, \infty)$. Since $\varsigma(z) \to -\infty$ as $z \to 0$ and $\varsigma(z) \to \infty$ as $z \to \infty$, there exists a $l_0 > 0$ such that $\varsigma(l_0) = 0$.

(2) By direct computation, $g'(z) = 2\frac{z\varsigma'(z) - \varsigma(z) - \mu}{z^2} = 2\frac{ln\frac{1+\sqrt{1+z^2}}{z} - \mu}{z^2}$ for $\mu > 0$. Then there exists a unique critical point $z_0 = \frac{2}{e^{\mu} - e^{-\mu}}$ such that $g'(z_0) = 0$. We can verify that $g(z)$ obtains an absolute maximum at z_0 by first derivative test. Since $ln\frac{1+\sqrt{1+z^2}}{z}$ is a strictly decreasing function with the range from positive infinity to 0, $g'(z) > 0$ for $z < z_0$ and $g'(z) < 0$ for $z > z_0$. Plugging z_0 into $\frac{\varsigma(z)+\mu}{z}$,

$$\begin{aligned}
\frac{\varsigma(z_0) + \mu}{z_0} &= \frac{\varsigma(z_0)}{z_0} + \frac{\mu}{z_0} \\
&= \frac{\varsigma(csch(\mu))}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= \frac{\sqrt{1 + csch^2(\mu)} + ln\frac{csch(\mu)}{1+\sqrt{1+csch^2(\mu)}}}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= \frac{\sqrt{coth^2(\mu)} + ln\frac{csch(\mu)}{1+\sqrt{coth^2(\mu)}}}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= \frac{coth(\mu) + ln\frac{csch(\mu)}{1+coth(\mu)}}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= \frac{coth(\mu) + ln\frac{1}{sinh(\mu)+cosh(\mu)}}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= \frac{coth(\mu) - \mu}{csch(\mu)} + \frac{\mu}{csch(\mu)} \\
&= cosh(\mu)
\end{aligned}$$

Thus, $g(z_0) = -1 + 2cosh(\mu) = e^{\mu} + e^{-\mu} - 1 = \lambda(\mu)$.

(3) We can prove it by direct computation of solving $c'(\mu) = 0$ and verifying $c''(\mu) > 0$.

(4) Let $h(\mu) = c(\mu) - \frac{2}{z_0}$. Then $h(\mu) = \frac{\lambda(\mu)}{\mu} - (e^{\mu} - e^{-\mu}) = \frac{\lambda(\mu) - \mu(e^{\mu} - e^{-\mu})}{\mu} = -\mu c'(\mu)$ and so $h(\mu)$ has an opposite sign as $c'(\mu)$. By (3), $c'(\mu) < 0$ for $\mu \in (0, \mu^*)$, $c'(\mu^*) = 0$ and $c'(\mu) > 0$ for $\mu > \mu^*$, as required.

(5) Since $(\frac{\varsigma(z)}{z})' = \frac{\varsigma'(z)z - \varsigma(z)}{z^2} = -\frac{1}{z^2} ln\frac{z}{1+\sqrt{1+z^2}} > 0$, $\frac{\varsigma(z)}{z}$ is a strictly increasing function on $(0, \infty)$. The limit $\lim_{z \to \infty} \frac{\varsigma(z)}{z} = 1$ follows easily. □

3. Existence of transition fronts and their asymptotic behaviors

This section is devoted to investigating the existence of transition fronts of (1.2) for wave speed $c \in [c^*, \hat{c}]$ when $\lambda \in [1, \lambda^*)$ and $c^* < \hat{c}$. By Lemma 2.1(3), the wave speed interval $c \in [c^*, \hat{c}]$ corresponds to the interval of $\mu \in [\hat{\mu}, \mu^*]$. To prove the existence of transition fronts, we apply fundamental tools such as comparison principles and constructions of super- and sub-solutions. First we introduce principal eigenvalue theory for Jacobi operators, that will play a central and important role in these processes.

3.1. Principal eigenvalue theory for Jacobi operators

Sometimes we want to consider the truncated eigenvalue problem of (1.6):

$$\phi_{j+1} - 2\phi_j + \phi_{j-1} + a_j\phi_j = \lambda_M \phi_j, \tag{3.1}$$

where $j \in [-M, M]$, and $\phi_{M+1} = \phi_{-M-1} = 0$ for $M > N$. If we write it in a matrix form, and let A_M be

$$\begin{pmatrix} a_{-M}-2 & 1 & 0 & \dots & 0 \\ 1 & a_{-M+1}-2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & a_{M-1}-2 & 1 \\ 0 & \dots & 0 & 1 & a_M-2 \end{pmatrix},$$

then

$$A_M\phi^M = \lambda_M \phi^{(M)}, \tag{3.2}$$

where $\phi^{(M)} = (\phi_{-M}, \dots, \phi_M)^T$. By Perron–Frobenius theorem, there exists a principal eigenvalue and an associated positive eigenvector. We let $(\lambda_M, \phi_j^{(M)})$ be the pair of corresponding l^∞ normalized principal eigenvalue and eigenvector, that is, $(\lambda_M, \phi_j^{(M)})$ satisfies (1.7) with $\|\phi^{(M)}\|_\infty = 1$ and $\phi_j^{(M)} > 0$ for $j \in [-M, M]$. Let M go to infinity and the limit of λ_M exists. If $\lambda > 1$, λ is a principal eigenvalue of (1.5) with $\lambda = \lim_{M\to\infty} \lambda_M$.

Lemma 3.1. *If $\lambda > 1$, λ is a principal eigenvalue of (1.5). Moreover, $\lambda = \lim_{M\to\infty} \lambda_M$.*

Proof. Without loss of generality, let both M and N be even and $M > N$. We let $\tilde{A}_M$ be an $2M+3$ by $2M+3$ matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a_{-M}-2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & a_M-2 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Then we have that $A_{M+1} \geq \tilde{A}_M$ and so $\rho(A_{M+1}) \geq \rho(\tilde{A}_M) = \rho(A_M)$, where $\rho(\#)$ is the spectral radius of the matrix $\#$. Thus, $\lambda_M = \rho(A_M)$ is non-decreasing in M. On the other hand, $\|A_M\|_{max} = \max_{i,j} |A_M(i,j)|$, where $A_M(i,j)$ is the element of A_M at the ith row and jth column. Then $0 < \lambda_M \leq \|A_M\|_{max} \leq \max_j\{|a_j| + 2\}$, that is, λ_M is uniformly bounded. Therefore, the limit $\lim_{M\to\infty} \lambda_M$ exists and it is denoted by $\lambda_\infty = \lim_{M\to\infty} \lambda_M$. Let $\phi^{(M)}$ be the positive eigenvector of A_M with $\|\phi^{(M)}\|_\infty = 1$. For each j, there exists a subsequence M_j of M such that $\lim_{M_j\to\infty} \phi_j^{(M_j)}$ exists and let $\phi_j^{(\infty)} = \lim_{M_j\to\infty} \phi_j^{(M_j)}$. For each $M \gg N$, let j_M be such that $\phi_{j_M}^{(M)} = 1$. We claim that there exists a $j_M \in [-N, N]$. For $j < -N$, we write (3.1) as the following,

$$\phi_{j+1} = (1 + \lambda_M)\phi_j - \phi_{j-1}. \tag{3.3}$$

Let $c_1 = 1 + \lambda_M$ and $c_2 = -1$. We can solve a recursive sequence $\phi_{j+1} = c_1\phi_j + c_2\phi_{j-1}$. To this end, we use an auxiliary equation $x^2 - c_1x - c_2 = 0$. Then solve it to have two roots $d_1 = \frac{1+\lambda_M+\sqrt{(1+\lambda_M)^2-4}}{2}$ and $d_2 = \frac{1+\lambda_M-\sqrt{(1+\lambda_M)^2-4}}{2}$. Note that $d_1d_2 = 1$ and so $d_1 \geq 1$ and $d_2 \leq 1$. Therefore, we have either $d_1 = d_2 = 1$ or $0 < d_2 < 1 < d_1$. Moreover,

$$\phi_{j+1} - d_1\phi_j = d_2(\phi_j - d_1\phi_{j-1}),$$
$$\phi_{j+1} - d_2\phi_j = d_1(\phi_j - d_2\phi_{j-1}).$$

Thus, for $-M \le j < -N$, we have

$$\phi_{j+1} - d_1\phi_j = (d_2)^{j+M+1}(\phi_{-M} - d_1\phi_{-M-1}),$$
$$\phi_{j+1} - d_2\phi_j = (d_1)^{j+M+1}(\phi_{-M} - d_2\phi_{-M-1}).$$

Recall Eq. (3.1) that $\phi_{-M-1} = 0$ and $\phi_{-M} > 0$. Consider $0 < d_2 < 1 < d_1$ first. Subtract the above equations, divide by $d_2 - d_1$ and then with $d_1 d_2 = 1$, for $-M \le j < -N$, we have

$$\begin{aligned}\phi_j &= \frac{(d_2)^{j+M+1}(\phi_{-M} - d_1\phi_{-M-1}) - (d_1)^{j+M+1}(\phi_{-M} - d_2\phi_{-M-1})}{d_2 - d_1} \\ &= \frac{(d_2)^{j+M+1} - (d_1)^{j+M+1}}{d_2 - d_1}\phi_{-M} \\ &= \frac{(d_1)^{-(j+M+1)} - (d_1)^{j+M+1}}{d_2 - d_1}\phi_{-M}\end{aligned}$$

Note that $d_1 > 1$, and $-(d_1)^{-x} + (d_1)^x$ is increasing for $x > 0$. Thus the ϕ_j is increasing for $j < -N$. If $d_1 = d_2 = 1$, we must have $\phi_{j+1} - \phi_j = \phi_{-M} - \phi_{-M-1} = \phi_{-M}$ for $M \le j < -N$. We also have that ϕ_j is increasing for $j < -N$. Therefore it implies that $\max_{-M \le j < -N} \phi_j = \phi_{-N-1}$. Similarly, we have $\max_{N < j \le M} \phi_j = \phi_{N+1}$. Then we have one $j_M \in [-N, N]$. There exists a subsequence M_k of M such that $\bar{j} = \lim_{M_k \to \infty} j_{M_k}$ for some $\bar{j} \in [-N, N]$. Thus, $\phi^{(\infty)}_{\bar{j}} = 1$. Moreover, by taking the limit, we have that $\Lambda\phi^{(\infty)} = \lambda_\infty \phi^{(\infty)}$. By the strong positivity of the semigroup generated by Λ, $\phi^{(\infty)}_{\bar{j}} = 1 > 0$ implies that $\phi^{(\infty)}_j > 0$ for all j.

Next we will apply an extension of the Krein–Rutman theorem, Theorem 2.2 in [28], to show that λ is a principal eigenvalue of (1.5). To this end, we introduce the Kuratowski measure of non-compactness as for any $B \subset X$,

$$\kappa(B) = inf\{d > 0 : \text{there exist finitely many sets of diameter at most d which cover } B\}.$$

Let $\hat{\Lambda} = \Lambda + sI$ with $s \ge 2 + |a_i|$ for all $i \in \mathbb{Z}$ such that $\hat{\Lambda} \ge 0$. Denote $\sigma(\hat{\Lambda})$ the spectrum and $\sigma_{ess}(\hat{\Lambda})$ the essential spectrum of $\hat{\Lambda}$. Their respective spectral radii are $r(\sigma(\hat{\Lambda}))$ and $r(\sigma_{ess}(\hat{\Lambda}))$. Rewrite $\hat{\Lambda} = \hat{\Lambda}_1 + \hat{\Lambda}_2$ with $\hat{\Lambda}_1\phi = \phi_{j+1} - \phi_j + \phi_{j-1} + s\phi_j$ and $\hat{\Lambda}_2 = diag\{a_j - 1, j \in \mathbb{Z}\}$, that is a diagonal matrix with 0 for all $|j| > N$. Then $\hat{\Lambda}_2(B)$ is a subset of $\mathbb{R}^{2N+1}$ that implies $\kappa(\hat{\Lambda}_2(B)) = 0$. Therefore, $\kappa(\hat{\Lambda}(B)) \le \kappa(\hat{\Lambda}_1(B)) + \kappa(\hat{\Lambda}_2(B)) = \kappa(\hat{\Lambda}_1(B)) \le \|\hat{\Lambda}_1\|\kappa(B) = (s+1)\kappa(B)$. Thus $\kappa(\hat{\Lambda}(B)) \le (s+1)\kappa(B)$, that is, $\hat{\Lambda}$ is an (s+1)-set-contraction according to [28]. Let

$$\alpha(\hat{\Lambda}) = inf\{c \ge 0 : \hat{\Lambda} \text{ is a c-set-contraction}\}.$$

By the definition of (2.23) in [28], $r(\sigma_{ess}(\hat{\Lambda})) = \lim_{n\to\infty}(\alpha(\hat{\Lambda}^n))^{\frac{1}{n}}$. Thus $r(\sigma_{ess}(\hat{\Lambda})) \le s + 1$. Since $r(\sigma(\hat{\Lambda})) = \lambda + s$, $\lambda > 1$ implies that $r(\sigma(\hat{\Lambda})) > r(\sigma_{ess}(\hat{\Lambda}))$. Therefore, by Theorem 2.2 in [28], $\lambda + s$ is a (principal) eigenvalue of $\hat{\Lambda}$ with a nontrivial nonnegative eigenvector ϕ and thus λ is a (principal) eigenvalue of (1.5) with the identical eigenfunction ϕ. Moreover, ϕ must be strictly positive. If not, suppose there is a $\phi_k = 0$. Then $\phi_{k\pm1} > 0$ otherwise $\phi \equiv 0$. Then $0 = (\lambda + 2 - a_k)\phi_k = \phi_{k+1} + \phi_{k-1} > 0$, which leads to a contradiction. Now let $v(t) = \phi^{(\infty)} - t\phi$, $t \ge 0$. There exists $t_0 > 0$ such that $v_k(t_0) = 0$ for some $k \in \mathbb{Z}$, while $v(t) \ngeq 0$ for $t > t_0$. Moreover,

$$0 \le \hat{\Lambda}(\phi^{(\infty)} - t_0\phi) = (\lambda_\infty + s)(\phi^{(\infty)} - \frac{\lambda + s}{\lambda_\infty + s}t_0\phi),$$

which implies that $\lambda + s \le \lambda_\infty + s$ due to the definition of t_0. Thus, $\lambda \le \lambda_\infty$. By interchanging (λ, ϕ) and $(\lambda_\infty, \phi^{(\infty)})$, we have $\lambda \ge \lambda_\infty$. To conclude, $\lambda = \lambda_\infty = \lim_{M\to\infty} \lambda_M$. □

Then we consider Eq. (1.7), $\Lambda_\mu \phi = \gamma_\mu \phi$ for $\gamma_\mu \in \mathbb{R}$. It is easy to see that a solution ϕ is uniquely determined by the values ϕ_k and ϕ_{k+1} at two consecutive points k_0, k_0+1. This results in a two-dimensional space of fundamental solutions $Span\{u, v\}$, where u and v satisfy (1.7) with the initial conditions $u_k = v_{k+1} = 1$ and $v_k = u_{k+1} = 0$ for $k = k_0$. In applications, we are interested in a positive solution $\phi \in X$ to (1.7). Before showing the existence of a positive solution of (1.7), we prove a lemma first.

Lemma 3.2. *Every solution of (1.7) can change sign at most once for any $\gamma_\mu > \lambda$.*

Proof. We modify the arguments in section 2.3 of [26] where they are concerned with Jacobi operators in a Hilbert space. Let $(\delta_n)_{n\in\mathbb{Z}}$ be the standard unit vectors of X, where $(\delta_n)_i = 1$ if $i = n$ otherwise zeros. We define the following restrictions Λ^+_{μ,k_0} and Λ^-_{μ,k_0} of Λ_μ on (k_0, ∞) and $(-\infty, k_0)$ respectively:

$$(\Lambda^+_{\mu,k_0}\phi)_j = \begin{cases} e^{-\mu}\phi_{j+1} - 2\phi_j + a_j\phi_j, j = k_0+1, \\ e^{-\mu}\phi_{j+1} - 2\phi_j + e^{\mu}\phi_{j-1} + a_j\phi_j, j > k_0+1, \end{cases}$$

and

$$(\Lambda^-_{\mu,k_0}\phi)_j = \begin{cases} e^{\mu}\phi_{j-1} - 2\phi_j + a_j\phi_j, j = k_0-1, \\ e^{-\mu}\phi_{j+1} - 2\phi_j + e^{\mu}\phi_{j-1} + a_j\phi_j, j < k_0-1. \end{cases}$$

For $\lambda \in \rho(\Lambda_\mu)$, the matrix $G := (\lambda - \Lambda_\mu)^{-1}$ is called Green's function with matrix elements

$$G_{ij} = (\lambda - \Lambda_\mu)^{-1}_{ij}.$$

Let $\overrightarrow{G}_j$ be the jth column vector of $(\lambda - \Lambda_\mu)^{-1}$ and then

$$(\lambda - \Lambda_\mu)\overrightarrow{G}_j = \delta_j.$$

Let

$$u(\lambda, \cdot) = (\lambda - \Lambda_\mu)^{-1}\delta_0(\cdot), \lambda \in \rho(\Lambda_\mu).$$

Let $u^{\pm}(\gamma_\mu, k)$ denote the solutions that coincide with $u(\gamma_\mu, k)$ for $k > 0$ and $k < 0$ respectively. We complete the proof of the lemma in a couple of steps.

Step 1. We have that $(k - k_0)s(\gamma_\mu, k, k_0) > 0, k \neq k_0$, where $s(\gamma_\mu, k, k_0)$ is the solution with initial conditions $s(\gamma_\mu, k_0, k_0) = 0$ and $s(\gamma_\mu, k_0+1, k_0) = 1$. Denote

$$G^+_{ij} = (\gamma_\mu - \Lambda^+_{\mu,k_0})^{-1}_{ij}.$$

and

$$\bar{G}^+_{ij} = \begin{cases} \frac{e^{\mu}u^+(\gamma_\mu,i)s(\gamma_\mu,j,k_0)}{u^+(\gamma_\mu,k_0)}, i \geq j \\ \frac{e^{\mu}u^+(\gamma_\mu,j)s(\gamma_\mu,i,k_0)}{u^+(\gamma_\mu,k_0)}, i \leq j. \end{cases}$$

We claim that $G^+_{ij} = \bar{G}^+_{ij}$. We note that the Wronskian of u^+ and s is a constant and

$$W(u^+, s) = e^{-\mu}(u^+(\gamma_\mu, k)s(\gamma_\mu, k+1, k_0) - u^+(\gamma_\mu, k+1)s(\gamma_\mu, k, k_0)) = e^{-\mu}u^+(\gamma_\mu, k_0).$$

Let $\phi = \overrightarrow{\bar{G}^+_j} = (\bar{G}^+_{ij})_{i\in\mathbb{Z}}$. For $k < j$, we have

$$\begin{aligned} ((\gamma_\mu - \Lambda^+_{\mu,k_0})\overrightarrow{\bar{G}^+_j})_k &= \gamma_\mu\phi_k - (e^{-\mu}\phi_{k+1} - 2\phi_k + e^{\mu}\phi_{k-1} + a_k\phi_k) \\ &= \frac{e^{\mu}u^+(\gamma_\mu, j)}{u^+(\gamma_\mu, k_0)}((\gamma_\mu - \Lambda^+_{\mu,k_0})s(\gamma_\mu, \cdot, k_0))_k \\ &= 0. \end{aligned}$$

Similarly, for $k>j$ we also have $((\gamma_\mu - \Lambda^+_{\mu,k_0})\overrightarrow{\bar{G}^+_j})_k = 0$ by interchanging u^+ and s in the above arguments for $k<j$. For $k=j$, we have

$$\begin{aligned}((\gamma_\mu - \Lambda^+_{\mu,k_0})\overrightarrow{\bar{G}^+_j})_k &= \gamma_\mu\phi_k - (e^{-\mu}\phi_{k+1} - 2\phi_k + e^{\mu}\phi_{k-1} + a_k\phi_k)\\ &= \frac{e^{\mu}u^+(\gamma_\mu,k)}{u^+(\gamma_\mu,k_0)}((\gamma_\mu - \Lambda^+_{\mu,k_0})s(\gamma_\mu,\cdot,k_0))_k\\ &+ \frac{[u^+(\gamma_\mu,k)s(\gamma_\mu,k+1,k_0) - u^+(\gamma_\mu,k)s(\gamma_\mu,k,k_0)]}{u^+(\gamma_\mu,k_0)}\\ &= \frac{W(u^+,s)}{e^{-\mu}u^+(\gamma_\mu,k_0)}\\ &= 1.\end{aligned}$$

Therefore, $(\gamma_\mu - \Lambda^+_{\mu,k_0})\overrightarrow{\bar{G}^+_j} = \delta_j$ and thus $G^+_{ij} = \bar{G}^+_{ij}$.

Thus,

$$0 < (\gamma_\mu - \Lambda^+_{\mu,k})^{-1}_{k+1,k+1} = \frac{e^{\mu}u^+(\gamma_\mu,k+1)}{u^+(\gamma_\mu,k)}.$$

Then $u^+(\gamma_\mu,k)$ and $u^+(\gamma_\mu,k+1)$ have the same signs. We can assume that $u^+(\gamma_\mu,k)>0$. On the other hand, by similar arguments above, for $k>k_0$

$$0 < (\gamma_\mu - \Lambda^+_{\mu,k_0})^{-1}_{k,k} = \frac{e^{\mu}u^+(\gamma_\mu,k)s(\gamma_\mu,k,k_0)}{u^+(\gamma_\mu,k_0)}.$$

Hence, $(k-k_0)s(\gamma_\mu,k,k_0)>0, k>k_0$. We can show the case of $k<k_0$ similarly.

Step 2. Suppose $u_j(\gamma_\mu,k)$, $j=1,2$, are two solutions of (1.7) with $u_1(\gamma_\mu,k_0) = u_2(\gamma_\mu,k_0)$ for some $k_0 \in \mathbb{Z}$. Then $u_1(\gamma_\mu,k) - u_2(\gamma_\mu,k) = cs(\gamma_\mu,k,k_0)$ for some $c\in\mathbb{R}$. Therefore, by Step 1, either $(k-k_0)(u_1(\gamma_\mu,k)-u_2(\gamma_\mu,k))>0$ for $k\neq k_0$ or $u_1(\gamma_\mu,k)=u_2(\gamma_\mu,k)$ for all k. The solutions $u(\gamma_\mu,k)$ can change sign at most once since $s(\gamma_\mu,k,k_0)$ does. □

Now we are in a position to prove a lemma that there exists a positive solution $\phi\in X$ to (1.7) for $\gamma_\mu = \lambda(\mu)$ for $\mu\in[\hat{\mu},\mu^*)$, and $\phi_k = 1$ for $k=n_0,n_0+1$ and $n_0>N$.

Lemma 3.3. *There exists a positive solution $\phi\in X$ to (1.7) for $\gamma_\mu=\lambda(\mu)$ for $\mu\in[\hat{\mu},\mu^*)$, and $\phi_k=1$ for $k=n_0,n_0+1$ and $n_0>N$. Moreover, $\phi_j=1$ for $j>N$ and there is a positive number l such that $\lim_{j\to-\infty}\phi_j = l$.*

Proof. For $|j|>N$, $a_j=1$. Thus, recalling (1.7), for $|j|>N$, we have

$$e^{-\mu}\phi_{j+1} - \phi_j + e^{\mu}\phi_{j-1} = \lambda(\mu)\phi_j.$$

Thus, recalling $\lambda(\mu) = e^{\mu} - 1 + e^{-\mu}$, for $|j|>N$, we have

$$e^{-\mu}\phi_{j+1} + e^{\mu}\phi_{j-1} = (e^{\mu}+e^{-\mu})\phi_j. \tag{3.4}$$

Since $\phi_k=1$ for $k=n_0,n_0+1$, with (3.4), for $j>N$, $\phi_j=1$. On the other hand, for $j<-N$, we write (3.4) as the following,

$$\phi_{j-1} = (1+e^{-2\mu})\phi_j - e^{-2\mu}\phi_{j+1}.$$

Let $c_1 = 1+e^{-2\mu}$ and $c_2 = -e^{-2\mu}$. We can solve a recursive sequence $\phi_{j-1} = c_1\phi_j + c_2\phi_{j+1}$. To this end, we use an auxiliary equation $x^2 - c_1x - c_2 = 0$. Then solve it to have two roots $d_1 = 1$ and $d_2 = e^{-2\mu}$. Therefore, we have

$$\begin{aligned}\phi_{j-1} - d_1\phi_j &= d_2(\phi_j - d_1\phi_{j+1}),\\ \phi_{j-1} - d_2\phi_j &= d_1(\phi_j - d_2\phi_{j+1}).\end{aligned}$$

Thus, for $j < -N$, we have

$$\phi_{j-1} - d_1\phi_j = (d_2)^{-N+1-j}(\phi_{-N} - d_1\phi_{-N+1}),$$
$$\phi_{j-1} - d_2\phi_j = (d_1)^{-N+1-j}(\phi_{-N} - d_2\phi_{-N+1}).$$

Subtract the above equations, divide by $d_2 - d_1$ and then for $j < -N$, we have

$$\begin{aligned}\phi_j &= \frac{(d_2)^{-N+1-j}(\phi_{-N} - d_1\phi_{-N+1}) - (d_1)^{-N+1-j}(\phi_{-N} - d_2\phi_{-N+1})}{d_2 - d_1}\\ &=: C_1 + C_2 e^{2\mu j}.\end{aligned} \tag{3.5}$$

Since $\lim\limits_{j\to-\infty}(d_2)^{-N+1-j} = \lim\limits_{j\to-\infty} e^{-2\mu(-N+1-j)} = 0$ and $(d_1)^{-N+1-j} = 1$,

$$\lim_{j\to-\infty}\phi_j = \frac{-(\phi_{-N} - d_2\phi_{-N+1})}{d_2 - d_1} := l.$$

Thus, $\phi \in X$. Next, we prove that $\phi > 0$. Suppose that $\phi_{k_0} \leq 0$ for some $k_0 < N$ while $\phi_j > 0$ for $j > k_0$ (i.e. k_0 is the first oscillation point around 0 from the right). Let $\hat{\phi}$ be a solution with $\hat{\phi}_k = e^{2\mu k}$ for $k = n_0, n_0+1$ and $n_0 > N$. Then for $j > N$, $\hat{\phi}_j = e^{2\mu j}$ for all $j > N$. There is an $\epsilon > 0$ small enough such that k_0 is also an oscillation point for $\phi - \epsilon\hat{\phi}$. Then $\phi - \epsilon\hat{\phi} > 0$ for $N < j < -\frac{ln(\epsilon)}{2\mu}$ and $\phi - \epsilon\hat{\phi} < 0$ for $j > -\frac{ln(\epsilon)}{2\mu}$. Thus there exists another oscillation point for $\phi - \epsilon\hat{\phi}$. This causes a contradiction with **oscillation theory**, every solution can change sign at most once (Lemma 3.2), and so $\phi_j > 0$. □

3.2. Sub/super-solutions

In this subsection, we construct a super-solution and a sub-solution with Lemma 3.3. By Lemma 3.3, the principal eigenvalue pair, denoted by $(\lambda_\mu, \phi_j^\mu)$, exists for Eq. (1.7), where $\lambda_\mu = \lambda(\mu)$ for $\mu \in [\hat{\mu}, \mu^*)$.

Let

$$\bar{u}_j = e^{-\mu(j-ct)}\phi_j^\mu. \tag{3.6}$$

Lemma 3.4. *$\{\bar{u}_j\}_{j\in\mathbb{Z}}$ is a super-solution of (1.2).*

Proof. By (H1), we have $f_j(\bar{u}_j) - f_j(0) \leq 0$. Recall that $a_j = f_j(0)$. By direct calculation, we have

$$\begin{aligned}&(\bar{u}_j)_t - [\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1} + f_j(\bar{u}_j)\bar{u}_j]\\ \geq &(\bar{u}_j)_t - [\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1} + a_j\bar{u}_j]\\ = &0.\end{aligned}$$ □

Let

$$\underline{u}_j = e^{-\mu(j-ct)}\phi_j^\mu - d_1 e^{-\mu_1(j-ct)}\phi_j^{\mu_1}. \tag{3.7}$$

for $\hat{\mu} \leq \mu < \mu_1 < \min\{2\mu, \mu^*\}$.

Lemma 3.5. *$\{\underline{u}_j\}_{j\in\mathbb{Z}}$ is a sub-solution of (1.2) for any* $d_1 > \max\left\{\frac{\sup\limits_j \phi_j^\mu}{\inf\limits_j \phi_j^{\mu_1}}, \frac{L(\sup\limits_j \phi_j^\mu)^2}{(\mu_1 c - \lambda(\mu_1))\inf\limits_j \phi_j^{\mu_1}}\right\}$.

Proof. Let $f_j(\underline{u}_j) = f_j(0)$ if $\underline{u}_j \leq 0$. By Lemma 2.1(3), for $\hat{\mu} \leq \mu < \mu_1 < \min\{2\mu, \mu^*\}$,

$$c(\mu_1) = \frac{\lambda(\mu_1)}{\mu_1} \leq \frac{\lambda(\mu)}{\mu} = c. \tag{3.8}$$

Then if $\underline{u}_j \leq 0$, we have

$$\begin{aligned}&(\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + f_j(\underline{u}_j)\underline{u}_j] \\ &= (\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + f_j(0)\underline{u}_j] \\ &= -(\mu_1 c - \lambda(\mu_1))d_1 e^{-\mu_1(j-ct)}\phi_j^{\mu_1} \\ &\leq 0.\end{aligned}$$

By Lemma 3.3 and $\hat{\mu} \leq \mu < \mu_1$, both $\sup_j \phi_j^{\mu}$ and $\inf_j \phi_j^{\mu_1}$ are positive. Let

$$d_0 = \max\left\{\frac{\sup_j \phi_j^{\mu}}{\inf_j \phi_j^{\mu_1}}, \frac{L(\sup_j \phi_j^{\mu})^2}{(\mu_1 c - \lambda(\mu_1))\inf_j \phi_j^{\mu_1}}\right\}.$$

Note that $d_1 > d_0$. If $\underline{u}_j > 0$, we have $e^{-\mu(j-ct)}\phi_j^{\mu} - d_1 e^{-\mu_1(j-ct)}\phi_j^{\mu_1} > 0$ and then

$$e^{-(\mu-\mu_1)(j-ct)} > d_1 \frac{\phi_j^{\mu_1}}{\phi_j^{\mu}} \geq \frac{d_1}{d_0} \geq 1,$$

that implies that $j - ct \geq 0$. For $\underline{u}_j > 0$, $\underline{u}_j^2 \leq e^{-2\mu(j-ct)}(\phi_j^{\mu})^2$. Then together with (3.8), for $\underline{u}_j > 0$, we have

$$\begin{aligned}&(\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + f_j(\underline{u}_j)\underline{u}_j] \\ &= (\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + a_j\underline{u}_j] + f_j(0)\underline{u}_j - f_j(\underline{u}_j)\underline{u}_j \\ &= (\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + a_j\underline{u}_j] - f_j'(y)\underline{u}_j^2 \\ &\leq -(\mu_1 c - \lambda(\mu_1))d_1 e^{-\mu_1(j-ct)}\phi_j^{\mu_1} - f_j'(y)e^{-2\mu(j-ct)}(\phi_j^{\mu})^2 \\ &\leq e^{-\mu_1(j-ct)}[-(\mu_1 c - \lambda(\mu_1))d_1\phi_j^{\mu_1} - f_j'(y)e^{-(2\mu-\mu_1)(j-ct)}(\phi_j^{\mu})^2] \\ &\leq e^{-\mu_1(j-ct)}[-(\mu_1 c - \lambda(\mu_1))d_1\phi_j^{\mu_1} - f_j'(y)(\phi_j^{\mu})^2],\end{aligned}$$

where y is such that $f_j(0) - f_j(\underline{u}_j) = -f_j'(y)\underline{u}_j$. Recall that $\mu < \mu_1 < \min\{2\mu, \mu^*\}$ and in (H1), $-L < \inf_{j\in\mathbb{Z}, u_j \geq 0}\{f_j'(u_j)\} \leq \sup_{j\in\mathbb{Z}, u_j\geq 0}\{f_j'(u_j)\} < 0$ for some $L > 0$. Since $d_1 > d_0 \geq \frac{L(\sup_j \phi_j^{\mu})^2}{(\mu_1 c - \lambda(\mu_1))\inf_j \phi_j^{\mu_1}}$, we have $[-(\mu_1 c - \lambda(\mu_1))d_1\phi_j^{\mu_1} - f_j'(y)(\phi_j^{\mu})^2] \leq 0$ and thus

$$(\underline{u}_j)_t - [\underline{u}_{j+1} - 2\underline{u}_j + \underline{u}_{j-1} + f_j(\underline{u}_j)\underline{u}_j] \leq 0$$

for $\underline{u}_j > 0$. This completes the proof of the proposition. □

Remark 3.1. For $\mu = \mu^*$, Lemma 3.4 holds. However, Lemma 3.5 does not hold for $\mu = \mu^*$, because valid positive eigenvector must locate in $(0, \mu^*]$ and there is no room for the choice of μ_1 that has to be bigger than μ. For another critical number $\mu = \hat{\mu}$, it is fine to be included because μ_1 can be chosen as long as $\hat{\mu} \leq \mu < \mu_1 < \min\{2\mu, \mu^*\}$.

3.3. Existence of transition fronts

In the last subsection we constructed the super/sub-solutions on the interval $[\hat{\mu}, \mu^*)$ of μ. In this subsection, we can obtain the existence of transition fronts to (1.2) for $c \in (c^*, \hat{c}]$ by the comparison principle. After that, with the limiting argument, we can have the existence of transition fronts to (1.2) of $c = c^*$. The proof of existence of transition fronts in part (1) of main theorem is completed by the following proposition.

Proposition 3.1. *Assume (H1)–(H2). If $\lambda \in [1, \lambda^*)$ and $\hat{c} > c^*$, then transition fronts exist for any speed $c \in [c^*, \hat{c}]$. Moreover, for $c^* < c < \hat{c}$, the constructed transition front $u_j(t)$ satisfy*

$$\lim_{j-ct\to\infty} \frac{u_j(t)}{e^{-\mu(j-ct)}} = \phi_j^\mu. \tag{3.9}$$

To prove Proposition 3.1, we will apply the following lemma. Let (λ_M, ϕ_M) be principal eigenvalue and eigenvector pair of (3.1) with $\|\phi_M\|_\infty = 1$ and $\tilde{u} = \delta\phi_M$ for $\delta > 0$.

Lemma 3.6. *For any given $M \gg 1$, there is a small enough $\delta_0 > 0$ such that $\tilde{u}$ is a sub-solution of (1.2) for any $\delta \in (0, \delta_0)$.*

Proof. Recall that $a_j = f_j(0)$. Choose δ_0 small enough such that

$$f_j(0) - f_j(\tilde{u}_j) \le f_j(0) - f_j(\delta_0) < \lambda_M, \forall \delta \in (0, \delta_0).$$

By direct calculation, we have

$$\begin{aligned} &(\tilde{u}_j)_t - [\tilde{u}_{j+1} - 2\tilde{u}_j + \tilde{u}_{j-1} + f_j(\tilde{u}_j)\tilde{u}_j] \\ =&(\tilde{u}_j)_t - [\tilde{u}_{j+1} - 2\tilde{u}_j + \tilde{u}_{j-1} + a_j\tilde{u}_j] + (f_j(0) - f_j(\tilde{u}_j))\tilde{u}_j \\ =&(-\lambda_M + (f_j(0) - f_j(\tilde{u}_j)))\tilde{u}_j \\ \le&0. \quad \square \end{aligned}$$

Proof of Proposition 3.1. As long as we have the required super/sub-solutions, the existence of transition fronts can be obtained by the standard "squeeze" techniques. Indeed, if $\lambda \in [1, \lambda^*)$ and $\hat{c} > c^*$, then we have a positive principal eigenvector to (1.7) for any speed $c \in (c^*, \hat{c}]$. Let $\bar{u}$ and $\underline{u}$ be chosen as in (3.6) and (3.7), $v = \min\{\bar{u}, u^*\}$ and $w = \max\{\underline{u}, 0\}$. Following arguments similar to [20], we have an entire solution that is sandwiched between v and w. In fact, for each $n \in \mathbb{N}$, let $\{u_j^n\}_{j\in\mathbb{Z}}$ be a solution of (1.2) with initial condition $u_j^n(-n) = v_j(-n)$. With the comparison principle, we have that for any $n \in \mathbb{N}$, and $(t, j) \in (-n, \infty) \times \mathbb{R}$,

$$0 \le w_j(t) \le u_j^n(t) \le v_j(t) \le u_j^*.$$

In particular, letting $t = -n+1$, we have $u_j^n(-n+1) \le v_j(-n+1) = u_j^{n-1}(-n+1)$, for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. With the comparison principle again, we have that for any $n \in \mathbb{N}$, and $(t, j) \in (-n+1, \infty) \times \mathbb{R}$,

$$0 \le u_j^n(t) \le u_j^{n-1}(t) \le u_j^*.$$

Note that $|u_j^n(t)| \le u_j^*$ and $|\dot{u}_j^n(t)| \le C\|\varLambda\| + \max_{0\le v\le u_j^*} |f_j(v)| \max_j u_j^*$ because $\varLambda$ is a bounded operator with operator norm $\|\varLambda\|$. By Arzelà−Ascoli theorem, there exists a subsequence $\{u_j^{n_k}(t)\}_{j\in\mathbb{Z}}$ with $n_k > |t|+1$, such that it converges uniformly on bounded sets. Letting $n_k \to \infty$, $u_j(t) := \lim_{n_k\to\infty} u_j^{n_k}(t)$ for all $(t, j) \in \mathbb{R} \times \mathbb{Z}$. Integrating (1.2) over $[0, t]$ with each $u_j^n(t)$ for $n \in \mathbb{N}$, we have

$$u_j^n(t) = u_j^n(0) + \int_0^t [u_{j+1}^n - 2u_j^n + u_{j-1}^n + f_j(u_j^n)u_j^n]ds.$$

Letting $n \to \infty$, we have

$$u_j(t) = u_j(0) + \int_0^t [u_{j+1} - 2u_j + u_{j-1} + f_j(u_j)u_j]ds,$$

which implies that $u_j \in C^1$ and also satisfies (1.2). Moreover, we also have that

$$0 \leq w_j(t) \leq u_j(t) \leq v_j(t) \leq u_j^*.$$

Thus, it yields $\lim_{j\to\infty} u_j(t) = 0$. It remains to show that $\lim_{j\to-\infty} u_j(t) = u_j^*$. By strong comparison principle, we have $u_j(\tau) > 0$ for $\tau > 0$. Let $\tilde{u}$ be as in Lemma 3.6. Since $\tilde{u}$ is compactly supported on $[-M, M]$, there exists a $\delta \in (0, \delta_0)$, such that $u_j(\tau) > \tilde{u}$. With the comparison principle again, $u_j(t) \geq u_j(t-\tau;\tilde{u})$ for $t > \tau$, where $\{u_j(t-\tau;\tilde{u})\}_{j\in\mathbb{Z}}$ is the solution of (1.2) with initial $\tilde{u}$ at $t = \tau$. Due to uniqueness of positive stationary solution of (1.2), we must have $\lim_{t\to\infty} u_j(t-\tau;\tilde{u}) = u_j^*$. Then for all $j \in \mathbb{Z}$,

$$\liminf_{t\to\infty} u_j(t) \geq \lim_{t\to\infty} u_j(t-\tau;\tilde{u}) = u_j^*,$$

that implies that $\lim_{t\to\infty} u_j(t) = u_j^*$. By the definition of $w(t)$ (sub-solution), there exist positive large L and small σ such that, for $j - ct > L$,

$$w(t) \geq \sigma e^{-\mu(j-ct)} > \sigma e^{-\mu L} \geq \tilde{u}.$$

In particular, let $\tilde{t} = \frac{j-L}{c}$ and we have $u_j(\tilde{t}) \geq w(\tilde{t}) \geq \sigma e^{-\mu L} \geq \tilde{u}$. Since $\lim_{t\to\infty} u_j(t;\tilde{u}) = u_j^*$, for any $\epsilon > 0$, there exists a $T_0 > 0$ such that $u_j(t;\tilde{u}) > u_j^* - \epsilon$, for all $t > T_0$ and $j \in \mathbb{Z}$. Note that as $j \to -\infty$, $t - \tilde{t} \to \infty$ for given $t \in \mathbb{R}$. Then for $t - \tilde{t} > T_0$,

$$u_j(t) = u_j(t - \tilde{t} + \tilde{t}) \geq u_j(t - \tilde{t};\tilde{u}) > u_j^* - \epsilon,$$

thus implies that $\lim_{j\to-\infty} u_j(t) = u_j^*$.

For $c = c^*$ ($\mu = \mu^*$), we claim that the transition front also exists and shall prove it by limit arguments due to the invalid sub-solutions in Remark 3.1. To prove the case with $c = c^*$, pick a sequence $\hat{c} > c_{\bar{n}} > c^*$ such that $c_{\bar{n}} \to c^*$. We simply denote the transition fronts of speed $c_{\bar{n}}$ by $\{u_j^{\bar{n}}(t)\}_{j\in\mathbb{Z}}$. By similar limiting arguments above for $\{u_j^n(t)\}_{j\in\mathbb{Z}}$, let the transition front of speed c^* be $u_j^\dagger(t) := \lim_{\bar{n}_k\to\infty} u_j^{\bar{n}_k}(t)$.

Finally, for $c^* < c < \hat{c}$, the limit (3.9) follows from $w_j(t) \leq u_j(t) \leq v_j(t)$ for all j and $t > 0$ with the comparison principle. This completes the proof. □

By Proposition 3.1, we have the following exponential tail estimates for the constructed transition fronts.

Corollary 3.1. *For the constructed transition fronts of $c^* < c < \hat{c}$ in Proposition 3.1, they own exponential tail estimates: for any $\epsilon > 0$, there exist $C_1,\ C_2,\ T > 0$ such that for $t > T$ and $j > ct$,*

$$C_1 e^{-(\mu+\epsilon)(j-ct)} \leq u_j(t) \leq C_2 e^{-(\mu-\epsilon)(j-ct)}. \tag{3.10}$$

Remark 3.2. If $\lambda = 1$, $\hat{c} = \infty$ ($\hat{\mu} = 0$). This includes the case of homogeneous equation with $f_j(u_j) = 1 - u_j$. In these cases, the required positive eigenvectors to (1.7) are always available for any $\mu \in (0, \mu^*)$. For $c = c^*$, since comparison principle does not work due to invalid sub-solutions, the tail estimate remains an open question and we should pay special attention to the critical speed c^*.

3.4. Asymptotic behaviors of transition fronts

In the last subsection, for any constructed transition fronts, they satisfy an exponential tail estimate (3.10). In this subsection, we will prove that if transition fronts exist, then they must own similar exponential tail estimates, that completes the proof of part (1) of main theorem. Recall that $\lambda(\mu) = e^\mu - 1 + e^{-\mu}$ and $c(\mu) = \frac{\lambda(\mu)}{\mu}$ for $\mu > 0$, then we have the following propositions about the asymptotic behaviors of transition fronts.

Proposition 3.2. *Let $c > c^*$, and $u_j(t)$ be a transition front of (1.2) with speed c. Then for any $\epsilon > 0$, there exists a $\hat{K}_\epsilon > 0$ such that*

$$u_k(t_j) \le \hat{K}_\epsilon e^{-(\mu-\epsilon)(k-j)},$$

for $k \ge j$ with j, t_j as in Definition 1.2 of Mean Wave Speed.

Proof. Suppose not, then there exist ϵ, j_n, t_{j_n}, k_n and $x_n := k_n - j_n \to \infty$ such that

$$u_{k_n}(t_{j_n}) \ge \hat{K}_\epsilon e^{-(\mu-\epsilon)x_n}. \tag{3.11}$$

For simplicity, we denote $T = t_{j_n}$.

Recall (2.5) that for $T \ge 0$,

$$\begin{aligned} u_j(t) &= e^{t-T}\sum_k h^{\mathbb{Z}}_{2(t-T)}(j-k)u_k(T) - \int_T^t e^{(t-s)}\sum_k h^{\mathbb{Z}}_{2(t-s)}(j-k)g_k(s)ds \\ &:= A(t) - B(t), \end{aligned}$$

and $g_j(t) = (1 - f_j(u_j))u_j$.

Let $z_0 = csch(\mu)$. Recall that $c > \frac{2}{z_0}$ by Lemma 2.1 (4). Let $\tilde{t}$ be such that $x_n = (c - \frac{2}{z_0})\tilde{t}$ and $j = k_n + \frac{2}{z_0}\tilde{t} = j_n + c\tilde{t} > N$. Choose $t = \tilde{t} + T$, that is, $\tilde{t} = t - T$. Then as $x_n \to \infty$, $\tilde{t} \to \infty$ and thus $t \to \infty$. By Lemma 2.1 (2), $g(z_0) = \lambda(\mu)$ and thus $-1 + 2\frac{\varsigma(z_0)+\mu}{z_0} = \mu c$. By heat kernel estimate (2.2) and with (3.11), we have

$$\begin{aligned} A(t) &= e^{\tilde{t}}\sum_k h^{\mathbb{Z}}_{2\tilde{t}}(j-k)u_k(T) \\ &\ge Ce^{\tilde{t}}\sum_k F(2\tilde{t}, j-k)u_k(T) \\ &= C\frac{1}{\sqrt{2\pi}}\Big(\frac{e^{\tilde{t}}}{(1+4\tilde{t}^2)^{\frac14}}u_j(T) + \sum_{k\neq j}\frac{exp[-\tilde{t} + |j-k|\varsigma(2\tilde{t}/|j-k|)]}{(1+4\tilde{t}^2+|j-k|^2)^{\frac14}}u_k(T)\Big) \\ &\ge C\frac{1}{\sqrt{2\pi}}\frac{exp[-\tilde{t} + |j-k_n|\varsigma(2\tilde{t}/|j-k_n|)]}{(1+4(\tilde{t})^2+(j-k_n)^2)^{\frac14}}u_{k_n}(T) \\ &\ge C\hat{K}_\epsilon\frac{1}{\sqrt{2\pi}}\frac{exp[-\tilde{t} + |j-k_n|\varsigma(2\tilde{t}/|j-k_n|)]}{(1+4(\tilde{t})^2+(j-k_n)^2)^{\frac14}}e^{-(\mu-\epsilon)x_n} \\ &= C\hat{K}_\epsilon\frac{1}{\sqrt{2\pi}}\frac{exp[-\tilde{t} + \frac{2\tilde{t}}{z_0}\varsigma(z_0) - (\mu-\epsilon)(c-\frac{2}{z_0})\tilde{t}]}{(1+4(\tilde{t})^2+(j-k_n)^2)^{\frac14}} \\ &= C\hat{K}_\epsilon\frac{1}{\sqrt{2\pi}}\frac{exp[-\tilde{t} + \frac{2\tilde{t}}{z_0}\varsigma(z_0) - \mu(c-\frac{2}{z_0})\tilde{t}]}{(1+4(\tilde{t})^2+(j-k_n)^2)^{\frac14}}e^{\epsilon(c-\frac{2}{z_0})\tilde{t}} \\ &= C\hat{K}_\epsilon\frac{1}{\sqrt{2\pi}}\frac{exp[\epsilon(c-\frac{2}{z_0})\tilde{t}]}{(1+4(\tilde{t})^2+(j-k_n)^2)^{\frac14}} \\ &\ge e^{\tilde{\epsilon}\tilde{t}}, \end{aligned}$$

where $\tilde{\epsilon}$ is chosen such that $\epsilon(c - \frac{2}{z_0}) > \tilde{\epsilon} > 0$ for $\mu \in (0, \mu^*)$ and the above inequality holds as $\tilde{t}$ is chosen large enough. On the other hand, we have that

$$\begin{aligned} B(t) &= \int_T^t e^{t-s}\sum_k h^{\mathbb{Z}}_{2(t-s)}(j-k)g_k(s)ds \\ &= \int_0^{\tilde{t}} e^{\tilde{t}-s}\sum_k h^{\mathbb{Z}}_{2(\tilde{t}-s)}(j-k)g_k(s)ds \end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{\tilde t} e^{\tilde t-s}\sum_k F(2(\tilde t-s),j-k)g_k(s)ds\\
&= C\int_0^{\tilde t} e^{\tilde t-s}\Big(\sum_{k\le -c(\tilde t-s)+cs+j_n}+\sum_{-c(\tilde t-s)+cs+j_n<k\le cs+j_n}+\sum_{k>cs+j_n}\Big)F(2(\tilde t-s),j-k)g_k(s)ds\\
&:= B_1+B_2+B_3.
\end{aligned}
$$

Since u_j is bounded, there exists a positive M such that

$$|g_j(s)| = |f_j(u_j)u_j - u_j| < M. \tag{3.12}$$

For $k > cs + j_n > N$, $f_k(0) = a_k = 1$. For any given positive η, there exists an n_0 such that for $n > n_0$, we have $u_k(s) < \eta$ whenever $k > cs + j_n$. This can be done because $\lim_{k-cs\to\infty} u_k(s) = 0$ and $j_n \approx ct_{j_n}$ large. Then for any $\delta \in (0, \tilde\epsilon)$, there exists an η such that $f_k(0) - f_k(u_k) < \delta$ and then

$$g_k(s) = (1 - f_k(u_k))u_k(s) < \delta u_k(s). \tag{3.13}$$

Without loss of generality, let $j_n = 0$ by translation and so $j = c\tilde t$. Recall that, in Lemma 2.1 (1), numerical computation shows that $l_0 > 0.66$. On the other hand, we have $c \ge c^* = \inf_{\mu>0} \frac{e^\mu - 1 + e^{-\mu}}{\mu} \approx 2.073$, and thus $\frac{1}{c} < 0.5 < l_0$. Therefore by Lemma 2.1 (1), $\varsigma(\frac{1}{c}) < \varsigma(l_0)$, that is, $\varsigma(\frac{1}{c}) < 0$. For $k \le -c(\tilde t - s) + cs$,

$$\varsigma\Big(\frac{2(\tilde t-s)}{|j-k|}\Big) \le \varsigma\Big(\frac{1}{c}\Big) < 0. \tag{3.14}$$

Therefore, with (3.12) and (3.14), for B_1 we have

$$
\begin{aligned}
B_1 &= C\int_0^{\tilde t} e^{\tilde t-s}\Big(\sum_{k\le -c(\tilde t-s)+cs} F(2(\tilde t-s),j-k)g_k(s)\Big)ds\\
&\le CM\int_0^{\tilde t} e^{\tilde t-s}\Big(\sum_{k\le -c(\tilde t-s)+cs} F(2(\tilde t-s),j-k)\Big)ds\\
&= C_1\int_0^{\tilde t}\sum_{k\le -c(\tilde t-s)+cs}\frac{e^{-(\tilde t-s)+|j-k|\varsigma(\frac{2(\tilde t-s)}{|j-k|})}}{\sqrt{2\pi}(1+4(\tilde t-s)^2+(j-k)^2)^{\frac14}}ds\\
&\le C_1\int_0^{\tilde t}\sum_{k\le -c(\tilde t-s)+cs}\frac{e^{-(\tilde t-s)+|j-k|\varsigma(\frac{1}{c})}}{\sqrt{2\pi}(1+4(\tilde t-s)^2+(j-k)^2)^{\frac14}}ds\\
&\le C_1\int_0^{\tilde t}\sum_{k\le min\{-c(\tilde t-s)+cs,0\}}\frac{e^{-(\tilde t-s)+|j-k|\varsigma(\frac{1}{c})}}{\sqrt{2\pi}(1+4(\tilde t-s)^2+(j-k)^2)^{\frac14}}ds\\
&\quad + C_1\int_0^{\tilde t}\sum_{0<k\le|-c(\tilde t-s)+cs|}\frac{e^{(-1+2c\varsigma(\frac{1}{c}))(\tilde t-s)}}{\sqrt{2\pi}(1+4(\tilde t-s)^2+(j-k)^2)^{\frac14}}ds\\
&\le C_1\int_0^{\tilde t}\sum_{k\le 0}\frac{e^{-(\tilde t-s)+|j-k|\varsigma(\frac{1}{c})}}{\sqrt{2\pi}(1+4(\tilde t-s)^2+(j-k)^2)^{\frac14}}ds + C_1\int_0^{\tilde t}\sum_{0<k\le c(\tilde t+2s)} 1ds\\
&\le C_1\int_0^{\tilde t}\Big(\sum_{k\le 0} e^{(-k)\varsigma(\frac{1}{c})} + c(\tilde t+2s)\Big)ds, \text{by } -(\tilde t-s)+j\varsigma\Big(\frac{1}{c}\Big) < 0
\end{aligned}
$$

$$= C_1 \int_0^{\tilde{t}} \Big(\frac{e^{\varsigma(\frac{1}{c})}}{1-e^{\varsigma(\frac{1}{c})}} + c(\tilde{t}+2s)\Big)ds$$
$$\leq P_1(\tilde{t}),$$

where $C_1 = CM$ and $P_1(\tilde{t}) = C_1(2c\tilde{t}^2 + (\frac{e^{\varsigma(\frac{1}{c})}}{1-e^{\varsigma(\frac{1}{c})}})\tilde{t})$ that is a quadratic equation.

Let $-\sigma = -1 + 2\frac{\varsigma(z_1)}{z_1} = \max_{-c(\tilde{t}-s)+cs<k\leq cs} -1 + 2\frac{\varsigma(z)}{z}$ with $z = \frac{2(\tilde{t}-s)}{|j-k|}$ and $z_1 = \frac{2}{c}$. We remark that $c \geq c^* \approx 2.073$ and $z_1 \leq \frac{2}{c^*} \approx 0.9648$. Thus,

$$\sigma = 1 - 2\frac{\varsigma(z_1)}{z_1} \geq 1 - 2\frac{\varsigma(0.9648)}{0.9648} \approx 0.0355793 > 0.$$

Then, for B_2 we have that

$$B_2 = C\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\sum_{-c(\tilde{t}-s)+cs<k\leq cs} F(2(\tilde{t}-s), j-k)g_k(s)\Big)ds$$
$$\leq CM\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\sum_{-c(\tilde{t}-s)+cs<k\leq cs} F(2(\tilde{t}-s), j-k)\Big)ds$$
$$= C_1\int_0^{\tilde{t}} \sum_{-c(\tilde{t}-s)+cs<k\leq cs} \frac{e^{-(\tilde{t}-s)+|j-k|\varsigma(\frac{2(\tilde{t}-s)}{|j-k|})}}{\sqrt{2\pi}(1+4(\tilde{t}-s)^2+(j-k)^2)^{\frac{1}{4}}}ds$$
$$\leq C_1\int_0^{\tilde{t}} \sum_{-c(\tilde{t}-s)+cs<k\leq cs} \frac{e^{(-1+c\varsigma(\frac{2}{c}))(\tilde{t}-s)}}{\sqrt{2\pi}(1+4(\tilde{t}-s)^2+(j-k)^2)^{\frac{1}{4}}}ds$$
$$\leq C_1\int_0^{\tilde{t}} \sum_{-c(\tilde{t}-s)+cs<k\leq cs} \frac{e^{-\sigma(\tilde{t}-s)}}{\sqrt{2\pi}(1+4(\tilde{t}-s)^2+(j-k)^2)^{\frac{1}{4}}}ds$$
$$\leq C_1\int_0^{\tilde{t}} \sum_{-c(\tilde{t}-s)+cs<k\leq cs} 1ds$$
$$= C_1\int_0^{\tilde{t}} c(\tilde{t}-s)ds$$
$$\leq P_2(\tilde{t}),$$

where $C_1 = CM$ and $P_2(\tilde{t}) = C_1(\frac{c}{2}\tilde{t}^2)$ that is a quadratic equation.

Finally, with (3.13), for B_3 we have that

$$B_3 = C\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\sum_{k>cs} F(2(\tilde{t}-s), j-k)g_k(s)\Big)ds$$
$$\leq \delta C\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\sum_{k>cs} F(2(\tilde{t}-s), j-k)u_k(s)\Big)ds$$
$$\leq \delta C\int_0^{\tilde{t}} e^{\tilde{t}-s}\Big(\sum_{k} F(2(\tilde{t}-s), j-k)u_k(s)\Big)ds.$$

Note that

$$A - B_3 \geq (S(t-T)u(T))_j - \delta\Big(\int_T^t S(t-s)u(s)ds\Big)_j$$
$$= e^{-\delta\tilde{t}}(S(t-T)u(T))_j$$

$$
\begin{aligned}
&= e^{-\delta\tilde{t}}A(t) \\
&\geq e^{(\tilde{\epsilon}-\delta)\tilde{t}},
\end{aligned}
$$

which is an exponential equation. On the other hand, $B_1+B_2 \leq P_1(\tilde{t})+P_2(\tilde{t})$, which is a quadratic equation. Thus, $u_j(t) \to \infty$ as $t \to \infty$ which contradicts that $u_j(t)$ is bounded. □

Proposition 3.3. *Let $u_j(t)$ be a transition front of (1.2) with speed c larger than c^*. Then for any $\epsilon > 0$, there exists a $\tilde{K}_\epsilon > 0$ and $T > 0$ such that*

$$u_k(t_j) \geq \tilde{K}_\epsilon e^{-(\mu+\epsilon)(k-j)},$$

for $t_j > T$ and $k \geq j$ with j, t_j as in Definition 1.2 of Mean Wave Speed.

Proof. We prove this lemma by contradiction. Assume the proposition to be false. Then for given ϵ, there exist sequences $t_{j_n} \in \mathbb{R}^+$, $k_n \in \mathbb{Z}^+$ and $j_n \in \mathbb{Z}^+$ such that $k_n \geq j_n$ and

$$u_{k_n}(t_{j_n}) \leq \tilde{K}_\epsilon e^{-(\mu+\epsilon)(k_n-j_n)}. \tag{3.15}$$

By applying Harnack inequality and shifting the origin of time and space, we can have a $q > 0$ such that

$$u_k(t_{j_n}) \leq Ce^{-(\mu+\frac{\epsilon}{2})(k_n-j_n)}, \forall k \in [(1-q\epsilon)k_n, (1+q\epsilon)k_n]. \tag{3.16}$$

Let $j = ct_{j_n} > N$, where $t_{j_n} \in \mathbb{R}^+$ is chosen such that $j \in \mathbb{Z}^+$ and N is as in (H2). For simplicity, we let $t = t_{j_n}$. We remark that t is a sequence and $n \to \infty$ implies that $t \to \infty$. Recall (2.5) that for $T = 0$,

$$
\begin{aligned}
u_j(t) &= e^t \sum_k h^{\mathbb{Z}}_{2t}(j-k)u_k(0) - \int_0^t e^{(t-s)} \sum_k h^{\mathbb{Z}}_{2(t-s)}(j-k)g_k(s)ds \\
&:= A(t) - B(t),
\end{aligned}
$$

where $g_j(t) = (1 - f_j(u_j))u_j$, and $\begin{cases} A(t) = e^t \sum_k h^{\mathbb{Z}}_{2t}(j-k)u_k(0) \\ B(t) = \int_0^t e^{(t-s)} \sum_k h^{\mathbb{Z}}_{2(t-s)}(j-k)g_k(s)ds. \end{cases}$.

We claim that $u_j(t) = A(t) - B(t) < 0$ as $t \to \infty$, which causes a contradiction.

For any $\delta > 0$, there exists a $l, j_\delta > 0$, such that $u_k(s) \geq l$ for $N < k \leq (c-\delta)s - j_\delta$ and $s \geq 0$. Then for $N < k \leq (c-\delta)s - j_\delta$ and $s \geq 0$,

$$g_k(s) = (f_k(0) - f_k(u_k))u_k(s) \geq (1 - \sup_k f_k(l))l := \hat{l}. \tag{3.17}$$

Thus, letting $\hat{k} = (c-\delta)s - j_\delta$ and $0 < \sigma_2 < \sigma_1 \ll 1$, let

$$C(s,t) = -(t-s) + |ct - \hat{k}|\varsigma(2(t-s)/|ct-\hat{k}|).$$

For $0 < (1-\sigma_1)t \leq s \leq (1-\sigma_2)t$, choosing $\delta = \sigma_1$, we have

$$
\begin{aligned}
C(s,t) &= -(t-s) + |ct-\hat{k}|\varsigma(2(t-s)/|ct-\hat{k}|) \\
&= -(t-s) + (c(t-s)+\delta s+j_\delta)\varsigma(2(t-s)/(c(t-s)+\delta s+j_\delta)) \\
&\geq -\sigma_1 t + (c(t-s)+\delta s+j_\delta)\varsigma\Big(\frac{2}{c+\delta(\frac{s}{t-s})+\frac{j_\delta}{t-s}}\Big) \\
&\geq -\sigma_1 t + (c\sigma_2 t+\delta(1-\sigma_1)t+j_\delta)\varsigma\Big(\frac{2}{c+\delta(\frac{1-\sigma_2}{\sigma_1})+\frac{j_\delta}{\sigma_2 t}}\Big)
\end{aligned}
$$

$$= -\sigma_1 t + (c\sigma_2 t + \sigma_1(1-\sigma_1)t + j_\delta)\varsigma(\frac{2}{c+1-\sigma_2+\frac{j_\delta}{\sigma_2 t}})$$
$$= (-\sigma_1 + (c\sigma_2 + \sigma_1(1-\sigma_1))\varsigma(\frac{2}{c+1-\sigma_2+\frac{j_\delta}{\sigma_2 t}}))t + j_\delta\varsigma(\frac{2}{c+1-\sigma_2+\frac{j_\delta}{\sigma_2 t}}).$$

For $t > \frac{j_\delta}{\sigma_2^2}$, we have

$$\begin{aligned} C(s,t) &\geq (-\sigma_1 + (c\sigma_2 + \sigma_1(1-\sigma_1))\varsigma(\frac{2}{c+1-\sigma_2+\frac{j_\delta}{\sigma_2 t}}))t + j_\delta\varsigma(\frac{2}{c+1-\sigma_2+\frac{j_\delta}{\sigma_2 t}}) \\ &\geq (-\sigma_1 + (c\sigma_2 + \sigma_1(1-\sigma_1))\varsigma(\frac{2}{c+1}))t + j_\delta\varsigma(\frac{2}{c+1}) \\ &:= -\hat{\sigma}_1 t - \hat{\sigma}_2. \end{aligned} \tag{3.18}$$

We remark that as $\sigma_1 \to 0$, $\hat{\sigma}_1 \to 0$, that is, $\hat{\sigma}_1$ can be chosen as small as required by choosing small enough $\sigma_{1,2}$. Let

$$\begin{aligned} \tilde{C}_B &:= \lim_{t\to\infty}(C\hat{l}\frac{(\sigma_1-\sigma_2)\sqrt{t}}{(1+4\sigma_1^2t^2+((\delta-\sigma_1(c-\delta))t+j_\delta)^2)^{\frac{1}{4}}}) \\ &= C\hat{l}\frac{(\sigma_1-\sigma_2)}{(1+4\sigma_1^2+((\delta-\sigma_1(c-\delta)))^2)^{\frac{1}{4}}} \end{aligned}$$

and $C_B = \tilde{C}_B/2$. Then, there exists a T_B such that for $t > T_B$,

$$C\hat{l}\frac{(\sigma_1-\sigma_2)\sqrt{t}}{(1+4\sigma_1^2t^2+((\delta-\sigma_1(c-\delta))t+j_\delta)^2)^{\frac{1}{4}}} \geq C_B.$$

Therefore, with (2.2), (3.17) and (3.18), for $t > \max\{\frac{j_\delta}{\sigma_2^2}, T_B\}$, we have

$$\begin{aligned} B(t) &= \int_0^t e^{t-s}\sum_k h^{\mathbb{Z}}_{2(t-s)}(j-k)g_k(s)ds \\ &\geq C\int_0^t e^{t-s}\sum_k F(2(t-s), j-k)g_k(s)ds \\ &\geq C\int_0^t e^{t-s}F(2(t-s), j-\hat{k})g_{\hat{k}}(s)ds \\ &\geq C\hat{l}\int_0^t \frac{exp[-(t-s)+|j-\hat{k}|\varsigma(2(t-s)/|j-\hat{k}|)]}{(1+4(t-s)^2+|j-\hat{k}|^2)^{\frac{1}{4}}}ds \\ &\geq C\hat{l}\int_{(1-\sigma_1)t}^{(1-\sigma_2)t} \frac{exp[-(t-s)+|j-\hat{k}|\varsigma(2(t-s)/|j-\hat{k}|)]}{(1+4(t-s)^2+|j-\hat{k}|^2)^{\frac{1}{4}}}ds \\ &\geq C\hat{l}e^{-\hat{\sigma}_1 t-\hat{\sigma}_2}\int_{(1-\sigma_1)t}^{(1-\sigma_2)t} \frac{1}{(1+4(t-s)^2+|j-\hat{k}|^2)^{\frac{1}{4}}}ds \\ &\geq C\hat{l}e^{-\hat{\sigma}_1 t-\hat{\sigma}_2}\int_{(1-\sigma_1)t}^{(1-\sigma_2)t} \frac{1}{(1+4\sigma_1^2t^2+((\delta-\sigma_1(c-\delta))t+j_\delta)^2)^{\frac{1}{4}}}ds \\ &= C\hat{l}e^{-\hat{\sigma}_1 t-\hat{\sigma}_2}\frac{(\sigma_1-\sigma_2)t}{(1+4\sigma_1^2t^2+((\delta-\sigma_1(c-\delta))t+j_\delta)^2)^{\frac{1}{4}}} \\ &= (C\hat{l}\frac{(\sigma_1-\sigma_2)\sqrt{t}}{(1+4\sigma_1^2t^2+((\delta-\sigma_1(c-\delta))t+j_\delta)^2)^{\frac{1}{4}}})\sqrt{t}e^{-\hat{\sigma}_1 t-\hat{\sigma}_2} \\ &\geq C_B\sqrt{t}e^{-\hat{\sigma}_1 t-\hat{\sigma}_2}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
A(t) &= e^t \sum_k h^{\mathbb{Z}}_{2t}(j-k)u_k(0)\\
&\le C_1 e^t \sum_k F(2t, j-k)u_k(0)\\
&= C_1 e^t \Big[\Big(\sum_{k\le -ct} + \sum_{-ct<k\le 0} + \sum_{1\le k\le (1-q\epsilon)k_n} + \sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}\\
&\qquad + \sum_{(1+q\epsilon)k_n<k\le j-1} + \sum_{k=j} + \sum_{j<k\le 3j} + \sum_{k>3j}\Big)F(2t,j-k)u_k(0)\Big]\\
&:= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8.
\end{aligned}$$

For $k \le -ct$, we have that $\varsigma(\frac{2t}{|j-k|}) \le \varsigma(\frac{1}{c}) < 0$. Then for A_1 and t large, we have

$$\begin{aligned}
A_1 :&= C_1 e^t \sum_{k\le -ct} F(2t, j-k)u_k(0)\\
&= C_1 \sum_{k\le -ct} \frac{e^{-t+|j-k|\varsigma(\frac{2t}{|j-k|})}}{\sqrt{2\pi}(1+4t^2+(j-k)^2)^{\frac{1}{4}}} u_k(0)\\
&\le C_1 \frac{e^{-t}}{\sqrt{2\pi}(1+4t^2+j^2)^{\frac{1}{4}}} \sum_{k\le 0} e^{|j-k|\varsigma(\frac{1}{c})} u_k(0)\\
&\le C_1 \sup_{k\le 0}\{u_k^*\} \frac{e^{-t}}{\sqrt{2\pi}(1+4t^2+j^2)^{\frac{1}{4}}} \sum_{k\le 0} e^{(-k)\varsigma(\frac{1}{c})}\\
&= \tilde{C}_1 \frac{e^{-t}}{\sqrt{2\pi}(1+(4+c^2)t^2)^{\frac{1}{4}}}\\
&\le \tilde{C}_1 e^{-t/2},
\end{aligned}$$

where $\tilde{C}_1 = C_1 \dfrac{e^{\varsigma(\frac{1}{c})}}{1-e^{\varsigma(\frac{1}{c})}} \sup_{k\le 0}\{u_k^*\}$.

Let $-\sigma = -1 + 2\frac{\varsigma(z_1)}{z_1} = \max_{-ct<k\le 0} -1 + 2\frac{\varsigma(z)}{z}$ with $z = \frac{2t}{|j-k|}$ and $z_1 = \frac{2}{c}$. We remark that $c \ge c^* \approx 2.073$ and $z_1 \le \frac{2}{c^*} \approx 0.9648$. Thus,

$$\sigma = 1 - 2\frac{\varsigma(z_1)}{z_1} \ge 1 - 2\frac{\varsigma(0.9648)}{0.9648} \approx 0.0355793 > 0.$$

Then, for A_2 and large t, with $z = \frac{2t}{|j-k|}$ and σ above we have

$$\begin{aligned}
A_2 :&= C_1 e^t \sum_{-ct<k\le 0} F(2t, j-k)u_k(0)\\
&= C_1 \sum_{-ct<k\le 0} \frac{e^{-t+|j-k|\varsigma(\frac{2t}{|j-k|})}}{\sqrt{2\pi}(1+4t^2+(j-k)^2)^{\frac{1}{4}}} u_k(0)\\
&= C_1 \sum_{-ct<k\le 0} \frac{e^{(-1+2\frac{\varsigma(z)}{z})t}}{\sqrt{2\pi}(1+4t^2+(j-k)^2)^{\frac{1}{4}}} u_k(0)\\
&\le C_1 \frac{e^{-\sigma t}}{\sqrt{2\pi}(1+4t^2+j^2)^{\frac{1}{4}}} \sum_{-ct<k\le 0} u_k(0)\\
&\le C_1 \sup_{k\le 0}\{u_k^*\} \frac{cte^{-\sigma t}}{\sqrt{2\pi}(1+(4+c^2)t^2)^{\frac{1}{4}}}\\
&\le C_1 e^{-\frac{\sigma}{2}t}.
\end{aligned}$$

For A_3, on $[1,(1-q\epsilon)k_n]$, $-t+|j-k|\varsigma(2t/|j-k|)$ obtains a maximum at $k=(1-q\epsilon)k_n$. Therefore, by Proposition 3.2, there exists a positive real C_δ such that $u_k(0)\le \frac{C_\delta}{C_1}e^{-(\mu-\delta)k}$ for $\delta>0$ and for $k_n=(c-\frac{2}{z_0})t$ and $j=ct$, $j-k\ge j-(1-q\epsilon)k_n$ for $k\in[1,(1-q\epsilon)k_n]$. Then, letting $z=\frac{2t}{|j-k|}$ we have $z\le \frac{2}{c-(1-q\epsilon)(c-\frac{2}{z_0})}=\frac{2}{\frac{2}{z_0}+q\epsilon(c-\frac{2}{z_0})}:=z_2<z_0$. By Lemma 2.1(2), with $z_2<z_0$ and $-1+2\frac{\varsigma(z_0)+\mu}{z_0}-\mu c=\lambda(\mu)-\mu c=0$, then we can let

$$\tilde{\epsilon}_3=-(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)>0.$$

Thus, for A_3

$$\begin{aligned}
A_3&=C_1e^t\sum_{1\le k\le(1-q\epsilon)k_n}F(2t,j-k)u_k(0)\\
&\le C_\delta\sum_{1\le k\le(1-q\epsilon)k_n}F(2t,j-k)e^{-(\mu-\delta)k}\\
&=C_\delta e^t\sum_{1\le k\le(1-q\epsilon)k_n}\frac{1}{\sqrt{2\pi}}\frac{exp[-t+|j-k|\varsigma(\frac{2t}{|j-k|})-(\mu-\delta)k]}{(1+4t^2+(j-k)^2)^{\frac14}}\\
&\le C_\delta\frac{1}{\sqrt{2\pi}}\frac{1}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}\sum_{1\le k\le(1-q\epsilon)k_n}exp[-t+|j-k|\varsigma(\frac{2t}{|j-k|})-(\mu-\delta)k]\\
&=C_\delta\frac{1}{\sqrt{2\pi}}\frac{1}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}\sum_{1\le k\le(1-q\epsilon)k_n}exp[(-1+2\frac{\varsigma(z)+\mu}{z})t-\mu j+\delta k]\\
&\le C_\delta\frac{1}{\sqrt{2\pi}}\frac{1}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}\sum_{1\le k\le(1-q\epsilon)k_n}exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2})t-\mu j+\delta k]\\
&=C_\delta\frac{1}{\sqrt{2\pi}}\frac{1}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)t]\sum_{1\le k\le(1-q\epsilon)k_n}e^{\delta k}\\
&\le C_\delta\frac{1}{\sqrt{2\pi}}\frac{1}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)t](1-q\epsilon)k_ne^{\delta(1-q\epsilon)k_n}\\
&=C_\delta\frac{1}{\sqrt{2\pi}}\frac{(1-q\epsilon)(c-\frac{2}{z_0})t}{(1+4t^2+(\frac{2}{z_0}t)^2)^{\frac14}}exp[(-\tilde{\epsilon}_3+\delta(1-q\epsilon)(c-\frac{2}{z_0}))t]\\
&\le C_\delta e^{-\epsilon_3 t},
\end{aligned}$$

where $\epsilon_3=\tilde{\epsilon}_3/2$ and choose δ such that $\delta(1-q\epsilon)(c-\frac{2}{z_0})<\tilde{\epsilon}_3/2$.

For A_4, with $k\in((1-q\epsilon)k_n,(1+q\epsilon)k_n]$, recalling that (3.16), $u_{k_n}(0)\le e^{-(\mu+\frac{\epsilon}{2})k_n}$,

$$\begin{aligned}
A_4&=C_1e^t\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}F(2t,j-k)u_k(0)\\
&=C_1\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}\frac{1}{\sqrt{2\pi}}\frac{exp[-t+|j-k|\varsigma(2t/|j-k|)]}{(1+4t^2+(j-k)^2)^{\frac14}}u_k(0)\\
&\le C_1\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}\frac{1}{\sqrt{2\pi}}\frac{exp[-t+|j-k|\varsigma(2t/|j-k|)]}{(1+4t^2+(j-k)^2)^{\frac14}}e^{-(\mu+\frac{\epsilon}{2})k_n}\\
&=C_1\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}\frac{1}{\sqrt{2\pi}}\frac{exp[-t+|j-k|\varsigma(2t/|j-k|)-(\mu+\frac{\epsilon}{2})k_n]}{(1+4t^2+(j-k)^2)^{\frac14}}\\
&\le C_1\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}exp[-t+|j-k|\varsigma(2t/|j-k|)-(\mu+\frac{\epsilon}{2})k_n]\\
&=C_1e^{-\frac{\epsilon}{2}k_n}\sum_{(1-q\epsilon)k_n<k\le(1+q\epsilon)k_n}exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)t+\mu(k-k_n)]
\end{aligned}$$

$$
\begin{aligned}
&\leq C_1 e^{-\frac{\epsilon}{2}k_n} \sum_{(1-q\epsilon)k_n<k\leq(1+q\epsilon)k_n} exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)t+\mu q\epsilon k_n] \\
&\leq C_1 e^{(-\frac{1}{2}+\mu q)\epsilon k_n} \sum_{(1-q\epsilon)k_n<k\leq(1+q\epsilon)k_n} exp[(-1+2\frac{\varsigma(z_2)+\mu}{z_2}-\mu c)t] \\
&\leq C_1 e^{-(\frac{1}{2}-\mu q)\epsilon(c-\frac{2}{z_0})t} \\
&= C_1 e^{-\epsilon_4 t},
\end{aligned}
$$

where $\epsilon_4 = (\frac{1}{2}-\mu q)\epsilon(c-\frac{2}{z_0})$ and choose $q < \frac{1}{2\mu}$.

For A_5 with $k \in ((1+q\epsilon)k_n, j)$, by Proposition 3.2, there exists a positive real C_δ such that $u_k(0) \leq \frac{C_\delta}{C_1}e^{-(\mu-\delta)k}$ for $\delta > 0$. Recall that $k_n = (c-\frac{2}{z_0})t$, $j = ct$, $j-k > 0$ for $k \in ((1+q\epsilon)k_n, j)$. Then, letting $z = \frac{2t}{|j-k|}$, $z_3 := \frac{2}{c-(1+q\epsilon)(c-\frac{2}{z_0})} = \frac{2}{\frac{2}{z_0}-q\epsilon(c-\frac{2}{z_0})} > z_0$ for ϵ small and we have $z \geq z_3$ for $k \in ((1+q\epsilon)k_n, j)$. Let $\tilde{\epsilon}_5 = -(-1+2\frac{\varsigma(z_3)+\mu}{z_3}-(\mu-\delta)c)$. Choose δ such that $\tilde{\epsilon}_5 > 0$, which can be done because of Lemma 2.1(2) with $z_3 > z_0$ and $-1+2\frac{\varsigma(z_0)+\mu}{z_0}-\mu c = \lambda(\mu)-\mu c = 0$. Then, letting $z = \frac{2t}{|j-k|}$ we have

$$
\begin{aligned}
A_5 &= C_1 e^t \sum_{(1+q\epsilon)k_n<k<j} F(2t, j-k)u_k(0) \\
&\leq C_\delta \sum_{(1+q\epsilon)k_n<k<j} F(2t, j-k)e^{-(\mu-\delta)k} \\
&= C_\delta e^t \sum_{(1+q\epsilon)k_n<k<j} \frac{1}{\sqrt{2\pi}} \frac{exp[-t+|j-k|\varsigma(\frac{2t}{|j-k|})-(\mu-\delta)k]}{(1+4t^2+(j-k)^2)^{\frac{1}{4}}} \\
&\leq C_\delta \sum_{(1+q\epsilon)k_n<k<j} exp[-t+|j-k|\varsigma(\frac{2t}{|j-k|})-(\mu-\delta)k] \\
&= C_\delta \sum_{(1+q\epsilon)k_n<k<j} exp[(-1+2\frac{\varsigma(z)+\mu}{z})t-\mu j+\delta k] \\
&\leq C_\delta \sum_{(1+q\epsilon)k_n<k<j} exp[(-1+2\frac{\varsigma(z_3)+\mu}{z_3})t-\mu j+\delta k] \\
&\leq C_\delta \sum_{(1+q\epsilon)k_n<k<j} exp[((-1+2\frac{\varsigma(z_3)+\mu}{z_3})-(\mu-\delta)c)t] \\
&\leq C_\delta c t e^{-\tilde{\epsilon}_5 t} \\
&\leq C_\delta e^{-\epsilon_5 t},
\end{aligned}
$$

where $\epsilon_5 = \tilde{\epsilon}_5/2$.

By Proposition 3.2, there exists a positive real C_δ such that $u_j(0) \leq \frac{C_\delta}{C_1}e^{-(\mu-\delta)j}$ for $\delta > 0$. Let $\tilde{\epsilon}_6 = -(1-(\mu-\delta)c) = \lambda(\mu)-1-\delta c$. δ can be chosen such that $\tilde{\epsilon}_6 > 0$ since $\lambda(\mu) > 1$ for $\mu > 0$. For A_6, we have

$$
\begin{aligned}
A_6 &= C_1 e^t \frac{1}{\sqrt{2\pi}} \frac{1}{(1+t^2)^{\frac{1}{4}}} u_j(0) \\
&\leq C_\delta e^t \frac{1}{\sqrt{2\pi}} \frac{1}{(1+t^2)^{\frac{1}{4}}} e^{-(\mu-\delta)j} \\
&\leq C_\delta \frac{1}{\sqrt{2\pi}} \frac{1}{(1+t^2)^{\frac{1}{4}}} e^{(1-(\mu-\delta)c)t} \\
&= C_\delta \frac{1}{\sqrt{2\pi}} \frac{1}{(1+t^2)^{\frac{1}{4}}} e^{-\tilde{\epsilon}_6 t} \\
&\leq C_\delta e^{-\epsilon_6 t},
\end{aligned}
$$

where $\epsilon_6 = \tilde{\epsilon}_6/2$. By similar arguments of A_3 and A_1 respectively, we have $A_7 \le C_7 e^{-\epsilon_7 t}$ and $A_8 \le C_8 e^{-\epsilon_8 t}$. Finally, we have that

$$\begin{aligned} u_j(t) &= A(t) - B(t) \\ &\le (\tilde{C}_1 e^{-\frac{t}{2}} + C_1 e^{-\frac{\sigma}{2}t} + C_\delta e^{-\epsilon_3 t} + C_1 e^{-\epsilon_4 t} + C_\delta e^{-\epsilon_5 t} \\ &\qquad + C_\delta e^{-\epsilon_6 t} + C_7 e^{-\epsilon_7 t} + C_8 e^{-\epsilon_8 t}) - C_B \sqrt{t} e^{-\hat{\sigma}_1 t - \hat{\sigma}_2}. \end{aligned}$$

As $\sigma_{1,2} \to 0$, $\hat{\sigma}_1 \to 0$. Thus, $\hat{\sigma}_1$ can be chosen such that $\hat{\sigma}_1 < \min\{\frac{1}{2}, \frac{\sigma}{2}, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8\}$. Therefore, as t is large enough, $A(t) - B(t) < 0$, which causes a contradiction. □

4. Nonexistence of transition fronts

In this section, we shall investigate the conditions under which transition fronts do not exist. We shall prove part (2) of the main theorem (Theorem 1.1) in the following proposition.

Proposition 4.1. *Transition fronts do not exist under the following conditions:*

(1) $\lambda > \lambda^*$;
(2) for $\lambda \in [1, \lambda^*)$, either [i] $c < c^*$ or [ii] $c > \hat{c}$.

It is known that there is a minimal speed (spreading speed) c^* such that transition fronts may exist, that is, transition fronts do not exist for $c < c^*$ (See Proposition 4.2). Thus, all μs of valid positive eigenvectors are located in $(0, \mu^*)$. However, if $\lambda \in (1, \lambda(\mu^*))$, there are also no valid positive eigenvectors for $\mu \in (0, \hat{\mu})$. Proposition 4.1 shows that there is a maximal speed $\hat{c} = \frac{\lambda}{\hat{\mu}}$ to prevent the existence of transition fronts, that is, transition fronts do not exist for $c > \hat{c}$ or $\mu < \hat{\mu}$. If $\lambda > \lambda(\mu^*)$, then $\hat{\mu} > \mu^*$ and there are none valid positive eigenvectors at all.

From Fig. 1 and Propositions 3.1–3.3, we have the following facts for transition fronts if they exist:

(1) For any $\epsilon > 0$, there exists a $T > 0$ such that for $t > T$ and $j > ct$,

$$C_1 e^{-(\mu+\epsilon)(j-ct)} \le u_j(t) \le C_2 e^{-(\mu-\epsilon)(j-ct)} (\text{ see Propositions 3.2–3.3}).$$

(2) Due to the spreading properties of transition fronts, the lower bound of speed (minimal wave speed) is c^*, corresponding to the upper bound of μ (i.e. μ^*). Then we must have $u_j(t) \ge K e^{-\mu^*(j-ct)}$ for t large and some $K > 0$.
(3) The upper bound of speed (maximal wave speed) is given by $\hat{c}$, corresponding to the lower bound of μ (i.e. $\hat{\mu}$). Thus, we must have $u_j(t) \le K e^{-\hat{\mu}(j-ct)}$ for t large and some $K > 0$, that is controlled by the spectral bound $\lambda = \lambda(\hat{\mu})$.

We see that if $\lambda > \lambda^*$ and $\hat{\mu} > \mu^*$, this causes a contradiction of (2) and (3). If $c > \hat{c}$ and $\mu < \hat{\mu}$, this causes a contradiction of (1) and (3).

4.1. Spreading speeds and the lower bound of wave speeds c^*

First, we shall show that c^* is the lower bound of the speeds (minimal wave speed) in this subsection. For simplicity, we write $u_j(t)$ for $u_j(t; z)$ if no confusion occurs with the initial z. Define

$$\hat{X}^+ = \{v_j \ge 0 \mid \liminf_{r\to-\infty} \inf_{j\le r} v_j > 0, v_j = 0 \text{ for } j \in \mathbb{Z} \text{ with } j > N_0, \text{ for some } N_0 > 0\}. \tag{4.1}$$

Definition 4.1. A number c^* is called the spreading speed of (1.2) if for any $z \in \hat{X}^+$,

$$\liminf_{j\le ct, t\to\infty} u_j(t) > 0, \forall c < c^*,$$

and

$$\limsup_{j\ge ct, t\to\infty} u_j(t) = 0, \forall c > c^*,$$

where $u_j(0) = z_j$ is the initial condition.

We remark that for homogeneous and periodically heterogeneous KPP–Fisher equations, the spreading speed exists. For (1.2), we have the following:

Lemma 4.1 (*See Theorem 2.2 in [23,24]*). *The spreading speed of (1.2) c^* exists. Moreover, c^* of (1.2) with localized periodic inhomogeneity coincides with that of (1.2) with corresponding periodic inhomogeneity.*

Lemma 4.2 (*See Theorem 2.3 in [23,24]*). *For each $\delta > 0$, $r > 0$, and $z \in X^+$ satisfying that $z_j \ge \delta$ for $|j| \le r$,*

$$\limsup_{|j|\le ct, t\to\infty} (u_j(t) - u_j^*) = 0, \forall 0 < c < c^*,$$

We remark that, in particular, for our main equation (1.2), the spreading speed coincides with the definition $c^* = \dfrac{\lambda(\mu^*)}{\mu^*} = \inf\limits_{\mu>0} \dfrac{\lambda(\mu)}{\mu}$ in the introduction of the current paper.

Proposition 4.2 (*Minimal Wave Speed*). *There does not exist a transition front of (1.2) with speed less than c^*.*

Proof. We prove this lemma by contradiction. Suppose that there is a transition front with speed $c < c^*$. Pick $c < c_1 < c^*$. Choose t_n such that $j_n = c_1 t_n \in \mathbb{Z}$. By Lemma 4.1, $\liminf\limits_{j_n \le c_1 t_n, t_n\to\infty} u_{j_n}(t_n) > 0$. On the other hand, $j_n - ct_n = (c_1 - c)t_n \to \infty$, by the definition of transition front, $\lim\limits_{j_n - ct_n\to\infty} u_{j_n}(t_n) = 0$, which causes a contradiction. □

4.2. Nonexistence of transition fronts for $\lambda > \lambda^*$

In this subsection, we will show that if $\lambda > \lambda(\mu^*)$, there are no transition fronts. In biological sense, transition fronts will not exist in strongly localized spatial inhomogeneous environments. We shall prove the following proposition.

Proposition 4.3. *If $\lambda > \lambda(\mu^*)$, any entire solution $u_j(t)$ of (1.2) such that $0 < u_j(t) < u_j^*$ satisfies that for any $c < \hat{c}$, there exists a $K > 0$ such that for all $(t, j) \in \mathbb{R}^- \times \mathbb{Z}$,*

$$u_j(t) \le K e^{-\mu^*(|j|-ct)},$$

where μ^* is such that $c^* = \frac{\lambda(\mu^*)}{\mu^*} = \inf\limits_{\mu>0} \dfrac{\lambda(\mu)}{\mu}$. In particular, no transition fronts exist if $\lambda > \lambda^*$.

To prove Proposition 4.3, we show the following Lemmas 4.3 and 4.4.

Lemma 4.3. *For each $m \in \mathbb{Z}$ and $\epsilon > 0$ there exist $k_\epsilon, \delta > 0$ such that if $u_j(t)$ solves (1.2) with $u_{j_0}(0) \ge \gamma$, for any given j_0 and $\gamma \le \frac{\delta}{2}$, then for $t \ge 0$ and $j \le j_0 + m - c^* t$,*

$$u_j(t) \ge k_\epsilon \gamma e^{(1-\epsilon)t} F(2t, j - j_0).$$

Proof. Without loss of generality, set $j_0 = 0$. Let $l = \min_j\{a_j\} \le 1$. Note that $v_j(t) = \gamma k_\epsilon e^{(l-\epsilon)t} h^{\mathbb{Z}}_{2t}(j)$ is a solution of

$$\dot v_j(t) = v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) + (l-\epsilon)v_j(t),$$

with initial $v_j(0) = \gamma k_\epsilon$ for $j = 0$ and $v_j(0) = 0$ for $j \ne 0$. Since $\|v(t)\|_\infty \le \gamma k_\epsilon e^{(1-\epsilon)t}$, we have $\|v(t)\| < \gamma$ if $k_\epsilon = e^{-2t_\epsilon}$ for $t \le t_\epsilon$. Let $\bar j = -\bar c t + 2m$ for some $\bar c < c^*$. Since by Lemma 2.1 (2) and (4), c^* satisfies $-1 + 2\frac{\zeta(2/c^*)}{2/c^*} = 0$, there are $\bar c \in (0, c^*)$ and t_ϵ such that for $t > t_\epsilon$,

$$\begin{aligned}
&-1 + 2\frac{\zeta(2t/|\bar j|)}{2t/|\bar j|}\\
&= -1 + 2\frac{\zeta(2t/|\bar c t - 2m|)}{2t/|\bar c t - 2m|}\\
&= -1 + 2\frac{\zeta(2/|\bar c - 2\frac{m}{t}|)}{2/|\bar c - 2\frac{m}{t}|}\\
&\le -1 + 2\frac{\zeta(2/|\bar c - 2\frac{|m|}{t_\epsilon}|)}{2/|\bar c - 2\frac{|m|}{t_\epsilon}|}\\
&< \epsilon/2.
\end{aligned}$$

Then for $t > t_\epsilon$, $-(1+\epsilon) + 2\frac{\zeta(2t/|\bar j|)}{2t/|\bar j|} < -\epsilon/2$. Therefore, for $t > t_\epsilon$, with (2.2)

$$\begin{aligned}
v_{\bar j}(t) &= \gamma k_\epsilon e^{(1-\epsilon)t} h^{\mathbb{Z}}_{2t}(\bar j)\\
&\le (1+\epsilon)\gamma k_\epsilon e^{(-(1+\epsilon)+2\frac{\zeta(2t/|\bar j|)}{2t/|\bar j|})t}\\
&\le (1+\epsilon)\gamma k_\epsilon.
\end{aligned}$$

Then for $t \ge t_\epsilon$, replace k_ϵ with $\frac{1}{(1+2\epsilon)}$ if the original k_ϵ is bigger than $\frac{1}{(1+2\epsilon)}$, and thus $(1+\epsilon)k_\epsilon < 1$, that is, we also have $v_{\bar j}(t) \le \gamma$. Furthermore, for $t > t_\epsilon \ge \frac{2|m|}{\bar c}$ such that $\bar j < 0$, $v_j(t) \le \gamma$ holds for all $j \le \bar j$, because $v_j(t) \le \gamma(1+\epsilon)k_\epsilon e^{(1-\epsilon)t} F(2t, j)$ and $F(2t, j)$ is increasing in $j \in (-\infty, \bar j)$ by Lemma 2.1 (5). Thus we have either

$$v_j(t) \le \gamma, \forall t \in [0, t_\epsilon), j \in \mathbb{Z},$$

or

$$v_j(t) \le \gamma, \forall t \ge t_\epsilon, j < -\bar c t + 2\,m.$$

Let $\Omega = \left\{(t, j) \in \mathbb{R} \times \mathbb{Z} | t \in [0, t_\epsilon) \times \mathbb{Z} \cup [t_\epsilon, \infty) \times (-\infty, -\bar c t + 2\,m)\right\}$.

By spreading properties in Lemma 4.2, for any given $0 < \gamma < \min\{u^*_j\}$, there exists a t_ϵ (if necessary, replace the original t_ϵ with the larger number) such that $u_{\bar j}(t) \ge \gamma$ and thus $v_{\bar j}(t) \le \gamma \le u_{\bar j}(t)$. Moreover, for any given ϵ, there exists a γ such that $f_j(\gamma) - l > 0$ and thus v is a sub-solution of (1.2) on Ω. Indeed,

$$\begin{aligned}
&\dot v_j(t) - [v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) + f(v_j(t))v_j(t)]\\
&= \dot v_j(t) - [v_{j+1}(t) - 2v_j(t) + v_{j-1}(t) + (l-\epsilon)v_j(t)] + (-f(v_j(t)) + l - \epsilon)v_j(t)\\
&= (-f(v_j(t)) + l - \epsilon)v_j(t)\\
&\le (-f(\gamma) + l - \epsilon)v_j(t)\\
&\le 0.
\end{aligned}$$

Since $v_j(0) \le u_j(0)$ for $j \in \mathbb{Z}$, by **comparison principle** (Lemma 2.1 in [15]), $v_j(t) \le u_j(t)$ on $\bar\Omega$. Note that $t \ge 0$ and $j \le j_0 + m - c^* t$ is a subset of $\bar\Omega$ and this completes the proof. □

Let $(\lambda_M, \phi^{(M)})$ be the principal eigenvalue and eigenvector to (3.2) with $\|\phi^{(M)}\|_\infty = 1$.

Lemma 4.4. *For every $\epsilon \in (0,1)$, there exists a $K_\epsilon > 1$ such that*

$$u_j(t) \leq K_\epsilon u_0(0)\sqrt{|t|}e^{\hat{\mu}j+(\lambda_M-\epsilon)t},$$

for all $t \leq -1$ and $j \in [M, -c_\epsilon t]$, with $c_\epsilon = \frac{\lambda_M-\epsilon}{\hat{\mu}}$.

Proof. Suppose the lemma to be false. Then there exist $\bar{t} \leq -1$ and $j_0 \in [M, -c_\epsilon \bar{t}]$ such that

$$u_{j_0}(\bar{t}) \geq K_\epsilon u_0(0)\sqrt{|\bar{t}|}e^{\hat{\mu}(j_0+c_\epsilon \bar{t})}. \tag{4.2}$$

Let $\tilde{u} = \delta\phi^{(M)}$ be a sub-solution of (1.2) as in Lemma 3.6, where $\phi^{(M)}$ is the principal eigenvector to (3.2) with $\|\phi^{(M)}\|_\infty = 1$. Let $\underline{v}_j$ be given by

$$\underline{v}_j = \min\{\delta, Ae^{(\lambda_M-\epsilon)t}\}\phi_j^{(M)} \tag{4.3}$$

Thus there exists a δ such that $\underline{v}$ is also a sub-solution of (1.2) for any $A > 0$. Indeed, if $\tilde{v}_j := Ae^{(\lambda_M-\epsilon)t}\phi_j^{(M)} < \delta\phi_j^{(M)}$, choosing a δ such that $f_j(0) - f_j(\delta) < \epsilon$, we have

$$\begin{aligned}
&(\tilde{v}_j)_t - [\tilde{v}_{j+1} - 2\tilde{v}_j + \tilde{v}_{j-1} + f_j(\tilde{v}_j)\tilde{v}_j] \\
=&(\tilde{v}_j)_t - [\tilde{v}_{j+1} - 2\tilde{v}_j + \tilde{u}_{j-1} + a_j\tilde{v}_j] + (f_j(0) - f_j(\tilde{v}_j))\tilde{v}_j \\
=&(\lambda_M - \epsilon - \lambda_M + (f_j(0) - f_j(\tilde{v}_j)))\tilde{v}_j \\
=&(-\epsilon + (f_j(0) - f_j(\tilde{v}_j)))\tilde{v}_j \\
\leq&(-\epsilon + (f_j(0) - f_j(\delta)))\tilde{v}_j \\
\leq&0.
\end{aligned}$$

For $\tilde{v} \geq \delta\phi^{(M)}$, the above inequality holds for $\underline{v}_j = \tilde{u}_j$ by the calculation in Lemma 3.6.

With a possible translation, we assume that $u_0(0) < \tilde{u}_0$ and $f_j(0) > 1$ for $j = 0$. Let β be chosen later such that $0 < \beta < 1$. For $-M \leq j \leq M$, let $z_1 = 2\beta|\bar{t}|/|j-j_0|$ and $z = 2\beta|\bar{t}|/j_0$, we have $z_1 > z$ and then by the monotonicity of $\frac{\varsigma(z)}{z}$ in Lemma 2.1, we have $\frac{\varsigma(z_1)}{z_1} > \frac{\varsigma(z)}{z}$. By **Heat Kernel Estimate** (2.2) $h_t^{\mathbb{Z}}(j) \asymp F(t,j)$, there exist positive C_1 and C_2 such that $C_1F(t,j) \leq h_t^{\mathbb{Z}}(j) \leq C_2F(t,j)$. Therefore, together with (4.2) and Lemma 4.3, we have

$$\begin{aligned}
u_j(\bar{t}+\beta|\bar{t}|) &\geq C_1e^{(1-\epsilon)\beta|\bar{t}|}\sum_k F(2\beta|\bar{t}|, j-k)u_k(\bar{t}) \\
&\geq C_1e^{(1-\epsilon)\beta|\bar{t}|}F(2\beta|\bar{t}|, j-j_0)u_{j_0}(\bar{t}) \\
&\geq C_1e^{(1-\epsilon)\beta|\bar{t}|}F(2\beta|\bar{t}|, j-j_0)K_\epsilon u_0(0)\sqrt{|\bar{t}|}e^{\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}} \\
&= C_1e^{(1-\epsilon)\beta|t|}\frac{1}{\sqrt{2\pi}}\frac{exp[-2\beta|\bar{t}| + |j-j_0|\varsigma(2\beta|\bar{t}|/|j-j_0|)]}{(1+4\beta^2\bar{t}^2+(j-j_0)^2)^{\frac{1}{4}}}K_\epsilon u_0(0)\sqrt{|\bar{t}|}e^{\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}} \\
&= C_1K_\epsilon u_0(0)\frac{1}{\sqrt{2\pi}}\frac{exp[(-2+2\frac{\varsigma(z_1)}{z_1})\beta|\bar{t}|]}{(1+4\beta^2\bar{t}^2+(j-j_0)^2)^{\frac{1}{4}}}\sqrt{|\bar{t}|}e^{(1-\epsilon)\beta|t|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}} \\
&\geq C_1K_\epsilon u_0(0)\frac{1}{\sqrt{2\pi}}\frac{exp[(-2+2\frac{\varsigma(z)}{z})\beta|\bar{t}|]}{(1+4\beta^2\bar{t}^2+j_0^2)^{\frac{1}{4}}}\sqrt{|\bar{t}|}e^{(1-\epsilon)\beta|t|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}} \\
&\geq C_1K_\epsilon u_0(0)\frac{1}{\sqrt{2\pi}}\frac{exp[(-2+2\frac{\varsigma(z)}{z})\beta|\bar{t}|]}{(1+4\bar{t}^2+(c_\epsilon\bar{t})^2)^{\frac{1}{4}}}\sqrt{|\bar{t}|}e^{(1-\epsilon)\beta|t|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}} \\
&\geq C_1K_\epsilon u_0(0)\frac{1}{\sqrt{2\pi}}\frac{exp[(-2+2\frac{\varsigma(z)}{z})\beta|\bar{t}|]}{(5+c_\epsilon^2)^{\frac{1}{4}}}e^{(1-\epsilon)\beta|t|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}}
\end{aligned}$$

$$\geq K'_\epsilon u_0(0)exp\{(-2+2\frac{\varsigma(z)}{z}+(1-\epsilon))\beta|\bar{t}|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}\}$$
$$:= K''_\epsilon,$$

where $K'_\epsilon = C_1 K_\epsilon \frac{1}{\sqrt{2\pi}(5+c_\epsilon^2)^{\frac{1}{4}}}$. Thus, with **Comparison Principle**, choosing $A = K''_\epsilon$ in (4.3), we have

$$u_j(t+\bar{t}+\beta|\bar{t}|) \geq \underline{v}_j(t)$$

for $t > 0$. In particular, letting $t = (1-\beta)|\bar{t}|$, we have

$$u_j(0) \geq \min\{\delta, K''_\epsilon e^{(\lambda_M-\epsilon)(1-\beta)|\bar{t}|}\}\phi_j^{(M)}.$$

By choosing β such that $K''_\epsilon e^{(\lambda_M-\epsilon)(1-\beta)|\bar{t}|} = K'_\epsilon u_0(0)$, that is,

$$exp\{(-2+2\frac{\varsigma(z)}{z}+(1-\epsilon))\beta|\bar{t}|+\hat{\mu}j_0+(\lambda_M-\epsilon)\bar{t}\} \times e^{(\lambda_M-\epsilon)(1-\beta)|\bar{t}|} = 1.$$

Therefore

$$(-2+2\frac{\varsigma(z)}{z}+(1-\epsilon)-(\lambda_M-\epsilon))\beta|\bar{t}|+\hat{\mu}j_0 = 0.$$

Recalling that $z = 2\beta|\bar{t}|/j_0$, $(-2+2\frac{\varsigma(z)}{z}+(1-\epsilon)-(\lambda_M-\epsilon))\beta|\bar{t}|+2\hat{\mu}\beta|\bar{t}|/z = 0$, that is,

$$-1+2\frac{\varsigma(z)+\hat{\mu}}{z}-(\lambda_M-\epsilon)-\epsilon = 0.$$

Thus, let $g(z)$ be as in Lemma 2.1 and we have

$$g(z) = \lambda_M < \lambda = \lambda(\hat{\mu}).$$

By the concavity of $g(z)$ in Lemma 2.1 (2), and $M \gg 1$, there exists at least one $\bar{z} < z_0 = csch(\hat{\mu})$ such that $g(\bar{z}) = 0$. Recall that $\hat{c} < \frac{2}{z_0}$ for $\hat{u} > \mu^*$ by Lemma 2.1 (4). With $c_\epsilon < \hat{c}$, $j_0 \leq c_\epsilon|\bar{t}|$ and $\bar{z} = 2\beta|\bar{t}|/j_0$, we have

$$\beta = \frac{\bar{z}j_0}{2|\bar{t}|} \leq \frac{z_0 j_0}{2|\bar{t}|} \leq \frac{z_0 c_\epsilon}{2} \leq \frac{z_0\hat{c}}{2} < 1.$$

Thus, $\beta < 1$ as required. Finally, by taking K_ϵ large enough and $j = 0$,

$$u_0(0) \geq \min\{\delta, K'_\epsilon u_0(0)\}\phi_0^{(M)} \geq \min\{\delta\phi_0^{(M)}, 2u_0(0)\},$$

which causes a contradiction. This completes the proof. □

Remark 4.1. Lemma 4.4 holds for fixed j and so for any j on a compact set without the assumption $\lambda > \lambda^*$. Indeed, in this case, we can remove the restriction of $c_\epsilon = \frac{\lambda_M-\epsilon}{\hat{\mu}}$ and freely choose $c_\epsilon < \frac{2}{z_0}$ in the lemma.

Lemma 4.5. *Assume that $c, c_1 \in (c^*, \hat{c})$ with $c < c_1$, there exists a $K_0 > 0$ and $\tau > 0$ such that*

$$u_j(t) \leq K_0 u_0(0) e^{\mu^*(j+ct)},$$

for all $(t,j) \in (-\infty,-1) \times [M, -c_1 t]$ as well as $(t,j) \in (-\infty,-t_0) \times [M,\infty)$.

Proof. Pick $\epsilon > 0$ such that $c_\epsilon = c_1$. Note that $\lambda(\hat{\mu}) = \lambda > \lambda^* = \lambda(\mu^*)$ implies that $\hat{\mu} > \mu^*$. With Lemma 4.4, there is $K_\epsilon > 0$ such that for all $t \leq -1$ and $j \in [M, -c_\epsilon t]$,

$$u_j(t) \leq K_\epsilon u_0(0)\sqrt{|t|}e^{\hat{\mu}(j+c_1 t)} \leq K_\epsilon u_0(0)\sqrt{|t|}e^{\mu^*(j+c_1 t)} \leq K_0 u_0(0) e^{\mu^*(j+ct)},$$

where $K_0 = \max_{t\leq -1} K_\epsilon\sqrt{|t|}e^{\mu^*(c_1-c)t}$. We can let $t_0 \equiv \dfrac{ln(K_0 u_0(0)) - ln(\max_j u_j^*)}{\mu^*(c_1-c)} > 0$ to complete the proof of the second part. Indeed, for $t < -t_0$, we have $K_0 u_0(0)e^{\mu^*(j+ct)} \geq u_j^*$ for all $j > -c_1 t$. Since $u_j(t) \leq u_j^*$ for all $(t,j) \in \mathbb{R}\times\mathbb{Z}$, the inequality holds for all $(t,j) \in (-\infty,-t_0)\times[M,\infty)$. □

Proof of Proposition 4.3. Given $c, c_1 \in (c^*, \hat{c})$ with $c < c_1$. Let $\tau_1 \equiv M/c_1$ and so $M \leq -c_1 t$ for $t \leq -\tau_1$. By Lemma 4.5, for $t \leq -\tau_1$,

$$u_M(t) \leq K_0 u_0(0) e^{\mu^*(M+ct)}.$$

Next, for $t \leq -\tau_0$, we let

$$v_j(t;t_0) \equiv K_0 u_0(0)[e^{\mu^*(j+ct_0+c^*(t-t_0))} + e^{\mu^*(2M-j+ct)}].$$

Then $v_j(t;t_0)$ is a super-solution on $(t_0,\infty)\times(M,\infty)$. Moreover, for $t \leq -\tau_0$ and $j > M$, we have

$$u_j(t_0) \leq K_0 u_0(0) e^{\mu^*(j+ct_0)} \leq v_j(t_0;t_0).$$

Since $c > c^*$, we have $u_M(t) \leq v_M(t;t_0)$ for all $t \in (t_0, -\tau_1)$. By comparison principle, $u_j(t) \leq v_j(t;t_0)$ for all $t \in [t_0, -\tau_1]$ and $j \geq M$. Letting $t_0 \to -\infty$, we have for all $t \leq -\tau_1$ and $j \geq M$,

$$u_j(t) \leq K_0 u_0(0) e^{\mu^*(2M-j+ct)}.$$

Similarly, we have for all $t \leq -\tau_1$ and $j \leq -M$,

$$u_j(t) \leq K_0 u_0(0) e^{\mu^*(2M+j+ct)}.$$

Thus, for all $t \leq -\tau_1$ and $j \leq \mathbb{Z} \setminus (-M, M)$,

$$u_j(t) \leq K_0 e^{2\mu^* M} u_0(0) e^{-\mu^*(|j|-ct)}.$$

The Harnack inequality extends this bound to all $t \leq -\tau_1 - 1$ and $j \in \mathbb{Z}$:

$$u_j(t) \leq K_1 u_0(0) e^{-\mu^*(|j|-ct)}.$$

Thus,

$$u_j(t) \leq K_1 u_0(0) e^{-\mu^*(|j|-c(-\tau_1-1))+(1+\|a\|_\infty)(t-(-\tau_1-1))},$$

for $t \geq -\tau_1 - 1$, where $\|a\|_\infty = \max_j a_j$. We note that the right-hand side is a super-solution. Thus, for $t \leq 0$ and $j \in \mathbb{Z}$, we have

$$u_j(t) \leq K_2 u_0(0) e^{-\mu^*(|j|-ct)}.$$

Finally, we have the non-existence of transition fronts if $\lambda > \lambda^*$, because $\lim_{j\to-\infty} u_j(t) = 0$ under the above inequality. □

4.3. The upper bound of wave speeds $\hat{c}$

Finally, we shall show $\hat{c}$ is the upper bound of the speeds (maximal wave speed) by investigating the nonexistence of transition fronts to (1.2) for $c > \hat{c}$, which is corresponding to $\mu \in (0, \hat{\mu})$ where no valid positive eigenvectors of (1.7) can be located.

Lemma 4.6. *For all $\epsilon > 0$, there exists $K_\epsilon > 0$ such that*

$$u_j(t) \leq K_\epsilon e^{\lambda(\hat{\mu}-\epsilon)t-(\hat{\mu}-\epsilon)j}, \text{ for all } j \geq 0 \text{ and } t \leq 0. \tag{4.4}$$

Proof. First, there exist M and $\bar{\epsilon}$ such that $\lambda(\hat{\mu}-\epsilon) = \lambda_M - \bar{\epsilon}$. By Remark 4.1 of Lemma 4.4, we have that (4.4) holds for fixed j and thus also for j in a bounded set $[0, M_0]$ of $\mathbb{Z}^+$, where $M_0 > N$. Second, we show that (4.4) holds for $j > M_0$. To this end, we claim that

$$u_j(t) \leq Ce^t \text{ for all } j \geq M_0 \text{ and } t \leq 0. \tag{4.5}$$

Let $\rho(t) = \sum_{j\geq M_0} u_j(t)$, which is well-defined due to Proposition 3.2. Then

$$\begin{aligned}\dot\rho(t) &= \sum_{j\geq M_0} \dot u_j(t)\\ &= \sum_{j\geq M_0}(u_{j+1} - 2u_j + u_{j-1} + f_j(u_j)u_j(t))\\ &= u_{M_0-1}(t) - u_{M_0}(t) + \sum_{j\geq M_0}(f_j(u_j)u_j(t))\end{aligned}$$

Therefore,

$$\dot\rho(t) - \rho(t) = u_{M_0-1}(t) - u_{M_0}(t) + \sum_{j\geq M_0}(f_j(u_j)-1)u_j(t).$$

For $j > M_0$ and $t \ll -1$, $f_j(u_j) - 1 = f_j(u_j) - f_j(0) < 0$ and thus

$$\sum_{j\geq M_0}(f_j(u_j)-1)u_j(t) < 0.$$

Then,

$$\begin{aligned}\frac{d}{dt}(-e^{-t}\rho(t)) &= -e^{-t}(\dot\rho(t) - \rho(t))\\ &= -e^{-t}(u_{M_0-1}(t) - u_{M_0}(t) + \sum_{j\geq M_0}(f_j(u_j)-1)u_j(t))\\ &\leq -e^{-t}(u_{M_0-1}(t) - u_{M_0}(t))\\ &\leq e^{-t}(u_{M_0}(t) + u_{M_0-1}(t))\\ &\leq e^{-t}K_\epsilon(1+e^{(\hat\mu-\epsilon)})e^{\lambda t-(\hat\mu-\epsilon)M_0}\\ &= K_\epsilon(1+e^{(\hat\mu-\epsilon)})e^{(\lambda-1)t-(\hat\mu-\epsilon)M_0}.\end{aligned}$$

Integrate both sides from $t(\leq 0)$ to 0, and we have $e^{-t}\rho(t) - \rho(0) \leq \frac{K_\epsilon(1+e^{(\hat\mu-\epsilon)})}{(\lambda-1)}e^{-(\hat\mu-\epsilon)M_0}$. Let $C = \rho(0) + \frac{K_\epsilon(1+e^{(\hat\mu-\epsilon)})}{(\lambda-1)}e^{-(\hat\mu-\epsilon)M_0}$. For $t \leq 0$, $\rho(t) \leq Ce^t$. Therefore, (4.5) holds for $j \geq M_0$. We let

$$w_j(t) = e^{-t}u_j(t) - K_\epsilon e^{(\lambda(\hat\mu-\epsilon)-1)t-(\hat\mu-\epsilon)(j-M-1)}. \tag{4.6}$$

Then, for $j > N$, we have

$$\begin{aligned}\dot w_j(t) &= e^{-t}\dot u_j(t) - e^{-t}u_j(t) - (\lambda(\hat\mu-\epsilon)-1)K_\epsilon e^{(\lambda(\hat\mu-\epsilon)-1)t-(\hat\mu-\epsilon)(j-M-1)}\\ &= e^{-t}(u_{j+1} - 2u_j + u_{j-1} + (f_j(u_j)-1)u_j(t))\\ &\qquad - (\lambda(\hat\mu-\epsilon)-1)K_\epsilon e^{(\lambda(\hat\mu-\epsilon)-1)t-(\hat\mu-\epsilon)(j-M-1)}\\ &= e^{-t}(u_{j+1} - 2u_j + u_{j-1}) - (\lambda(\hat\mu-\epsilon)-1)K_\epsilon e^{(\lambda(\hat\mu-\epsilon)-1)t-(\hat\mu-\epsilon)(j-M-1)}\\ &\qquad + e^{-t}(f_j(u_j)-1)u_j(t).\end{aligned}$$

On the other hand, we have

$$w_{j+1} - 2w_j + w_{j-1} = e^{-t}(u_{j+1} - 2u_j + u_{j-1}) - (\lambda(\hat\mu-\epsilon)-1)K_\epsilon e^{(\lambda(\hat\mu-\epsilon)-1)t-(\hat\mu-\epsilon)(j-M-1)}$$

Thus,

$$\begin{aligned}&\dot w_j(t) - (w_{j+1} - 2w_j + w_{j-1})\\ &\quad = e^{-t}(f_j(u_j)-1)u_j(t)\\ &\quad \leq 0.\end{aligned} \tag{4.7}$$

Note that $w_{M_0}(t) \leq 0$ for $t \leq 0$ because (4.4) holds for fixed $j = M_0$ that has been proved previously. By (4.5), $w_j(t) < e^{-t}u_j(t) < C$ for all $j \geq 0$ and $t \leq 0$. For $t \leq 0$ and $j \geq M_0$, choose ϵ such that $\lambda(\hat{\mu} - \epsilon) > 1$, then $w_j(t)$ is bounded. Furthermore we claim that for $j > M_0$ and $t \leq 0$, $w_j(t)$ cannot attain a positive maximum, and there cannot be a sequence (t_n, j_n) such that $w_{j_n}(t_n)$ tends to a positive supremum. Suppose that it obtains a positive maximum at (t_0, j_0) and for $M_0 \leq j < j_0$, $w_j(t_0) < w_{j_0}(t_0)$. Then $\dot{w}_{j_0}(t_0) - (w_{j_0+1} - 2w_{j_0} + w_{j_0-1}) > 0$, which contradicts (4.7). Suppose that there is a sequence (t_n, j_n) such that $w_{j_n}(t_n)$ tends to a positive supremum for $j > M_0$. Then $j_n \to \infty$ as $n \to \infty$, otherwise j_n goes to some fixed $\hat{j}$ as $n \to \infty$ that contradicts with (4.4) holds for fixed $\hat{j}$. And then $t \to -\infty$ as $n \to \infty$. Otherwise, $t \to -T$ as $n \to \infty$ for some $T > 0$. With (4.6), we have $\lim_{n\to\infty} w_{j_n}(T) = 0$. Therefore, $w_j(t) \leq 0$ for all $j \geq M_0$ and $t \leq 0$, which implies that (4.4) holds for all $j \geq M_0$ and $t \leq 0$. This completes the proof. □

Proof of Proposition 4.1. First, we proved the case with $\lambda > \lambda^*$ by Proposition 4.3. Next, it has been shown in Proposition 4.2 that there are no transition fronts for $c < c^*$ due to the properties of spreading speeds. Finally we prove the nonexistence for $c > \hat{c}$. Assume there exists a transition front $u_j(t)$ for $c > \hat{c}$. Let μ and $\hat{\mu}$ be such that $c = \frac{\lambda(\mu)}{\mu}$ and $\hat{c} = \frac{\lambda(\hat{\mu})}{\hat{\mu}}$. Then we have that $0 < \mu < \hat{\mu} < \mu^*$. By Proposition 3.3, we have

$$u_j(t) \geq K_\epsilon e^{-(\mu+\epsilon)(j-ct)}, \text{ for all } j \geq ct + M_0 \text{ and } \ t \geq 0. \tag{4.8}$$

By Lemma 4.6, in particular for $t = 0$, we have that

$$u_j(0) \leq K_\epsilon e^{-(\hat{\mu}-\epsilon)j}, \text{ for all } j \geq 0. \tag{4.9}$$

Consider the linear periodic equation restricted on $[M_0, M_0 + p]$ for $p \gg 1$ (i.e. p is as large as required), that is, $v_{j+p} = v_j$ for any $j \in \mathbb{Z}$.

$$\dot{v}_j = v_{j+1} - v_j + v_{j-1}, \quad M_0 \leq j \leq M_0 + p. \tag{4.10}$$

Let $\bar{v}_j(t) = K_\epsilon e^{-(\hat{\mu}-\epsilon)(j-ct)}$ for $j \in [M_0, M_0 + p)$ and $\underline{v}_j(t) = u_j(t)$ for $j \in [M_0, M_0 + p)$. Then by (4.9) $\underline{v}_j(0) \leq \bar{v}_j(0)$.

By direct calculation, we have

$$\begin{aligned}
&(\bar{v}_j)_t - [\bar{v}_{j+1} - \bar{v}_j + \bar{v}_{j-1}] \\
&= \bar{v}_j((\hat{\mu} - \epsilon)c - (e^{(\hat{\mu}-\epsilon)} - 1 + e^{-(\hat{\mu}-\epsilon)})) \\
&= \bar{v}_j(\hat{\mu} - \epsilon)\big(c - \frac{(e^{(\hat{\mu}-\epsilon)} - 1 + e^{-(\hat{\mu}-\epsilon)})}{(\hat{\mu} - \epsilon)}\big) \\
&= \bar{v}_j(\hat{\mu} - \epsilon)\big(c - c(\hat{\mu} - \epsilon)\big) \\
&\geq 0.
\end{aligned}$$

Choose $\epsilon = (\hat{\mu} - \mu)/3$ and we have $c > c(\hat{\mu} - \epsilon) = \frac{(e^{(\hat{\mu}-\epsilon)} - 1 + e^{-(\hat{\mu}-\epsilon)})}{(\hat{\mu}-\epsilon)}$. Thus, $\bar{v}$ is a super-solution of (4.10). In addition, for $j \geq M_0$, we have

$$\dot{u}_j - (u_{j+1} - 2u_j + u_{j-1}) = (f_j(u_j) - f_j(0))u_j \leq 0.$$

Thus, $\underline{v}$ is a sub-solution of (4.10). By the comparison principle and letting $p \to \infty$, for $j > M_0 + ct$, we have that

$$u_j(t) \leq K_\epsilon e^{-(\hat{\mu}-\epsilon)(j-ct)}. \tag{4.11}$$

However, this contradicts with (4.8) by choosing $\epsilon = (\hat{\mu} - \mu)/3$. □

Proof of Theorem 1.1.

(1) Existence of transition fronts and asymptotic behaviors (1.8) have been shown in Propositions 3.1–3.3.
(2) Non-existence of transition fronts follows by Proposition 4.1. □

5. Example

In this section, we provide an example for localized perturbations in homogeneous media of (1.2) with $f_j(u_j) = a_j(1-u_j)$ and $a_j = 1$ for $j \neq 0$. Thus (1.2) becomes the following,

$$\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + a_j u_j(1-u_j), \quad j \in \mathbb{Z}, \tag{5.1}$$

with $a_j = 1$ for $j \neq 0$. It is easy to see that $u_j^* = 1$. The corresponding linearized equation is given by

$$\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + a_j u_j, \quad j \in \mathbb{Z}. \tag{5.2}$$

The eigenvalue problem is given by

$$\lambda(\mu)u_j = e^{\mu}u_{j+1} - 2u_j + e^{-\mu}u_{j-1} + a_j u_j, \quad j \in \mathbb{Z}. \tag{5.3}$$

For homogeneous case, all a_js are ones. By observation, $\lambda(\mu) = e^{\mu} - 1 + e^{-\mu}$ with constant eigenvector 1. It is easy to verify that $u_j(t) = e^{-\mu(j-ct)}$ is a solution of (5.2) with $c = \frac{\lambda(\mu)}{\mu}$. Next we investigate the existence of the positive eigenvectors of (5.3) for the localized perturbation case $a_0 \neq 1$. We assume that one solution to localized perturbation case coincides with homogeneous case at the right with $u_j(t) = e^{-\mu(j-ct)}$ for $j \geq 0$. From (5.2), we have

$$u_{j-1} = \dot{u}_j + (2-a_j)u_j - u_{j+1}, \quad j \in \mathbb{Z}. \tag{5.4}$$

Thus, by induction, for $j < 0$,

$$\begin{aligned}
u_{-1} &= \dot{u}_0 + (2-a_0)u_0 - u_1 = (1 + (1-a_0)e^{-\mu})e^{-\mu(-1-ct)}, \\
u_{-2} &= \dot{u}_{-1} + u_{-1} - u_0 = (1 + (1-a_0)e^{-\mu} + (1-a_0)e^{-3\mu})e^{-\mu(-2-ct)}, \\
u_{-3} &= \dot{u}_{-2} + u_{-2} - u_{-1} = (1 + (1-a_0)(e^{-\mu} + e^{-3\mu} + e^{-5\mu}))e^{-\mu(-3-ct)}, \\
&\vdots \\
u_j &= (1 + (1-a_0)e^{-\mu}\frac{1-e^{2\mu j}}{1-e^{-2\mu}})e^{-\mu(j-ct)}.
\end{aligned}$$

Therefore, the eigenvector to (5.3) is given by $\phi_j = (1 + (1-a_0)e^{-\mu}\frac{1-e^{2\mu j}}{1-e^{-2\mu}})e^{-\mu j}$ for $j < 0$ and $\phi_j = e^{-\mu j}$ for $j \geq 0$. Note that if $a_0 \leq 1$, $\phi_j > 0$ for all $j \in \mathbb{Z}$. That means the positive eigenvectors always exist for $a_0 \leq 1$ and so do the transition fronts of speed c no less than c^*. The minimal speed c^* is given by $c^* = \dfrac{e^{\mu^*} + e^{-\mu^*} - 1}{\mu^*} = \inf\limits_{\mu>0} \dfrac{e^{\mu} + e^{-\mu} - 1}{\mu} \approx 2.073$ at $\mu^* \approx 0.9071$.

For $a_0 > 1$, $\phi_j > 0$ for all $j \in \mathbb{Z}$ whenever $a_0 \leq e^{\mu} - e^{-\mu} + 1$, which implies that $\mu \geq ln[\frac{-(1-a_0)+\sqrt{(1-a_0)^2+4}}{2}]$ that gives the $\hat{\mu} = ln[\frac{-(1-a_0)+\sqrt{(1-a_0)^2+4}}{2}]$. By $\lambda = \lambda(\hat{\mu})$, we have $\lambda = \sqrt{(1-a_0)^2+4} - 1$. On the interval $[\hat{\mu}, \mu^*]$ whenever $\hat{\mu} \leq \mu^*$, the speed is well defined by $c = \frac{\lambda(\mu)}{\mu} = \frac{e^{\mu}+e^{-\mu}-1}{\mu}$ and we can have both a minimal and a maximal speed on this closed interval. Outside this interval for $\mu < \hat{\mu}$, since the components of the eigenvector are mixed with negative and positive signs, we fail to obtain the transition fronts. If $\hat{\mu} > \mu^*$, that is, $a_0 > e^{\mu^*} - e^{-\mu^*} + 1$, then the existence interval $[\hat{\mu}, \mu^*]$ will be empty.

In summary, we have the following facts,

(1) If $a_0 \leq 1$, then $\lambda = 1$. In this case, $\hat{c} = \infty$, that is, the existence interval of speeds for transition fronts is $[c^*, \infty)$.

(2) If $a_0 > 1$, then $\lambda > 1$. If $\lambda < \lambda^*$, then transition front exists for any speed $c \in [c^*, \hat{c}]$. However, transition fronts do not exist under three cases: $c < c^*$, $c > \hat{c}$ and $\lambda > \lambda^*$. The c^*, $\hat{c}$, λ and λ^* are given by as follows:

[i] The minimal wave speed c^* is given by

$$c^* = \frac{e^{\mu^*} + e^{-\mu^*} - 1}{\mu^*} = \inf_{\mu>0} \frac{e^{\mu} + e^{-\mu} - 1}{\mu} \approx 2.073 \text{ at } \mu^* \approx 0.9071.$$

[ii] The maximal wave speed $\hat{c}$ is given by

$$\hat{c} = \frac{e^{\hat{\mu}} + e^{-\hat{\mu}} - 1}{\hat{\mu}} \text{ at } \hat{\mu} = ln[\frac{-(1-a_0) + \sqrt{(1-a_0)^2 + 4}}{2}].$$

Note that $a_0 > 1$, $\hat{c}$ depending on a_0 is finite, that is, $\hat{c} < \infty$.

[iii] $\lambda = \sqrt{(1-a_0)^2 + 4} - 1$ and $\lambda^* = e^{\mu^*} + e^{-\mu^*} - 1 \approx 1.8808$. No transition fronts exist for $\lambda > \lambda^*$. Under this case, we must have $a_0 > e^{\mu^*} - e^{-\mu^*} + 1 \approx 3.073$.

6. Concluding remarks

We studied the existence and non-existence of transition fronts for monostable lattice differential equations in locally spatially inhomogeneous patchy environments. We collected fundamental tools such as discrete heat kernel estimates and discrete parabolic Harnack inequality. We proved that Poincaré inequality holds on a 2-regular graph and so does a discrete parabolic Harnack inequality. Under the assumptions (H1)–(H2), there is a positive principal eigenvector for $\lambda^* > \lambda(\mu) > \lambda$. This positive principal eigenvector is the main ingredient in constructions of super/sub-solutions. The right end (i.e. $j > N$) of positive principal eigenvector is one, that coincides with the principal eigenvector in homogeneous media. There are significant differences on the middle localized perturbation part (i.e, $j \in [-N, N]$) and the left end (i.e. $j < -N$). However, this impact declines to 0 for $j < -N$ as $|j| \to \infty$. With comparison principles and the super/sub-solutions, we obtained the transition fronts of mean wave speed on a finite range $(c^*, \hat{c}]$ and then pass the limit to have the case of $c = c^*$. For $c \in (c^*, \hat{c}]$, the profiles of transition fronts are highly related to the graphs of super-solutions $e^{-\mu(j-ct)}\phi_j^{\mu}$. Note that, in the right end for $j > N$, we have $e^{-\mu(j-ct)}$ that is moving at the exact speed c. For $j < N$, the profiles will change with amplitude ϕ_j^{μ}. If $c \in (c^*, \hat{c}]$ and $j \ll -N$, $\lim_{j\to-\infty} \phi_j^{\mu} = l > 0$, that means they are essentially constant profiles $le^{-\mu(j-ct)}$.

We proved transition fronts if they exist must possess the exponential tail properties. There are no transition fronts at all if $\lambda > \lambda^*$, the mean wave speed is slower than the minimal speed c^*, or faster than the maximal wave speed $\hat{c}$. The strongly localized spatial inhomogeneous patchy environments prevent the existence of transition fronts. The proof of minimal wave speed c^* follows from the work of Shen and Kong [23,24], where they also studied the localized perturbation with periodic media for both nonlocal problem and lattice differential equations. The proof of maximal wave speed $\hat{c}$ relies heavily on discrete heat kernel estimates, comparison principles and discrete parabolic Harnack inequality. We leave the uniqueness and stability of transition fronts to (1.2) and transition fronts to lattice differential equations with the localized perturbation of periodic media for future study.

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References

[1] R. Fisher, The wave of advance of advantageous genes, Ann. Eugen. 7 (1937) 335–369.
[2] A. Kolmogorov, I. Petrowsky, N. Piscunov, A Study of the Equation of Diffusion with Increase in the Quantity of Matter, and Its Application To a Biological Problem, Vol. 1, Bjul. Moskovskogo Gos. Univ, 1937, pp. 1–26.
[3] W. Hudson, B. Zinner, Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media, World Sci. Publ., River Edge, NJ, 1995, pp. 187–199.
[4] H.F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, J. Math. Biol. 45 (2002) 511–548.
[5] H. Berestycki, F. Hamel, N. Nadirashvili, The speed of propagation for KPP type problems, I - Periodic framework, J. Eur. Math. Soc. 7 (2005) 172–213.
[6] H. Berestycki, F. Hamel, Generalized transition waves and their properties, Comm. Pure Appl. Math. 65 (2012) 592–648.
[7] W. Shen, Traveling waves in diffusive random media, J. Dynamics and Diff. Eqns. 16 (2004) 1011–1060.
[8] H. Berestycki, F. Hamel, H. Matano, Bistable traveling waves around an obstacle, Comm. Pure Appl. Math. 62 (2009) 729–788.
[9] G. Nadin, L. Rossi, Propagation phenomenas for time heterogeneous KPP reaction diffusion equations, J. Math. Pures Appl. 98 (6) (2012) 633–653.
[10] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, A. Zlatoš, Existence and non-existence of Fisher–KPP transition fronts, Arch. Ration. Mech. Anal. 203 (2012) 217–246.
[11] A. Zlatoš, Transition fronts in inhomogeneous Fisher–KPP reaction–diffusion equations, J. de Mathématiques Pures Et Appliquées 98 (1) (2012) 89–102.
[12] J.W. Cahn, J. Mallet-Paret, E.S. Van Vleck, Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice, SIAM J. Appl. Math. 59 (2) (1999) 455–493.
[13] X. Chen, J.-S. Guo, Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations, J. Differential Equations 184 (2002) 549–569.
[14] X. Chen, J.-S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics, Math. Ann. 326 (2003) 123–146.
[15] X.F. Chen, J.S. Guo, C.C. Wu, Traveling waves in discrete periodic media for bistable dynamics, Arch. Ration. Mech. Anal. 189 (2) (2008) 189–236.
[16] S.N. Chow, J. Mallet-Paret, W. Shen, Traveling waves in lattice dynamical systems, J. Differential Equations 149 (1998) 249–291.
[17] J.-S. Guo, C.-C. Wu, Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system, J. Differential Equations 246 (2009) 3818–3833.
[18] A. Hoffman, H.J. Hupkes, E.S. Van Vleck, Multi-dimensional stability of waves traveling through rectangular lattices in rational directions, Trans. Amer. Math. Soc. 367 (12) (2015) 8757–8808.
[19] A. Hoffman, H.J. Hupkes, E.S. Van Vleck, Entire solutions for bistable lattice differential equations with obstacles, Mem. Amer. Math. Soc. 250 (1188) (2017) v+119.
[20] J.-S. Guo, F. Hamel, Front propagation for discrete periodic monostable equations, Math. Ann. 335 (3) (2006) 489–525.
[21] W. Hudson, B. Zinner, Existence of traveling waves for a generalized discrete Fisher's equation, Commun. Appl. Nonlinear Anal. 1 (1994) 23–46.
[22] A.R. Humphries, B.E. Moore, E.S. Van Vleck, Front solutions for bistable differential-difference equations with inhomogeneous diffusion, SIAM J. Appl. Math. 71 (4) (2011) 1374–1400.
[23] L. Kong, W. Shen, Positive stationary solutions and spreading speeds of KPP equations in locally spatially inhomogeneous media, Methods Appl. Anal. 18 (4) (2011) 427–456.
[24] L. Kong, W. Shen, Liouville type property and spreading speeds of KPP equations in periodic media with localized spatial inhomogeneity, J. Dynam. Differential Equations 26 (1) (2014) 181–215.
[25] H. Berestycki, F. Hamel, Generalized traveling waves for reaction–diffusion equations. Perspectives in nonlinear partial differential equations. In honor of H. Brezis, Contemp. Math, Amer. Math. Soc. 446 (2007) 101–123.
[26] G. Teschl, Jacobi operators and completely integrable nonlinear lattices, in: Mathematical Surveys and Monographs, vol. 72, American Mathematical Society, Providence, R.I, 2000.
[27] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Uspehi Matem. Nauk (N. S.) (in Russian) 3 (1(23)) (1948) 1–95, MR 0027128;
M.G. Krein, M.A. Rutman (Eds.), English translation, in: Linear operators leaving invariant a cone in a Banach space, Vol, 6, Amer. Math. Soc. Transl., 1950, MR 0038008.
[28] R. Nussbaum, Eigenvectors of nonlinear positive operators and the linear krein–rutman theorem, in: Fixed Point Theory, 1981, pp. 309–330.
[29] A. Pazy, Semigroups of Linear Operators and Applications To Partial Differential Equations, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1983.
[30] B.V. Pal'tsev, On two-sided estimates, uniform with respect to the real argument and index, for modified bessel functions, Mat. Zametki 65 (5) (1999) 681–692.
[31] G. Chinta, J. Jorgenson, A. Karlsson, Heat kernels on regular graphs and generalized ihara zeta function formulas, Monatsh. Math. 178 (2) (2015) 171–190.
[32] M. Cowling, S. Meda, A.G. Setti, Estimates for functions of the Laplace operator on homogeneous trees, Trans. Amer. Math. Soc. 352 (2000) 4271–4293.
[33] J. Moser, A harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964) 101–134, Correction 20 (1967) 231-236.
[34] T. Delmotte, Parabolic harnack inequality and estimates of Markov chains on graphs, Rev. Mat. Iberoam. 15 (1999) 181–232.