Approximating Observables Is as Hard as Counting

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Abstract

We study the computational complexity of estimating local observables for Gibbs distributions. A simple combinatorial example is the average size of an independent set in a graph. A recent work of Galanis et al (2021) established NP-hardness of approximating the average size of an independent set utilizing hardness of the corresponding optimization problem and the related phase transition behavior. We instead consider settings where the underlying optimization problem is easily solvable. Our main contribution is to classify the complexity of approximating a wide class of observables via a generic reduction from approximate counting to the problem of estimating local observables. The key idea is to use the observables to interpolate the counting problem.

Using this new approach, we are able to study observables on bipartite graphs where the underlying optimization problem is easy but the counting problem is believed to be hard. The most-well studied class of graphs that was excluded from previous hardness results were bipartite graphs. We establish hardness for estimating the average size of the independent set in bipartite graphs of maximum degree 6; more generally, we show tight hardness results for general vertex-edge observables for antiferromagnetic 2-spin systems on bipartite graphs. Our techniques go beyond 2-spin systems, and for the ferromagnetic Potts model we establish hardness of approximating the number of monochromatic edges in the same region as known hardness of approximate counting results.

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1 Introduction

Can we efficiently estimate the average size of an independent set in an input graph $G = (V, E)$? Moreover, can we do so without utilizing a sampling algorithm for generating a random independent set?

In this paper, for a broad class of problems captured by Gibbs distributions, we address the relationship between the computational complexity of approximating local observables (such as estimating the average size of an independent set) and the computational complexity of approximating the partition function (such as estimating the total number of independent sets). It is a standard technique in the area to reduce estimating observables to approximate counting, by first implementing an approximate sampler and then using an unbiased estimator of the desired observable. The focus of this paper is the converse, where there is no previously

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known technique to answer the following question: does an algorithm for local observables yield an algorithm for the partition function? We prove, in a broad setting, that these two genres of problems are computationally equivalent.

Previous work of [9] only achieved this indirectly; they showed hardness of approximating local observables (in fact, only for a certain observable, called magnetization, see below for definitions) utilizing the hardness of MaxCut. Here, we show a direct reduction from the observable problem to the partition-function problem, relating therefore more crisply the two problems. This allows us to obtain hardness results in several new regimes (in particular,not covered by [9]) where the counting problem is hard but there is no underlying hard optimization problem.

An interesting setting to highlight the usefulness of our reduction is *bipartite* independent sets. In this example there is no corresponding hard optimization problem (as the maximum independent set problem is poly-time solvable in bipartite graphs), and hence to prove hardness we need to utilize hardness of approximate counting results. Another pertinent example for our results are attractive graphical models, these are equivalent to ferromagnetic spin systems in statistical physics. The simplest case is the ferromagnetic Ising model and its generalization known as the Potts model. In the Ising/Potts model on a graph (see Section 1.1 for more precise definitions), the configurations of the model are the collection of labellings σ of the vertices with q spins (colours), each weighted as $\beta^{m(\sigma)}$ where $m(\sigma)$ is the number of monochromatic edges and β is a parameter > 1 (so that labellings with many monochromatic edges are favored). Because of the attractiveness assumption that $\beta > 1$, once again, there is no corresponding hard optimization problem for this problem (contrast this with the case $\beta < 1$ where the largest weight labellings have the smallest number of monochromatic edges). Nevertheless, using our new reduction, we show that hardness of the associated approximate counting problem implies hardness of estimating the (weighted) average of the monochromatic edges in the Potts model.

Our two illustrative examples, the average size of an independent set and the number of monochromatic edges in the Ising/Potts model, are instances of a local observable in statistical physics; specifically they correspond to the magnetization and susceptibility, respectively. The behavior of observables is fundamental to the study of phase transitions, e.g., see [1, 4].

We begin giving more precise definitions for our initial example of *bipartite* independent sets, before considering the ferromagnetic Potts model, and finally generalizing to arbitrary local observables in general 2-spin systems. For a graph $G = (V, E)$ let $\mathcal{I}_G$ denote the set of independent sets (of all sizes) of G, and let $\mu := \mu_G$ denote the uniform distribution over $\mathcal{I}_G$. Denote the average independent set size by $\mathcal{M}(G) = \mathbf{E}_{\sigma\sim\mu}\big[|\sigma|\big]$; this corresponds to the magnetization in statistical physics (and hence the choice of notation $\mathcal{M}$). We say that an algorithm is an FPRAS for the average independent set size if given a graph $G = (V, E)$ and parameters $\epsilon, \delta > 0$, the algorithm outputs an estimate EST which is within a multiplicative factor $(1 \pm \epsilon)$ of the desired quantity $\mathcal{M}(G)$, with probability $\geq 1 - \delta$, and runs in time $\mathrm{poly}(|V|, 1/\epsilon, \log(1/\delta))$. One can also consider an FPRAS for estimating $|\Omega|$, the number of independent sets of the input graph G; we refer to this as an efficient approximate counting algorithm.

It is a classical result [14] that an efficient approximate counting algorithm is polynomial-time interreducible with an efficient algorithm for approximate sampling from μ. In turn, efficiently estimating the average independent set size of a graph G is easily reduced to approximate sampling from the uniform distribution μ_G. The challenging aspect, and the focus of this paper, is the *reverse* implication. Can we estimate the typical size of an independent set without utilizing an approximate sampling algorithm? We will show it is *not* possible, i.e., hardness of approximate counting (and hence approximate sampling) implies hardness of estimating the average independent set size.

For graphs of maximum degree 5, Weitz [20] presented an FPTAS for approximating the number of independent sets, which yields an efficient approximate sampling scheme; note an FPTAS is the deterministic analog of an FPRAS, i.e., it achieves $\delta = 0$. Very recently, Chen et al. [5] proved that the simple MCMC algorithm known as the Gibbs sampler (or Glauber dynamics) has $O(n \log n)$ mixing time for this same class of graphs of maximum degree 5. Hence, one immediately obtains an FPRAS for the average independent set size $\mathcal{M}(G)$.

On the other side, for graphs of maximum degree 6, Sly [18] proved that approximating the number of independent sets is NP-hard, by a reduction from max-cut. Schulman et al. [16] showed #P-hardness for exact computation of the average independent set size. Moreover, recent work of Galanis et al. [9] shows that approximating the average independent-set size is also NP-hard for graphs of maximum degree 6. The proof of [9] does not directly relate approximate counting and estimating the average independent set size; instead [9] also shows a (more sophisticated) reduction from max-cut and utilizes the associated gadgets used in Sly's inapproximability result [18].

This begets the question: are these problems still intractable when restricted to *bipartite graphs*? For bipartite graphs there is no longer a hard optimization problem, such as max-cut, that one can utilize as a starting point for a hardness reduction. However, approximately counting independent sets is considered to be intractable on bipartite graphs of maximum degree 6; in particular, it is is #BIS-hard [3] where #BIS refers to the problem of approximately counting independent sets on general bipartite graphs (with potentially unbounded degree). There are now a multitude of approximate counting problems which share the same #BIS-hardness status or are even #BIS-equivalent, e.g., see [6, 3, 11, 7].

We present a general approach for reducing approximate counting to approximating averages. This yields hardness for approximating the average independent-set size in *bipartite graphs* of maximum degree 6.

▶ **Theorem 1.** *Let $\Delta \geq 6$ be an integer. There is no* FPRAS *for the average independent-set size on* bipartite *graphs of maximum degree Δ unless* #BIS *admits an* FPRAS*.*

Note that the #BIS-hardness result of Theorem 1 gives a weaker guarantee than those shown in [9] where they obtain in some cases constant-factor inapproximability results (using the constant-factor NP-hardness of the optimization problem). This difference is inherent with the #BIS-hardness assumption, i.e., that there is no FPRAS for #BIS. Moreover, an algorithm which approximates #BIS within any poly(n)-factor implies an FPRAS, and obtaining constant-factor inapproximability results for magnetization on bipartite graphs would require (among other things) hardness of #BIS within an exponential-factor.

Our results extend to the hard-core model on weighted independent sets, and to general 2-spin antiferromagnetic models. These more general results are detailed in Section 1.2.

1.1 Ferromagnetic Potts Model

Ferromagnetic spin systems, which are equivalent to attractive undirected graphical models, are an interesting class of models to illustrate our new proof technique on. In ferromagnetic models there is no hard optimization problem as the maximum likelihood configurations are trivial assignments (setting all vertices to the same spin/label). Consequently, to obtain hardness results for computing averages in ferromagnetic models we need to work directly from hardness of approximate counting results, which we can do using our new approach.

The most well-studied examples of ferromagnetic models are the Ising and Potts models. Given a graph G and an integer $q \geq 2$, configurations of the Ising/Potts model are the collection Ω of assignments $\sigma : V(G) \to [q]$ where $[q] = \{1, \ldots, q\}$ are the labels of the q spins. The case $q = 2$ corresponds to the Ising model and the case $q \geq 3$ is the Potts model.

The models are parameterised by an edge activity[1] $\beta > 0$. The weight of an assignment σ is defined as $w_{G;q,\beta}(\sigma) = \beta^{m_G(\sigma)}$ where $m_G(\sigma) = |\{(u,v) \in E : \sigma(u) = \sigma(v)\}|$ is the number of edges which are monochromatic in σ. Finally, the Gibbs distribution is defined as $\mu_{G;q,\beta}(\sigma) = w_{G;q,\beta}(\sigma)/Z_{G;q,\beta}$ where the normalising factor $Z_{G;q,\beta} := \sum_{\tau:V(G)\to[q]} w(\tau)$ is the partition function. In this paper, we restrict attention to the case $\beta > 1$ which is the ferromagnetic (attractive) case, and hence the most likely configurations are the q monochromatic configurations (all vertices are assigned the same spin).

For the Ising and Potts models, the analog of the average independent set size is the average number of vertices assigned spin 1. This quantity $\mathcal{M}_{q,\beta}(G)$, known as the magnetization, is trivial in these cases since, due to the Ising/Potts models symmetry among spins, it holds that $\mathcal{M}_{q,\beta}(G) = n/q$. The simplest and most natural observable to consider is the average number of monochromatic edges under the Potts distribution, i.e., the quantity

$$\mathcal{S}_{q,\beta}(G) := \mathbf{E}_{\sigma\sim\mu_{G;q,\beta}}[m_G(\sigma)]$$

which is known as the *susceptibility*. Sinclair and Srivastava [17] showed that exact computation of the susceptibility in the ferromagnetic Ising model is #P-hard.

For the Ising model a classical result of Jerrum and Sinclair [13] presents an efficient sampling scheme for all G, all β. This yields an efficient algorithm for approximating averages in the Ising model (this holds for any local observables as defined subsequently in Section 1.2). In contrast for the Potts model (for any $q \geq 3$) approximating the partition function becomes computationally intractable for large β as we detail below.

The Potts model with $q \geq 3$ spins undergoes a computational phase transition on bipartite graphs of maximum degree Δ at the following critical point $\beta_c(q,\Delta) = \frac{q-2}{(q-1)^{1-2/\Delta}-1}$. In [10] it was established that for all $q, \Delta \geq 3$ and $\beta > \beta_c(q,\Delta)$ approximating the partition function of the ferromagnetic Potts model is #BIS-hard on bipartite graphs of maximum degree Δ. Using our general counting-to-observables reduction we show that approximating the average number of monochromatic edges under the Potts distribution is as hard as approximating the partition function for the ferromagnetic Potts model.

▶ **Theorem 2.** *Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q,\Delta)$. There is no* FPRAS *for the susceptibility in the q-state Potts model on bipartite graphs of maximum degree Δ, unless* #BIS *admits an* FPRAS.

1.2 General 2-spin systems

Theorem 1 for independent sets is a special case of a general result for arbitrary 2-spin antiferromagnetic systems. Such spin systems are parameterized by three parameters, β, γ and λ; the first two are edge activities and control the strength of the spin interactions between neighboring vertices, and the third is a vertex activity (a.k.a. external field) that favors one spin over the other.

More precisely, for a graph $G = (V, E)$, $\beta, \gamma \geq 0$ which are not both equal to zero and $\lambda > 0$, let $\mu_{G;\beta,\gamma,\lambda}$ denote the Gibbs distribution on G with edge activities β, γ and external field λ, i.e., for $\sigma : V \to \{0,1\}$ we have

$$\mu_{G;\beta,\gamma,\lambda}(\sigma) = \frac{\lambda^{|\sigma|}\beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}}{Z_{G;\beta,\gamma,\lambda}},$$

[1] We remark that β is usually used to denote the so-called inverse temperature of the Potts model; here to have consistent notation with general 2-spin systems presented in Section 1.2 we take β to be the exponent of the inverse temperature.

where $|\sigma|$ is the number of vertices with spin 1, and $m_0(\sigma), m_1(\sigma)$ denote the number of edges in G whose endpoints are assigned under σ the pair of spins $(0,0)$ and $(1,1)$, respectively.

The parameter pair (β,γ) is called *antiferromagnetic* if $\beta\gamma \in [0,1)$ and at least one of β,γ is non-zero, and it is called ferromagnetic, otherwise. Note that the hard-core model on independent sets weighted by $\lambda > 0$ is the case $\beta = 1, \gamma = 0$ (under the convention that $0^0 \equiv 1$). Our earlier example of unweighted independent sets corresponds to the hard-core model with $\lambda = 1$. The antiferromagnetic Ising model is the special case $0 < \beta = \gamma < 1$.

Our results apply to general "vertex-edge observables" defined as follows.

▶ **Definition 3.** *Let (β,γ) be antiferromagnetic and $\lambda > 0$. For real numbers a, b, c, the (a,b,c)* vertex-edge observable *of a graph G in the 2-spin system corresponding to (β,γ,λ) is given by*

$$\mathcal{O}_{\beta,\gamma,\lambda}(G) = \mathbf{E}_{\sigma\sim\mu_{G;\beta,\gamma,\lambda}}\big[o_G(\sigma)\big], \text{ where } o_G(\sigma) = a|\sigma| + bm_0(\sigma) + cm_1(\sigma).$$

The observable is trivial on general graphs *if any of the following hold: (i) $a = b = c = 0$, (ii) $\beta = 0$ and $a = c = 0$, (iii) $\gamma = 0$ and $a = b = 0$, (iv) $\beta = \gamma$, $\lambda = 1$ and $b + c = 0$. We say that the observable is* trivial on bipartite graphs *if either any of the above hold, or $\beta = \gamma$ and $\lambda = 1$. Otherwise, we say that the observable is* non-trivial.

Notice that by setting $(a,b,c) = (1,0,0)$ we obtain the magnetization $\mathcal{M}_{\beta,\gamma,\lambda}(G)$, which in the special case of the hard-core model with $\lambda = 1$ is the average size of an independent set. Furthermore, by setting $(a,b,c) = (0,1,1)$ we obtain the susceptibility, denoted by $\mathcal{S}_{\beta,\gamma,\lambda}(G)$, which is the average number of monochromatic edges.

The terminology "trivial" is applied liberally here and meant to convey that there is an efficient algorithm for the relevant parameters. In particular, while cases (i)-(iii) are degenerate, case (iv) corresponds to the Ising model without an external field. A classical (and highly non-trivial) result of Jerrum and Sinclair [13] presented an FPRAS for the ferromagnetic Ising model on any graph, any $\beta > 1$. Moreover, for bipartite graphs, the subcase $\beta < 1$ (antiferromagnetic) can be reduced to an equivalent $\beta > 1$ (ferromagnetic) system.

We next define the range of parameters (β,γ,λ) where our inapproximability results for vertex-edge observables apply; these are precisely the parameters where the hard-core and the antiferromagnetic Ising models exhibit non-uniqueness on the infinite Δ-regular tree (for general 2-spin systems this threshold corresponds to what is known as up-to-Δ non-uniqueness, which captures the computational phase transition).

▶ **Definition 4.** *Let $\Delta \geq 3$ be an integer. We let $\mathcal{N}_\Delta$ be the set of (β,γ,λ) such that (β,γ) is antiferromagnetic, and the (unique) fixpoint $x^* > 0$ of the function $f(x) = \frac{1}{\lambda}\big(\frac{\beta x+1}{x+\gamma}\big)^{\Delta-1}$ satisfies $|f'(x^*)| > 1$. The region $\mathcal{N}_\Delta$ is known as the* non-uniqueness region.

Note there is an efficient sampling/counting algorithm for graphs of maximum degree Δ, roughly[2], for (β,γ,λ) outside the parameter region $\mathcal{N}_\Delta$ [15, 5]. Inside $\mathcal{N}_\Delta$, it is NP-hard to approximate the partition function on graphs of maximum degree Δ [19] and it is #BIS-hard to approximate the partition function on *bipartite* graphs of maximum degree Δ [3]. We prove that it is also hard to compute any non-trivial vertex-edge observable in exactly this same region where the corresponding counting problem is hard.

[2] More precisely, the (strict) uniqueness region is defined as those $\beta,\gamma,\lambda)$ where the fixpoint x^* in Definition 4 satisfies the (strict) inequality $|f'(x^*)| < 1$. For certain monotonicity reasons, the algorithm for max-degree Δ graphs demands that (β,γ,λ) lie in the intersection of these uniqueness regions for all degrees $d \leq \Delta$, see [15] for details.

▶ **Theorem 5.** *Let $\Delta \geq 3$ be an integer and $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$. Then, for any vertex-edge observable that is non-trivial on bipartite graphs, there is no* FPRAS *on bipartite graphs of maximum degree Δ unless #BIS admits an* FPRAS*.*

We stress that the above result holds for bipartite graphs. The previous work of Galanis et al. [9] showed hardness for general antiferromagnetic 2-spin systems in the same non-uniqueness region but on general graphs, only for the magnetization, and only achieved the stronger constant-factor hardness for a dense set of λ.

We begin by establishing Theorem 2 for hardness of approximating the susceptibility for the ferromagnetic Potts model, see Section 2. We then present the refinements to establish Theorems 5 for general 2-spin antiferromagnetic systems in Section 3; Theorem 1 follows as a corollary of Theorem 5.

2 Hardness of Susceptibility for the Ferromagnetic Potts model

Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q, \Delta)$. To prove Theorem 5, we will assume the existence of an FPRAS for the susceptibility of Potts with parameters q, β on maximum degree Δ graphs and show how to obtain an FPRAS for the partition function of the Potts model with parameters q, β^* on bipartite graphs of maximum degree 3 for some $\beta^* > \beta_c(q, 3)$; the latter problem is #BIS-hard by [10].

To aid the presentation, it will be convenient to consider the following computational problems and use the notion of AP-reduction between counting problems [6]; roughly, for two problems A, B, the notation $A \leq_{\mathrm{AP}} B$ means that the existence of an FPRAS for B implies the existence of an FPRAS for A. In the first computational problem that will be relevant, the parameters are q, β, Δ as detailed below.

Name #$\mathsf{Susc}(q, \beta, \Delta)$.
Instance A bipartite graph G with max degree Δ.
Output The susceptibility on G with parameters q, β, i.e., the value $\mathcal{S}_{q,\beta}(G)$.

In the second, the parameter is going to be just q; note that the problem allows the edge activity to be part of the input.

Name #$\mathsf{SuscCubic}(q)$.
Instance A cubic bipartite graph H, and a rational edge activity $\hat{\beta} \geq 1$.
Output The susceptibility on H with parameters $q, \hat{\beta}$, i.e., the value $\mathcal{S}_{q,\hat{\beta}}(H)$.

The key ingredient underpinning our proof approach is captured by the following lemma, whose proof is given in Section 2.3. Roughly, the lemma asserts that, despite the fact that the parameter β is fixed, with appropriate gadget constructions we can "shift" it in a fine-tuned way to any desired $\hat{\beta}$. In turn, this allows us to do an appropriate integration of the observable (viewed as a function of the parameter $\hat{\beta}$) to recover the partition function of a #BIS-hard problem; we will refer loosely to this integration technique as *interpolation*.

▶ **Lemma 6.** *Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q, \Delta)$ be an arbitrary real. Then,*

$$\#\mathsf{SuscCubic}(q) \leq_{\mathrm{AP}} \#\mathsf{Susc}(q, \beta, \Delta).$$

Before proceeding with outlining the proof of the key Lemma 6, we first present the interpolation-scheme idea that allows us to conclude Theorem 2 from Lemma 6.

Proof of Theorem 2 (assuming Lemma 6). Let $\beta^* > \beta_c(q,3)$ be an arbitrary rational number and consider the problem $\#\mathsf{PottsCubic}(q,\beta^*)$, i.e., the problem of approximating the partition function $Z_{G;q,\beta^*}$ on cubic bipartite graphs G. From [10, Theorem 3], we have that $\#\mathsf{PottsCubic}(q,\beta^*)$ is #BIS-hard. From Lemma 6, we have that for $\beta > \beta_p(q,\Delta)$ it holds that $\#\mathsf{SuscCubic}(q) \leq_{\mathrm{AP}} \#\mathsf{Susc}(q,\beta,\Delta)$, so to prove the theorem it suffices to show that $\#\mathsf{PottsCubic}(q,\beta^*) \leq_{\mathrm{AP}} \#\mathsf{SuscCubic}(q)$.

Let G be an instance of $\#\mathsf{PottsCubic}(q,\beta^*)$ with n vertices and m edges, and $\epsilon > 0$ be the desired relative error that we want to approximate $Z_{G;q,\beta^*}$. Since $\frac{\partial \log Z_{G;q,\hat{\beta}}}{\partial \hat{\beta}} = \frac{1}{\hat{\beta}}\mathcal{S}_{q,\hat{\beta}}(G)$, we have

$$\log Z_{G;q,\beta^*} = \int_1^{\beta^*} \tfrac{1}{\hat{\beta}}\mathcal{S}_{q,\hat{\beta}}(G)\mathrm{d}\hat{\beta}. \tag{1}$$

Let $M = \lceil (10q\hat{\beta}m/\epsilon)^4 \rceil$ and for $i = 0,1,\dots,M$, consider the sequence of edge parameters $\hat{\beta}_i = 1 + i\frac{\beta^*-1}{M}$. It is a standard fact that the function $\log Z_{G;\hat{\beta}}$ is convex with respect to $\hat{\beta}$ (the second derivative is equal to the variance of the number of monochromatic edges) and therefore the function $\frac{1}{\hat{\beta}}\mathcal{S}_{q,\hat{\beta}}(G)$ is increasing. Therefore, from the standard technique of approximating integrals with rectangles, we obtain from (1) that

$$\frac{1}{M}\sum_{i=0}^{M-1} \frac{\mathcal{S}_{G;q,\hat{\beta}_i}}{\hat{\beta}_i} \leq \log Z_{G;\beta^*} \leq \frac{1}{M}\sum_{i=1}^{M} \frac{\mathcal{S}_{q,\hat{\beta}_i}(G)}{\hat{\beta}_i}.$$

Using the bound $m/q \leq \mathcal{S}_{q,\hat{\beta}}(G) \leq m$ that holds for all $\hat{\beta} \geq 1$, we obtain that

$$\log Z_{G;q,\beta^*} = \big(1 \pm \frac{\epsilon}{10}\big)\sum_{i=1}^{M} \frac{\mathcal{S}_{q,\hat{\beta}_i}(G)}{\hat{\beta}_i}.$$

Using the presumed oracle for $\#\mathsf{SuscCubic}(q)$ we can compute $\hat{S}_i$ such that $\hat{S}_i = (1 \pm \frac{\epsilon}{10Mm})\mathcal{S}_{q,\hat{\beta}_i}(G)$ for $i \in [M]$, and therefore the quantity $\hat{Z} = \exp\big(\sum_{i\in[M]} \frac{\mathcal{S}_{q,\hat{\beta}_i}(G)}{\hat{\beta}_i}\big)$ is a $(1 \pm \epsilon)$-factor approximation to $Z_{G;q,\beta^*}$. This completes the AP-reduction, and therefore the proof as well. ◀

In the rest of Section 2, we focus on proving Lemma 6.

2.1 Proof overview of Lemma 6

In this section, we give the proof overview of Lemma 6 which as we saw in the previous section is the key ingredient to carry out the interpolation-scheme idea. We highlight here some of the key ideas (with a non-technical overview), which are also used to prove the analogous Lemma for obtaining our inapproximability results for 2-spin systems.

To prove Lemma 6, we will use three different types of gadgets.

The first type of gadgets, that have also been used in previous inapproximability results, are the so-called "phase gadgets", which are almost Δ-regular bipartite graphs with a relatively small number of degree $\Delta - 1$ vertices (the so-called "ports"). This type of gadget exploits the phase transitions of the model and has q-ary behaviour, in the sense that a typical sample from their Gibbs distribution is in one of the q ordered phases, favoring one spin over the others. Aside from this q-ary behaviour, another feature of these gadgets is that they are convenient to maintain the degree of the vertices in our constructions small, using the ports to make connections between gadgets.

The second type of gadgets are paths; these allow us to interpolate the edge activity β. The key point is that long paths induce some small edge-interaction β between their endpoints (bigger than but close to 1) and by using a big number of them (in parallel-style connections) we can achieve a target edge activity $\hat{\beta}$ with arbitrary good precision; here, the ports of the phase gadgets allow us to do these parallel connections without exceeding the degree bound Δ. This is a crucial ingredient in implementing the new reduction idea.

The final type of gadgets consists of the so-called edge-interaction gadgets. Each such gadget has two vertices, say ρ, ρ', which we also refer to as ports. We are interested in two quantities of these gadgets (cf. Definition 9):

- the effective edge activity, i.e., the relative ratio of the aggregate weight of configurations where $\sigma(u) = \sigma(v)$ versus $\sigma(u) \neq \sigma(v)$. Note that this ratio is always bigger than 1, due to the ferromagnetic interaction.
- the susceptibility gap, i.e., the difference between the expected susceptibility conditioned on $\sigma(u) = \sigma(v)$ and the susceptibility conditioned on $\sigma(u) \neq \sigma(v)$.

We prove the existence of pairs of susceptibility gadgets which have roughly equal induced edge parameters but different susceptibility gaps. The equality between the induced edge parameters allows us to use them as probes (without changing the underlying distribution) for "susceptibility" between two vertices s, t, i.e., the probability that s, t have the same colour, in a graph G. That is, we can invoke a presumed oracle for susceptibility when we use the first gadget (by identifying s, t with the terminals) and get a "reading" for susceptibility, and do the same for the second and get a second "reading"; the difference between the two readings gives us information about the probability that s, t have the same colour in the original graph G.

The reason that these susceptibility gadgets are useful is that analysing the susceptibility of the other two types of gadgets is deeply unpleasant and, in fact, it is not even known how to obtain susceptibility estimates for the phase gadgets (since their analysis in earlier works builds upon second moment methods that give rather crude bounds in our setting). Hence, by the subtraction trick above, we have the required modularity to avoid such refined considerations.

That said, establishing the existence of pairs of susceptibility gadgets with the required properties has various challenges and the proof is based on an elaborate construction which finishes by a contradiction argument via Cauchy's functional equation. Fortunately, this ground work has been largely done in [9], though in our setting we need to consider edge gadgets instead of vertex gadgets, which complicates the underlying functions involved in the proofs. We believe that these constructions can be used to strengthen the results of [9] and obtain inapproximability for multi-spin systems such as colourings or the antiferromagnetic Potts model.

These ideas suitably adapted apply to obtain our inapproximability results for antiferromagnetic 2-spin systems. The difference for 2-spin systems is that the interpolation is quite trickier, since in the setting there it is harder to make vertex or edge activities that are close to 1 and do the interpolation (in contrast to the paths used above which is the fairly natural choice). Instead, to do the interpolation, we use a pair of trees whose induced vertex activities (at the root) are sufficiently close and which are attached (in appropriate numbers) to the ports of the phase gadgets to imitate the effect of an external field close to 1. We are then able to interpolate in terms of λ by a suitable implementation of the subtraction trick idea; we again need to depart from [9] (where 2-spin models were also considered) since the construction there does not yield a suitable interpolation parameter. The final new ingredient is to account for the general vertex-edge observables, since a key fact used in [9] is that the magnetization is an appropriate derivative of the log-partition function, which is no longer the case for general vertex-edge observables.

We now state more formally the above ingredients and show how to combine these and conclude the proof of Lemma 6.

2.2 The gadgets

2.2.1 Bipartite phase gadgets for the Potts model

For integers t, n, Δ, we let $\mathcal{G}^t_{n,\Delta}$ be the distribution on bipartite graphs where there are n vertices with degree Δ on each side, and t vertices of degree $\Delta - 1$ on each side. For a graph $G \in \mathcal{G}^t_{n,\Delta}$, we denote the set of vertices with degree Δ by U and by W those with degree $\Delta - 1$, so that $|U| = 2n$ and $|W| = 2t$. We will refer to set W as the *ports* of G. For $\sigma : U \to [q]$, we define the *phase* $\mathcal{Y}(\sigma)$ of the configuration σ as the most frequent color (breaking ties arbitrarily), i.e., which has the most occupied vertices under σ, i.e.,

$$\mathcal{Y}(\sigma) = \arg\max_{i\in[q]} |\sigma^{-1}(i)|.$$

Let $p > 1/q$ be given from $p = \frac{x}{x+q-1}$ where $x > 1$ is the largest solution of $x = \big(\frac{\beta x+q-1}{x+\beta+q-2}\big)^{\Delta-1}$, cf. [10, Footnote 5]. For a colour $i \in [q]$, we define the product measure $Q^i_W(\cdot)$ on configurations $\tau : W \to [q]$, given by

$$Q^i_W(\tau) = p^{|\tau^{-1}(i)|}\big(\tfrac{1-p}{q-1}\big)^{|W|-|\tau^{-1}(i)|}.$$

We will need the following two properties from the phase gadget G for some sufficiently small $\epsilon > 0$. Let $\mu := \mu_{G;q,\beta}$.

1. The q phases appear with roughly equal probability, i.e., $\big|\mu(\mathcal{Y}(\sigma) = i) - \frac{1}{q}\big| \le \epsilon$ for $i \in [q]$.
2. For $i \in [q]$ and any $\tau : W \to [q]$, $\Big|\frac{\mu\left(\sigma_W=\tau \mid \mathcal{Y}(\sigma)=i\right)}{Q^i_W(\tau)} - 1\Big| \le \epsilon$.

Let $\mathcal{G}^{t,\epsilon}_{n,\Delta}$ denote the set of graphs $G \in \mathcal{G}^t_{n,\Delta}$ satisfying Items 1 and 2. The following lemma is shown in [10].

▶ **Lemma 7** ([10, Lemma 28]). *Let $q, \Delta \ge 3$ be integers and $\beta > \beta_c(q,\Delta)$. Then, there is a randomized algorithm that, on input integer $t \ge 1$ and $\epsilon > 0$, outputs in time* $\mathrm{poly}(t, \frac{1}{\epsilon})$ *an integer n and a graph G that belongs to $\mathcal{G}^{t,\epsilon}_{n,\Delta}$, with probability $\ge 3/4$.*

2.2.2 Edge-interaction/susceptibility gadgets

▶ **Definition 8.** *An* edge-interaction gadget *is a connected series-parallel graph $\mathcal{E}$ with two distinct vertices ρ, ρ' that have degree one. We will refer to ρ, ρ' as the* ports *of $\mathcal{E}$.*

▶ **Definition 9.** *Let $\mathcal{E}$ be an edge-interaction gadget with ports ρ, ρ', and $\mu = \mu_{\mathcal{E};\beta}$. We denote by $B_{\mathcal{E}} = B_{\mathcal{E}}(\beta)$ the* effective interaction *of the gadget, i.e., $B_{\mathcal{E}} = \frac{\mu(\sigma_\rho=\sigma_{\rho'}=1)}{\mu(\sigma_\rho=1,\sigma_{\rho'}=2)}$ and by $S_{\mathcal{E}} = S_{\mathcal{E}}(\beta)$ the* susceptibility gap *of the gadget, i.e., $S_{\mathcal{E}} = \mathbf{E}_{\sigma\sim\mu}[m_{\mathcal{E}}(\sigma) \mid \sigma_\rho = \sigma_{\rho'}] - \mathbf{E}_{\sigma\sim\mu}[m_{\mathcal{E}}(\sigma) \mid \sigma_\rho \neq \sigma_{\rho'}]$.*

The following "interaction" gadget will allow us to change the edge interaction parameter to any desired value.

▶ **Lemma 10.** *Let $q \ge 2$ be an integer and $\beta > 1$ be a real. There is an algorithm, which, on input a rational $r \in (0, 1/2)$, outputs in time poly*$(\mathrm{bits}(r))$ *a path $\mathcal{P}$ of length $O(|\log r|)$, such that $0 < B_{\mathcal{P}} - 1 < r$.*

The proof of Lemma 10 is given in Section C.1.2 of the full version. The following lemma gives pairs of edge-interaction gadgets which have almost the same edge interaction but different susceptibility gaps; this difference in the susceptibility gaps while maintaining the edge interaction will be the key to read off the susceptibility by subtraction.

▶ **Lemma 11.** *Let $q \geq 3$ be an integer and $\beta > 1$ be a real. For any arbitrarily small constant $\delta > 0$, there exist constants $S, \Xi > 0$ and $B \in (1, 1+\delta)$ such that the following holds. There is an algorithm, which, on input a rational $r \in (0, 1/2)$, outputs in time $poly(\mathrm{bits}(r))$ a pair of edge-interaction gadgets $\mathcal{E}_1, \mathcal{E}_2$, each of maximum degree 3 and size $O(|\log r|)$, such that*

$$|B_{\mathcal{E}_1} - B|, |B_{\mathcal{E}_2} - B| \leq r, \text{ but } |S_{\mathcal{E}_1} - S_{\mathcal{E}_2}| \geq S.$$

Moreover, the susceptibility gaps $|S_{\mathcal{E}_1}|, |S_{\mathcal{E}_2}|$ are upper-bounded in absolute value by the constant Ξ, i.e., $|S_{\mathcal{E}_1}|, |S_{\mathcal{E}_2}| \leq \Xi$.

The proof of Lemma 11 generalises the techniques from [9], and is given in Section C.3 of the full version.

2.3 The reduction – proof of Lemma 6

Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q, \Delta)$. Let H be a cubic bipartite graph which is input to the problem $\#\mathsf{Susc}(q)$ of Section 2. For integers $n, t \geq 1$ and rational $\epsilon > 0$, let $G \in \mathcal{G}_{n,\Delta}^{t,\epsilon}$ be a bipartite phase gadget satisfying Items 1 and 2 of Section 2.2.1. Let $\mathcal{E}$ be an edge-interaction gadget with effective interaction $B_{\mathcal{E}}$ and susceptibility gap $S = S_{\mathcal{E}}$. Let $\mathcal{P}$ be a path with effective edge interaction $B_{\mathcal{P}}$.

For an integer ℓ satisfying $\ell < t/3$, we define the graph $H_{G,\mathcal{E},\mathcal{P}}^{\ell}$ as follows. For each vertex v of H replace it with a distinct copy of G, denoted by G_v; we also use U_v, W_v to denote the sets corresponding to U, W in G_v. Moreover for each $\{u, v\}$ of H, add a matching of size $\ell + 1$ between W_u and W_v, and replace ℓ edges of the matching by (distinct) copies of the path $\mathcal{P}$ and the last edge of the matching by the gadget $\mathcal{E}$. Since H is bipartite, this constuction can clearly be done so that the final graph $H_{G,\mathcal{E},\mathcal{P}}^{\ell}$ obtained this way is bipartite. Let $H_{G,\mathcal{P}}^{\ell}$ be the graph with the copies of the susceptibility gadget removed.

The lemma below relates the susceptibility $\mathcal{S}_{q,\beta}(H_{G,\mathcal{E},\mathcal{P}}^{\ell})$ with the susceptibility of $\mathcal{S}_{q,\hat{\beta}}(H)$, for some appropriate $\hat{\beta}$ that is a function of the parameters q, Δ, β and $\ell, B_{\mathcal{E}}, B_{\mathcal{P}}$; we expain how the lemma corresponds to the overview of Section 2.1 right after its statement. The following piece of notation will be useful: for a graph J and a subgraph J' of J, given a configuration $\sigma : V(J) \to [q]$, it will be convenient to denote by $m_{J'}(\sigma) = \sum_{e=\{u,v\} \in E(J')} \mathbf{1}\{\sigma(u) = \sigma(v)\}$ the number of monochromatic edges of J' under σ.

▶ **Lemma 12.** *Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q, \Delta)$. There are constants $1 > R_0 > R_1 > 0$ so that the following holds for any path $\mathcal{P}$ with edge interaction $B_{\mathcal{P}}$, any edge-interaction gadget $\mathcal{E}$ with effective interaction $B_{\mathcal{E}}$ and susceptibility gap $S_{\mathcal{E}}$, and any integers ℓ, t with $t \geq 3(\ell + 1)$.*

For a cubic bipartite graph H, for any $\epsilon \leq \frac{1}{(5q|V(H)|)^2}$, any integer n and phase gadget $G \in \mathcal{G}_{n,\Delta}^{t,\epsilon}$, for $\mu := \mu_{H_{G,\mathcal{E},\mathcal{P}}^{\ell}}$ and $\epsilon' = 10q|V(H)|\epsilon$, it holds that

$$\mathcal{S}_{q,\beta}(H_{G,\mathcal{E},\mathcal{P}}^{\ell}) = \mathcal{A}_{\mathcal{E}}|E(H)| + \mathbf{E}_{\sigma\sim\mu}[m_{H_{G,\mathcal{P}}^{\ell}}(\sigma)] + (1 \pm \epsilon')S_{\mathcal{E}}\left[(A_0 - A_1)\mathcal{S}_{q,\hat{\beta}}(H) + A_1|E(H)|\right],$$

where $A_{\mathcal{E}} = \mathbf{E}_{\sigma\sim\mu_{\mathcal{E}}}[m_{\mathcal{E}}(\sigma) \mid \sigma_\rho = \sigma_{\rho'}]$ and

$$\hat{\beta} := \left(\tfrac{1+(B_{\mathcal{P}}-1)R_0}{1+(B_{\mathcal{P}}-1)R_1}\right)^{\ell}\left(\tfrac{1+(B_{\mathcal{E}}-1)R_0}{1+(B_{\mathcal{E}}-1)R_1}\right), \qquad A_j := \frac{B_{\mathcal{E}}}{B_{\mathcal{E}} + \frac{1-R_j}{R_j}} \text{ for } j \in \{0, 1\}.$$

To give a bit of intuition behind the expression of $\mathcal{S}_{q,\beta}(H^{\ell}_{G,\mathcal{E},\mathcal{P}})$, recall that the vertices of H are replaced with copies of the bipartite phase gadgets G and that for each pair of neighboring vertices in H we connect the corresponding copies of G using the appropriate number of the gadgets $\mathcal{E},\mathcal{P}$. The point here is that the bipartite gadgets are so large that each one of them is with very high probability in one of the q phases (cf. Item 1 in Section 2.2.1) and therefore in the Gibbs distribution of $H^{\ell}_{G,\mathcal{E},\mathcal{P}}$ (with parameters q,β) they behave like meta-vertices which are in one of q states, analogously to a Potts model on H with q spins and a new edge activity $\hat{\beta}$, which is ultimately determined by the $\mathcal{E},\mathcal{P}$-connections and the (induced) probability distributions on the ports of the bipartite phase gadgets (conditioned on the phase, cf. Item 2 in Section 2.2.1). This explains (at an intuitive level) the presence of the quantity $\mathcal{S}_{q,\hat{\beta}}(H)$; the remaining terms are offsets to account for the addition of the various gadgets. Of those, the most complicated is the term $\mathbf{E}_{\sigma\sim\mu}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)]$ which involves the contribution to the susceptibility from edges in the graph $\mathbf{E}_{\sigma\sim\mu}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)]$ which is hard to get a neat expression since the average is taken over the complicated distribution μ. This is where the idea of having a pair of susceptibility gadgets $(\mathcal{E}_1,\mathcal{E}_2)$ with the same effective interaction but different susceptibility gaps will come into play (in the proof of Lemma 6 below): by subtracting the susceptibilities for the graphs $H^{\ell}_{G,\mathcal{E}_1,\mathcal{P}}$ and $H^{\ell}_{G,\mathcal{E}_2,\mathcal{P}}$ between these, the terms corresponding to $\mathbf{E}_{\sigma\sim\mu}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)]$ will cancel (since $\mathcal{E}_1,\mathcal{E}_2$ have roughly the same effective interaction $B_{\mathcal{E}_1},B_{\mathcal{E}_2}$) allowing us to approximate the target quantity $\mathcal{S}_{q,\hat{\beta}}(H)$ (since $\mathcal{E}_1,\mathcal{E}_2$ have substantially different susceptibility gaps $S_{\mathcal{E}_1},S_{\mathcal{E}_2}$). That said, the proof of Lemma 12 is on the technical side and is deferred to Section B of the full version.

To finish the proof of Lemma 6, we need the following crude bound on the change of susceptibility when we slightly change the values of the edge activities on a subset of the edges. To state the lemma, for a graph G with edge-activity vector $\boldsymbol{\beta}=\{\beta_e\}_{e\in E(H)}$, define the weight of an assignment $\sigma: V(G)\to[q]$ as $w(\sigma)=\prod_{e=\{u,v\}\in E(G)}(\beta_e)^{\mathbf{1}\{\sigma(u)=\sigma(v)\}}$, and let $\mu_{G;q,\boldsymbol{\beta}}(\sigma)=w(\sigma)/Z_{G;q,\boldsymbol{\beta}}$ denote the corresponding Gibbs distribution, where $Z_{G;q,\boldsymbol{\beta}}$ is the normalizing constant.

▶ **Lemma 13.** *Let H be a graph and F be a subgraph of H on the same set of vertices. Suppose that $\boldsymbol{\beta}=\{\beta_e\}_{e\in E(H)},\boldsymbol{\beta}'=\{\beta'_e\}_{e\in E(H)}$ are edge activity vectors such that $\beta_e=\beta'_e=\beta$ for $e\in E(F)$, and $\beta_e=\beta_0$, $\beta'_e=\beta_1$ for $e\notin E(F)$. Then, for $\mu:=\mu_{H;q,\boldsymbol{\beta}}$ and $\mu':=\mu'_{H;q,\boldsymbol{\beta}}$, it holds that*

$$\Big|\mathbf{E}_{\sigma\sim\mu}[m_F(\sigma)]-\mathbf{E}_{\sigma\sim\mu'}[m_F(\sigma)]\Big|\leq|E(H)|^2|\beta_0-\beta_1|.$$

Proof. Suppose without loss of generality that $\beta_0\geq\beta_1$; by the monotonicity of the ferromagnetic Potts model we have that $\mathbf{E}_{\sigma\sim\mu}[m_F(\sigma)]\geq\mathbf{E}_{\sigma\sim\mu'}[m_F(\sigma)]$ (see, e.g., [12, Theorems 1.16 & 3.21]). For a configuration $\sigma:V(H)\to[q]$, let $w(\sigma),w'(\sigma)$ denote its weight under the edge activity vectors $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$, respectively. Consider an edge $e\in E(F)$. Then, using that for reals $a>b>0$ it holds that $|a^k-b^k|\leq k|a-b|a^k$, we obtain that for every σ it holds that

$$0<w(\sigma)-w'(\sigma)=\beta^{m_F(\sigma)}\big(\beta_0^{|E(H)|-m_F(\sigma)}-(\beta_1)^{|E(H)|-m_F(\sigma)}\big)\leq|E(H)|(\beta_0-\beta_1)w(\sigma)$$

By summing over σ, it follows also that $Z_{G;q,\boldsymbol{\beta}'}\leq Z_{G;q,\boldsymbol{\beta}}$, and combining these we obtain that

$$\mathbf{E}_{\sigma\sim\mu}[m_F(\sigma)]-\mathbf{E}_{\sigma\sim\mu'}[m_F(\sigma)]\leq\frac{\sum_\sigma m_F(\sigma)\big(w(\sigma)-w'(\sigma)\big)}{Z_{G;q,\boldsymbol{\beta}}}\leq|E(H)|^2(\beta_0-\beta_1).\qquad◀$$

We now give the proof of Lemma 6 which we restate here.

▶ **Lemma 6.** *Let $q, \Delta \geq 3$ be integers and $\beta > \beta_c(q, \Delta)$ be an arbitrary real. Then,*

$$\#\mathsf{SuscCubic}(q) \leq_{\mathrm{AP}} \#\mathsf{Susc}(q, \beta, \Delta).$$

Proof. Let H be a cubic bipartite graph and $\hat{\beta} > 1$ be the inputs to $\#\mathsf{SuscCubic}(q)$, and let $\eta \in (0,1)$ be the desired relative error that we want to approximate $\mathcal{S}_{q,\hat{\beta}}(H)$. We may assume that $\hat{\beta} \geq \beta_0 = \big(\frac{q-1}{3}\big)^{1/\Delta}$; for $\hat{\beta} < \beta_0$, a fairly standard coupling argument shows that Glauber dynamics converges rapidly to the Gibbs distribution $\mu_{H;q,\hat{\beta}}$, see for example [2, Theorem 1.1], and therefore it can be used to approximate $\mathcal{S}_{q,\hat{\beta}}(H)$ in time $poly\big(V(H), \frac{1}{\eta}, \mathrm{bits}(\hat{\beta})\big)$ using rejection sampling. For some of the bounds below, it will also be convenient to assume that $|V(H)|, |E(H)|$ are bigger than a sufficiently large constant (otherwise, we can just brute-force).

Let $1 > R_0 > R_1 > 0$ be the constants in Lemma 12, and let $\delta \in (0,1)$ be a rational constant such that for all $B \in (1, 1+\delta)$, it holds that $\frac{1+(B-1)R_0}{1+(B-1)R_1} \leq \beta_0 < \hat{\beta}$. Note that the choice of δ is a constant depending on q, Δ but independent of H and $\hat{\beta}$. By Lemma 11, there are constants $B \in (1, 1+\delta)$, $S > 0$ and an algorithm, which, on input a rational $r \in (0, 1/2)$, outputs in time $poly(\mathrm{bits}(r))$ a pair of susceptibility gadgets $\mathcal{E}_1, \mathcal{E}_2$, each of maximum degree 3 and size $O(|\log r|)$, such that

$$|B_{\mathcal{E}_1} - B|, |B_{\mathcal{E}_2} - B| \leq r, \text{ but } |S_{\mathcal{E}_1} - S_{\mathcal{E}_2}| \geq S. \tag{2}$$

Let $\epsilon = \frac{\eta}{|E(H)|^5}$ and $t = \Big\lceil \big(\frac{|E(H)| \log \hat{\beta}}{\epsilon\delta(B-1)}\big)^4 \Big\rceil$. By Lemma 7, there is an algorithm that outputs in time $poly(t, \frac{1}{\epsilon})$ an integer n and a graph $G \in \mathcal{G}^{t,\epsilon}_{n,\Delta}$ (satisfying Items 1 and 2). Use the algorithm of Lemma 11 to obtain gadgets $\mathcal{E}_1, \mathcal{E}_2$ satisfying (2) for $r = \frac{\epsilon^4}{10\delta(R_0-R_1)\beta_0}$. Moreover, use Lemma 10, to obtain in time $poly(\mathrm{bits}(r))$ an edge interaction gadget with $1 < B_{\mathcal{P}} < 1+\epsilon$. Let ℓ be the smallest positive integer such that

$$\big(\tfrac{1+(B_{\mathcal{P}}-1)R_0}{1+(B_{\mathcal{P}}-1)R_1}\big)^{\ell}\big(\tfrac{1+(B-1)R_0}{1+(B-1)R_1}\big) > \hat{\beta}$$

and note that such an integer exists by the choice of δ since the l.h.s. for $\ell = 0$ is smaller than $\hat{\beta}$, and each of the fractions is bigger than 1 from $R_0 > R_1$ and $B > 1$. In fact, we have that $\ell = O(\frac{1}{\epsilon}\log\hat{\beta})$, where the implicit constants depend only on q, Δ. It follows in particular that $\ell < t/3$.

For $i \in \{1,2\}$, consider now the graphs $\widehat{H}_i = H^{\ell}_{G,\mathcal{P},\mathcal{E}_i}$ and let $\mu_i = \mu_{\widehat{H}_i;q,\beta}$. From Lemma 12, we have that

$$\mathcal{S}_{q,\beta}(\widehat{H}_i) = \mathcal{A}_{\mathcal{E}_i}|E(H)| + \mathbf{E}_{\sigma\sim\mu_i}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)] + (1\pm\eta^2)S_{\mathcal{E}_i}\Big[\big(A^{(i)}_0 - A^{(i)}_1\big)\mathcal{S}_{q,\hat{\beta}_i}(H) + A^{(i)}_1|E(H)|\Big],$$

where $A_{\mathcal{E}_i} = \mathbf{E}_{\sigma\sim\mu_{\mathcal{E}_i}}[m_{\mathcal{E}_i}(\sigma) \mid \sigma_\rho = \sigma_{\rho'}]$ and

$$\hat{\beta}_i := \big(\tfrac{1+(B_{\mathcal{P}}-1)R_0}{1+(B_{\mathcal{P}}-1)R_1}\big)^{\ell}\big(\tfrac{1+(B_{\mathcal{E}_i}-1)R_0}{1+(B_{\mathcal{E}_i}-1)R_1}\big), \qquad A^{(i)}_j := \frac{B_{\mathcal{E}_i}}{B_{\mathcal{E}_i} + \frac{1-R_j}{R_j}} \text{ for } j \in \{0,1\}.$$

From (2), we have that $\hat{\beta}_i = (1\pm\epsilon^3)\hat{\beta}$, and therefore from Lemma 13, we have that

$$\big|\mathbf{E}_{\sigma\sim\mu_1}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)] - \mathbf{E}_{\sigma\sim\mu_2}[m_{H^{\ell}_{G,\mathcal{P}}}(\sigma)]\big| \leq |E(H^{\ell}_{G,\mathcal{E},\mathcal{P}})|\epsilon^3 \leq \epsilon^2,$$
$$|\mathcal{S}_{q,\hat{\beta}_i}(H) - \mathcal{S}_{q,\hat{\beta}}(H)| \leq |E(H)|^2\epsilon^3 \leq \epsilon^2$$

Using (2), we also have that

$$A^{(i)}_0 = (1\pm\epsilon)A_0, A^{(i)}_1 = (1\pm\epsilon^2)A_i, \text{ where } A_j := \frac{B}{B+\frac{1-R_j}{R_j}} \text{ for } j \in \{0,1\}.$$

We can invoke the oracle for $\mathcal{S}_{q,\beta}(\widehat{H}_i)$ to compute $\hat{\mathcal{S}}_i$ such that $\hat{\mathcal{S}}_i = (1 \pm \epsilon^2)\mathcal{S}_{q,\beta}(\widehat{H}_i)$. Note also that $\mathcal{E}_i$ has size $poly(\text{bits}(r))$ and therefore we can invoke the oracle for $\#\mathsf{Susc}(q, \Delta, \beta)$ to compute $\hat{A}_{\mathcal{E}_i}, \hat{S}_{\mathcal{E}_1}$ such that $\hat{A}_{\mathcal{E}_i} = (1 \pm \epsilon^2)A_{\mathcal{E}_i}$ and $\hat{S}_{\mathcal{E}_i} = (1 \pm \epsilon^2)S_{\mathcal{E}_i}$.[3] It follows that

$$\hat{S} = \frac{1}{A_0 - A_1}\Big(\frac{(\hat{\mathcal{S}}_1 - \hat{\mathcal{S}}_2) - |E(H)|(\hat{A}_{\mathcal{E}_1} - \hat{A}_{\mathcal{E}_2})}{\hat{S}_{\mathcal{E}_1} - \hat{S}_{\mathcal{E}_2}} - A_1|E(H)|\Big)$$

is within a factor of $(1 \pm \eta)$ of the susceptibility $\mathcal{S}_{q,\hat{\beta}}(H)$, as needed. This finishes the reduction and therefore the proof of Lemma 6. ◀

3 Hardness of vertex-edge observables for 2-spin systems

Throughout this section, we will fix integer $\Delta \geq 3$, and antiferromagnetic $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$ in the non-uniqueness region. We will also fix an (a, b, c) vertex-edge observable that is non-trivial on bipartite graphs.

3.1 The interpolation scheme

Analogously to Section 2, it will be convenient to consider the following computational problems.

Name $\#\mathsf{Observable2Spin}(\beta, \gamma, \lambda, a, b, c)$.
Instance A bipartite graph G with max degree Δ.
Output The (a, b, c) vertex-edge observable on G with parameters β,γ,λ, i.e., the value $\mathcal{O}_{\beta,\gamma,\lambda}(G)$.

In the second, the parameter is going to be the edge activity $\alpha < 1$ of an antiferromagnetic Ising model; note that the problem allows the vertex activity to be part of the input.

Name $\#\mathsf{MagnetIsingCubic}(\alpha)$.
Instance A cubic bipartite graph H, and a rational vertex activity $\hat{\lambda} > 0$.
Output The magnetization on H for the Ising model with parameters $\alpha, \hat{\lambda}$, i.e., the value $\mathcal{M}_{\alpha,\alpha,\hat{\lambda}}(H)$.

We now show the following analogue of the interpolation scheme in Lemma 6.

▶ **Lemma 14.** *Let $\Delta \geq 3$ be an integer and $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$. Then, there is $\alpha \in (0, 1)$ such that for any (a, b, c) vertex-edge observable that is not trivial on bipartite graphs,*

$$\#\mathsf{MagnetIsingCubic}(\alpha) \leq_{\mathrm{AP}} \#\mathsf{Observable2Spin}(\beta, \gamma, \lambda, a, b, c).$$

Assuming the key Lemma 14, the proof of Theorem 5 can be done analogously to Theorem 2, interpolating now in terms of the vertex activity $\hat{\lambda}$. We defer the proof to Section A of the full version.

[3] For $\hat{A}_{\mathcal{E}_i}$, just invoke the oracle on the graph obtained from $\mathcal{E}_i$ by identifying ρ_i and ρ'_i; this graph has maximum degree at most 3 since ρ_i, ρ'_i both have degree 1. Observe that $S_{\mathcal{E}_i} = 2A_{\mathcal{E}_i} - \mathcal{S}_{q,\beta}(\mathcal{E}_i)$ and therefore we can obtain the desired $\hat{S}_{\mathcal{E}_i}$ by using a further oracle call on $\mathcal{E}_i$ to approximate $\mathcal{S}_{q,\beta}(\mathcal{E}_i)$.

3.2 The gadgets

In this section, we outline the gadgets that will be used to prove Lemma 14. These are analogous to those presented in the case of the Potts model, especially the phase gadgets. To account for general vertex-edge observables, we refine appropriately the field-gadget idea of [9], by now paying attention to the so-called observable gap (cf. Definition 17).

3.2.1 Bipartite phase gadgets for antiferromagnetic 2-spin systems

We follow the same notation as in Section 2.2.1 to denote for integers t, n, Δ the class $\mathcal{G}^t_{n,\Delta}$ of bipartite graphs where there are n vertices with degree Δ on each side, and t vertices of degree $\Delta - 1$ on each side. For a graph $G \in \mathcal{G}^t_{n,\Delta}$, we denote its bipartition by (U^+, U^-) where U^+, U^- are vertex sets with $|U^+| = |U^-| = n$, and we denote by W^+, W^- the sets of vertices with degree $\Delta - 1$ on each side of the bipartition, so that $|W^+| = |W^-| = t$. We will refer to set $W = W^+ \cup W^-$ as the *ports* of G. For $\sigma : U \to \{0, 1\}$, we define the *phase* $\mathcal{Y}(\sigma)$ of the configuration σ as the side of the bipartite graph which has the most occupied vertices under σ, i.e.,

$$\mathcal{Y}(\sigma) = \arg\max_{i \in \{+,-\}} |\sigma^{-1}(1) \cap U^i|.$$

It is known that for $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$ the system of equations $x = \frac{1}{\lambda}\left(\frac{\beta y+1}{y+\gamma}\right)^{\Delta-1}, y = \frac{1}{\lambda}\left(\frac{\beta x+1}{x+\gamma}\right)^{\Delta-1}$ has a unique solution with $y > x > 0$, see, e.g., [8, Lemma 7]. Let $q_+ = \frac{1}{1+x}$, $q_- = \frac{1}{1+y}$ and note that q_+, q_- are distinct numbers in the interval $(0, 1)$. Define the product distributions $Q^+_W(\cdot), Q^-_W(\cdot)$ by

$$Q^\pm_W(\tau) = (q_\pm)^{|\tau^{-1}(1)\cap W^+|}(1-q_\pm)^{|\tau^{-1}(0)\cap W^+|}(q_\mp)^{|\tau^{-1}(1)\cap W^-|}(1-q_\mp)^{|\tau^{-1}(0)\cap W^-|}. \tag{3}$$

We will need the following two properties from the phase gadget G for some sufficiently small $\epsilon > 0$. Let $\mu := \mu_{G;\beta,\gamma,\lambda}$.

1. The phases appear with roughly equal probability, i.e., $\left|\mu(\mathcal{Y}(\sigma) = \pm) - \frac{1}{2}\right| \le \epsilon$.
2. For any $\tau : W \to \{0, 1\}$, $\left|\dfrac{\mu\big(\sigma_W = \tau \mid \mathcal{Y}(\sigma) = \pm\big)}{Q^\pm_W(\tau)} - 1\right| \le \epsilon$.

Let $\mathcal{G}^{t,\epsilon}_{n,\Delta}$ denote the set of graphs $G \in \mathcal{G}^t_{n,\Delta}$ satisfying Items 1 and 2. The following lemma is is shown in [3].

▶ **Lemma 15** ([3, Lemma 9]). *Let $\Delta \ge 3$ and $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$. Then, there is a randomized algorithm that, on input integer $t \ge 1$ and $\epsilon > 0$, outputs in time* $\mathrm{poly}(t, \frac{1}{\epsilon})$ *an integer n and a graph G that belongs to $\mathcal{G}^{t,\epsilon}_{n,\Delta}$, with probability $\ge 3/4$.*

3.2.2 Field gadgets with observable gaps

We adopt the following definition of "field" gadgets from [9].

▶ **Definition 16.** *For $\lambda \ne \frac{1-\beta}{1-\gamma}$, a* field gadget *is a rooted tree $\mathcal{T}$ whose root ρ has degree one. When $\lambda = \frac{1-\beta}{1-\gamma}$, a field gadget consists of a rooted bipartite graph obtained from a rooted tree where some of the leaves have been replaced by a distinct cycle of length four (by identifying the leaf with a vertex of the cycle).*

▶ **Definition 17.** *Let $\mathcal{T}$ be a field gadget rooted at ρ, and $\mu = \mu_{\mathcal{T};\beta,\gamma,\lambda}$. We denote by $R_\mathcal{T} = R_\mathcal{T}(\beta, \gamma, \lambda)$ the* effective field *of the gadget, i.e., $R_\mathcal{T} = \frac{1}{\lambda}\frac{\mu(\sigma_\rho=1)}{\mu(\sigma_\rho=0)}$ and by $O_\mathcal{T} = O_\mathcal{T}(\beta, \gamma, \lambda)$ the* observable gap *of the gadget, i.e., $O_\mathcal{T} = \mathbf{E}_{\sigma\sim\mu}[\, o_\mathcal{T}(\sigma) \mid \sigma_\rho = 1] - a - \mathbf{E}_{\sigma\sim\mu}[\, o_\mathcal{T}(\sigma) \mid \sigma_\rho = 0]$.*

The division by λ in the definition of the effective field of a gadget is to avoid double-counting the contribution of the root later on.

▶ **Lemma 18.** *Let (β,γ,λ) be antiferromagnetic such that at least one of $\beta\neq\gamma$ or $\lambda\neq 1$ holds. There are constants $C,\tilde{R}>0$ with $\tilde{R}\neq 1$ and an algorithm which, on input a rational $r\in(0,1/2)$, outputs in time poly$(\mathrm{bits}(r))$ field gadgets $\mathcal{T}_+,\mathcal{T}_-$, each of maximum degree 3 and size $O(|\log r|)$, such that*

$$R_{\mathcal{T}_+} > R_{\mathcal{T}_-} + r/2 \text{ and } |R_{\mathcal{T}_+}-\tilde{R}|,\ |R_{\mathcal{T}_-}-\tilde{R}|\leq r.$$

▶ **Theorem 19.** *Let (β,γ,λ) be antiferromagnetic, and consider any non-trivial vertex-edge observable on general graphs. There exist constants $\hat{R},\hat{O},\Xi>0$ and an algorithm, which, on input a rational $r\in(0,1/2)$, outputs in time poly$(\mathrm{bits}(r))$ a pair of field gadgets $\mathcal{T}_1,\mathcal{T}_2$, each of maximum degree 3 and size $O(|\log r|)$, such that*

$$|R_{\mathcal{T}_1}-\hat{R}|,\,|R_{\mathcal{T}_2}-\hat{R}|\leq r,\ \text{ but } |O_{\mathcal{T}_1}-O_{\mathcal{T}_2}|\geq\hat{O}.$$

Moreover, the observable gaps $O_{\mathcal{T}_1},O_{\mathcal{T}_2}$ are upper-bounded in absolute value by the constant Ξ.

The proofs of Lemma 18 and Theorem 19 follow closely the approach in [9], and are therefore deferred to Section C.3 of the full version.

3.3 The reduction

Let H be a cubic bipartite graph which is input to the problem #MagnetIsingCubic(α) of Section 3.1, for some constant $\alpha\in(0,1)$ to be specified. For integers $n,t\geq 1$ and rational $\epsilon>0$, let $G\in\mathcal{G}^{t,\epsilon}_{n,\Delta}$ be a bipartite phase gadget satisfying Items 1 and 2 of Section 3.2.1. Let $\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}$ be field gadgets. Note that the gadgets $\mathcal{T}_+,\mathcal{T}_-$ serve a different role to that of $\mathcal{T}$, and in particular they will be used to interpolate over the vertex activity $\hat{\lambda}$.

To achieve this, for integers ℓ_+,ℓ_- satisfying $t\geq 5+\max\{\ell_+,\ell_-\}$, we define the graph $H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}}$ as follows. For each vertex v of H replace it with a distinct copy of G, denoted by G_v; we denote by $U^{\pm}_v,W^{\pm}_v$ the sets corresponding to $U^{\pm},W^{\pm}$ in G_v. Moreover, for each $v\in V(H)$, attach one copy of the gadget $\mathcal{T}$ and ℓ_+ copies of the gadget $\mathcal{T}_+$ on mutually distinct vertices of W^+ by identifying them with the corresponding roots. Similarly, attach ℓ_- copies of the gadget $\mathcal{T}_-$ on mutually distinct vertices of W^-. Let $\mathcal{T}_v$ be the copy of $\mathcal{T}_v$ corresponding to v, and w_v be its root. Let $W_{\mathcal{T}}=\{w_v\mid v\in V(H)\}$ be the set of all these roots. Further, for each edge $\{u,v\}$ of H, add an edge between W^+_u and W^+_v, and an edge between W^-_u and W^-_v.

Let $H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-}$ be the graph without the internal vertices and edges of the copies of gadget $\mathcal{T}$, i.e., we keep only the roots $W_{\mathcal{T}}$ of the gadgets in $H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-}$. The following piece of notation will be useful: for a graph J and a subgraph J' of J, given a configuration $\sigma: V(J)\to[q]$, it will be convenient to denote by $m_{J'}(\sigma)=\sum_{e=\{u,v\}\in E(J')}\mathbf{1}\{\sigma(u)=\sigma(v)\}$ the number of monochromatic edges of J' under σ.

The following lemma relates the value of the observable $\mathcal{O}_{\beta,\gamma,\lambda}(H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}})$ with the magnetization $\mathcal{S}_{\alpha,\hat{\lambda}}(H)$, for some appropriate $\hat{\lambda}$ that is a function of the parameters β,γ,λ and $\ell_+,\ell_-,R_{\mathcal{T}_+},R_{\mathcal{T}_+},R_{\mathcal{T}}$. Analogously to Section 2.3, for a graph J and a subgraph J' of J, given a configuration $\sigma: V(J)\to[q]$, it will be convenient to denote by $o_{J'}(\sigma)=a|\sigma_{V(J')}|+bm_0(\sigma_{V(J')})+cm_1(\sigma_{V(J')})$ the contribution of J' to the value of the observable on J.

▶ **Lemma 20.** *Let $\Delta \geq 3$ be an integer, $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$, and (a, b, c) be a vertex-edge observable. Then, there are constants $q_+, q_- \in (0, 1)$ with $q_+ > q_-$ and $\alpha \in (0, 1)$ so that the following holds for any field gadgets $\mathcal{T}_+, \mathcal{T}_-, \mathcal{T}$ with effective fields R_+, R_-, R and observable gaps O_1, O_2, O, and any positive integers ℓ_+, ℓ_-, t with $t \geq 5 + \max\{\ell_+, \ell_-\}$.*

For a cubic bipartite graph H, for any $\epsilon \leq \frac{1}{(5|V(H)|)^2}$, any integer n and phase gadget $G \in \mathcal{G}^{t,\epsilon}_{n,\Delta}$, for $\mu := \mu_{H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}}}$ and $\epsilon' = 10|V(H)|\epsilon$, it holds that

$$\mathcal{O}_{\beta,\gamma,\lambda}(H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}}) = \mathcal{A}|V(H)| + \mathbf{E}_{\sigma\sim\mu}\big[o_{H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-}}(\sigma)\big] + (1 \pm \epsilon')O\big[(q_+ - q_-)\mathcal{M}_{\alpha,\hat{\lambda}}(H) + q_-|V(H)|\big],$$

where $\mathcal{A} = \mathbf{E}_{\sigma\sim\mu_\mathcal{T}}[o_\mathcal{T}(\sigma) \mid \sigma_\rho = 0]$ and $\hat{\lambda} := \big(\frac{q_+R+1-q_+}{q_-R+1-q_-}\big)\big(\frac{q_+R_++1-q_+}{q_-R_++1-q_-}\big)^{\ell_+}/\big(\frac{q_+R_-+1-q_+}{q_-R_-+1-q_-}\big)^{\ell_-}$.

The proof of Lemma 20 builds upon similar ideas to that of Lemma 12 (see in particular the discussion around there for how this blends with the overview of Section 2.1) and is deferred to Section B of the full version.

We will need the following bound on the change of the observable value when we change the vertex activities of a subset of the vertices. Namely, let $G = (V, E)$ be a graph and (β, γ) be antiferromagnetic. For a vertex-activity vector $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V}$, define the Gibbs distribution $\mu_{G;\beta,\gamma,\boldsymbol{\lambda}}(\sigma) \propto \beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}\prod_{v\in V;\,\sigma(v)=1}\lambda_v$ for $\sigma : V \to \{0, 1\}$.

▶ **Lemma 21** (Minor adaptation of [9, Lemma 35])**.** *Let (β, γ) be antiferromagnetic, $\lambda, \lambda_1, \lambda_2 > 0$, and (a, b, c) be a vertex-edge observable. Let $G = (V, E)$ be a graph and $S \subseteq V$. For $i \in \{1, 2\}$, let $\boldsymbol{\lambda}_i$ be the field vector on V, where every $v \in S$ has activity λ_i, whereas every $v \in V\backslash S$ has activity λ. Let μ_i be the Gibbs distribution on G with parameters $\beta, \gamma, \boldsymbol{\lambda}_i$. Then, for every subgraph F of G, it holds that*

$$\big|\mathbf{E}_{\sigma\sim\mu_2}[o_F(\sigma)] - \mathbf{E}_{\sigma\sim\mu_1}[o_F(\sigma)]\big| \leq 2\big(|a| + |b| + |c|\big)\big(|V(G)|^2 + |E(G)|^2\big)\left|\frac{\lambda_2}{\lambda_1} - 1\right|.$$

We now have all the ingredients to prove Lemma 14.

▶ **Lemma 14.** *Let $\Delta \geq 3$ be an integer and $(\beta, \gamma, \lambda) \in \mathcal{N}_\Delta$. Then, there is $\alpha \in (0, 1)$ such that for any (a, b, c) vertex-edge observable that is not trivial on bipartite graphs,*

$$\#\mathsf{MagnetIsingCubic}(\alpha) \leq_{\mathrm{AP}} \#\mathsf{Observable2Spin}(\beta, \gamma, \lambda, a, b, c).$$

Proof. Let $K = |a| + |b| + |c|$ and q_+, q_-, α be the constants from Lemma 20; recall that $\alpha \in (0, 1)$ and $1 > q_+ > q_- > 0$. Let H be a cubic bipartite graph and $\hat{\lambda} > 1$ be the inputs to $\#\mathsf{MagnetIsingCubic}(\alpha)$, and let $\eta \in (0, 1)$ be the desired relative error that we want to approximate $\mathcal{M}_{\alpha,\hat{\lambda}}(H)$.

By Lemma 18 and Theorem 19, there exist constants $\tilde{R}, \hat{R}, \hat{O}, C > 0$ with $\tilde{R} \neq 1$ and polynomial time algorithms, which on input rationals $r, r' \in (0, 1/2)$, output in time $poly(\mathrm{bits}(r), \mathrm{bits}(r'))$ pairs of field gadgets $(\mathcal{T}_+, \mathcal{T}_-)$ and $(\mathcal{T}_1, \mathcal{T}_2)$ satisfying

$$\begin{aligned} R_+ > R_- + Cr \text{ and } |R_+ - \tilde{R}|,\ |R_- - \tilde{R}| \leq r,\\ |R_1 - \hat{R}|, |R_2 - \hat{R}| \leq r, \text{ but } |O_1 - O_2| \geq \hat{O}, \end{aligned} \tag{4}$$

where R_+, R_-, R_1, R_2 are the effective fields of $\mathcal{T}_+, \mathcal{T}_-, \mathcal{T}_1, \mathcal{T}_2$ and O_1, O_2 are the observable gaps of $\mathcal{T}_1, \mathcal{T}_2$, respectively.

Let $\epsilon = \frac{\eta}{|V(H)|^8}$ and $t = \big\lceil\big(\frac{1}{\epsilon}|V(H)|^2\log\hat{\lambda}\big)^6\big\rceil$. By Lemma 7, there is an algorithm that outputs in time $poly(t, \frac{1}{\epsilon})$ an integer n and a graph $G \in \mathcal{G}^{t,\epsilon}_{n,\Delta}$ (satisfying Items 1 and 2). Let also $\mathcal{T}_+, \mathcal{T}_-$ be field gadgets satisfying (4) for $r = \frac{|\tilde{R}-1|}{10}\epsilon^4$. Consider also the integers $\ell_+, \ell_- = \ell$, where ℓ is an integer specified according to whether $\hat{\Lambda} = \hat{\lambda}/\big(\frac{q_+\hat{R}+1-q_+}{q_-\hat{R}+1-q_-}\big)$ is bigger than 1. Suppose first that $\hat{\Lambda} \geq 1$. Since $R_+ > R_-$ and $q_+ > q_-$, we have that $\frac{q_+R_++1-q_+}{q_-R_++1-q_-} > \frac{q_+R_-+1-q_+}{q_-R_-+1-q_-}$, and we pick ℓ to be the smallest positive integer such that

$$\big(\tfrac{q_+R+1-q_+}{q_-R+1-q_-}\big)\big(\tfrac{q_+R_++1-q_+}{q_-R_++1-q_-}\big)^{\ell}/\big(\tfrac{q_+R_-+1-q_+}{q_-R_-+1-q_-}\big)^{\ell} \geq \hat{\lambda}. \tag{5}$$

If $\hat{\Lambda} < 1$, then we pick ℓ to be the small positive integer so that

$$\big(\tfrac{q_+R+1-q_+}{q_-R+1-q_-}\big)\big(\tfrac{q_+R_-+1-q_+}{q_-R_-+1-q_-}\big)^{\ell}/\big(\tfrac{q_+R_++1-q_+}{q_-R_++1-q_-}\big)^{\ell} \leq \hat{\lambda},$$

In either case, using the lower bound $R_+ - R_- > Cr$ from (4), we have that $\ell = O(\frac{1}{r}\log\hat{\lambda})$ where the implicit constant depends only on $\beta, \gamma, \lambda, \Delta$. In particular, we have that $t \geq 5 + \max\{\ell_+, \ell_-\}$. In the argument below, we assume w.l.o.g. that $\hat{\Lambda} \geq 1$; otherwise, just apply the same argument by swapping the roles of the gadgets $\mathcal{T}_+, \mathcal{T}_-$ in the construction below.

For $i \in \{1, 2\}$, consider now the graphs $\widehat{H}_i = H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-,\mathcal{T}_i}$ and let $\mu_i = \mu_{\widehat{H}_i;\beta,\gamma,\lambda}$. For convenience, let also F denote the graph $H^{\ell_+,\ell_-}_{G,\mathcal{T}_+,\mathcal{T}_-}$, and note that F is a subgraph of both $\widehat{H}_1, \widehat{H}_2$. From Lemma 20, we have that for $i \in \{1, 2\}$, for $\epsilon' = 10|V(H)|\epsilon$, it holds that

$$\mathcal{O}_{\beta,\gamma,\lambda}(\widehat{H}_i) = \mathcal{A}_i|V(H)| + \mathbf{E}_{\sigma\sim\mu_i}\big[o_F(\sigma)\big] + (1 \pm \epsilon')O_i\big[(q_+ - q_-)\mathcal{M}_{\alpha,\hat{\lambda}_i}(H) + q_-|V(H)|\big], \tag{6}$$

where $\mathcal{A}_i = \mathbf{E}_{\sigma\sim\mu_{\mathcal{T}_i}}[o_{\mathcal{T}_i}(\sigma) \mid \sigma_{\rho_i} = 0]$ and $\hat{\lambda}_i := \big(\frac{q_+R_i+1-q_+}{q_-R_i+1-q_-}\big)\big(\frac{q_+R_++1-q_+}{q_-R_++1-q_-}\big)^{\ell_+}/\big(\frac{q_+R_-+1-q_+}{q_-R_-+1-q_-}\big)^{\ell_-}$. From (2), we have that $\hat{\lambda}_i = (1 \pm \epsilon^3)\hat{\lambda}$, and therefore from Lemma 21, we have that

$$\begin{aligned}&\big|\mathbf{E}_{\sigma\sim\mu_1}[o_F(\sigma)] - \mathbf{E}_{\sigma\sim\mu_2}[o_F(\sigma)]\big| \leq |E(H^{\ell}_{G,\mathcal{E},\mathcal{P}})|\epsilon^3 \leq \epsilon^2,\\ &|\mathcal{M}_{\alpha,\hat{\lambda}_1}(H) - \mathcal{M}_{\alpha,\hat{\lambda}_2}(H)| \leq 2K\big(|V(H)|^2 + |E(H)|^2\big)\epsilon^3 \leq \epsilon^2.\end{aligned}$$

We now invoke the oracle for $\mathcal{M}_{\beta,\gamma,\lambda}(\widehat{H}_i)$ to compute $\hat{\mathcal{M}}_i$ such that $\hat{\mathcal{M}}_i = (1 \pm \epsilon^2)\mathcal{M}_{\beta,\gamma,\lambda}(\widehat{H}_i)$. By exploiting the tree structure of the field gadgets $\mathcal{T}_1, \mathcal{T}_2$ (cf. Definition 16), and since they both have size $poly(\text{bits}(r))$, we can compute the values $\mathcal{A}_1, \mathcal{A}_2$ exactly in time $poly(|V(H)|, \frac{1}{\eta})$ by fairly routine dynamic programming techniques. Combining these with (6), it follows that

$$\hat{M} = \frac{1}{q_+ - q_-}\Big(\frac{(\hat{\mathcal{M}}_1 - \hat{\mathcal{M}}_2) - |V(H)|(\mathcal{A}_1 - \mathcal{A}_2)}{O_1 - O_2} - q_-|E(H)|\Big)$$

is within a factor of $(1 \pm \eta)$ of the susceptibility $\mathcal{M}_{\alpha,\hat{\lambda}}(H)$, as needed. This finishes the reduction and therefore the proof of Lemma 6. ◀

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