Characterizing covariational reasoning in physics modeling

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Covariational reasoning—considering how changes in one quantity affect another, related quantity—is a foundation of quantitative modeling in physics. Understanding quantitative models is a learning objective of introductory physics instruction at the college level. Prior work suggests that covariational reasoning in physics contexts differs from the reasoning about functions and graphs in purely mathematical contexts that students develop in math courses; this reasoning is effortful in physics even for mathematically well-prepared students. In order to improve physics students' covariational reasoning, we must first characterize covariational reasoning with physics quantities. To this end, we present a framework of covariational reasoning in physics contexts, to describe the ways that covariational reasoning is used in physics modeling. The framework can be used as a lens through which to analyze student reasoning, and can help inform instructional interventions. We describe an application of this framework in the development of a set of computer-based training assignments.

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I. INTRODUCTION AND BACKGROUND

Covariational reasoning is defined by mathematics education researchers as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" [1]. It is considered an essential tool for pre-calculus and calculus courses, and has thus been extensively characterized in mathematics contexts [1–3]. Covariational reasoning is also at the heart of quantitative modeling in physics, but prior work indicates that physics experts reason mathematically in different ways than mathematics experts [4, 5]. While students in introductory physics courses may have experience with covariational reasoning in mathematical contexts from prerequisite math courses, covariational reasoning in physics contexts is not simply doing math with physics quantities; rather, it requires an understanding of both the mathematics and the quantities themselves [6–10]. This suggests that characterization of *physics* covariational reasoning is necessary to provide guidance for leveraging the experience that students have from math courses for more productive reasoning and quantitative modeling in physics contexts. In this paper, we present a framework that characterizes how covariational reasoning is used in quantitative modeling with physics quantities. The framework is a lens through which student reasoning can be viewed and can provide guidance for the development of instructional interventions.

The covariational reasoning framework we present is based largely on three areas of research: work by mathematics education researchers on covariational reasoning in mathematical contexts; interviews with physics graduate students completing tasks requiring covariational reasoning; and work by physics education researchers on student understanding of mathematics, quantities, and variables.

Mathematics education researchers Carlson et al. developed frameworks to describe levels of development of covariational reasoning, and "mental actions" that are supported by those levels [1]. The framework of mental actions describes five modes of reasoning about relationships between variables that effectively operationalize covariational reasoning in mathematics. An abbreviated version of the mental actions framework is shown in Table I.

TABLE I. Framework of covariational reasoning "Mental Actions" in mathematics contexts developed by Carlson et al. [1]

Mental Action	Description
MA1	Coordinating the value of one variable with changes in another
MA2	Coordinating the direction of change of one variable with changes in another variable
MA3	Coordinating the amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the domain of the function

The mental actions described by Carlson et al. are often considered in the context of graphing. Graphing is covariational reasoning: a graph is an important and ubiquitous representation of a relationship between quantities. Mathematics education researchers Hobson and Moore reported on the results of interviews with mathematics graduate students performing graphing tasks [11]. To investigate if and how physics experts might use covariational reasoning differently from mathematics experts when graphing, Zimmerman et al. replicated their study by interviewing physics graduate students ("experts" in introductory-level physics) using the same tasks [4]. It was found that physics experts were more likely than mathematics experts to: (1) rely on familiarity and facility with physics quantities; (2) rely on experience with physics models to approximate or describe a physics context quantitatively, rather than engage in novel covariatonal reasoning; and (3) consider the covariational relationship between quantities near important points of change, rather than continuously (as in MA5 by Carlson et al.). For more details, see Ref. [4].

Students may enter physics courses comfortable with graphing in mathematical contexts, learned in prerequisite math courses; but substantial research by physics education researchers suggests that facility with graphing in purely mathematical contexts may not transfer to productive interpretation of graphs in physics contexts. For example, McDermott et al. found that students had difficulty connecting features of a graph to the physical quantities they represent [12]. Others have found that the physics content of the graph affected how students interpret graphs and influences approaches to calculations of slope [13, 14].

Research suggests introductory physics students struggle to emulate quantitative reasoning used by physics experts [9, 15–17]. Introductory students may have difficulty with unfamiliar notation though the underlying mathematics is familiar [18, 19]. Sherin's "symbolic forms" were developed to explain how successful physics students interpret and use equations [6]; Sherin suggested that successful students associate symbolic patterns with physical and mathematical meaning. Study of introductory-level physics students indicated that physics learners engage in similar behaviors as physics experts when engaged in covariational reasoning tasks, but do so less productively [20, 21]. Differences between introductory and graduate students in physics led us to consider what introductory physics students bring in from their prerequisite courses in mathematics, and what additional resources graduate students are using to engage in the behaviors more productively.

Here, we present a framework of covariational reasoning in physics contexts, to describe the ways covariational reasoning is used in physics modeling. To demonstrate the applicabil-

TABLE II. Current version of a framework to describe the use of covariational reasoning in physics modeling

PROCEPTUAL UNDERSTANDING	
I. Mathematics Resources	**II. Physics Quantities**
A. Common function behavior	A. Constructing quantity
B. Common function rules	B. Mathematical structure
C. Use of common operations	C. Constraints of quantities
D. Use of common procedures	D. Symbolizing
	E. Combining quantities
PHYSICS MENTAL ACTIONS	EXPERT BEHAVIORS
I. Related Quantities	**I. Compiled Relationships**
II. Trend of Change	A. Proxy Quantities
III. Discrete Change	B. "Goes Like"
IV. Small Chunks of Change	**II. Simplification Techniques**
V. Functional Reasoning	A. Limiting Cases
	B. Physically Significant Points
	C. Symmetry

ity of this framework, we describe the development of a set of computer-based training assignments intended to improve introductory physics students' covariational reasoning.

II. FRAMEWORK

In physics contexts, covariational reasoning is done in the service of quantitative modeling, and is used in conjunction with understanding of physics quantities and familiarity with physics content. The framework presented in Table II characterizes the use of covariational reasoning in physics modeling. We developed this framework by combining previously published findings [1, 7, 9, 11] with additional results from interviews with physics graduate students engaged in covariational reasoning graphing tasks [4]. Generally speaking, the items under the heading *Proceptual Understanding* describe foundational reasoning that is required for productive covariational reasoning; *Physics Mental Actions* describe reasoning that is directly related to novel covariation of quantities; and *Expert Behaviors* describe behaviors observed in physics experts that either facilitate covariational reasoning with physics quantities, or limit the amount of novel covariational reasoning necessary in a given context. Though the framework is presented as having these three distinct parts, we stress that there is significant interaction between the three parts when physics experts use covariational reasoning.

A. Proceptual understanding

Proceptual understanding is defined by Gray and Tall as a combination of *pro*cedural mastery and con*ceptual* understanding [22]. They explain that in the context of fractions, for example, "the symbol $\frac{3}{4}$ stands for both the process of division and the concept of fraction." A student with a proceptual understanding of fractions would move fluidly between the procedure of dividing 3 by 4, and the instantiation of the fraction $\frac{3}{4}$ as a precise quantification of portion. In our framework, we identify elements of proceptual understanding as it relates to covariational reasoning in physics contexts in two broad areas: mathematics resources and physics quantities. We first consider four aspects of proceptual understanding of mathematics resources that define understanding of the underlying mathematics in math contexts, which is a basis for applying the mathematics to physics contexts.

A. Common function behavior: familiarity with the overall behavior of relevant functions. There are a handful of relevant functions in introductory-level physics: linear, quadratic, trigonometric, exponential, and rational functions.

B. Rules for common functions: familiarity with range, domain, etc.

C. Use of common operations: facility with mathematical operations such as addition/subtraction, multiplication/division, and taking derivatives/integrals.

D. Use of common procedures: facility with mathematical procedures such as taking a limit or finding extrema. Procedures often involve using mathematical operations to act on a function.

Physics graduate students may rely on facility with physics quantities when engaging in covariational reasoning [4]. Here, we define proceptual understanding of quantities.

A. Constructing quantity: Generating a quantity to represent a physical quality, or understanding how and why a quantity is generated. The quantity "velocity," for example, can be understood as a quantity that relates a change in position to a time interval through the operation of division, in order to characterize the rate at which an object moves.

B. Mathematical structure: Recognition of features (e.g., scalar or vector, units, sign) of a quantity, and why those features are appropriate for that quantity. In addition, understanding the meaning of the features of a quantity. For example, the negative sign associated with a quantity carries meaning that may be specific to that quantity—the sign associated with electric charge has a different meaning than the sign associated with mechanical work.

C. Constraints of physics quantities: Understanding of the constraints of a quantity. Not all quantities can be negative, and some can only be integers, or particular, discrete values. Some quantities are bounded—speed, for example, cannot exceed c, the speed of light in vacuum. Understanding of constraints also includes understanding of appropriate scale for a given context: a car will not move at an appreciable fraction of c, and a small metal sphere will not have a charge of 5 Coulombs.

D. Symbolizing: Understanding of multiple representations of a quantity (algebraic, graphical, or diagrammatic), as well as translate between them.

E. Combining quantities: Reasoning around how and why mathematical operations are used to combine quantities, including reasoning about combining quantities with different properties or units. For example, recognition that a velocity vector and speed value cannot be added because they are different kinds of quantities.

B. Physics mental actions

The physics mental actions (PMA) are derived from and closely aligned with the mental actions described by Carlson et al. [1]. There are a number of small but important differences between the two sets of mental actions, based on observed differences between mathematics and physics experts. For all of the PMA, we stress that the variables are physics quantities, rather than just numeric values. Moreover, the PMA do not entail reasoning about continuous changes, instead focusing on small "chunks" of change. PMA V does not have an analog in the mental actions by Carlson et al., and seems to stem from knowledge of familiar physics models. Physics mental actions describe direct covariation of quantities; they are explicit instantiations of considering how quantities relate to each other or how changes in one quantity lead to changes in the other.

I. The recognition that one quantity changes with another; describing one quantity as a function of another is a behavior associated with this PMA.

II. Connecting the trend of change of one quantity with the trend of change of the other quantity. One associated behavior is an explicit statement of the trend, e.g., "the electric force decreases as separation increases."

III. Consideration of the change in quantity due to a change of a specified amount in another; the statement "the electric field decreases by a factor of 4 when the separation doubles" is an example of an associated behavior.

IV. An "almost continuous" consideration of change in one quantity due to a small (almost infinitesimal) change in another. Recognition that the electric force between two point changes more quickly when the charges are closer together than when they are farther apart.

V. Using a functional relationship between quantities to consider how two quantities change with respect to each other. An associated behavior would be recognition that the force between two point charges goes as $1/r^2$, with the exact value at any point determined by physical constants and the magnitude of the charges.

C. Expert behaviors

The expert behaviors were informed by the graphing task interviews with physics graduate students by Zimmerman, et al. These interviews started with the Hobson and Moore replication study, and continued with novel graphing tasks developed specifically for physics experts (see Ref. [4] for more detail). In general, expert behaviors are used in ways that tend to limit or guide the use of PMA, though we don't claim that this is a conscious process. Here, we give an overview of the behaviors, in the context of the *Ferris wheel* item. The *Ferris wheel* interview task, developed by Hobson and Moore and used to study covariational reasoning with both mathematics and physics experts [4, 11], used an animation of a Ferris wheel rotating at constant angular speed, and asked subjects to create a graph to relate the height of a Ferris wheel car to the total distance the car had traveled around the wheel. A still of the item, and a correct response produced by a physics graduate student is shown in Fig. 1.

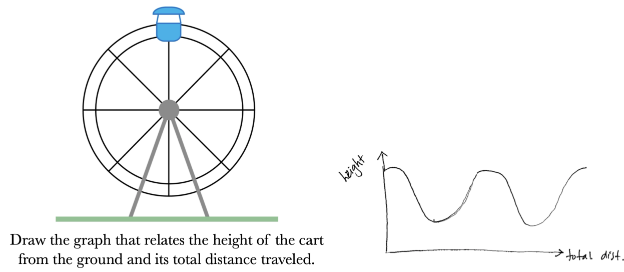


FIG. 1. Left: animation still of the *Ferris wheel* interview task given to mathematics and physics graduate students. Right: graph deemed correct created by a physics expert.

Compiled Relationships describes behaviors that use understanding of physics quantities or familiarity with physics contexts, and minimize mental effort. *Proxy quantities* is the use of a different, more familiar quantity substituted for another when covarying two quantities. *"Goes Like"* reasoning refers to ways physicists relate two quantities through a simplified function. These behaviors are illustrated below.

When creating a graph for the *Ferris wheel* item, some physics experts began by creating a graph that related the height of the car to the time elapsed rather than total distance traveled. They used time as a *proxy quantity* for total distance; for uniform circular motion, time and total distance traveled are directly proportional. In addition, multiple physics experts noted that height *goes like* a trigonometric function; that is, they identified a direct functional relationship between quantities based on familiarity with the context.

Use of Simplification Techniques describes behaviors which physics experts engaged in that both guide and limit the amount of novel covariational reasoning that is required in a given physics context. *Limiting cases* describes the habit-of-mind of taking a limit, as well as the physics content knowledge to choose a productive limit to take. Experts often choose *physically significant points* around which to engage in the physics mental actions. For example, some physics experts did not use "goes like" reasoning when completing the *Ferris wheel* task; these students tended to engage in covariational reasoning about the motion of the car at the sides and top of the Ferris wheel (i.e., at the points at which the height of the car changes the most and least rapidly as a function of total distance traveled). Finally, many experts simplify problems using symmetry to reduce complexity. As an example, some physics graduate students recognized that the height of the Ferris wheel car would decrease in the same way that it increased, and used this knowledge to quickly complete their graphs.

III. APPLICATION OF THE FRAMEWORK

Our framework lays out a number of foundational "skills" that underlie productive covariational reasoning in physics contexts. We describe here an example of using the framework to inform the development of instructional interventions. The tasks are designed to strengthen student facility with these skills with the intent of supporting students' use of covariational reasoning in physics modeling.

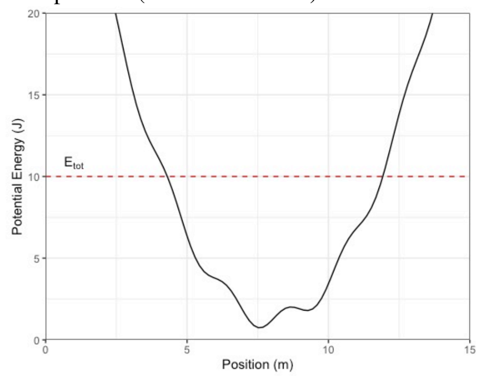
Physics education researchers Mikula and Heckler described a framework for improving physics essential skills (ES) [23], fundamental skills that are necessary for problem-solving in STEM fields. They are largely mathematical and procedural in nature, such as vector superposition and algebraic manipulation of variables. While such skills are easy and automatic for physics experts, they can be more time-consuming and effortful for students. The ES framework describes a structure for computer-based assignments to improve students' fluency with these skills; when students are able to do the math quickly and accurately (i.e., *fluently*), they are able to dedicate more cognitive effort to understanding physics content and reasoning. Similarly, we believe that ES assignments featuring more conceptual essential skills can also facilitate physics problem solving. Recently developed ES items that focus on interpretation of the meaning of the negative sign in physics contexts [17], and proportional reasoning [24] may be effective ways to increase students' quantitative reasoning [25].

To this end, we are using the framework described in this paper to identify essential skills related to covariational reasoning, and developing ES assignments that target those skills. Our first set of covariational reasoning ES items focuses on graphing, and targets proceptual understanding of physics quantities.

As described in Sec. II, using covariational reasoning in physics modeling requires a proceptual understanding (PU) of physics quantities, including their graphical representations (PU II.D in the framework). Interpreting a graph in a physics context entails understanding what various graphical features mean in terms of the physics quantities they represent. For example, for an object moving in one dimension, the slope of a position vs. time graph is a named physics quantity (velocity), and has a physical interpretation (the time rate of change of position, PU II.A). We consider interpretation of graphical features, such as the slope, to be an essential skill, and developed a set of ES items to improve student fluency.

The graphical features ES items involve a wide range of physics topics (e.g., mechanics, temperature, volume) and non-physics topics. Items may ask for interpretation of a graphical feature such as a point, the slope of a line, or the area under the curve (PU II.A), or the identification of a quantity associated with a given graphical feature (PU II.D). Other items ask students to identify the graphical feature that is associated with a given interpretation or quantity. Because proceptual understanding of physics quantities involves facility with units and unit manipulations (PU II.E), many of the items focus on the units of quantities represented in a graph. An example graphical feature ES item is shown in Fig. 2.

Graphs like the one below are referred to as "potential energy diagrams" and show how potential energy varies with position (in one dimension).



How could you use the potential energy diagram above to find a quantity that has units of Newtons?

(a) Find the slope of the graph
(b) Find the area under the curve
(c) Find the vertical coordinate of a point on the graph
(d) Find the vertical intercept

FIG. 2. Essential skills item intended to improve student fluency with interpretation of graphical features. The correct answer is (a), and uses PU II. A, D, and E.

The covariational reasoning framework may be particularly useful for understanding the gap between the reasoning students develop in math courses and the reasoning they are expected to use in physics courses. The intervention described here may help bridge that gap by developing reasoning about physics quantities in the context of covariational reasoning.

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