Abstract

The remarkable performance of deep learning models and their applications in consequential domains (e.g., facial recognition) introduces important challenges at the intersection of equity and security. Fairness and robustness are two desired notions often required in learning models. Fairness ensures that models do not disproportionately harm (or benefit) some groups over others, while robustness measures the models’ resilience against small input perturbations.

This paper shows the existence of a dichotomy between fairness and robustness, and analyzes when achieving fairness decreases the model robustness to adversarial samples. The reported analysis sheds light on the factors causing such contrasting behavior, suggesting that distance to the decision boundary across groups as a key explainer for this behavior. Extensive experiments on non-linear models and different architectures validate the theoretical findings in multiple vision domains. Finally, the paper proposes a simple, yet effective, solution to construct models achieving good tradeoffs between fairness and robustness.

1. Introduction

Data-driven learning systems have become instrumental for decision-making in a variety of consequential contests. They include assistance in legal decisions [16], lending [33], hiring [30], performing personalized ads targeting [6], and providing personalized recommendations [4]. As a result, fairness has become a crucial requirement for their successful use and adoption. Various notions of fairness drawing from legal and philosophical doctrine have been proposed to ensure that the models errors do not disproportionately affect the decisions of some groups over others [25]. In general, fair models attempt at constraining their hypothesis space so that the errors of the reported outcomes are distributed uniformly across different protected groups.

When these fairness constraints are enforced in learning systems, a commonly observed behavior is an overall degradation of the model accuracy. Thus, a growing body of research has been focusing on striking the right balance between fairness and accuracy [29]. This paper shows that fairness may have another important consequence on the deployed models: a reduction of the model robustness. This aspect is important as the vulnerability of deep learning models to adversarial examples hinders their application in many security-sensitive domains. However, these behaviors are currently not fully understood and have not received the attention they deserve given the significant equity and security consequences they have on the final decisions.

This paper addresses this important gap and shows that enforcing fairness may negatively affect the robustness of a model. In particular, the paper makes the following contributions: (1) it analyzes when and why fairness and robustness may be misaligned in their objectives, (2) it provides an analysis on the relationship between fair, robust, and "natural" (e.g., non-fair non-robust) models, and (3) it identifies the distance to the decision boundary as a key aspect linking fairness and robustness. Moreover, (4) the paper shows how the distance to the decision boundary can explain the increase of adversarial vulnerability of fair models, providing extensive experiments and validation over a variety of vision tasks and architectures, and verifying the presence of the fairness/robustness dichotomy for multiple techniques aimed at achieving fairness and measuring robustness. Finally, (5) building from the reported theoretical observations, the paper also proposes a simple, yet effective, strategy to find a good tradeoff between accuracy, fairness, and robustness.

To the best of the authors’ knowledge, this is the first work showing that enforcing fairness may negatively affect the robustness of a model. It is important to note that this result should not be read as an endorsement to avoid constructing fairer or safer models; rather it should be understood as a call for additional research at the intersection of fairness and robustness to achieve appropriate tradeoffs.

2. Related work

Fairness. Models that learn from rich datasets have been shown to carry over bias which may induce to disproportionately harm some groups of individuals (often identified
by their race or gender) over others [3, 19, 39]. These observations have resulted in a whole new research area that has focused on defining, analyzing, and mitigating unfairness [8, 45]. The source of the observed unfairness has been often attributed to data properties [5, 25, 34] or different aspects of the model’s properties [35, 38]. For example, imbalance in groups’ size is commonly argued to create disparities in the task’s performance [25]. It has also been shown that constraining the model’s hypothesis space to satisfy privacy [2], sparsity [14, 15], or robustness [26, 44] can result in disparate outcomes.

**Robustness.** Deep Neural Networks (DNNs) have been shown to be susceptible to carefully crafted adversarial perturbations which—imperceptible to a human—result in a misclassification by the model [36]. The literature on the topic attributes the reason for such behavior to arise to three key factors: data properties, network architecture, and model training. For example, [11, 31] observed that the input dimension plays a decisive role in the robustness of a model, with larger inputs yielding more brittle models. Likewise, the ability of some architectures to capture high frequency spectrum of image data (that are almost imperceptible to a human) relates to the success of devising adversarial examples in the underlying models [41]. With respect to the network architecture, it has been observed that some network architectures may render the underlying model more or less brittle to adversarial inputs. For example, batchnormalization is cited to the models’ robustness [9], some shift-invariant architectures, such as CNNS, are more vulnerable to adversarial inputs compared to other architectures, e.g., fully connected networks. Finally, Yao et al. [46] empirically observed that model trained with large batch-sizes can be more easily fooled by adversarial inputs when compared to models trained with smaller batch sizes.

**Robustness and fairness.** This work lies in the intersection of fairness and robustness. Within this context, recently Xu et al. [44] has shown that adversarially robust models exhibit remarkable disparity of natural accuracy and robust accuracy metrics among different classes, compared to those exhibited by their standard counterpart. Khani and Liang [17] analyze why noise in features can cause disparity in error rates when learning a regression.

We believe this is the first work to show that enforcing fairness may negatively affect a model’s robustness and hope that this result will lead to further strengthening the interconnection of these two important machine learning areas.

### 3. Problem Settings

The paper considers a typical multi-class classification problem, whose input is a dataset $D$ consisting of $n$ data points $(X_i, A_i, Y_i)$, each of which drawn i.i.d. from an unknown distribution $\Pi$ and where $X_i \in \mathcal{X}$ is a feature vector, $A_i \in \mathcal{A}$ is a protected attribute, and $Y_i \in \mathcal{Y} = [C]$ is a label, with $C$ being the number of possible class labels. For example, consider the case of a classifier to predict the age range of an individual. The features $X_i$ may describe the pixels associated with the individual headshot and their demographics, the protected attribute $A_i$ may describe the individual gender or ethnicity, and $Y_i$ represents the age range. The goal is to learn a classifier $f_\theta: \mathcal{X} \rightarrow \mathcal{Y}$, where $\theta$ is a vector of real-valued parameters. The model quality is assessed in terms of a non-negative loss function $\ell: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_+$, and the training aims at minimizing the empirical risk function:

$$\hat{\theta} = \arg\min_{\theta} \mathcal{L}_\theta(D) \left( = \frac{1}{n} \sum_{i=1}^{n} \ell(f_\theta(X_i), Y_i) \right)$$

(1)

For a group $a \in \mathcal{A}$, notation $D_a$ is used to denote the subset of $D$ containing exclusively samples $i$ with $A_i = a$. Importantly, the paper assumes that the attribute $A$ is not part of the model input during inference. The paper focuses on learning classifiers that satisfy group fairness (to be defined shortly) and on analyzing the robustness impact of fairness.

### 4. Preliminaries

#### 4.1. Fairness and fair learning

This paper considers a classifier $f$ satisfying accuracy parity [49], a group fairness notion commonly adopted in machine learning requiring model misclassification rates to be conditionally independent of the protected attribute. That is, $\forall (X, Y, A) \sim \Pi$ and $\forall a \in \mathcal{A}$,

$$|\Pr(f_\theta(X) \neq Y \mid A = a) - \Pr(f_\theta(X) \neq Y)| \leq \alpha, \quad (2)$$

where $\alpha$ denotes the allowed fairness violation. In practice, the above is expressed as a difference of empirical expectations of the group and population misclassification rates. That is, $\forall a \in \mathcal{A}$:

$$\left| \frac{1}{|D_a|} \sum_{(X, A, Y) \in D_a} 1\{f_\theta(X) \neq Y\} - \frac{1}{n} \sum_{(X, A, Y) \in D} 1\{f_\theta(X) \neq Y\} \right| \leq \alpha.$$

Several approaches have been proposed in the literature to encourage the satisfaction of accuracy parity. They can be summarized in methods that use penalty terms into the empirical risk loss function to capture the fairness violations, which—imperceptible to a human—result in a group and population misclassification rates. The above is expressed as a difference of empirical expectations of the group and population misclassification rates.

#### 4.1. Penalty-based methods

In this category, the model loss function (Equation (1)) is augmented with penalty fairness constraint terms [1, 40] as follows:

$$\theta_i(\lambda) = \arg\min_{\theta} \mathcal{L}_\theta(D) + \lambda \left( \sum_{a \in \mathcal{A}} |\mathcal{L}_\theta(D_a) - \mathcal{L}_\theta(D)| \right)$$

(3)
where $L_\theta(D_\alpha) = \frac{1}{|D_\alpha|} \sum_{(X,A,Y)\in D_\alpha} \ell(f_\theta(X), Y)$ is the empirical risk loss associated with protected group $\alpha \in A$. In addition, $\lambda > 0$ is the fairness penalty parameter that enforces a tradeoff between fairness and accuracy.

### 4.2. Robustness and Robust Learning

This paper analyzes the effect of enforcing fairness on adversarial robustness, a key property of trustworthy machine-learning systems. In this work, and following robust learning conventions, the robustness of a model $f$ is measured in terms of the **robust error**:

$$L_\theta^{\text{rob}}(\epsilon) = \Pr(\exists \tau, \|\tau\|_p \leq \epsilon, f_\theta(X + \tau) \neq Y),$$

(4)

which measures the sensitivity of the model errors to small input perturbations $\|\tau\|_p \leq \epsilon$ in $\ell_p$ norms, with $p$ often considered in $\{0, 1, 2, \infty\}$. The robust error can be decomposed into two components [47]:

$$L_\theta^{\text{rob}}(\epsilon) = L_\theta^{\text{nat}} + L_\theta^{\text{bdy}}(\epsilon),$$

(5)

where the first denotes the **natural error** and the second the **boundary error**. The natural error measures the standard model performance when exposed to unperturbed samples $(X, A, Y)$:

$$L_\theta^{\text{nat}} = \Pr(f_\theta(X) \neq Y),$$

(6)

whose empirical version is defined in Equation (1) with a 0/1 loss function. The boundary error measures the probability that the model predictions change on perturbed samples $(X + \|\tau\|_p, A, Y)$:

$$L_\theta^{\text{bdy}}(\epsilon) = \Pr(\exists \|\tau\|_p \leq \epsilon, f_\theta(X + \tau) \neq f_\theta(X), f_\theta(X) = Y).$$

(7)

The boundary error implicitly introduces a notion of **decision boundary** and a **distance between an input sample and this decision boundary**. For instance, in linear classifiers, the decision boundary is represented by an hyperplane. The distance of a sample $X$ to the decision boundary for a classifier $f_\theta$ can be formalized as

$$\Delta(X, f_\theta) = \max \epsilon \text{ s.t. } f_\theta(X + \tau) = f_\theta(X), \forall \|\tau\| \leq \epsilon.$$

Samples close to the decision boundary will be less tolerant to noise than those lying far from it. The analysis in this paper regarding the impact of fairness on robustness is based on this concept. In particular, the results show that imposing fairness constraints may reduce the distance to the decision boundary of the samples $(X, A, Y) \sim \Pi$.

### 5. Real-World Implications

Prior diving into the analysis, the paper provides an example showing how robustness errors can be exacerbated when a image classifier is trained to satisfy fairness. Deep neural networks have been used in many real-world applications, including image facial recognition and object detection. When perturbations (either due to noise or by malicious adversaries) are introduced in the model inputs, they may cause harmful effects as they lead the classifier to misclassify targeted inputs.

Figure 1 shows an example of inputs from the UTKFace dataset where a classifier is trained to minimize the regular empirical risk loss of equation (1) (top) or the fair empirical risk loss of equation (3) (bottom). Both inputs are perturbed with the same amount of $\ell_\infty$ noise, but the fair network is much more brittle than its regular counterpart, inducing errors in the classifier outputs. It is important to note that this
Consider a binary classification setting (i.e., \( Y = \{-1, 1\} \)) with data drawn from a mixture of Gaussian distributions, so that \( \Pr(X | Y = -1) \propto \mathcal{N}(\mu_-, K) \) and \( \Pr(X | Y = 1) \propto \mathcal{N}(\mu_+, K) \), with \( \mu_- < \mu_+ \) and different variances \((K > 1)\). The analysis can be easily extended to higher-dimensional cases, but these non-restrictive assumptions help simplifying and clarifying exposition. An illustration of this setting is reported in Figure 2 (top) where the data distributions are highlighted with black dashed curves.

The following analysis poses no restrictions on the relative subgroup sizes \(|D_1|\) and \(|D_{-1}|\) and focuses on the less-restrictive balanced data setting, in which data samples from different protected groups are equally likely.

The paper studies a family of parametric classifiers \(\{f_\theta\}_\theta\) with \(\theta \in [\mu_-, \mu_+] \subseteq \mathbb{R}\), where \(f_\theta(X) = \mathbb{1}\{X > \theta\}\) denotes the classification output of the classifier. The optimal models with respect to the natural, fair, and robust losses can be specified as follows:

**Optimal natural model** \(f_{\theta^*}\). It is the Bayes classifier which minimizes the natural classification error as defined in Equation \((1)\). In Figure 2 (top), this classifier is represented by vertical blue lines.

**Optimal fair model** \(f_{\theta_1}\). Intuitively, this classifier is \(\theta_1(\infty)\) as defined in Equation \((3)\). Formally speaking, this classifier minimizes a lexicographic function whose first component is \(\sum_{a \in A} (\mathcal{L}_a(D_a) - \mathcal{L}_a(D))\) and second component is \(\mathcal{L}_\theta(D)\). In Figure 2 (top), this classifier is represented by vertical red lines.

**Optimal robust model** \(f_{\theta_1(\epsilon)}\). This classifier minimizes the robust classification error in Equation \((5)\), for a given \(\epsilon\). In Figure 2 (top), it is depicted by vertical green lines.

**Relationships Between the Optimal Models**

The next result characterizes the positional relationship among the three optimal models mentioned above, which can be observed in Figure 2.

**Theorem 6.1.** For any \(\epsilon \in \left[0, \frac{\mu_+ - \mu_-}{2}\right]\) and \(K \in (1, B_K]\), where \(B_K = \min \left\{ \exp \left(\frac{\mu_+ - \mu_-, \epsilon}{\epsilon}\right), \frac{\mu_+ - \mu_-}{\epsilon} - 1 \right\}\),

\[ \mu_- + \epsilon \leq \theta_1 \leq \theta_1(\epsilon) \leq \mu_+ - \epsilon . \]  

Besides, \(\theta_1(\epsilon)\) is an increasing function of \(\epsilon\) over \([0, \frac{\mu_+ - \mu_-}{2}]\).

The result follows from the observation that the optimal natural model \(f_{\theta^*}\) can be expressed as

\[ \hat{\theta} = \mu_- - \frac{\mu_+ - \mu_-}{K^2 - 1} + \frac{K}{K^2 - 1} \sqrt{2(K^2 - 1) \ln(K) + (\mu_+ - \mu_-)^2}; \]

the fair classifier \(f_{\theta_1}\) as:

\[ \theta_1 = \mu_- + \frac{\mu_+ - \mu_-}{K + 1} \]

and the robust classifier \(f_{\theta_1(\epsilon)}\) as

\[ \theta_1(\epsilon) = \mu_- + \frac{\mu_+ - \mu_- - (K^2 + 1)\epsilon}{K^2 - 1} + \frac{K}{K^2 - 1} \sqrt{2(K^2 - 1) \ln(K) + (\mu_+ - \mu_- - 2\epsilon)^2}. \]

From the result above, it follows that (1) the fair classifier achieves the largest robust error while the robust classifier...
results in the least error, and (2) the fair classifier achieves
the largest boundary error while the robust classifier results in
the smallest boundary error, as expressed by the following
Corollaries.

**Corollary 6.2.** For any \( \epsilon \in \left[ 0, \frac{\mu_- - \mu_+}{2} \right] \) and \( K \in (1, B_K) \),

\[
L_{\theta_1}^{\text{rob}} (\epsilon) \geq L_{\theta_1}^{\text{rob}} (\epsilon) \geq L_{\theta_1}^{\text{rob}} (\epsilon) .
\]

**Corollary 6.3.** For any \( \epsilon \in \left[ 0, \frac{\mu_- - \mu_+}{4} \right] \) and \( K \in (1, \bar{B}_K) \),

\[
L_{\theta_1}^{\text{bdy}} (\epsilon) \geq L_{\theta_1}^{\text{bdy}} (\epsilon) \geq L_{\theta_1}^{\text{bdy}} (\epsilon) ,
\]

where \( \bar{B}_K = \min \left\{ \exp \left( \frac{2 \epsilon}{\mu_- - \mu_+ - 2} \right), \phi^{-1} \left( \frac{\mu_- - \mu_+ - 2}{\mu_- - \mu_+} \right) \right\} \)
and \( \phi^{-1} \) is the inverse function associated with \( \phi : [1, +\infty) \rightarrow [2, +\infty) \) such that \( \phi(x) = x + 1/x \).

These results highlight the impossibility of achieving fairness
and robustness simultaneously in this classification task.
Fairness and robustness are pulling the classifier in opposite
directions.

**The Role of the Decision Boundary**

Building on the previous results, this section provides the key
theoretical intuitions to explain why fairness increases adversarial vulnerability. It identifies the average distance to
the decision boundary as the central aspect linking fairness
and robustness, which is formalized in Theorem 6.4.

**Theorem 6.4.** For any \( \epsilon \in \left[ 0, \frac{\mu_- - \mu_+}{2} \right] \) and \( K \in (1, B_K) \),

\[
\mathbb{E} \left[ \Delta \left( X, f_{\theta_1}(\epsilon) \right) \right] \geq \mathbb{E} \left[ \Delta \left( X, f_{\theta_1} \right) \right] \geq \mathbb{E} \left[ \Delta \left( X, f_{\theta_1} \right) \right].
\]

In addition, the fair model minimizes the average distance
to its decision boundary over all valid classifiers, i.e.,

\[
\theta_1 = \arg\min_{\theta \in [\mu_- , \mu_+]} \mathbb{E} \left[ \Delta \left( X, f_{\theta} \right) \right].
\]

This result indicates that, among the three considered optimal
models, the fair model has the smallest average distance to
the decision boundary, while the robust model has the largest
distance. The result above is exemplified in Figure 2. The
bottom plots show the losses associated with the optimal
natural, fair, and robust models for two choices of \( K \) (left
and right) while the top plots show the optimal decision
boundaries associated with each of the three models – notice
that they correspond to the minima of their relative losses.

Observe how class class \( Y = 1 \) has a higher classification
error than class \( Y = -1 \) under the natural (and thus unfair) classifier \( f_{\theta} \). This is intuitive since the conditional
distribution \( \Pr ( X \mid Y = 1 ) \) has much higher variance than
\( \Pr ( X \mid Y = -1 ) \). Hence, to balance the classification errors, the fair classifier pushes the decision boundary towards the mean of class \( Y = -1 \). This increases the error of class \( Y = -1 \) while decreasing the error of class \( Y = 1 \). In
contrast, the robust classifier pushes the decision boundary
far away from the dense input region, i.e., the mean of the
data associated with class \( Y = -1 \).

There are a few points worth emphasizing. First, robustness
and fairness pull the decision boundary into two opposite
directions. Second, the fair model \( f_{\theta_1} \) results in predictions with higher robust errors, when compared to the optimal natural model \( f_{\theta} \), and it also increases adversarial vulnerability as the variance \( K \) increases. The variance \( K \)
regulates the difference in the standard deviation of the underlying distributions associated with the protected groups
and thus controls the overall distance to the decision boundary.
In summary, fairness can reduce the average distance of the training samples to the decision boundary which, in
turn, makes the model less tolerant to adversarial noise.

This section concludes with another important result. The
previous relationships continue to hold even when the optimality
conditions of the fair classifier are relaxed, i.e., when \( \lambda \) is taking values different from \( \infty \). Moreover, the fairness
constraints always reduce the distance to the decision
boundary among protected groups and this reduction is propor
tional to the strength of the fairness constraints (or the
tightness of the required fairness bound \( \alpha \)).

**Theorem 6.5.** Consider the fair classifier \( f_{\theta_1(\lambda)} \) that optimizes Equation (3). It follows that, for any \( \lambda \in \left( \frac{K-1}{K+1}, +\infty \right) \),

\[
\theta_1(\lambda) = \theta_1
\]

while for any \( \lambda \in \left[ 0, \frac{K-1}{K+1} \right] \), \( \theta_1(\lambda) = \mu_+ - \frac{\mu_- - \mu_+}{K+1} + \frac{K}{K^2 - 1} \sqrt{2(K^2 - 1) \ln \left( \frac{1+K}{1+K} \cdot K \right) + (\mu_+ - \mu_-)^2} \). Moreover, the parameter \( \theta \) associated with the fair classifier and the average distance to its decision boundary
\( \mathbb{E} \left[ \Delta \left( X, f_{\theta_1(\lambda)} \right) \right] \) are both decreasing as \( \lambda \) increases.

Informally speaking, Theorem 6.5 states that applying fairness
constraint with large enough penalty \( \lambda \) will push the decision boundary towards the negative class (group with smallest variance). As a result, the average distance to the decision boundary of all samples will be reduced.

While the analysis above applies to the linear setting considered in this section, the results were empirically validated
on large non-linear models. For example, Figure 3 compares the performance of a penalty based fair CNN model (bot
tom plots) with \( \lambda = 1.0 \) against a natural (non-fair) CNN
classifier (top plots). The left plots report the task accuracy
by each subgroup (denoting races) and average distance to
decision boundary (right) of each subgroup. Note how the
fair classifier reduces the disparities in task accuracy experienced by the various subgroups. This effect, however, also reduces the overall average distance to the decision boundary. As a consequence, fair models will be more vulnerable to adversarial perturbations.

The next sections focus on assessing these theoretical intuitions onto general non-linear classifiers in a variety of settings and on devising a possible mitigation strategy to balance a good tradeoff between fairness and robustness.

7. Beyond the Linear Case

This section validates the theoretical intuitions presented above on much more complex architectures, datasets, and loss functions. The experiments focus on highlighting fairness, robustness, errors, and their relation to the distance to the decision boundary. When $f_\theta$ is a non-linear model, computing the distance to the decision boundary becomes a computational challenge. Thus, this section uses a commonly adopted proxy metric that measures the difference between the first two order statistics of the softmax outputs in the model [21,42].

Datasets. The experiments of this section focus on three vision datasets: UTK-Face [48], FMNIST [43] and CIFAR-10 [20]. The adopted protected groups and labels in the UTK-Face datasets are ethnicity (White/Black/Indian/Asian/Others) or age (nine age bins), resulting in two distinct tasks. For FMNIST and CIFAR, the experiments use their standard labels and assume that labels are also protected groups, mirroring the setting of previous work [23,39,44]. A complete description of the dataset and settings is found in Appendix D.

Settings. The experiments consider several deep neural network architectures, including CNN [27], ResNet 50 [13] and VGG-13 [32]. The former uses 3 convolutional layers followed by 3 fully connected layers. Models trained on the UTK-Face data use a learning rate of $1e^{-3}$ and 70 epochs. Those trained on FMNIST and CIFAR, use a learning rate of $1e^{-1}$ and 200 epochs, as suggested in previous work [44]. For all datasets and models, unless otherwise specified, a batch size of 32 is used. The experiments analyze penalty-based fairness method, RFGSM attacks [37], and the VGG-13 network, unless specified otherwise. Additional experiments using group-loss focused method (see Appendix B), additional network architectures, and adversarial attacks are reported in Appendix D.

Fairness impacts on the decision boundary

As shown by Theorem 6.5, fairness reduces the average distance of the testing samples to the decision boundary. This section illustrates how this result carries over to larger non-linear models. Figure 4 reports results obtained by executing the penalty-based fair models on the UTK-Face datasets for ethnicity (top) and age (middle) classification and on CIFAR (bottom). A clear trend emerges: As more fairness is enforced (larger $\lambda$ values), the natural errors (left plots) increase, while the fairness violations (center plots) decrease. Importantly, and in agreement with the theoretical results, the experiments report a sharp reduction to the average distance to the decision boundary (right plots). This behavior renders fair models more vulnerable to adversarial attacks, as will be highlighted shortly. Similar results are also observed for the group-loss based models and other architectures.

Boundary errors increase as fairness decreases

This section highlights the key consequence of the sharp reduction to the average distance to the decision boundary: the increase of the vulnerability to adversarial attacks. Figure 5 (top) reports the natural errors (left), boundary errors (middle), and fairness violations (right) for a VGG-13 model trained on UTKFace dataset on the ethnicity task using a standard cross-entropy (CE) loss. Once again, other archi-
The natural errors and fairness violations are reported for fair classifiers, at varying of the fairness violation parameter $\lambda$. The boundary errors (middle) are reported for classifiers satisfying various fairness levels (i.e., using different $\lambda$ values) and at varying of the strength $\varepsilon$ of the desired robustness level (see Equation (4)).

Notice how, compared to the natural models, the fair models incur much higher natural and boundary errors. In particular, the relative increase in boundary errors are significant: The fairness models have boundary errors that are up to 9% larger than their natural counterparts. These observations match the theoretical analysis and highlight a significant increase in vulnerability to adversarial examples by the fair models, even for moderate selections of the fairness violation parameters $\lambda$.

**Enforcing Fairness and Robustness Simultaneously**

This section considers an additional experiment to show how fairness may negatively impact robustness. It reports results of a classifier trying to enforce both fairness and robustness, similarly to the proposal of Xu et al. [44]. The resulting model aims at solving the following regularized ERM problem:

$$\min_{\theta} \frac{1}{n} \sum_{i=1}^{n} \max_{\|\tau\|_p \leq \varepsilon_s} \ell(f_\theta(X_i + \tau), Y_i) + \lambda \left[ \frac{1}{|D_+|} \sum_{(X, A, Y) \in D_+} \ell(f_\theta(X), Y) - \frac{1}{n} \sum_{i=1}^{n} \ell(f_\theta(X_i), Y_i) \right]$$  \hspace{1cm} (9)$$

using stochastic gradient descent. The first component aims at increasing the robustness of the classifier under a margin perturbation $\varepsilon_s$, following the PGD training [24] with perturbation norm $p = \infty$. It works by first generating adversarial samples $X_i + \tau$, where $\|\tau\|_\infty \leq \varepsilon_s$, and then the learning progress aims at minimizing the loss between the model prediction for that adversarial samples and the ground-truth $\ell(f_\theta(X_i + \tau), Y_i)$. The larger the margin perturbation $\varepsilon_s$, the more robust the resulting classifier. The second component implements a penalty-based fairness strategy [1], which promotes fairness by penalizing the difference among each groups’ average loss and the overall’s average loss.

The experiments vary the margin perturbation $\varepsilon_s$ (robustness) and the penalty value $\lambda$ (fairness). Figure 6 reports the (natural) error (left), fairness violations (middle) and average distance to the decision boundary (right) at varying of the margin perturbation $\varepsilon$ and fairness parameters $\lambda$. 

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1With the caveat that VGG-13 could not be used for FMNIST since the 28x28 pixel resolution of FMNIST is smaller than that required by some VGG filters.
Figure 7. Robust errors for different attack levels $\epsilon$ of a robust and fair classifier at varying of the margin perturbation $\epsilon_\alpha$ and fairness parameters $\lambda$.

Figure 8. Classifiers obtained using different loss functions (top) and the associated natural and robust error obtained by such losses.

average distance to the decision boundary (right) for different levels of the margin perturbation $\epsilon_\alpha$ on the UTK-Face (ethnicity) dataset. As expected, enforcing larger margin perturbations $\epsilon_\alpha$ increases the average distance to the decision boundary (thus improving robustness), but at the cost of significantly increasing the natural errors. Increasing the fairness parameter $\lambda$ decreases the average distance to the decision boundary.

Figure 7 reports the robust errors under different levels of adversarial attacks, which are specified by the level of perturbation $\epsilon$. Notice how the level of defense $\epsilon_\alpha$ correlates with higher robustness (and smaller average distances to the decision boundary) for all fairness parameters $\lambda$ tested. These results show the challenge to achieving simultaneously robustness, fairness, and accuracy.

Overall, the results show that, without a careful consideration, inducing a desired equity property on a learning task may create significant security challenges. This should not be read as an endorsement to satisfy a single property, but as a call for additional research at the intersection of fairness and robustness in order to design appropriate tradeoffs.

8. A Mitigating Solution with Bounded Losses

While the previous sections have shown that the conflict between fairness and robustness is unavoidable, this section proposes a theoretically motivated solution attempting to attenuate this tension. The proposed solution relies on the observation that, using standard (unbounded) loss functions, misclassified samples lying far away from the decision boundary are associated to much larger losses than those which are closer to it. Recall also that the decision boundary was found as the predominant factor linking fairness and robustness. This key observation suggests the use of a bounded loss function, defined as \cite{7, 10}:

$$\ell_{\text{Ramp}}(f_\theta(X), Y) = \min(1, \max(0, 1 - Y f_\theta(X))).$$

and referred to as \textit{Ramp loss}, with domain $[0, 1]$. The proposed strategy simply applies this loss function to a fair classifier (Equation 3). Its benefits can be appreciated in Figure 8, which reports the results for the same setting used in the previous section and compares a fair classifier trained using the ramp loss with one trained using a 0/1-loss (which is also bounded but not differentiable), a log-loss, and an exponential loss (both unbounded) (top). The results show that the fair classifier trained using a ramp-loss is the least impacted by misclassified samples, resulting in lower robust errors compared to unbounded losses. It can be observed in the bottom subplot, where its associated loss is the closest, among all differentiable losses, to the local minima. The observed benefits of the ramp loss also carry over high-dimensional data and non-linear models, as shown in Figure 5 bottom, and further reported in Appendix D.

9. Conclusions

This paper was motivated by two key challenges brought by the adoption of modern machine learning systems in consequential domains: fairness and robustness. The paper observed and analyzed the relationship between these two important machine-learning properties and showed that fairness increases vulnerability to adversarial examples. Through a theoretical analysis on linear models, this work provided a new understanding of why such tension arises and identified the distance to the decision boundary as a key explanation factor linking fairness and robustness. These theoretical findings were validated on non-linear models through extensive experiments on a variety of vision tasks. Finally, building
from this new understanding, the paper proposed a simple, yet effective, strategy to find a better balance between accuracy and robustness. We hope these results could stimulate a needed discussion and research at the intersection of fairness and robustness to achieve appropriate tradeoffs.

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A. Missing Proofs

Proof of Theorem 6.1

Proof. \( i \) Notice that the natural classification error and its derivative can be expressed as

\[
\mathcal{L}^\text{nat}_\theta = \Pr (f_\theta (X) \neq Y)
\]

\[
= \frac{1}{2} \Pr (f_\theta (X) \neq 1 \mid Y = 1) + \frac{1}{2} \Pr (f_\theta (X) \neq -1 \mid Y = -1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}K} \exp \left( -\frac{(x - \mu_+)^2}{2K^2} \right) dx + \frac{1}{2} \int_{\theta}^{+\infty} \frac{1}{\sqrt{2\pi}K} \exp \left( -\frac{(x - \mu_-)^2}{2} \right) dx.
\]

and

\[
(\mathcal{L}^\text{nat}_\theta)' = \left[ \exp \left( -\frac{(\theta - \mu_+)^2}{2K^2} \right) - K \exp \left( -\frac{(\theta - \mu_-)^2}{2} \right) \right] - \frac{2K^2}{2K^2} \sqrt{2\pi}.
\]

The derivative \((\mathcal{L}^\text{nat}_\theta)'\) turns out to be an increasing function over the interval \((\mu_-, \mu_+)\) with \((\mathcal{L}^\text{nat}_\mu_-)' < 0\) and \((\mathcal{L}^\text{nat}_\mu_+)' > 0\) due to the assumption that \(K < B_K < \exp \left( -\frac{(\mu_+ - \mu_-)^2}{2K^2} \right) \).

Since the Bayes classifier is to minimize the natural classification error, \(\hat{\theta}\) is supposed to be the unique root of \((\mathcal{L}^\text{nat}_\theta)'\), i.e.,

\[
\left[ \exp \left( -\frac{(\hat{\theta} - \mu_+)^2}{2K^2} \right) - K \exp \left( -\frac{(\hat{\theta} - \mu_-)^2}{2} \right) \right] = 0.
\]

By solving the equation above, we end up with the following.

\[
\hat{\theta} = \mu_- - \mu_+ - \mu_- \frac{K}{K^2 - 1} \sqrt{2(K^2 - 1) \ln(K) + (\mu_+ - \mu_-)^2},
\]

which belongs to the open interval \((\mu_-, \mu_+)\).

\( ii \) Equalized classification errors require the following equations hold.

\[
\Pr (f_\theta (X) \neq 1 \mid Y = 1)
\]

\[
= \int_{-\infty}^{\theta_t} \frac{1}{\sqrt{2\pi}K} \exp \left( -\frac{(x - \mu_+)^2}{2K^2} \right) dx
\]

\[
= \int_{\theta_t}^{+\infty} \frac{1}{\sqrt{2\pi}K} \exp \left( -\frac{(x - \mu_-)^2}{2} \right) dx
\]

\[
= \Pr (f_\theta (X) \neq -1 \mid Y = -1),
\]

which leads to the result that

\[
\theta_t = \mu_- + \mu_+ - \frac{\mu_-}{K + 1} > \mu_- + \epsilon,
\]

where the inequality is due to the assumption that \(K < \frac{(\mu_+ - \mu_-)}{\epsilon - 1} \leq B_K\).

\( iii \) The robust classification error and its partial derivative can then be given by the following:

\[
\mathcal{L}^\text{rob}_\theta (\epsilon) = \Pr (\exists |\tau| \leq \epsilon, f_\theta (X + \tau) \neq Y)
\]

\[
= \frac{1}{2} \Pr (\exists |\tau| \leq \epsilon, f_\theta (X + \tau) \neq 1 \mid Y = 1) + \frac{1}{2} \Pr (\exists |\tau| \leq \epsilon, f_\theta (X + \tau) \neq -1 \mid Y = -1)
\]

\[
= \frac{1}{2} \Pr (X \leq \theta + \epsilon \mid Y = 1) + \Pr (X > \theta - \epsilon \mid Y = -1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\theta + \epsilon} \frac{1}{\sqrt{2\pi}K} \exp \left( -\frac{(x - \mu_+)^2}{2K^2} \right) dx + \frac{1}{2} \int_{\theta - \epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x - \mu_-)^2}{2} \right) dx,
\]

and

\[
\frac{\partial}{\partial \theta} \mathcal{L}^\text{rob}_\theta (\epsilon) = \frac{\left[ \exp \left( -\frac{(\theta + \epsilon - \mu_+)^2}{2K^2} \right) - K \exp \left( -\frac{(\theta - \epsilon - \mu_-)^2}{2} \right) \right]}{2K^2}.
\]

By the assumptions made about \(K\) and \(\epsilon\), there exists a unique root \(\theta_t^{(\epsilon)} \in (\mu_-, \mu_+)\) of \(\frac{\partial}{\partial \theta} \mathcal{L}^\text{rob}_\theta (\epsilon) = 0\) such that the robust error \(\mathcal{L}^\text{rob}_\theta (\epsilon)\) is decreasing over \((\mu_-, \theta_t^{(\epsilon)})\) while increasing over \((\theta_t^{(\epsilon)}, \mu_+)\), i.e.,

\[
\frac{\partial}{\partial \theta} \mathcal{L}^\text{rob}_\theta (\epsilon) \begin{cases} < 0, & \theta \in (\mu_-, \theta_t^{(\epsilon)}) \\ > 0, & \theta \in (\theta_t^{(\epsilon)}, \mu_+) \end{cases},
\]

which indicates that \(\theta_t^{(\epsilon)}\) essentially minimizes the robust classification error.

Therefore, by solving \(\frac{\partial}{\partial \theta} \mathcal{L}^\text{rob}_\theta (\epsilon) \bigg|_{\theta = \theta_t^{(\epsilon)}} = 0\), we are able to derive the robust classifier as follows.

\[
\theta_t^{(\epsilon)} = \mu_- + \mu_+ - \frac{\mu_- - (K^2 + 1)\epsilon}{K^2 - 1} + \frac{K}{K^2 - 1} \sqrt{2(K^2 - 1) \ln(K) + (\mu_+ - \mu_- - 2\epsilon)^2},
\]

which satisfies the following

\[
\theta_t^{(\epsilon)} \leq \mu_- - \epsilon \leq \mu_+.
\]
iv) The next step is to compare the three different classifiers we just obtained. We start with the Bayes and fair classifiers.

\[
\frac{d - d_t}{(2 - 1) \ln(K) + (\mu_+ - \mu_-)^2 - \mu_+ - \mu_-)}
\]

\[
\frac{K}{(\mu_+ - \mu_-)^2 - \mu_+ - \mu_-)}
\]

\[
0
\]

where the inequality comes from the assumption that \( K \) is strictly larger than 1. It indicates that the threshold of the Bayes classifier is greater than that of the fair classifier. Then, we move on to the comparison between the Bayes and robust classifier. Note that the robust classifier is identical to the Bayes classifier when \( \epsilon = 0 \), i.e., \( \theta_t^{(0)} = \hat{\theta} \). Besides, we have that

\[
\frac{\partial \hat{\theta}(e)}{\partial e} = \frac{K^2 + 1 - \frac{2(\mu_+ - \mu_- - 2e)K}{K^2 - 1}}{\sqrt{\mu_+ - \mu_-}^2 + 2(K^2 - 1) \ln(K)}
\]

\[
\frac{K^2 + 1 - \frac{2(\mu_+ - \mu_- - 2e)K}{K^2 - 1}}{\sqrt{\mu_+ - \mu_-}^2 + 2(K^2 - 1) \ln(K)}
\]

\[
0
\]

where Equation (12) is due to the fact that \( K > 1 \) and Equation (13) comes from the assumption that \( \epsilon \leq \frac{\mu_+ - \mu_-}{2} \), which ensures that \( \mu_+ - \mu_- - 2\epsilon \) is strictly positive. Therefore, the partial derivative of \( \theta_t^{(e)} \) in \( \epsilon \) is strictly positive over the interval \([0, \frac{\mu_+ - \mu_-}{2}]\). As a consequence, for any \( \epsilon \in [0, \frac{\mu_+ - \mu_-}{2}] \), the following relation always holds that \( \mu_+ + \epsilon \leq \theta_t^{(e)} \leq \theta_t^{(0)} = \hat{\theta} \). Putting things together, we end up with following relation: for any \( \epsilon \in [0, \frac{\mu_+ - \mu_-}{2}] \) and \( K \in (1, B_K) \),

\[
\mu_+ + \epsilon \leq \theta_t \leq \theta_t^{(e)} \leq \mu_+ - \epsilon
\]

\[
\text{Proof of Corollary 6.2}
\]

\[
\text{Proof.} \quad \text{Note that, by Equation (11), the robust classification error is strictly decreasing over } (\mu_-, \theta_t^{(e)}). \quad \text{Then, by Equation (8) in Proposition 6.1, the three classifiers satisfy the following relation}
\]

\[
\mu_+ \leq \theta_t \leq \theta_t^{(e)}.
\]

Due to contiguity of \( L_{\theta}^{rob} (\epsilon) \), we can argue that, for any \( \epsilon \in [0, \frac{\mu_+ - \mu_-}{2}] \) and \( K \in (1, B_K) \),

\[
L_{\theta_t}^{rob} (\epsilon) \geq L_{\theta_t}^{rob} (\epsilon) \geq L_{\theta_t}^{rob} (\epsilon).
\]

\[
\square
\]

\[
\text{Proof of Corollary 6.3}
\]

\[
\text{Proof.} \quad \text{The boundary error and its partial derivative are presented in the following:}
\]

\[
L_{\theta}^{bdy} (\epsilon) = \Pr (|\tau| \leq \epsilon, f_0(X + \tau) \neq Y, f_0(X) = Y)
\]

\[
= \frac{1}{2} \Pr (|\tau| \leq \epsilon, f_0(X + \tau) = 1, f_0(X) = 1 | Y = 1) + \frac{1}{2} \Pr (|\tau| \leq \epsilon, f_0(X + \tau) = 1, f_0(X) = -1 | Y = -1)
\]

\[
= \frac{1}{2} \Pr (\theta < X \leq \theta + \epsilon | Y = 1) + \frac{1}{2} \Pr (\theta > X > \theta - \epsilon | Y = -1)
\]

\[
= \frac{1}{2} \int_{0}^{\theta + \epsilon} \frac{1}{\sqrt{2\pi K}} \exp \left( -\frac{(x - \mu_+)^2}{2K^2} \right) dx + \frac{1}{2} \int_{\theta - \epsilon}^{\theta} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x - \mu_-)^2}{2} \right) dx,
\]

and

\[
\frac{\partial}{\partial \theta} L_{\theta}^{bdy} (\epsilon) = \frac{\exp \left( -\frac{(\theta + \epsilon - \mu_+)^2}{2K^2} \right)}{2\sqrt{2\pi K}} - \frac{\exp \left( -\frac{(\theta - \mu_+)^2}{2K^2} \right)}{2\sqrt{2\pi K}} + \frac{\exp \left( -\frac{(\theta - \mu_-)^2}{2} \right)}{2\sqrt{2\pi}} - \frac{\exp \left( -\frac{(\theta + \epsilon - \mu_-)^2}{2K^2} \right)}{2\sqrt{2\pi K}}
\]

\[
g(\epsilon; \theta, K) := \frac{\exp \left( -\frac{(\theta + \epsilon - \mu_+)^2}{2K^2} \right)}{2\sqrt{2\pi K}} - \frac{\exp \left( -\frac{(\theta - \mu_+)^2}{2K^2} \right)}{2\sqrt{2\pi K}} + \frac{\exp \left( -\frac{(\theta - \mu_-)^2}{2} \right)}{2\sqrt{2\pi}} - \frac{\exp \left( -\frac{(\theta + \epsilon - \mu_-)^2}{2K^2} \right)}{2\sqrt{2\pi K}}.
\]

By Proposition A.1, the partial derivative of the boundary error in \( \theta \) is always negative for any \( \theta \in [\theta_t, \theta_t^{(e)}] \) because

\[
\frac{\partial}{\partial \theta} L_{\theta}^{bdy} (\epsilon) = \frac{g(\epsilon; \theta, K) - g(0; \theta, K)}{2\sqrt{2\pi}} < 0.
\]
It leads to the following result that the boundary error \( L_{\theta_i}^{\text{rob}}(\epsilon) \) decreases in \( \theta \) over \([\theta_{i}, \theta_{i}(\epsilon)]\), and, therefore,
\[
L_{\theta_i}^{\text{bdy}}(\epsilon) \geq L_{\theta_i}^{\text{bdy}}(\epsilon) \geq e^{\text{bdy}}_{\theta_i}(\epsilon).
\]
\[\square\]

**Proposition A.1.** For any \( \epsilon \in \left[0, \frac{\mu_{+} - \mu_{-}}{4}\right] \), \( K \in (1, B_K) \), and \( \theta \in \left[\theta_{i}, \theta_{i}(\epsilon)\right] \), the following relation holds
\[
g(\epsilon; \theta, K) \leq g(0; \theta, K),
\]
where the function \( g \) is defined in Equation (14).

**Proof.** Note that the partial derivative of the function \( g \) in \( \epsilon \) can be given by
\[
\frac{\partial}{\partial \epsilon} g(\epsilon; \theta, K) = \frac{\mu_{+} - \theta - \epsilon}{K^{3}} \exp\left(-\frac{(\mu_{+} - \theta - \epsilon)^{2}}{2K^{2}}\right) - (\theta - \epsilon - \mu_{-}) \exp\left(-\frac{(\theta - \epsilon - \mu_{-})^{2}}{2}\right).
\]
In order to establish the result in Equation (15), it suffices to demonstrate that the partial derivative \( \frac{\partial}{\partial \epsilon} g(\epsilon; \theta, K) \) is negative, which implies that the function \( g(\epsilon; \theta, K) \) is strictly decreasing over \([0, \frac{\mu_{+} - \mu_{-}}{4}]\). Note that
\[
\ln\left(\frac{\mu_{+} - \theta - \epsilon}{K^{3}} \exp\left(-\frac{(\mu_{+} - \theta - \epsilon)^{2}}{2K^{2}}\right)\right) - \ln\left((\theta - \epsilon - \mu_{-}) \exp\left(-\frac{(\theta - \epsilon - \mu_{-})^{2}}{2}\right)\right)
= -\ln(\theta - \epsilon - \mu_{-}) - \ln(\mu_{+} - \theta - \epsilon) - \frac{[(\theta - \epsilon - \mu_{-})^{2} - (\mu_{+} - \theta - \epsilon)^{2}]}{2K^{2}} - 3 \ln(K)
= \frac{1}{2} \left(q(\theta; \epsilon, K) - p(\theta; \epsilon, K) - 4 \ln(K)\right)
\leq 0,
\]
where the last inequality comes from Proposition A.2. It follows that, due to monotonicity of the function \( x \mapsto \ln(x) \),
\[
\frac{\partial}{\partial \epsilon} g(\epsilon; \theta, K) = \frac{\mu_{+} - \theta - \epsilon}{K^{3}} \exp\left(-\frac{(\mu_{+} - \theta - \epsilon)^{2}}{2K^{2}}\right) - (\theta - \epsilon - \mu_{-}) \exp\left(-\frac{(\theta - \epsilon - \mu_{-})^{2}}{2}\right)
\leq 0,
\]
which completes our proof here. \(\square\)

**Proposition A.2.** For any \( \epsilon \in \left[0, \frac{\mu_{+} - \mu_{-}}{4}\right] \), \( K \in (1, B_K) \), and \( \theta \in \left[\theta_{i}, \theta_{i}(\epsilon)\right] \), the following relation always holds:
\[
p(\theta; \epsilon, K) \geq q(\theta; \epsilon, K) - 4 \ln(K),
\]
where
\[
p(\theta; \epsilon, K) = \ln\left((\theta - \epsilon - \mu_{-})^{2} - \ln\left((\mu_{+} - \theta - \epsilon)^{2}\right)\right),
\]
\[
q(\theta; \epsilon, K) = (\theta - \epsilon - \mu_{-})^{2} - (\mu_{+} - \theta - \epsilon)^{2}.
\]

**Proof.** First off, observe that the functions \( p \) and \( q \) are both increasing over \([\theta_{i}, \theta_{i}(\epsilon)]\). It follows that, for any \([\theta_{i}, \theta_{i}(\epsilon)]\),
\[
p(\theta; \epsilon, K) \geq p(\theta; \epsilon, K) \geq -2 \ln(K) = 2 \ln(K) - 4 \ln(K)
= q(\theta_{i}(\epsilon); \epsilon, K) - 4 \ln(K)
\geq q(\theta; \epsilon, K) - 4 \ln(K),
\]
where Equation (18) and (21) come from monotonicity of the functions \( p \) and \( q \). Equation (19) and (20) are due to Proposition A.4 and A.3 respectively. \(\square\)

**Proposition A.3.** For any \( \epsilon \in \left[0, \frac{\mu_{+} - \mu_{-}}{4}\right] \) and \( K \in (1, B_K) \),
\[
q\left(\theta_{i}(\epsilon); \epsilon, K\right) = 2 \ln(K),
\]
where the definition of the function \( q \) is given in Equation (17).

**Proof.** Due to optimality of \( \theta_{i}(\epsilon) \) in terms of robust error, \( \theta_{i}(\epsilon) \) should be a solution to Equation (10), i.e.,
\[
\exp\left(-\frac{(\theta_{i}(\epsilon) - \epsilon - \mu_{-})^{2}}{2K^{2}}\right) = K \exp\left(-\frac{(\theta_{i}(\epsilon) - \epsilon - \mu_{-})^{2}}{2K^{2}}\right) = 0,
\]
which leads to the following, by multiplying the both sides with the term \( \exp\left(-\frac{(\theta_{i}(\epsilon) - \epsilon - \mu_{-})^{2}}{2K^{2}}\right) \),
\[
\exp\left(q(\theta_{i}(\epsilon); \epsilon, K)\right) = K,
\]
and
\[
q(\theta_{i}(\epsilon); \epsilon, K) = 2 \ln(K).
\]
\(\square\)

**Proposition A.4.** For any \( \epsilon \in \left[0, \frac{\mu_{+} - \mu_{-}}{4}\right] \) and \( K \in (1, B_K) \),
\[
p(\theta_{i}; \epsilon, K) \geq -2 \ln(K),
\]
where the definition of the function \( p \) is given in Equation (16).
Proof. Equation (22) can be rewritten into the following equivalent form
\[
\exp(p(\theta_i; \epsilon, K) + 2 \ln(K)) = K - \frac{(K - 1)\epsilon}{\mu_+ - \mu_-} > 1.
\]
(23)

Note that
\[
K \leq B_K \implies \frac{1}{K} + \frac{\mu_+ - \mu_-}{\epsilon} = K + 1 - 2 = \frac{\mu_+ - \mu_-}{\epsilon} \geq K + 1 - 2,
\]
which leads to the following result
\[
\exp(p(\theta_i; \epsilon, K) + 2 \ln(K)) = K - \frac{(K - 1)\epsilon}{\mu_+ - \mu_-} \geq K - (K - 1) = 1.
\]

It helps complete our proof here. □

Proof of Theorem 6.4

Proof. Since \(f_\theta\) is essentially a linear classifier, the distance to the decision boundary of \(f_\theta\) is simply the absolute value between the feature \(X\) and the threshold \(\theta\), i.e., \(\Delta(X, f_\theta) = |X - \theta|\). Notice that the average distance to the decision boundary of \(f_\theta\) can then be given by
\[
E[\Delta(X, f_\theta)] = E[|X - \theta|]
\]
\[
= E[|X - \theta| | Y = 1] \cdot \Pr(Y = 1) + E[|X - \theta| | Y = -1] \cdot \Pr(Y = -1)
\]
\[
= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{|x - \theta|}{\sqrt{2\pi}K} \exp\left(-\frac{(x - \mu_+)^2}{2K^2}\right) dx + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{|x - \theta|}{\sqrt{2\pi}} \exp\left(-\frac{(x - \mu_-)^2}{2}\right) dx,
\]
whose derivative can be expressed as
\[
(E[\Delta(X, f_\theta)])' = 2\left(\Phi(\theta - \mu_-) - \Phi\left(\frac{\mu_+ - \theta}{K}\right)\right),
\]
where \(\Phi\) represents the cumulative distribution function associated with the standard normal distribution. Recall that
\[
\theta_i = \mu_- + \frac{\mu_+ - \mu_-}{K + 1}.
\]
It follows that
\[
(E[\Delta(X, f_\theta)])' \begin{cases} < 0, & \theta \in (\mu_-, \theta_i), \\ > 0, & \theta \in (\theta_i, \mu_+), \end{cases}
\]
which implies that the average distance strictly decreases over \((\mu_-, \theta_i)\) while increases over \((\theta_i, \mu_+)\). By the relation shown in Equation (8), we figure out that, for any \(\epsilon \in \left[0, \frac{\mu_+ - \mu_-}{2}\right]\) and \(K \in (1, B_K]\),
\[
E[\Delta(X, f_\theta)] \geq E[\Delta(X, f_\theta)] \geq E[\Delta(X, f_\theta)].
\]
Moreover, \(\theta_i\) is the minimizer of the average distance \(E[\Delta(X, f_\theta)]\) over the interval \([\mu_- , \mu_+]\). □

Proof of Theorem 6.5

Proof. First off, notice that \(\theta_i(\lambda)\) is the minimizer of \(\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}}\) over \([\mu_- , \mu_+]\), where \(\mathcal{L}_\theta^{\text{fair}}\) is a shorthand for
\[
|\Pr(f_\theta(X) \neq 1 | Y = 1) - \Pr(f_\theta(X) \neq -1 | Y = -1)|.
\]
Thus, \(\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}}\) can be presented in a piecewise way as follows:

1) if \(\theta \leq \theta_i\),
\[
\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}} = (1 - \lambda) \Pr(f_\theta(X) \neq 1 | Y = 1) + (1 + \lambda) \Pr(f_\theta(X) \neq -1 | Y = -1);
\]
2) if \(\theta > \theta_i\),
\[
\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}} = (1 + \lambda) \Pr(f_\theta(X) \neq 1 | Y = 1) + (1 - \lambda) \Pr(f_\theta(X) \neq -1 | Y = -1).
\]

We start with the first case where \(\theta\) is no greater than \(\theta_i\). The partial derivative of \(\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}}\) is then given by
\[
\frac{\partial (\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}})}{\partial \theta} = \frac{1 - \lambda}{2\sqrt{2\pi}K} \exp\left(-\frac{(\theta - \mu_+)^2}{2K^2}\right) - \frac{1 + \lambda}{2\sqrt{2\pi}} \exp\left(-\frac{((\lambda - \mu_-)^2}{2}\right). \tag{24}
\]
Observe that, when \(\lambda \in [0, 1],\) this partial derivative is increasing over \([\mu_- , \theta_i]\) with its value at \(\theta_i\) no greater than 0 while it is always non-positive for any \(\lambda \in (1, +\infty)\) and \(\theta \in [\mu_- , \theta_i]\). Therefore, the partial derivative presented in Equation (24) proves to be non-positive, whatever \(\lambda\), which implies that the function \(\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}}\) decreases over \([\theta, \theta_i]\) and we merely need to focus on the case of \(\theta \in [\theta_i, \mu_+\) in pursuit of its minimizer. Then, we continue to investigate the case where \(\theta\) is greater than \(\theta_i\). Likewise, the partial derivative of \(\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}}\) can be expressed as
\[
\frac{\partial (\mathcal{L}_\theta^{\text{nat}} + \lambda \cdot \mathcal{L}_\theta^{\text{fair}})}{\partial \theta} = \frac{1 + \lambda}{2\sqrt{2\pi}K} \exp\left(-\frac{(\theta - \mu_+)^2}{2K^2}\right) - \frac{1 - \lambda}{2\sqrt{2\pi}} \exp\left(-\frac{(\theta - \mu_-)^2}{2}\right). \tag{25}
\]
We split our studies into the following three scenarios:
1) if \( \lambda \in \left[ 0, \frac{K-1}{K+1} \right] \), \( \mathcal{L}_p^{\text{nat}} + \lambda \cdot \mathcal{L}_p^{\text{fair}} \) first decreases over \([\theta_1, \theta_1(\lambda)]\) and then increases over \([\theta_1(\lambda), \mu_+]\) where its minimum takes place at

\[
\mu_- = \frac{\mu_+ - \mu_-}{K^2 - 1} + \frac{K}{K^2 - 1} \left( 2(K^2 - 1) \ln \left( \frac{1 - \lambda}{1 + \lambda} \cdot K \right) + (\mu_+ - \mu_-)^2 \right)
\] (26)

2) if \( \lambda \in \left( \frac{K}{K+1}, 1 \right] \), the partial derivative in Equation (25) turns out to be increasing in \( \theta \) with its value at \( \theta_1 \) non-negative. It implies that the partial derivative is always non-negative and, thus, the function \( \mathcal{L}_p^{\text{nat}} + \lambda \cdot \mathcal{L}_p^{\text{fair}} \) increases over \([\theta_1, \mu_+]\). As a consequence, the minimizer \( \theta_1(\lambda) \) actually coincides with \( \theta_1 \).

3) if \( \lambda \in (1, +\infty) \), note that the partial derivative is always positive. Following the same reasoning in the previous scenario, we figure out that the minimizer \( \theta_1(\lambda) \) is identical to \( \theta_1 \).

Since the function \( \lambda \mapsto \frac{1 - \lambda}{1 + \lambda} \) is decreasing over \([0, \frac{K-1}{K+1}]\), by Equation (26), we can argue that \( \theta_1(\lambda) \) is decreasing in \( \lambda \) over \( \mathbb{R}_+ \) as well. Furthermore, by the proof of Theorem 6.4, the average distance to the decision boundary is an increasing function in \( \theta \) over \([\theta_1, \mu_+]\), which indicates that the average distance associated with the classifier \( f_{\theta_1(\lambda)} \) decreases, as \( \lambda \) increases.  

\[ \square \]

B. Fairness and Robustness Models

B.1. Fair models

The main text mainly discussed penalty-based methods as a way to encourage accuracy parity in a classifier. This section discusses a second methodology to achieve fairness denoted group-loss focused methods [22]. We will show that the main conclusion of this paper (e.g., that fairness increase adversarial vulnerability) holds regardless of the methodology adopted to achieve fairness.

Group-loss focused methods. Methods in this category force the training to focus on the loss component of worst performing groups. An effective method to achieve this goal was proposed in [22]:

\[
\theta_1 = \arg \min_{\theta} \sum_{a \in \mathcal{A}} \frac{1}{q + 1} \mathcal{L}_a(D_a)^{q + 1}, \quad (27)
\]

where \( q \) is a non-negative constant. The intuition behind powering the loss by positive number \( q + 1 \) is to penalize more the classes that have the larger losses. Thus, \( q \) plays the role of the fairness parameter, like \( \lambda \) in penalty-based methods: larger \( q \) or \( \lambda \) values are associated with fairer (but also often less accurate) models. The main differences between penalty-based methods and group-loss focused methods are the following. First, the loss function of group-loss focused methods is fully differentiable, in contrast to that of penalty-based methods, which is sub-differentiable when the group loss equals the population loss. Second, penalty-based methods try to equalize the losses across various subgroup, while group-focused based methods attempt at minimizing the maximum loss across all subgroups.

C. Datasets and Settings

Datasets. Experiments were performed using three benchmark datasets: UTK Face [48], CIFAR-10 [20], and Fashion MNIST (FMNIST) [43].

1. UTK Face [48]. It consists of more than 20,000 facial images of 48x48 pixels resolution. The experiments consider two learning tasks: (1) The first splits the data into five ethnicities: White, Black, Asian, Indian, and Others. (2) The second splits the data into nine age bins: under-ten years old, 10-14, 15-19, 20-24, 25-29, 30-39, 40-49, 50-59, and over 59 years old. The classes are not uniformly distributed per number of groups and do not contain the same number of images in each group. An 80/20 train-test split is performed.

2. CIFAR-10 [20]. It consists of 60,000 32x32 coloured images belonging to 10 classes, with 6000 images per class. The training set has 50,000 images while the test set has 10,000 images.

3. Fashion MNIST (FMNIST) [43]. It consists of 60,000 28x28 gray-colored images belonging to 10 classes, with 6000 images per class. The training set has 50,000 images while the test set has 10,000 images.

Network architectures. The experiments consider three network architectures of increasing complexity:

1. A CNN consisting of 3 convolutions layers followed by 3 fully connected layers.


We use CNN to provide some examples in the main text, but in this Appendix we will focus on the ResNet and VGG networks.

Fair models and parameter settings. For penalty-based methods the experiments vary the range of fairness parameter \( \lambda \in [0, 2] \). We note that larger \( \lambda \) values may have a detrimental effects to the models accuracy, which, we believe, limit
the applicability of the fairness methods in real use cases, thus we focus on these more realistic scenarios. However, we also note that appropriate choices of hyperparameters will necessarily depend on the task and architecture at hand and can be used to balance the trade-off between accuracy and fairness.

For group-loss focused methods, the experiments consider \(q \in [0, 2]\) for similar reasons as those stated above.

The set of hyper-parameters, learning rate (lr), batch-size (bs), and number of training epochs (epochs) adopted, for each dataset and architecture is reported in Table 1.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>lr</th>
<th>bs</th>
<th>epochs</th>
</tr>
</thead>
<tbody>
<tr>
<td>UTK Face</td>
<td>1e-3</td>
<td>32</td>
<td>70</td>
</tr>
<tr>
<td>CIFAR-10</td>
<td>1e-2</td>
<td>32</td>
<td>200</td>
</tr>
<tr>
<td>FMNIST</td>
<td>1e-2</td>
<td>32</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 1. Hyperparameters settings for each dataset.

For each setting, the experiments report the average results of 10 runs, each initializing the models parameters using a different random seed.

**Adversarial attacks.** The experiments also consider two classes of adversarial attacks to test the model robustness: (1) The \(l_\infty\) RFGSM attacks [37] and (2) The \(l_2\) PGD attacks [24]. The experiments adopt the implementations reported in the Python package torchattacks [18].

**Code and Preprocessing.** Follow standard setting, the range of the pixel values was normalized in [0, 1] for all datasets adopted.

All codes were written in Python 3.7 and in Pytorch 1.5.0. The library torchattacks [18] was adopted to generate different adversarial attacks. The repository contained the dataset and implementation will be released publicly upon paper acceptance.

**D. Additional experiments**

This section describes additional experiments to further support the claims reported in the main paper. In particular, the experiments report results for deeper networks (e.g., ResNet-50) and for additional fairness methods and adversarial attacks.

**D.1. Fairness impacts on the decision boundary**

Recall, from Theorem 6.5, that fairness reduces the average distance of the testing samples to the decision boundary. As a consequence the fair classifiers are more vulnerable against the adversarial attacks than the natural (unfair) classifiers. This section provides additional evidence to support this claim on a high-dimensional model (ResNet 50) and using both penalty based methods additional and group-loss focused methods (see Section B.1) to derive a fair classifier.

**Penalty-based methods.** Figure 9 and Figure 10 summarize the results obtained by a penalty-based fair model executed on different benchmark datasets. The experiments again report a consistent trend: as more fairness is enforced (increasing the values \(\lambda\) ), the natural errors (left plots) generally increase while both the fairness violations (middle plots) and the average distance to the decision boundary (right plots) decrease. Recall that the latter is a proxy for measuring the model robustness: the closer are samples to the decision boundary the less robust is a model. Thus the previous plots show that robustness decreases as fairness increase.

**Group-loss focused methods** A similar setting is reported for a model satisfying fairness using the group-loss focused method described in section B.1. This method maximizes the worst group accuracy, which, in turn, attempts at equalizing the accuracy across groups. Figure 11 reports the results obtained using VGG-13 and the UTK-Face dataset. The results again illustrate similar trends: as the fairness parameter \(q\) increases, the natural errors (right) tend to increase while both the fairness violations (middle) and the average distance to the decision boundary (left) decrease. Notice that small enough \(q\)-values may also act as a regularizer and have a beneficial effect toward the natural error, as observed for the UTK-age bin task (bottom-left plot).
D.2. Boundary errors increase as fairness decreases

This section provides additional experiments to illustrating that the result of the paper (fairness increases the adversarial vulnerability) is invariant across different fair classifier implementations. The experiments adopt the group-loss focused methods for different values of the fairness parameter $q$. Notice that a natural classifier is obtained when $q = 0$. Figure 12 displays the natural (left) and boundary (center) errors attained under different level of RFGSM attacks (regulated by parameter $\epsilon$) and the fairness violation (right) on CIFAR (top) F MNIST (middle) and UTK (bottom) datasets. Notice how increasing $q$ to large enough values typically decreases the fairness violations. However, this comes at the cost of increasing the natural error and the boundary errors, which, in turn, exacerbate the robust errors.

D.3. A Mitigating Solution with Bounded Losses

More intuition why bounded loss works. This section first provides more intuition about why bounded loss functions, such as the ramp loss adopted in the main text, can help reducing the impacts of fairness towards robustness. To guide the intuitions, we will refer to Figure 13, which plots the graph functions of the following loss function for binary classification tasks:

- Ramp loss [7, 10], defined by:
  $$\ell(f_\theta(X), Y) = \min(1, \max(0, 1 - Yf_\theta(X))).$$

- Log-loss [12], defined by:
  $$\ell(f_\theta(X), Y) = \log(1 + \exp(-Yf_\theta(X))).$$

- Exponential loss [12], defined by:
  $$\ell(f_\theta(X), Y) = \exp(-Yf_\theta(X))).$$

Notice that, as illustrated in Figure 13, unbounded losses (such as log-loss or exponential loss) amplify the classification errors of misclassified samples $X$, in a way proportionately to the distance of $X$ from the decision boundary. The misclassification of a sample is captured by the expression $f_\theta(X)Y < 0$, while the distance to the decision boundary...
by expression $|f_\theta(X)Y|$ (x-axis). Notably these losses are unbounded.

Now notice that a consequence of training a fair classifier is to push its decision boundary to dense region of the advantaged group. This is because such classifier attempts at aligning the groups classification losses, i.e., $E[\ell(f_\theta(X),Y)|Y=-1] = E[\ell(f_\theta(X),Y)|Y=1]$. When the decision boundary is moved closer to the input samples, the classifier will inevitably become less robust to small perturbations of adversarial noise.

On the contrary, using a bounded loss function, such as ramp loss, during fair learning, can greatly reduce the impact produced by such (outlier) samples.

Effectiveness of the mitigation solution. Next, this section provides additional experiments to demonstrate the effectiveness of the proposed solution to find a good trade-off between fairness and robustness. The generality of the proposed solution is demonstrated across several architectures (VGG-13 and ResNet 50) and adversarial attacks ($\ell_\infty$ RFGSM and $\ell_2$ PGD attacks under different level of attacks $\epsilon$).

In summary, the proposed mitigation solution—which uses a bounded loss—result in classifiers that, in the vast majority of the cases, are fairer and more robust that those produced by models using a standard (cross entropy) loss.

VGG-13 and $\ell_\infty$ RFGSM attacks. Figures 14 and 15 report the boundary errors attained using RFGSM attacks on fair ($\lambda > 0$) and regular ($\lambda = 0$) classifiers on CIFAR and UTK datasets, respectively and using a VGG 13. The plots compare models obtained using a cross entropy loss (top plots) and those using a Ramp loss (bottom plots).

ResNet 50 and $\ell_\infty$ RFGSM attacks Figures 18 and 19 report the boundary errors attained using a RFGSM attacks on fair ($\lambda > 0$) and regular ($\lambda = 0$) classifiers on UTK Face and FMNIST datasets, respectively and using a ResNet50. Once again, the plots compare models obtained using a cross entropy loss (top plots) and those using a Ramp loss (bottom plots).

ResNet 50 and $\ell_2$ PGD attacks Figures 20 and 21 report the boundary errors attained using a PGD attack on fair ($\lambda > 0$) and regular ($\lambda = 0$) classifiers on CIFAR10 and FMNIST datasets, respectively, and using a ResNet50. Once again, the plots compare models obtained using a cross entropy loss (top plots) and those using a Ramp loss (bottom plots).

VGG-13 and $\ell_2$ PGD attacks. Figures 16 and 17 report the boundary errors attained using a PGD attacks on fair ($\lambda > 0$) and regular ($\lambda = 0$) classifiers on UTK datasets and using a VGG 13. Once again, the plots compare models obtained using a cross entropy loss (top plots) and those using a Ramp loss (bottom plots).
Figure 14. **Top:** Natural errors (left) and fairness violations (right) on the CIFAR-10 *ethnicity* task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different RFGSM attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifiers are VGG-13.

Figure 15. **Top:** Natural errors (left) and fairness violations (right) on the UTKFace *age bins* task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different RFGSM attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifiers are VGG-13.
Figure 16. **Top:** Natural errors (left) and fairness violations (right) on the UTKFace *ethnicity* task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_2$ PGD attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifier are VGG-13.

Figure 17. **Top:** Natural errors (left) and fairness violations (right) on the UTKFace *age bins* task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_2$ PGD attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifiers are VGG-13.
Figure 18. Top: Natural errors (left) and fairness violations (right) on the UTKFace ethnicity task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_1$ RFGSM attack levels. Bottom: Mitigating solution using the bounded Ramp loss. The base classifier are Res Net 50.

Figure 19. Top: Natural errors (left) and fairness violations (right) on the FMNIST task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_\infty$ RFGSM attack levels. Bottom: Mitigating solution using the bounded Ramp loss. The base classifier are Res Net 50.
Figure 20. **Top:** Natural errors (left) and fairness violations (right) on the CIFAR 10 task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_2$ PGD attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifiers are ResNet 50.

Figure 21. **Top:** Natural errors (left) and fairness violations (right) on the FMNIST task at varying of the fairness parameters $\lambda$. The middle plots compares the robustness of fair ($\lambda > 0$) vs. natural ($\lambda = 0$) classifiers to different $l_2$ PGD attack levels. **Bottom:** Mitigating solution using the bounded Ramp loss. The base classifiers are ResNet 50.