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Persistent Laplacians: Properties, Algorithms and Implications*

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Abstract. We present a thorough study of the theoretical properties and devise efficient algorithms for the *persistent Laplacian*, an extension of the standard combinatorial Laplacian to the setting of pairs (or, in more generality, sequences) of simplicial complexes $K \hookrightarrow L$, which was recently introduced by Wang, Nguyen, and Wei. In particular, in analogy with the nonpersistent case, we first prove that the nullity of the qth persistent Laplacian $\Delta_q^{K,L}$ equals the qth persistent Betti number of the inclusion $(K \hookrightarrow L)$. We then present an initial algorithm for finding a matrix representation of $\Delta_q^{K,L}$, which itself helps interpret the persistent Laplacian. We exhibit a novel relationship between the persistent Laplacian and the notion of Schur complement of a matrix which has several important implications. In the graph case, it both uncovers a link with the notion of effective resistance and leads to a persistent version of the Cheeger inequality. This relationship also yields an additional, very simple algorithm for finding (a matrix representation of) the qth persistent Laplacian which in turn leads to a novel and fundamentally different algorithm for computing the qth persistent Betti number for a pair $K \hookrightarrow L$ which can be significantly more efficient than standard algorithms. Finally, we study persistent Laplacians for simplicial filtrations and establish novel functoriality properties and stability results for their eigenvalues. Our work brings methods from spectral graph theory, circuit theory, and persistent homology together with a topological view of the combinatorial Laplacian on simplicial complexes.

Key words. combinatorial Laplacian, persistent Laplacian, Schur complement, persistent homology, effective resistance, Cheeger inequality

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1. Introduction.

The combinatorial graph Laplacian, as an operator on functions defined on the vertex set of a graph, is a fundamental object in the analysis of and optimization on graphs. Its spectral properties are widely used in graph optimization problems (e.g., spectral clustering [8, 31, 40, 49]) and in the efficient solution of systems of equations; cf. [27, 34, 46, 48]. The graph Laplacian is also connected to network circuit theory via the notion of *effective resistance* [1, 10, 35, 45].

There is also an algebraic topology view of the graph Laplacian which arises through considering boundary operators and specific inner products defined on simplicial (co)chain

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groups [8]. This permits extending the graph Laplacian to a more general operator, the qth combinatorial Laplacian Δ_q^K on the qth (co)chain groups of a given simplicial complex K (see, e.g., [12, 11, 16, 23]), so that the standard graph Laplacian simply corresponds to the 0th case. These ideas connect to the topology of the input simplicial complex via the so-called combinatorial Hodge theorem [12], which states that the nullity of the qth combinatorial Laplacian is equal to the rank of the qth cohomology group of K with real coefficients, i.e., the qth Betti number of K. See also [23, 33] for thorough expositions.

The combinatorial Laplacian (and variants) have received a great deal of attention in recent years; see, e.g., [16, 17, 18, 38]. For example, [26] aims to extend the related concept, effective resistance from network circuit theory, to this "high dimensional" situation, whereas [20, 19] consider a spectral theory of cellular sheaves with applications to sparsification and synchronization problems.

Adopting the algebraic topology view of the qth combinatorial Laplacian, Wang, Nguyen, and Wei [51] introduced the so-called qth persistent Laplacian $\Delta_q^{K,L}$, which is an extension of the combinatorial Laplacian mentioned above to *a pair* of simplicial complexes $K \hookrightarrow L$ connected by an inclusion. To the best of our knowledge, [51] is the first work which establishes a link between *persistent* homology [14, 54], one of the most important developments in the field of applied and computational topology in the past two decades, with the Laplacian, a common and fundamental object with a vast literature, in both the theoretical and applied domains. These ideas surrounding the persistent Laplacian therefore have the potential to allow importing rich ideas from the toolset of analysis into the TDA field—a field which has so far been propelled mostly by algebraic methods. See also [9, 41] for other work in computational topology which leverages ideas connected to the (standard) combinatorial Laplacian.

It is thus natural and also highly desirable to achieve better understanding, as well as algorithmic developments, for this persistent Laplacian, all of which will help broaden its potential applications. The present paper aims to close this gap.

Contributions. In this paper, we carry out a thorough study of the properties of and develop algorithms for the persistent Laplacian. Our work brings together ideas and methods from several communities, including spectral graph theory, circuit theory, topological treatments of high-dimensional combinatorial Laplacians, together with a persistent homology perspective (both at the theoretical and algorithmic levels). For instance, we relate the computation of persistent homology with notions from network theory such as the Kron reduction (and also Schur complements) which have novel algorithmic implications; see below.

This is an overview of our results:

- In section 2, we present several results about the properties of the qth persistent Laplacian $\Delta_q^{K,L}$, including Theorem 2.7, which establishes that the nullity of $\Delta_q^{K,L}$ equals the qth *persistent* Betti number from K to L—a result analogous to the one that holds in the nonpersistent case.
- In section 3, we give a first algorithm (Algorithm 3.1) to compute a matrix representation of $\Delta_q^{K,L}$, which relies on matrix reduction ideas which are standard when computing persistent homology.
- In section 4, we establish our main observation Theorem 4.6, a relationship between the persistent Laplacian and the concept of *Schur complement* of a matrix. This observation has several immediate and important implications:

1. We establish a second, very simple algorithm (Algorithm 4.1) which computes the matrix representation of the persistent Laplacian $\Delta_q^{K,L}$ (for any q) efficiently, purely based on a linear algebraic formulation (Theorem 4.6).
2. This observation leads to a new algorithm to compute the qth persistent Betti number for a pair of spaces in a fundamentally different manner from extant algorithms in the computational topology literature. This new algorithm is, under mild conditions (e.g., as those commonly satisfied by Vietoris–Rips complexes) significantly more efficient than existing algorithms. We believe that this new algorithm for computing persistent Betti numbers is of independent interest.
3. In the graph case (i.e., when K and L are graphs and $q = 0$), this provides a direct connection with notions from network circuit theory such as the Kron reduction [10], a connection which reveals that the matrix representation of the persistent Laplacian permits recovering the effective resistance of pairs of vertices in K w.r.t. the larger graph L (cf. Proposition 4.10 and Theorem 4.11). The connection with network circuit theory leads to our definition of a "persistent" Cheeger constant as well as to a novel persistent Cheeger-like inequality for a pair of graphs $K \hookrightarrow L$ (cf. subsection 4.4).

- Finally, in section 5, we consider qth persistent Laplacians for filtrations of simplicial complexes (connected by inclusion morphisms). We first describe an efficient algorithm to *iteratively* compute the persistent Laplacian for all pairs of complexes in a filtration. We then discuss certain functoriality and stability results for the persistent Laplacian for filtrations of simplicial complexes.

Some technical details are relegated to the appendix and/or to the supplementary materials (M143547R_Supplementary_Materials_1.pdf [local/web 526KB]).

2. The persistent Laplacian for simplicial pairs $K \hookrightarrow L$.

In this section, after introducing some basic notions/definitions in subsection 2.1, we formulate the persistent Laplacian for simplicial pairs in subsection 2.2 and present some basic properties of persistent Laplacians in subsection 2.3.

2.1. Basics.

Simplicial complexes. An *(abstract) simplicial complex* K over a finite *ordered* set V is a collection of finite subsets of V such that for any $\sigma \in K$, if $\tau \subseteq \sigma$, then $\tau \in K$. Denote by $\mathbb{N}$ the set of nonnegative integers. For each $q \in \mathbb{N}$, an element $\sigma \in K$ is called a *q-simplex* if $|\sigma| = q + 1$, where we use $|A|$ to denote the cardinality of a set A. A 0-simplex, usually denoted by v, is also called a *vertex*. Denote by S_q^K the set of q-simplices of K. Note that $S_0^K \subseteq V$. The *dimension* of K, denoted by $\dim(K)$, is the largest q such that $S_q^K \neq \emptyset$. A 1-dim simplicial complex is also called a *graph* and we often use $K = (V^K, E^K)$ to represent a graph, where $V^K := S_0^K$ denotes the vertex set and $E^K := S_1^K$ denotes the edge set.

An *oriented* simplex, denoted by $[\sigma]$, is a simplex $\sigma \in K$ with an ordering on its vertices. For simplicity of our presentation, we always assume that the ordering is inherited from the ordering of V. Let $\bar{S}_q^K := \{[\sigma] : \sigma \in S_q^K\}$. The qth chain group $C_q^K := C_q(K, \mathbb{R})$ of K is the vector space over $\mathbb{R}$ with basis $\bar{S}_q^K$. Let $n_q^K := \dim C_q^K = |S_q^K|$. We define the boundary operator $\partial_q^K : C_q^K \to C_{q-1}^K$ by

$$\partial_q^K([v_0, \dots, v_q]) := \sum_{i=0}^{q} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_q] \tag{2.1}$$

for each $\sigma = [v_0, \dots, v_q] \in \bar{S}_q^K$, where $\hat{v}_i$ denotes the omission of the ith vertex. The qth homology group of K is $H_q(K) = \frac{\ker(\partial_q^K)}{\operatorname{im}(\partial_{q+1}^K)}$ and $\beta_q^K := \operatorname{rank}(H_q(K))$ is its qth Betti number.

A *weight function* on a simplicial complex K is any positive function $w^K : K \to (0, \infty)$. Throughout the paper, *every simplicial complex K is (implicitly) endowed with a weight function w^K.* We call K *unweighted* if $w^K \equiv 1$ (subsection 4.4.1 is the only place where we restrict ourselves to the unweighted case).

Combinatorial Laplacian. Let K be a simplicial complex with a weight function w^K. Given any $q \in \mathbb{N}$, let $w_q^K := w^K|_{S_q^K}$ and define the inner product $\langle \cdot, \cdot \rangle_{w_q^K}$ on C_q^K as follows:

$$\langle [\sigma], [\sigma'] \rangle_{w_q^K} := \delta_{\sigma\sigma'} \cdot \left(w_q^K(\sigma) \right)^{-1} \quad \forall \sigma, \sigma' \in S_q^K, \tag{2.2}$$

where $\delta_{\sigma\sigma'}$ is the Kronecker delta.

Remark 2.1. Consider the dual space of C_q^K: the cochain space $C^q(K) := \operatorname{Hom}(C_q(K), \mathbb{R})$. Then, $\langle \cdot, \cdot \rangle_{w_q^K}$ on $C_q(K)$ induces an inner product $\langle\langle \cdot, \cdot \rangle\rangle_{w_q^K}$ on $C^q(K)$ such that

$$\langle\langle f, g \rangle\rangle_{w_q^K} = \sum_{\sigma \in S_q^K} w_q^K(\sigma) f([\sigma]) g([\sigma]) \quad \forall f, g \in C^q(K).$$

This inner product $\langle\langle \cdot, \cdot \rangle\rangle_{w_q^K}$ on $C^q(K)$ coincides with the one defined in [23], which explains the reciprocal in the definition (2.2) of the inner product $\langle \cdot, \cdot \rangle_{w_q^K}$ on $C_q(K)$.

We denote by $(\partial_q^K)^* : C_{q-1}^K \to C_q^K$ the adjoint of ∂_q^K under these inner products. Then, we define the qth *(combinatorial) Laplacian* $\Delta_q^K : C_q^K \to C_q^K$ as follows:

$$\Delta_q^K := \underbrace{\partial_{q+1}^K \circ \left(\partial_{q+1}^K\right)^*}_{\Delta_{q,\mathrm{up}}^K} + \underbrace{\left(\partial_q^K\right)^* \circ \partial_q^K}_{\Delta_{q,\mathrm{down}}^K}, \tag{2.3}$$

where for convenience we have also defined the corresponding "up" and "down" Laplacians. By convention we let $\partial_0^K := 0$ and thus $\Delta_0^K = \partial_1^K \circ (\partial_1^K)^*$. When K is a graph and $w_0^K \equiv 1$, Δ_0^K reduces to the graph Laplacian of the weighted graph (K, w_1^K) [8].

Theorem 2.2 ([12]). *For each $q \in \mathbb{N}$, $\beta_q^K = \operatorname{nullity}(\Delta_q^K)$.*

Simplicial pairs and simplicial filtrations. A *simplicial pair*, denoted $K \hookrightarrow L$, consists of any pair K and L of simplicial complexes over the *same finite ordered set* V such that $K \subseteq L$, i.e., $S_q^K \subseteq S_q^L$ for all $q \in \mathbb{N}$, and $w^K = w^L|_K$. A *simplicial filtration* $\mathbf{K} = \{K_t\}_{t \in T}$ is a set of simplicial complexes over the same finite ordered set V indexed by a subset $T \subseteq \mathbb{R}$ such that for all $s \leq t \in T$, $K_s \hookrightarrow K_t$ is a simplicial pair. For an integer $q \geq 0$ and for any $s \leq t \in T$, via functoriality of homology [21] one obtains a map $f_q^{s,t} : H_q(K_s) \to H_q(K_t)$ and the qth *persistent homology groups* are defined as the images of these maps. The qth *persistent Betti numbers* $\beta_q^{s,t}$ of $\mathbf{K}$ are in turn defined as the ranks of these groups. Of course when one is just presented with a simplicial pair $K \hookrightarrow L$, for each q one also obtains the analogously defined qth persistent Betti number $\beta_q^{K,L}$.

2.2. Definition of the persistent Laplacian. Suppose that we have a simplicial pair $K \hookrightarrow L$ and that $q \in \mathbb{N}$. Consider the subspace

$$C_q^{L,K} := \left\{ c \in C_q^L \,:\, \partial_q^L(c) \in C_{q-1}^K \right\} \subseteq C_q^L$$

consisting of those q-chains in C_q^L such that their images under the boundary operator ∂_q^L is in the subspace C_{q-1}^K of C_{q-1}^L. $C_q^{L,K}$ is endowed with the inner product $\langle \cdot, \cdot \rangle_{w_q^{L,K}}$ which arises through restricting the inner product $\langle \cdot, \cdot \rangle_{w_q^L}$ on C_q^L by to $C_q^{L,K}$. Let $n_q^{L,K} := \dim(C_q^{L,K})$.

Now, for each q let $\partial_q^{L,K}$ denote the restriction of ∂_q^L to $C_q^{L,K}$ so that we obtain the "diagonal" operators $\partial_q^{L,K} : C_q^{L,K} \to C_{q-1}^K$. As we mentioned earlier, for each q both C_q^K and C_q^L are endowed with inner products $\langle \cdot, \cdot \rangle_{w_q^K}$ and $\langle \cdot, \cdot \rangle_{w_q^L}$ so that we can consider the adjoints of $\partial_{q+1}^{L,K}$ and ∂_q^L. See the diagram below for the construction where the blue arrows signal the important part of the diagram:

$$\begin{tikzcd}
C_{q+1}^K \arrow[rr, "\partial_{q+1}^K"] \arrow[dd, hook, dashed] & & C_q^K \arrow[r, shift left, "\partial_q^K"] \arrow[dl, shift left, "\left(\partial_{q+1}^{L,K}\right)^*"] \arrow[dd, hook, dashed] & C_{q-1}^K \arrow[l, shift left, "\left(\partial_q^K\right)^*"] \arrow[dd, hook, dashed] \\
& C_{q+1}^{L,K} \arrow[ur, shift left, "\partial_{q+1}^{L,K}"] \arrow[dl, hook, dashed] & & \\
C_{q+1}^L \arrow[rr, "\partial_{q+1}^L"] & & C_q^L \arrow[r, "\partial_q^L"] & C_{q-1}^L
\end{tikzcd}$$

One can then define the qth *persistent Laplacian* [51] $\Delta_q^{K,L} : C_q^K \to C_q^K$ by:

$$\Delta_q^{K,L} := \underbrace{\partial_{q+1}^{L,K} \circ \left(\partial_{q+1}^{L,K}\right)^*}_{\Delta_{q,\mathrm{up}}^{K,L}} + \left(\partial_q^K\right)^* \circ \partial_q^K, \tag{2.4}$$

where we have also defined the qth *up* persistent Laplacian $\Delta_{q,\mathrm{up}}^{K,L}$ with the same domain/codomain as $\Delta_q^{K,L}$. When $q = 0$, since $\partial_0^K = 0$, $\Delta_0^{K,L} = \partial_1^{K,L} \circ (\partial_1^{K,L})^* = \Delta_{0,\mathrm{up}}^{K,L}$.

Example 2.3 (trivial cases).

1. When $C_{q+1}^{L,K} = \{0\}$, $\partial_{q+1}^{L,K} = 0$ and thus $\Delta_{q,\mathrm{up}}^{K,L} = 0$.
2. When $K = L$, then obviously $\Delta_q^{K,L} = \Delta_q^L$, the usual Laplacian on L.
3. If $S_q^K = S_q^L$, then $\Delta_{q,\mathrm{up}}^{K,L} = \Delta_{q,\mathrm{up}}^L$. In particular, if $S_0^K = S_0^L$, then $\Delta_0^{K,L} = \Delta_0^L$. If furthermore $S_{q-1}^K = S_{q-1}^L$, then $\Delta_{q,\mathrm{down}}^K = \Delta_{q,\mathrm{down}}^L$ and thus $\Delta_q^{K,L} = \Delta_q^L$.

Obviously, $\Delta_q^{K,L}$ is a self-adjoint, nonnegative and compact operator on C_q^K and thus has nonnegative real eigenvalues. We denote by $0 \le \lambda_{q,1}^{K,L} \le \lambda_{q,2}^{K,L} \le \cdots \le \lambda_{q,n_q^K}^{K,L}$ the eigenvalues of $\Delta_q^{K,L}$ (including repetitions) sorted in increasing order.

2.3. Basic properties of the persistent Laplacian.

We now show some basic properties of $\Delta_q^{K,L}$. All proofs are given in Appendix A.

Lemma 2.4. *Suppose L has n connected components $L_1, \dots, L_n$. Suppose K only intersects the first m connected components. Let $K_i := K \cap L_i$ for each $i = 1, \dots, m$. Then, $\Delta_q^{K,L}$ is the direct sum of persistent Laplacians $\Delta_q^{K_i,L_i}$ on $C_q^{K_i}$ for $i = 1, \dots, m$, i.e., $\Delta_q^{K,L} = \bigoplus_{i=1}^m \Delta_q^{K_i,L_i}$.*

Given a graph K, the multiplicity of the 0 eigenvalue of Δ_0^K coincides with the number of connected components of K [37]. The following result is a persistent version of this.

Theorem 2.5. *The eigenvalues of $\Delta_0^{K,L}$ satisfy the following basic properties.*

1. *$\lambda_{0,1}^{K,L} = 0$, and if L is connected, then $\lambda_{0,2}^{K,L} > 0$.*
2. *Let m be the multiplicity of the 0 eigenvalue of $\Delta_0^{K,L}$, then K intersects exactly m connected components of L.*

We have a complete description of the behavior of the up persistent Laplacian on *interior simplices*, where a q-simplex $\sigma \in S_q^K$ is called an interior simplex if σ only shares cofaces with q-simplices in K, i.e., for all $\sigma' \in S_q^L$, if $\sigma \cup \sigma' \in S_{q+1}^L$, then $\sigma' \in S_q^K$.

Theorem 2.6. *Let $c^L \in C_q^L$ and let c^K be the image of c^L under the orthogonal projection $C_q^L \to C_q^K$. Then, for any interior simplex $\sigma \in S_q^K$, we have that*

$$\left\langle \Delta_{q,\mathrm{up}}^L c^L, [\sigma] \right\rangle_{w_q^L} = \left\langle \Delta_{q,\mathrm{up}}^{K,L} c^K, [\sigma] \right\rangle_{w_q^K}.$$

The following result showing persistent Laplacians recover persistent Betti numbers was mentioned in passing and without proof in [51]. We give a full proof in Appendix A.

Theorem 2.7. *For each integer $q \ge 0$, we have that $\beta_q^{K,L} = \mathrm{nullity}(\Delta_q^{K,L})$.*

3. A first algorithm for computing a matrix representation of $\boldsymbol{\Delta}_q^{K,L}$.

In this section, we first provide a matrix representation $\boldsymbol{\Delta}_q^{K,L}$ of $\Delta_q^{K,L}$ given the canonical basis $\bar{S}_q^K$ of C_q^K and then devise an algorithm for computing $\boldsymbol{\Delta}_q^{K,L}$.[1]

Note. For simplicity, given a simplicial pair $K \hookrightarrow L$, for each $q \in \mathbb{N}$ we assume an ordering $\bar{S}_q^L = \{[\sigma_i]\}_{i=1}^{n_q^L}$ on $\bar{S}_q^L$ such that $\bar{S}_q^K = \{[\sigma_i]\}_{i=1}^{n_q^K}$. Unless otherwise specified, matrix representations of operators between chain groups are *always* from such orderings on canonical bases $\bar{S}_q^K$ and $\bar{S}_q^L$ of C_q^K and C_q^L, respectively.

Theorem 3.1. *Assume that $n_{q+1}^{L,K} := \dim(C_{q+1}^{L,K}) > 0$. Choose any basis of $C_{q+1}^{L,K} \subseteq C_{q+1}^L$ represented by a column matrix $Z \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^{L,K}}$. Let B_q^K and $B_{q+1}^{L,K}$ be matrix representations*

[1] In [51] it is suggested that the qth persistent Laplacian $\Delta_q^{K,L}$ can be computed by (i) taking a certain submatrix of the boundary operator and then (ii) multiplying it by its transpose. However, simply following these two steps does not yield a correct algorithm. The calculation of the matrix form of the persistent Laplacian turned out to be rather subtle as shown in Theorem 3.1; see also section SM2 for details.

of boundary maps ∂_q^K *and* $\partial_{q+1}^{L,K}$*, respectively. Let* W_q^K *(or* W_q^L*) denote the diagonal weight matrix representation of* w_q^K *(or* w_q^L*). Then, the matrix representation* $\mathbf{\Delta}_q^{K,L}$ *of* $\Delta_q^{K,L}$ *is expressed as follows:*

$$\mathbf{\Delta}_q^{K,L} = \underbrace{B_{q+1}^{L,K}\left(Z^{\mathrm{T}}\left(W_{q+1}^{L}\right)^{-1}Z\right)^{-1}\left(B_{q+1}^{L,K}\right)^{\mathrm{T}}\left(W_q^K\right)^{-1}}_{\mathbf{\Delta}_{q,\mathrm{up}}^{K,L}} + \underbrace{W_q^K\left(B_q^K\right)^{\mathrm{T}}\left(W_{q-1}^K\right)^{-1}B_q^K}_{\mathbf{\Delta}_{q,\mathrm{down}}^{K}}. \tag{3.1}$$

Moreover, $\mathbf{\Delta}_q^{K,L}$ *is invariant under the choice of basis for* $C_{q+1}^{L,K}$*.*

Remark 3.2 (matrix representations of combinatorial Laplacians). When $K = L$, (3.1) reduces to the matrix representation of the combinatorial Laplacian:

$$\mathbf{\Delta}_q^K := \underbrace{B_{q+1}^K W_{q+1}^K\left(B_{q+1}^K\right)^{\mathrm{T}}\left(W_q^K\right)^{-1}}_{\mathbf{\Delta}_{q,\mathrm{up}}^{K}} + \underbrace{W_q^K\left(B_q^K\right)^{\mathrm{T}}\left(W_{q-1}^K\right)^{-1}B_q^K}_{\mathbf{\Delta}_{q,\mathrm{down}}^{K}}.$$

Since $B_{q+1}^K W_{q+1}^K (B_{q+1}^K)^{\mathrm{T}}(W_q^K)^{-1} = (W_q^K)^{\frac{1}{2}}((W_q^K)^{-\frac{1}{2}}B_{q+1}^K W_{q+1}^K(B_{q+1}^K)^{\mathrm{T}}(W_q^K)^{-\frac{1}{2}})(W_q^K)^{-\frac{1}{2}}$, $\mathbf{\Delta}_{q,\mathrm{up}}^K$ is of the form $W^{-1}PW$ where P is symmetric positive semidefinite and W is a positive diagonal matrix. The same result holds for down Laplacians, up persistent Laplacians, and (persistent) Laplacians. Note that if $w_q^K \equiv 1$, then $\mathbf{\Delta}_q^K = B_{q+1}^K W_{q+1}^K (B_{q+1}^K)^{\mathrm{T}} + (B_q^K)^{\mathrm{T}}(W_{q-1}^K)^{-1}B_q^K$ is itself a symmetric positive semidefinite matrix.

To prove the theorem, we need the following result.

Lemma 3.3. *Let* $f : (\mathbb{R}^n, W_n) \to (\mathbb{R}^m, W_m)$ *be a linear map where* $W_n \in \mathbb{R}^{n\times n}$ *and* $W_m \in \mathbb{R}^{m\times m}$ *denote the inner product matrices. Let* $F \in \mathbb{R}^{m\times n}$ *denote the matrix representation of* f*. Then, the matrix representation* F^* *of the adjoint* f^* *of* f *is* $W_n^{-1}F^{\mathrm{T}}W_m$*.*

Proof. For any $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$, we have that

$$\langle fx, y\rangle_{\mathbb{R}^m} = (Fx)^{\mathrm{T}} W_m y = x^{\mathrm{T}}F^{\mathrm{T}}W_m y, \text{ and } \langle x, f^*y\rangle_{\mathbb{R}^n} = x^{\mathrm{T}}W_n F^* y.$$

Since $\langle fx, y\rangle_{\mathbb{R}^m} = \langle x, f^*y\rangle_{\mathbb{R}^n}$ and x, y are arbitrary, we must have that $F^* = W_n^{-1}F^{\mathrm{T}}W_m$. ■

Proof of Theorem 3.1. Based on our choice of bases for $C_{q+1}^{L,K}$, C_q^K, and C_{q-1}^K, the corresponding inner product matrices are $Z^{\mathrm{T}}(W_{q+1}^L)^{-1}Z$, $(W_q^K)^{-1}$, and $(W_{q-1}^K)^{-1}$, respectively. By Lemma 3.3, the matrix representation for $(\partial_{q+1}^{L,K})^*$ is $(Z^{\mathrm{T}}(W_{q+1}^L)^{-1}Z)^{-1}(B_{q+1}^{L,K})^{\mathrm{T}}(W_q^K)^{-1}$ and the matrix representation for $(\partial_q^K)^*$ is $W_q^K(B_q^K)^{\mathrm{T}}(W_{q-1}^K)^{-1}$. By (2.4), we have

$$\mathbf{\Delta}_q^{K,L} = B_{q+1}^{L,K}\left(Z^{\mathrm{T}}\left(W_{q+1}^{L}\right)^{-1}Z\right)^{-1}\left(B_{q+1}^{L,K}\right)^{\mathrm{T}}\left(W_q^K\right)^{-1} + W_q^K\left(B_q^K\right)^{\mathrm{T}}\left(W_{q-1}^K\right)^{-1}B_q^K.$$

Since $\partial_{q+1}^{L,K}(\partial_{q+1}^{L,K})^*$ is a self-operator on C_q^K, its matrix representation $\mathbf{\Delta}_q^{K,L}$ only depends on the choice of basis of C_q^K and it is thus independent of the choice of basis of $C_{q+1}^{L,K}$. ■

An algorithm for computing the matrix representation of $\Delta_q^{K,L}$. We use the symbol $[n]$ to denote the set $\{1,\dots,n\}$ for a positive integer n. We first introduce a notation for representing submatrices. Let $M \in \mathbb{R}^{m\times n}$ be a real matrix and let $\emptyset \neq I \subseteq [m]$ and $\emptyset \neq J \subseteq [n]$. We denote by $M(I,J)$ the submatrix of M consisting of those rows and columns indexed by I and J, respectively. Moreover, we use $M(:,J)$ (or $M(I,:)$) to denote $M([m],J)$ (or $M(I,[n])$).

By Theorem 3.1, to compute a matrix representation of $\Delta_q^{K,L}$, the key is to produce a basis (i.e., Z) for $C_{q+1}^{L,K}$. Let $B_{q+1}^L \in \mathbb{R}^{n_q^L\times n_{q+1}^L}$ be the matrix representation of the boundary map ∂_{q+1}^L. We assume that $n_q^K < n_q^L$ since the case $n_q^K = n_q^L$ is trivial (cf. Example 2.3). Then, the following lemma (see the proof in Appendix A) suggests a way of constructing Z from B_{q+1}^L.

Lemma 3.4. *Let $D_{q+1}^L := B_{q+1}^L([n_q^L]\backslash[n_q^K],:)$. Then, there exists a nonsingular matrix $Y \in \mathbb{R}^{n_{q+1}^L\times n_{q+1}^L}$ such that $R_{q+1}^L := D_{q+1}^L Y$ is column reduced.*[2] *Moreover, let $I \subseteq [n_{q+1}^L]$ be the index set of 0 columns of R_{q+1}^L. The following hold:*

1. *If $I=\emptyset$, then $C_{q+1}^{L,K}=\{0\}$.*
2. *If $I\neq\emptyset$, let $Z := Y(:,I)$, then columns of Z constitute a basis of $C_{q+1}^{L,K}$.*

Moreover, if $I\neq\emptyset$, then $B_{q+1}^{L,K} := (B_{q+1}^L Y)([n_q^K],I)$ is the matrix representation of $\partial_{q+1}^{L,K}$.

We can apply a column reduction process (e.g., Gaussian elimination) to D_{q+1}^L to obtain $Y \in \mathbb{R}^{n_{q+1}^L\times n_{q+1}^L}$ and $R_{q+1}^L := D_{q+1}^L Y$ requested in Lemma 3.4. See Algorithm 3.1 for a

Algorithm 3.1. Persistent Laplacian: matrix representation

Data: $B_q^K, B_{q+1}^L, W_{q-1}^K, W_q^K$, and W_{q+1}^L
Result: $\boldsymbol{\Delta}_q^{K,L}$
compute $\boldsymbol{\Delta}_{q,\mathrm{down}}^K$ from B_q^K, W_{q-1}^K, and W_q^K
if $n_q^K == n_q^L$ **then**
    compute $\boldsymbol{\Delta}_{q,\mathrm{up}}^L$ from B_{q+1}^L, W_q^K, and W_{q+1}^L;
    return $\boldsymbol{\Delta}_{q,\mathrm{up}}^L + \boldsymbol{\Delta}_{q,\mathrm{down}}^K$
end if
$D_{q+1}^L = B_{q+1}^L\left([n_q^L]\backslash[n_q^K],:\right)$
$(R_{q+1}^L, Y) = \mathrm{ColumnReduction}(D_{q+1}^L)$
$I \leftarrow$ index set corresponding to the all-zero columns of R_{q+1}^L
if $I == \emptyset$ **then**
    return $\boldsymbol{\Delta}_{q,\mathrm{down}}^K$
end if
$Z = Y(:,I)$
$B_{q+1}^{L,K} = \left(B_{q+1}^L Y\right)\left([n_q^K],I\right)$

return $B_{q+1}^{L,K}\left(Z^{\mathrm{T}}\left(W_{q+1}^L\right)^{-1} Z\right)^{-1}\left(B_{q+1}^{L,K}\right)^{\mathrm{T}}\left(W_q^K\right)^{-1} + \boldsymbol{\Delta}_{q,\mathrm{down}}^K$

[2] We say a matrix is *column reduced* if for each two nonzero columns, their indices of the lowest nonzero elements are different.

pseudocode for computing $\mathbf{\Delta}_q^{K,L}$ based on Lemma 3.4. We remark that Algorithm 3.1 is conceptually important, as it connects to the standard matrix reduction algorithm for computing persistent homology [13].

Complexity analysis. The computation of $\mathbf{\Delta}_{q,\mathrm{down}}^{K}$ takes time $O((n_q^K)^2)$ (see section SM1 for details). The size of D_{q+1}^L is $(n_q^L - n_q^K) \times n_{q+1}^L$; thus the column reduction process takes time $O((n_q^L - n_q^K)(n_{q+1}^L)^2)$. Computing the product $B_{q+1}^L Y$ takes time $O(n_q^L (n_{q+1}^L)^2)$. The size of Z is $n_{q+1}^L \times |I|$, where $|I| \le n_{q+1}^L$. Then, computing $(Z^{\mathrm{T}}(W_{q+1}^L)^{-1}Z)^{-1}$ takes time $O((n_{q+1}^L)^3)$. The product $B_{q+1}^{L,K}(Z^{\mathrm{T}}(W_{q+1}^L)^{-1}Z)^{-1}(B_{q+1}^{L,K})^{\mathrm{T}}(W_q^K)^{-1}$ can be computed in time $O(n_q^K (n_{q+1}^L)^2)$. Hence Algorithm 3.1 takes $O(n_q^L (n_{q+1}^L)^2 + (n_{q+1}^L)^3 + (n_q^K)^2)$ total time. One can also improve this time complexity by using fast matrix multiplication to both perform reductions and compute multiplications/inverses. We omit the details.

4. Schur complement, persistent Laplacian, and implications.

Let $M \in \mathbb{R}^{n\times n}$ be a block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $D \in \mathbb{R}^{d\times d}$ is a square matrix. Then, *the (generalized) Schur complement of D in M* [4], denoted by M/D, is $M/D \coloneqq A - BD^{\dagger}C$, where $D^{\dagger}$ is the Moore–Penrose generalized inverse of D. Note that having D be the bottom right submatrix is done only for notational simplicity. Schur complement is defined for any principal submatrix. More precisely, let $\emptyset \neq I \subsetneq [n]$ be a proper subset. Then, the (generalized) Schur complement of $M(I,I)$ in M is defined as

$$M/M(I,I) \coloneqq M([n]\backslash I,[n]\backslash I) - M([n]\backslash I, I)M(I,I)^{\dagger}M(I,[n]\backslash I). \tag{4.1}$$

Now we introduce some useful properties of the Schur complement.

Definition 4.1 (proper submatrices). *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a square block matrix where both A and D are square matrices. The submatrix D is* proper *in M if* $\ker(D) \subseteq \ker(B)$ *and* $\ker(D^{\mathrm{T}}) \subseteq \ker(C^{\mathrm{T}})$.

Lemma 4.2 (positive semidefinite matrices). *Let P be a positive semidefinite block matrix $P = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that A and D are square matrices. Let W be a positive diagonal matrix and we write W as a block matrix $W = \left(\begin{smallmatrix} W_1 & 0 \\ 0 & W_2 \end{smallmatrix}\right)$ such that W_1 and W_2 have the same sizes as A and D, respectively. Consider $M \coloneqq W^{-1}PW = \left(\begin{smallmatrix} W_1^{-1}AW_1 & W_1^{-1}BW_2 \\ W_2^{-1}CW_1 & W_2^{-1}DW_2 \end{smallmatrix}\right)$. Then, $W_2^{-1}DW_2$ is proper in M and $M/(W_2^{-1}DW_2) = W_1^{-1}(P/D)W_1$.*

Lemma 4.3 ([4, Theorem 1]). *Let M be a square block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that A and D are square matrices. Then,* $\mathrm{rank}(M) \ge \mathrm{rank}(D) + \mathrm{rank}(M/D)$.

Lemma 4.4 (quotient formula [4, Theorem 4]). *Let M, D, and H be square matrices with the following block structures: $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right)$. If D is proper in M and H is proper in D, then D/H is proper in M/H and $M/D = (M/H)/(D/H)$.*

Lemma 4.5 (eigenvalue interlacing property). *Let $M = W^{-1}PW$ be as in Lemma 4.2. Suppose that the size of M is $n \times n$ and the size of D is $d \times d$. Then,*

$$\lambda_k(M) \le \lambda_k(M/(W_2^{-1}DW_2)) \le \lambda_k(W_1^{-1}AW_1) \quad \forall 1 \le k \le n-d, \tag{4.2}$$

where $\lambda_k(A)$ denotes the kth smallest eigenvalue of A (counted with multiplicity).

See section SM6 for proofs of Lemmas 4.2 and 4.5.

4.1. Up persistent Laplacian as a Schur complement. For a simplicial pair $K \hookrightarrow L$, recall from section 3 that for each $q \in \mathbb{N}$ we assume an ordering $\bar{S}_q^L = \{[\sigma_i]\}_{i=1}^{n_q^L}$ on $\bar{S}_q^L$ such that $\bar{S}_q^K = \{[\sigma_i]\}_{i=1}^{n_q^K}$. Given such orderings on canonical bases of C_q^K and C_q^L, the matrix representation $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L}$ of $\Delta_{q,\mathrm{up}}^{K,L} : C_q^K \to C_q^K$ is related to the matrix representation $\mathbf{\Delta}_{q,\mathrm{up}}^{L}$ of $\Delta_{q,\mathrm{up}}^{L} : C_q^L \to C_q^L$ via the Schur complement as follows.

Theorem 4.6 (up persistent Laplacian as Schur complement). *Let $K \hookrightarrow L$ be a simplicial pair. Assume that $n_q^K < n_q^L$ and let $I_K^L := [n_q^L]\backslash[n_q^K]$. Then,*

$$\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} = \mathbf{\Delta}_{q,\mathrm{up}}^{L} / \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L, I_K^L\right). \tag{4.3}$$

To prove the above theorem, we first need the following lemma (whose proof is given in section SM6) which relates Schur complements with a certain matrix operation.

Lemma 4.7. *Let $B \in \mathbb{R}^{n\times m}$ be a block matrix $B = \left(\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right)$, where $B_1 \in \mathbb{R}^{d\times m}$ for some $1 \le d < n$. Let $W_1 \in \mathbb{R}^{d\times d}$ and $W_2 \in \mathbb{R}^{(n-d)\times(n-d)}$ be nonsingular diagonal matrices and let $W = \left(\begin{smallmatrix} W_1 & 0 \\ 0 & W_2 \end{smallmatrix}\right)$. Let $M := BB^{\mathrm{T}}W$, which is a block matrix*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} B_1 B_1^{\mathrm{T}} W_1 & B_1 B_2^{\mathrm{T}} W_2 \\ B_2 B_1^{\mathrm{T}} W_1 & B_2 B_2^{\mathrm{T}} W_2 \end{pmatrix}.$$

If B_2 has full column rank, *then $M/M_{22} = 0$. Otherwise, for any nonsingular block matrix $Y = \left(\begin{smallmatrix} Y_1 & Y_2 \end{smallmatrix}\right) \in \mathbb{R}^{m\times m}$, if $B_2 Y_1 = 0$ and $B_2 Y_2$ has* full column rank, *then $M/M_{22} = B_1 Y_1 (Y_1^{\mathrm{T}} Y_1)^{-1} (B_1 Y_1)^{\mathrm{T}} W_1$.*

Proof of Theorem 4.6. Let $B := B_{q+1}^L (W_{q+1}^L)^{\frac{1}{2}}$, $W := W_q^L$ and $W_1 := W([n_q^K],[n_q^K]) = W_q^K$. Set $B_1 := B([n_q^K],:)$ and $B_2 := B([n_q^L]\backslash[n_q^K],:)$. Then, $B = \left(\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right)$. Note that $B_2 = D_{q+1}^L (W_{q+1}^L)^{\frac{1}{2}}$ using notation in Lemma 3.4. By Lemma 3.4, there exists a nonsingular matrix $\hat{Y} \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^L}$ such that $R_{q+1}^L := D_{q+1}^L \hat{Y}$ is column reduced. Let $Y := (W_{q+1}^L)^{-\frac{1}{2}} \hat{Y}$, which is still nonsingular. Then,

$$R_{q+1}^L = D_{q+1}^L \hat{Y} = D_{q+1}^L \left(W_{q+1}^L\right)^{\frac{1}{2}} \left(W_{q+1}^L\right)^{-\frac{1}{2}} \hat{Y} = B_2 Y.$$

Let $I \subseteq [n_{q+1}^L]$ be the index set of 0 columns of R_{q+1}^L. If $I = \emptyset$, then by Lemma 3.4 we have that $C_{q+1}^{L,K} = \{0\}$ and thus $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} = 0$. On the other hand, $I = \emptyset$ implies that B_2 has full column rank. Let $M := BB^{\mathrm{T}}W$. Then, we have that

$$M = B_{q+1}^L W_{q+1}^L \left(B_{q+1}^L\right)^{\mathrm{T}} W_q^L = \mathbf{\Delta}_{q,\mathrm{up}}^L$$

and thus $M_{22} = \mathbf{\Delta}_{q,\mathrm{up}}^L(I_K^L, I_K^L)$. Then by Lemma 4.7, we have that

$$\mathbf{\Delta}_{q,\mathrm{up}}^{L} / \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L, I_K^L\right) = M/M_{22} = 0 = \mathbf{\Delta}_{q,\mathrm{up}}^{K,L}.$$

Now, we assume that $I \neq \emptyset$. Without loss of generality, we assume that $I = [n_{q+1}^{L,K}] \subseteq [n_{q+1}^L]$ (otherwise we multiply Y by a permutation matrix). Let $Y_1 := Y(:,I) = (W_{q+1}^L)^{-\frac{1}{2}} Z$ where

Z is a column matrix representing a basis of $C_{q+1}^{L,K}$ (cf. Lemma 3.4). Let $Y_2 := Y(:, [n_{q+1}^L]\backslash I)$. Then, $Y = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$ is a block matrix such that $B_2 Y_1 = R_{q+1}^L(:, I) = 0$ and that $B_2 Y_2 = R_{q+1}^L(:, [n_{q+1}^L]\backslash I)$ has full column rank. Then, by Lemma 4.7, we have that

$$\begin{aligned}
&\mathbf{\Delta}_{q,\mathrm{up}}^L / \mathbf{\Delta}_{q,\mathrm{up}}^L \left(I_K^L, I_K^L\right) = M/M_{22} = B_1 Y_1 \left(Y_1^{\mathrm{T}} Y_1\right)^{-1} (B_1 Y_1)^{\mathrm{T}} W_1 \\
&= B_1 \left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \left(\left(\left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \right)^{\mathrm{T}} \left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \right)^{-1} \left(B_1 \left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \right)^{\mathrm{T}} W_q^K \\
&= B_1 \left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \left(Z^{\mathrm{T}} \left(W_{q+1}^L\right)^{-1} Z \right)^{-1} \left(B_1 \left(W_{q+1}^L\right)^{-\frac{1}{2}} Z \right)^{\mathrm{T}} W_q^K.
\end{aligned}$$

Note also that $B_{q+1}^{L,K} = B_{q+1}^L([n_q^K], :)Z = B_1 (W_{q+1}^L)^{-\frac{1}{2}} Z$. Then, by Lemma 3.4 we have that

$$\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} = B_{q+1}^{L,K} \left(Z^{\mathrm{T}} \left(W_{q+1}^L\right)^{-1} Z \right)^{-1} \left(B_{q+1}^{L,K} \right)^{\mathrm{T}} W_q^K = \mathbf{\Delta}_{q,\mathrm{up}}^L / \mathbf{\Delta}_{q,\mathrm{up}}^L \left(I_K^L, I_K^L\right).$$

This finishes the proof of Theorem 4.6. ∎

4.2. Fast computation of the matrix representation of $\mathbf{\Delta}_q^{K,L}$. For a simplicial pair $K \hookrightarrow L$, by Theorem 4.6, we now simply compute $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L}$ via (4.3) using only Schur complement computations, which then give us $\mathbf{\Delta}_q^{K,L} = \mathbf{\Delta}_{q,\mathrm{up}}^{K,L} + \mathbf{\Delta}_{q,\mathrm{down}}^K$. A pseudocode for this simple algorithm is given in Algorithm 4.1.

Algorithm 4.1. Persistent Laplacian: matrix representation via Schur complement

Data: $B_q^K, B_{q+1}^L, W_{q-1}^K, W_q^K, W_q^L$, and W_{q+1}^L
Result: $\mathbf{\Delta}_q^{K,L}$
Compute $\mathbf{\Delta}_{q,\mathrm{down}}^K$ from B_q^K, W_{q-1}^K, and W_q^K
Compute $\mathbf{\Delta}_{q,\mathrm{up}}^L$ from B_{q+1}^L, W_q^L, and W_{q+1}^L
if $n_q^K == n_q^L$ **then**
    return $\mathbf{\Delta}_{q,\mathrm{up}}^L + \mathbf{\Delta}_{q,\mathrm{down}}^K$
end if
$\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} = \mathbf{\Delta}_{q,\mathrm{up}}^L / \mathbf{\Delta}_{q,\mathrm{up}}^L \left(I_K^L, I_K^L\right)$

return $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} + \mathbf{\Delta}_{q,\mathrm{down}}^K$

Time complexity. Computing $\mathbf{\Delta}_{q,\mathrm{up}}^L$ takes time $O(n_{q+1}^L)$ and computing $\mathbf{\Delta}_{q,\mathrm{down}}^K$ takes $O((n_q^K)^2)$ (see section SM1 for details). The Schur complement $\mathbf{\Delta}_{q,\mathrm{up}}^L / \mathbf{\Delta}_{q,\mathrm{up}}^L(I_K^L, I_K^L)$ takes time $O((n_q^K)^2 + (n_q^L - n_q^K)^3 + n_q^K(n_q^L - n_q^K)^2) = O((n_q^L)^3)$ to compute. Hence the total time complexity of computing $\mathbf{\Delta}_q^{K,L}$ via (4.3) is $O((n_q^L)^3 + n_{q+1}^L)$. By using the fast matrix multiplication algorithm (which takes $O(r^\omega)$, $\omega < 2.373$, to multiply two $r \times r$ matrices), this time complexity can be improved to $O((n_q^L)^\omega + n_{q+1}^L)$.

Remark 4.8 (comparison with Algorithm 4.1). The time complexity of Algorithm 3.1 and that of Algorithm 4.1 are not directly comparable: which one is faster depends on the relationship between of the number of p-dimensional simplices in L and the corresponding number of $(p+1)$-dimensional simplices. Recall that the time complexity of Algorithm 3.1 is $O(n_q^L(n_{q+1}^L)^2+(n_{q+1}^L)^3+(n_q^K)^2)$. In a scenario when $n_q^L=O(n_{q+1}^L)$, this complexity is larger than $O((n_q^L)^3+n_{q+1}^L)$ which is the time complexity of Algorithm 4.1. In the case of clique complexes, which commonly arise when studying point cloud data via Rips complexes, n_{p+1}^L is often larger than n_p^L, rendering Algorithm 4.1 more efficient. However, this may not always be the case: given a graph (which can be viewed as a one-dimensional simplicial complex), the number of 2-simplices is 0, while the number of 1-simplices is much larger.

Computation of persistent Betti numbers. By Theorem 2.7, we can compute the persistent Betti number $\beta_q^{K,L}$ in the following manner: we first compute $\mathbf{\Delta}_q^{K,L}$ and then compute $\beta_q^{K,L}=\text{nullity}(\mathbf{\Delta}_q^{K,L})$. Since calculating the nullity of an $n_q^K\times n_q^K$ square matrix can be done in time $O((n_q^K)^\omega)=O((n_q^L)^\omega)$, we obtain a method for computing the persistent Betti number in time $O((n_q^L)^\omega+n_{q+1}^L)$ (which is $O((n_q^L)^\omega)$ if $n_q^L=O(n_{q+1}^L)$). Currently, the existing approach in the literature to compute the persistent Betti numbers is through computing the persistent homology of the pair $K\hookrightarrow L$ using boundary matrices B_{q+1}^L and B_q^K, which can be done in $O((n_q^L)^2n_{q+1}^L+(n_{q-1}^K)^2n_q^K)$ time or in $O((n_q^L)^{\omega-1}n_{q+1}^L+(n_{q-1}^K)^{\omega-1}n_q^K)$ (if we assume that $n_q^L=O(n_{q+1}^L)$ and $n_{q-1}^K=O(n_q^K)$) using the earliest basis (via fast matrix multiplication) approach [2]. Our new algebraic formulation of persistent Laplacian (via Schur complement) thus also leads to a faster algorithm to compute the persistent Betti number for a pair of spaces for the setting when $n_q^L=O(n_{q+1}^L)$. Note that the condition $n_q^L=O(n_{q+1}^L)$ holds in many practical scenarios, especially for the popular Rips or Čech complexes and their variants. Given that this new algorithm is fundamentally different from existing ones (using only simple Schur complement computations), we believe that this is of independent interest.

Remark 4.9. A MATLAB implementation of Algorithm 4.1 for unweighted simplicial pairs is given in [39]. A recent preprint [52] by some of the authors of [51] describes an alternative software implementation of the persistent Laplacian which is available at [53].

4.3. Relationship with the notion of effective resistance. Let $K=(V^K,E^K,w^K)$ be a connected weighted graph. Unless otherwise specified, for any weighted graph considered in this section, we assume that w^K satisfies that $w_0^K=w^K|_{S_0^K}\equiv 1$, i.e., the vertices of the graph are unweighted. For any two vertices $v,v'\in V^K$, we let $\partial_{[v,v']}:=-[v]+[v']\in C_0^K$. Let $D_{[v,v']}^K:=\chi_{v'}-\chi_v\in\mathbb{R}^{n_0^K}$ denote the vector representation of $\partial_{[v,v']}$ in C_0^K, where $\chi_v\in\mathbb{R}^{n_0^K}$ is the indicator vector of $v\in V^K$. We consider that each edge $e\in E^K$ has an *electrical conductance* $w^K(e)$. Then, the *effective resistance* $\mathfrak{R}_{v,v'}^K$ between v and v' is defined by

$$\mathfrak{R}_{v,v'}^K:=\left(D_{[v,v']}^K\right)^{\mathrm{T}}\left(\mathbf{\Delta}_0^K\right)^{\dagger}D_{[v,v']}^K. \tag{4.4}$$

Given a graph pair $K\hookrightarrow L$, by Theorem 4.6 the persistent Laplacian $\mathbf{\Delta}_0^{K,L}$ turns out to be the graph Laplacian of a new weighted graph.

Proposition 4.10 ([10, Lemma 2.1]). *Suppose that $K\hookrightarrow L$ is a graph pair. Assume that L is connected and $w_0^L\equiv 1$. Then, $\mathbf{\Delta}_0^{K,L}=\mathbf{\Delta}_{0,\mathrm{up}}^{K,L}$ is the graph Laplacian $\mathbf{\Delta}_0^{\tilde K}$ of a connected weighted graph $\tilde K=(V^{\tilde K},E^{\tilde K},w^{\tilde K})$ such that $V^{\tilde K}=V^K$.*

Figure 1. *On the left we show a weighted graph L consisting of four vertices with edge weights indicated next to each edge. Let K be the subgraph of L induced by the three red vertices (i.e., the fully disconnected graph on three vertices). Then, $\tilde{K}$ is the three-vertex weighted graph shown on the right (edge weights are indicated next to each edge). Note that $\tilde{K}$ has a totally different edge set from that of K.*

$\tilde{K}$ is known as the *Kron reduction* of L and $\mathbf{\Delta}_0^{\tilde{K}}$ is called the *Kron-reduced matrix*; see Figure 1 for an illustration. The Kron reduction [28] has been used in network circuit theory, and it preserves effective resistance (cf. [10, Theorem 3.8]). This in turn implies that the persistent Laplacian $\mathbf{\Delta}_0^{K,L}$ is able to recover the effective resistance $\mathfrak{R}_{v,v'}^L$ w.r.t. the larger graph L for all pairs of vertices $v, v' \in K$. The result below follows from Theorem 4.6 and [10, Theorem 3.8].

Theorem 4.11. *Let $K \hookrightarrow L$ be a graph pair where L is connected. Let $\tilde{K} = (V^K, E^{\tilde{K}}, w^{\tilde{K}})$ denote the weighted graph such that $\mathbf{\Delta}_0^{\tilde{K}} = \mathbf{\Delta}_0^{K,L}$. Then, $\tilde{K}$ is connected and for two distinct vertices $v, v' \in V^K$, we have that $\mathfrak{R}_{v,v'}^L = \mathfrak{R}_{v,v'}^{\tilde{K}}$.*

Remark 4.12 (higher dimensional generalization). The effective resistance has been generalized to the case of simplicial complexes in [26]. In section SM5, we show a higher dimensional extension of Theorem 4.11, i.e., that higher dimensional effective resistances are preserved by the up persistent Laplacian, the proof of which deals with the subtleties of the Moore–Penrose generalized inverse directly without resorting to a limiting argument as in the proof of [10, Theorem 3.8]. In addition, we provide an example illustrating the impossibility of a higher dimensional generalization of the Kron reduction in the current simplicial setting.

The following result controls the change of degrees after applying the Kron reduction.

Proposition 4.13. *Let $\tilde{K}$ be the graph described in Proposition 4.10. Then, for any $v \in V^K = V^{\tilde{K}}$, we have that $\deg^{\tilde{K}}(v) \leq \deg^L(v)$, where $\deg^G(v) := \sum_{v' \in G} w^G(\{v, v'\})$ is the weighted degree of a vertex in a graph G.*

Proof. We first observe that $\deg^G(v) := \sum_{v' \in G} w^G(\{v, v'\}) = \left(\chi_v^G\right)^{\mathrm{T}} \mathbf{\Delta}_0^G \chi_v$. Therefore,

$$\begin{aligned}
\deg^{\tilde{K}}(v) &= \left(\chi_v^K\right)^{\mathrm{T}} \mathbf{\Delta}_0^{\tilde{K}} \chi_v^{\tilde{K}} = \left(\chi_v^{\tilde{K}}\right)^{\mathrm{T}} \mathbf{\Delta}_0^{K,L} \chi_v^{\tilde{K}} = \left(\chi_v^{\tilde{K}}\right)^{\mathrm{T}} \mathbf{\Delta}_0^L / \mathbf{\Delta}_0^L (I_K^L, I_K^L) \chi_v^{\tilde{K}} \\
&= \left(\chi_v^{\tilde{K}}\right)^{\mathrm{T}} \mathbf{\Delta}_0^L \left([n_0^K], [n_0^K]\right) \chi_v^{\tilde{K}} - \left(\chi_v^{\tilde{K}}\right)^{\mathrm{T}} \mathbf{\Delta}_0^L \left([n_0^K], I_K^L\right) \left(\mathbf{\Delta}_0^L \left(I_K^L, I_K^L\right)\right)^{\dagger} \mathbf{\Delta}_0^L \left(I_K^L, [n_0^K]\right) \chi_v^{\tilde{K}} \\
&\leq \left(\chi_v^{\tilde{K}}\right)^{\mathrm{T}} \mathbf{\Delta}_0^L \left([n_0^K], [n_0^K]\right) \chi_v^{\tilde{K}} = \left(\chi_v^L\right)^{\mathrm{T}} \mathbf{\Delta}_0^L \chi_v^L = \deg^L(v).
\end{aligned}$$
■

In the case when K consists of only two points in L, we have the following explicit relation between the persistent Laplacian and the effective resistance.

Corollary 4.14. *Let L be a connected graph and let K be a two-vertex subgraph with vertex set $V^K = \{v, v'\}$. Then,*

$$\mathbf{\Delta}_0^{K,L} = \begin{pmatrix} \frac{1}{\mathfrak{R}^L_{v,v'}} & -\frac{1}{\mathfrak{R}^L_{v,v'}} \\ -\frac{1}{\mathfrak{R}^L_{v,v'}} & \frac{1}{\mathfrak{R}^L_{v,v'}} \end{pmatrix}.$$

Proof of Corollary 4.14. By Theorem 4.11 (or by [10, Lemma 3.10]), it is easy to show that

$$(\mathbf{\Delta}_0^{K,L})^\dagger = \begin{pmatrix} \frac{\mathfrak{R}^L_{v,v'}}{4} & -\frac{\mathfrak{R}^L_{v,v'}}{4} \\ -\frac{\mathfrak{R}^L_{v,v'}}{4} & \frac{\mathfrak{R}^L_{v,v'}}{4} \end{pmatrix}.$$

Therefore,

$$\mathbf{\Delta}_0^{K,L} = \begin{pmatrix} \frac{1}{\mathfrak{R}^L_{v,v'}} & -\frac{1}{\mathfrak{R}^L_{v,v'}} \\ -\frac{1}{\mathfrak{R}^L_{v,v'}} & \frac{1}{\mathfrak{R}^L_{v,v'}} \end{pmatrix}.$$

∎

4.3.1. Effective resistance between disjoint sets. The effective resistance between two vertices has been generalized to the case of two disjoint sets of vertices in [35, Exercise 2.13] via an energy minimization process. In [44], a formula invoking the graph Laplacian was used to define the effective resistance between disjoint sets. The two definitions are equivalent (see subsection SM3.1 for a proof) and in this section we adopt the definition from [44].

Let K be a connected weighted graph. For any nonempty disjoint subsets $A, B \subseteq V^K$, let $V^J := A \cup B$ and let J be the induced subgraph with vertex set V^J. Then, following [44], the effective resistance $\mathfrak{R}^K_{A,B}$ between A and B is defined as follows:

$$\mathfrak{R}^K_{A,B} := \left(\left(\chi_A^J\right)^{\mathrm{T}} \cdot \mathbf{\Delta}_0^K / \mathbf{\Delta}_0^K \left(V^J, V^J\right) \cdot \chi_A^J \right)^{-1}, \tag{4.5}$$

where $\mathbf{\Delta}_0^K(V^J, V^J)$[3] denotes the submatrix of $\mathbf{\Delta}_0^K$ with rows and columns indexed by V^J and $\chi_A^J \in \mathbb{R}^{n_0^J}$ denotes the indicator vector of $A \subseteq V^J$. By Theorem 4.6, we have that $\mathfrak{R}^K_{A,B} = ((\chi_A^J)^{\mathrm{T}} \cdot \mathbf{\Delta}_0^{J,K} \cdot \chi_A^J)^{-1}$. In particular, when $A \cup B = V^J = V^K$, $\mathfrak{R}^K_{A,B} = ((\chi_A^K)^{\mathrm{T}} \cdot \mathbf{\Delta}_0^K \cdot \chi_A^K)^{-1}$. We call $\mathfrak{C}^K_{A,B} := \frac{1}{\mathfrak{R}^K_{A,B}}$ the *effective conductance* between A and B.

Remark 4.15. Note that (a) when $A = \{v\}$ and $B = \{w\}$ are two singleton sets, it is easy to see that $\mathfrak{R}^K_{A,B} = \mathfrak{R}^K_{v,v'}$; (b) (4.5) might seem asymmetric with respect to A and B; in fact, we have that $\mathfrak{R}^K_{A,B} = -((\chi_A^J)^{\mathrm{T}} \cdot \mathbf{\Delta}_0^{J,K} \cdot \chi_B^J)^{-1}$ (see [44, Lemma 3]): (c) an explanation from the point of view of circuit theory is given in section SM3; see Figure 2.

As a generalization of Theorem 4.11, we establish the following result.

Theorem 4.16. *For a graph pair $K \hookrightarrow L$ where L is connected, let $\tilde{K} = (V^K, E^{\tilde{K}}, w^{\tilde{K}})$ denote the graph such that $\mathbf{\Delta}_0^{\tilde{K}} = \mathbf{\Delta}_0^{K,L}$. Then $\mathfrak{R}^{\tilde{K}}_{A,B} = \mathfrak{R}^L_{A,B}$ for any disjoint $A, B \subseteq V^K$.*

[3] $\mathbf{\Delta}_0^K(V^J, V^J)$ was required to be nonsingular in [44]. This holds automatically as long as K is connected; see [10, Lemma 2.1].

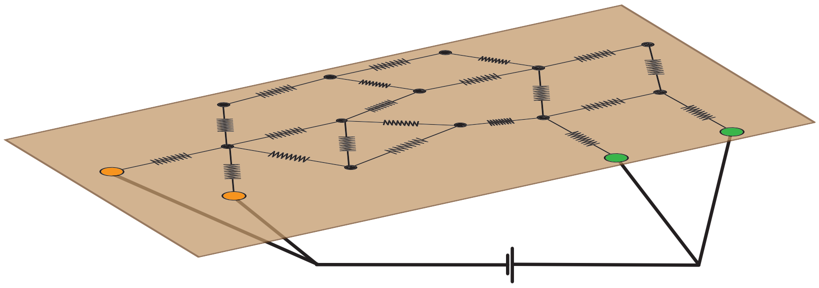

Figure 2. *The effective resistance between two sets of vertices from the circuit theory perspective. One set is represented by the orange vertices, the other one by the green vertices. See subsection SM3.1 for more details.*

Proof. By Theorem 4.11, $\tilde{K}$ is a connected graph and thus $\mathfrak{R}^{\tilde{K}}_{A,B}$ is well-defined. Then, let $V^J := A \cup B \subseteq V^K \subseteq V^L$ and let J denote the induced graph in L with vertex set V^J. By Lemma 4.4, we have that $\mathbf{\Delta}_0^{J,\tilde{K}} = \mathbf{\Delta}_0^{J,L}$. Then, by (4.5) we have that $\mathfrak{R}^{\tilde{K}}_{A,B} = \mathfrak{R}^L_{A,B}$. ∎

4.4. Persistent Cheeger inequality for graph pairs $K \hookrightarrow L$.

The *Cheeger constant* [7] h^K of a weighted graph $K = (V^K, E^K, w^K)$ is defined as follows:

$$h^K := \min_{\substack{\emptyset \neq A \subsetneq V^K \\ |A| \leq \frac{1}{2}|V^K|}} \frac{\left\|E^K(A, V^K\backslash A)\right\|_{w^K}}{|A|},$$

where $E^K(A,B)$ denotes the set of all edges $\{v,v'\} \in E^K$ such that $v \in A$ and $v' \in B$, $|A|$ denotes the cardinality of A, and $\|E^K(A,B)\|_{w^K} := \sum_{\{v,v'\}\in E^K(A,B)} w^K(\{v,v'\})$.

The Cheeger constant h^K measures the edge expansion [22] of K and it is related to the second smallest eigenvalue $\lambda^K_{0,2}$ of the graph Laplacian Δ^K_0 as follows:

$$\frac{\left(h^K\right)^2}{2\, d^K_{\max}} \leq \lambda^K_{0,2} \leq 2\, h^K, \tag{4.6}$$

where $d^K_{\max} := \max_{v\in V^K} \deg^K(v)$. Equation (4.6) is called the *discrete Cheeger inequality* [7, 18, 24], which is a discrete analogue to isoperimetric inequalities in Riemannian geometry [3, 5].

In this section, we define a *persistent Cheeger constant* for any graph pair $K \hookrightarrow L$ via the effective resistance and establish a corresponding *persistent Cheeger inequality* in analogy to (4.6).

To this end, for a subset $\emptyset \neq A \subsetneq V^K$, we first observe the following relationship between $\|E^K(A, V^K\backslash A)\|_{w^K}$ and the effective conductance $\mathfrak{C}^K_{A,V^K\backslash A}$ in a given weighted graph K.

Lemma 4.17. *Given a weighted graph K and any $\emptyset \neq A \subsetneq V^K$, we have that*

$$\left\|E^K(A, V^K\backslash A)\right\|_{w^K} = \mathfrak{C}^K_{A,V^K\backslash A}.$$

Proof. By Remark 4.15,

$$\mathfrak{C}^K_{A,V^K\backslash A} = -\left(\chi^K_A\right)^{\mathrm{T}} \cdot \mathbf{\Delta}^K_0 \cdot \chi^K_{V^K\backslash A} = \sum_{v\in A} \sum_{\substack{v'\in V^K\backslash A\\ \{v,v'\}\in S^K_1}} w^K(\{v,v'\}) = \left\|E^K(A, V^K\backslash A)\right\|_{w^K}. \qquad \blacksquare$$

Hence, the Cheeger constant of a weighted graph K can be equivalently expressed as

$$h^K = \min_{\substack{\emptyset\neq A\subsetneq V^K\\ |A|\leq\frac{1}{2}|V^K|}} \frac{\mathfrak{C}^K_{A,V^K\backslash A}}{|A|}. \tag{4.7}$$

We will use this expression to generalize the Cheeger constant to the case of graph pairs. In the case of a graph pair $K \hookrightarrow L$, we define a persistent Cheeger constant by replacing the right-hand side of (4.7) with the effective conductance between subsets of vertices of K *inside the ambient graph* L:

Definition 4.18 (Persistent Cheeger constant). *The* persistent Cheeger constant $h^{K,L}$ *for a graph pair* $K \hookrightarrow L$ *is defined as follows:*

$$h^{K,L} := \min_{\substack{\emptyset\neq A\subsetneq V^K\\ |A|\leq\frac{1}{2}|V^K|}} \frac{\mathfrak{C}^L_{A,V^K\backslash A}}{|A|}.$$

It is clear that when $K = L$, $h^{K,L}$ reduces to h^K. The following result (whose proof is postponed to section SM6) indicates the persistent Cheeger constant grows as the ambient graph becomes "more connected."

Proposition 4.19. *Consider three weighted graphs* $K \subseteq L_1 \subseteq L_2$. *Then,*

$$h^K \leq h^{K,L_1} \leq h^{K,L_2}.$$

See subsection 4.4.1 for comments about using other possible generalizations of the standard Cheeger constant to the case of graph pairs.

Remark 4.20 (probabilistic interpretation). Consider the canonical random walk $\{X_n\}_{n=0}^{\infty}$ defined on L with V^L being the set of states and the transition probability from v to one of its neighbors v' is $\frac{w^L(\{v,v'\})}{\deg^L(v)}$. For any $A \subsetneq V^K$, let $B := V^K\backslash A$. We establish in subsection SM3.2 that $\mathfrak{C}^L_{A,B}$ is proportional to the *escape probability from* A *to* B, i.e., the probability of the walk, starting randomly from a vertex in A, reaching B before returning to A. In this way, we see that $\mathfrak{C}^L_{A,B}$ measures whether A and B are well-separated in L, i.e., the larger $\mathfrak{C}^L_{A,B}$ is, the more connected A and B are. Thus, $h^{K,L}$ measures the capability of K being partitioned into two well-separated parts in L.

Our definition of persistent Cheeger constant naturally leads us to the following *persistent* Cheeger inequality.

Theorem 4.21 (persistent Cheeger inequality). *Let $K \hookrightarrow L$ be a weighted graph pair, then*

$$\frac{\left(h^{K,L}\right)^2}{2\, d_{\max}^{K,L}} \leq \lambda_{0,2}^{K,L} \leq 2\, h^{K,L}, \tag{4.8}$$

where $d_{\max}^{K,L} := \max_{v\in V^K} \deg^L(v)$ and $\lambda_{0,2}^{K,L}$ denotes the second smallest eigenvalue of $\Delta_0^{K,L}$.

Note that when $K = L$, (4.8) reduces to (4.6). So our persistent Cheeger inequality is a proper generalization of the standard discrete Cheeger inequality.

Proof. By Proposition 4.10, $\boldsymbol{\Delta}_0^{K,L}$ is the graph Laplacian $\boldsymbol{\Delta}_0^{\tilde{K}}$ of a weighted graph $\tilde{K} = (V^K, E^{\tilde{K}}, w^{\tilde{K}})$, so that $\lambda_{0,2}^{K,L} = \lambda_{0,2}^{\tilde{K}}$. By (4.6), we have that $\frac{(h^{\tilde{K}})^2}{2d_{\max}^{\tilde{K}}} \leq \lambda_{0,2}^{K,L} \leq 2h^{\tilde{K}}$. Note that by Lemma 4.17

$$h^{\tilde{K}} = \min_{\substack{\emptyset\neq A\subsetneq V^K\\ |A|\leq\frac{1}{2}|V^K|}} \frac{\left\|E^K(A, V^K\backslash A)\right\|_{w^K}}{|A|} = \min_{\substack{\emptyset\neq A\subsetneq V^K\\ |A|\leq\frac{1}{2}|V^K|}} \frac{\mathfrak{C}^{\tilde{K}}_{A,V^K\backslash A}}{|A|},$$

where we have used the fact that $V^K = V^{\tilde{K}}$. By Theorem 4.16, we have that $\mathfrak{R}^{\tilde{K}}_{A,V^K\backslash A} = \mathfrak{R}^{L}_{A,V^K\backslash A}$ and thus $\mathfrak{C}^{\tilde{K}}_{A,V^K\backslash A} = \mathfrak{C}^{L}_{A,V^K\backslash A}$. This implies that $h^{\tilde{K}} = h^{K,L}$.

For any $v \in V^K = V^{\tilde{K}}$, by Proposition 4.13 we have that

$$\sum_{v'\in V^K} w^{\tilde{K}}(\{v,v'\}) \leq \sum_{v'\in V^L} w^{L}(\{v,v'\})$$

and thus $d_{\max}^{\tilde{K}} \leq d_{\max}^{K,L}$. Therefore,

$$\frac{(h^{K,L})^2}{2d_{\max}^{K,L}} \leq \frac{(h^{K,L})^2}{2d_{\max}^{\tilde{K}}} = \frac{(h^{\tilde{K}})^2}{2d_{\max}^{\tilde{K}}} \leq \lambda_{0,2}^{K,L} \leq 2h^{\tilde{K}} = 2h^{K,L}.$$

■

4.4.1. A combinatorial upper bound for $\boldsymbol{\lambda_{0,2}^{K,L}}$. When graphs are unweighted, we provide a combinatorial upper bound for $\lambda_{0,2}^{K,L}$.

A *path* in a graph $K = (V^K, E^K)$ is a tuple $p = (v_0, \ldots, v_n)$ such that $v_i \in V^K$ for each $i = 0, \ldots, n$ and $\{v_i, v_{i+1}\} \in E^K$ for each $i = 0, \ldots, n-1$. For two nonempty disjoint subsets $A, B \subseteq V^K$, we denote by $P_K(A, B)$ the set of all paths $p = (v_0, \ldots, v_n)$ in K satisfying (i) $v_0 \in A, v_n \in B$ and $v_i \notin A \cup B$ for $i = 1, \ldots, n-1$; (ii) $\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\}$ for $i \neq j$. If $A = \{v\}$ and $B = \{w\}$ are one-point sets, then we also denote $P_K(v, v') := P_K(\{v\}, \{w\})$. The following Nash–Williams inequality [36, Lemma 2.1] permits relating $P_K(A, B)$ with $\mathfrak{R}^K_{A,B}$.

Lemma 4.22 (Nash–Williams inequality). *Let K be a weighted graph. Let A, B be nonempty disjoint subsets of V^K. A set $\Pi \subseteq E^K$ is called a* cut set *between A and B if for any $v \in A$ and $v' \in B$, every path from v to v' contains an edge in Π. Suppose $\Pi_1, \ldots, \Pi_n$ are disjoint cut sets between A and B. Then,*

$$\mathfrak{R}^K_{A,B} \geq \sum_{k=1}^{n} \left(\sum_{e\in\Pi_k} w^K(e)\right)^{-1}.$$

Now, consider a graph pair $K \hookrightarrow V$. Let $\emptyset \neq A \subseteq V^K$ and let $B := V^K \backslash A$. Then, let $p_1, \dots, p_N$ denote all the paths in $P_L(A, B)$. Choose an arbitrary edge e_i from each path p_i. The set $\Pi := \{e_i : i = 1, \dots, N\}$ is obviously a cut set between A and B. By Lemma 4.22 we have that $\mathfrak{C}^L_{A,B} \leq \sum_{e \in \Pi} w^L(e) \leq |P_L(A, B)|$. By Theorem 4.21 we have the following upper bound for $\lambda^{K,L}_{0,2}$ which arises by minimizing the number of paths in L connecting the two sets in a bipartition of V^K:

$$\frac{1}{2}\lambda^{K,L}_{0,2} \leq h^{K,L} \leq \min_{\substack{\emptyset \neq A \subsetneq V^K \\ |A| \leq \frac{1}{2}|V^K|}} \frac{\left|P_L(A, V^K \backslash A)\right|}{|A|} =: h^{K,L}_{\text{path}}.$$

A priori, it seems plausible that one could have used the right-hand side of the above inequality, $h^{K,L}_{\text{path}}$, as the definition of the persistent Cheeger constant. However, as we show in section SM4, this quantity does not have a good interplay with the second persistent eigenvalue, i.e., $h^{K,L}_{\text{path}}$ cannot be upper bounded by $\lambda^{K,L}_{0,2}$ in any suitable sense.

5. The persistent Laplacian for simplicial filtrations.

We now extend the setting of section 2 for simplicial pairs to a simplicial filtration.

5.1. Formulation.

Let $\mathbf{K} = \{K_t\}_{t \in T}$ be a simplicial filtration with an index set $T \subseteq \mathbb{R}$. For each $t \in T$ and $q \in \mathbb{N}$ we let $C^t_q := C^{K_t}_q$, $S^t_q := S^{K_t}_q$, and $w^t_q := w^{K_t}_q$. For $s \leq t \in T$ we let

$$C^{t,s}_q := \left\{c \in C^t_q : \partial^t_q(c) \in C^s_{q-1}\right\} \subseteq C^t_q.$$

Let $\partial^{t,s}_q$ be the restriction of ∂^t_q to $C^{t,s}_q$. Then, $\partial^{t,s}_q$ is a map from $C^{t,s}_q$ to C^s_{q-1}. Finally, we define the qth persistent Laplacian $\Delta^{s,t}_q : C^s_q \to C^s_q$ by

$$\Delta^{s,t}_q := \underbrace{\partial^{t,s}_{q+1} \circ \left(\partial^{t,s}_{q+1}\right)^*}_{\Delta^{s,t}_{q,\text{up}}} + \underbrace{\left(\partial^s_q\right)^* \circ \partial^s_q}_{\Delta^s_{q,\text{down}}}, \tag{5.1}$$

where we view C^t_q for each $t \in T$ as a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{w^t_q}$ and A^* means the adjoint of an operator A under these inner products. We also let Δ^t_q denote the qth Laplacian of K_t for $t \in T$. Note that $\Delta^{t,t}_q = \Delta^t_q$ (cf. Example 2.3).

5.2. An algorithm for $\boldsymbol{\Delta}^{s,t}_q$.

Consider the simplicial filtration $K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m$ where each K_{t+1} contains exactly one more simplex than K_t for $t = 1, \dots, m-1$. In this section, we show that, for a fixed index $t \in [m]$, we can compute the matrix representation $\boldsymbol{\Delta}^{s,t}_q$ of the persistent Laplacian $\Delta^{s,t}_q$, for all $1 \leq s \leq t$, in time $O(t(n^t_q)^2 + n^t_{q+1})$, where $n^t_q := n^{K_t}_q$ is the number of q-simplices in K_t. Note that this is more efficient than applying the Schur complement formula for $\boldsymbol{\Delta}^{s,t}_q$ (equation (4.3)) t times, which will lead to $O(t(n^t_q)^3 + t\, n^t_{q+1})$ total time. This result is again achieved via the relation between persistent Laplacian with Schur complement (cf. Theorem 4.6).

Recall from (5.1) that for any $1 \leq s \leq t$, $\boldsymbol{\Delta}^{s,t}_q = \boldsymbol{\Delta}^{s,t}_{q,\text{up}} + \boldsymbol{\Delta}^s_{q,\text{down}}$. Since $\boldsymbol{\Delta}^s_{q,\text{down}}$ can be constructed in time $O((n^s_q)^2) = O((n^t_q)^2)$ (cf. section SM1), the set of $\boldsymbol{\Delta}^s_{q,\text{down}}$ for all $1 \leq s \leq t$ can be computed in $O(t(n^t_q)^2)$ time.

For simplicity, we assume that $S_q^s = \{\sigma_1, \ldots, \sigma_s\}$ for each $s = 1, \ldots, t$, that is, K_{s+1} contains exactly one more q-simplex than K_s for $s = 1, \ldots, t-1$. It then follows that $\mathbf{\Delta}_{q,\mathrm{up}}^{s,t} = \mathbf{\Delta}_{q,\mathrm{up}}^{t}/\mathbf{\Delta}_{q,\mathrm{up}}^{t}(I_s^t, I_s^t)$, where I_s^t is the index set $I_s^t = \{s+1, s+2, \ldots, t\}$. By Remark 3.2 and Lemma 4.2, $\mathbf{\Delta}_{q,\mathrm{up}}^{t}(I_s^t, I_s^t)$ is proper in $\mathbf{\Delta}_{q,\mathrm{up}}^{t}$ for each $s = 1, \ldots, t-1$. Therefore, following the quotient formula (Lemma 4.4), to compute $\mathbf{\Delta}_{q,\mathrm{up}}^{s,t}$, one can perform an iterative reduction from $\mathbf{\Delta}_{q,\mathrm{up}}^{t}$ to $\mathbf{\Delta}_{q,\mathrm{up}}^{t-1,t}, \ldots, \mathbf{\Delta}_{q,\mathrm{up}}^{s+1,t}$, and down to $\mathbf{\Delta}_{q,\mathrm{up}}^{s,t}$. More precisely, for any $\ell \leq t$,

(5.2)
$$\mathbf{\Delta}_{q,\mathrm{up}}^{\ell-1,t}(i,j) = \begin{cases} \mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(i,j) - \frac{\mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(i,\ell)\mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(j,\ell)}{\mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(\ell,\ell)} & \text{if } \mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(\ell,\ell) \neq 0 \\ \mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(i,j) & \text{if } \mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}(\ell,\ell) = 0 \end{cases} \quad \text{for any } i,j \in [\ell-1].$$

Equation (5.2) coincides with the celebrated Kron reduction formula (see equation (16) of [10]) when K_t is a connected graph, $q = 0$, and $w_0^t \equiv 1$. In other words, $\mathbf{\Delta}_{q,\mathrm{up}}^{\ell-1,t}$ can be computed from $\mathbf{\Delta}_{q,\mathrm{up}}^{\ell,t}$ in time linear to the size of the matrix, which is bounded by $O((n_q^t)^2)$. Note that from section SM1 we know computing $\mathbf{\Delta}_{q,\mathrm{up}}^{t}$ takes time $O(n_{q+1}^t)$. It then follows that using (5.2), we can compute $\mathbf{\Delta}_{q,\mathrm{up}}^{s,t}$ for all $1 \leq s \leq t$ iteratively in $O(t(n_q^t)^2 + n_{q+1}^t)$ total time. We summarize our discussion into the following theorem.

Theorem 5.1. *Let $K_1 \hookrightarrow \cdots \hookrightarrow K_m$ be a simplicial filtration where each K_{t+1} contains exactly one more simplex than K_t for all $t \in [m-1]$. For any fixed $t \in [m]$, we can compute the whole set $\{\mathbf{\Delta}_q^{s,t}\}_{s=1}^t$ of persistent Laplacians in $O(t(n_q^t)^2 + n_{q+1}^t)$ time. This also implies that we can compute all $\mathbf{\Delta}_q^{i,j}$, for any $1 \leq i \leq j \leq m$, in $O(m^2(n_q^m)^2 + m\, n_{q+1}^m)$ total time.*

Remark 5.2. Note that if the input filtration is not simplexwise, namely if $K_{i+1} \backslash K_i$ is allowed to contain more than one simplex, then one can still use the above procedure by first refining the input filtration so as to produce a simplexwise one. However, this will increase the length of the filtration and thus impact the time complexity. Hence, if the size of $K_{i+1} \backslash K_i$ is large, then it may be more beneficial to instead perform multiple Schur complements in order to construct the set of persistent Laplacians.

5.3. Monotonicity, functoriality, and stability of (up) persistent eigenvalues. Recall from subsection 2.3 that for a simplicial pair $K \hookrightarrow L$, $\lambda_{q,k}^{K,L}$ denotes the kth smallest eigenvalue of $\Delta_q^{K,L}$. Now, given a simplicial filtration $\mathbf{K} = \{K_t\}_{t\in T}$, we define its *$k$th persistent eigenvalue* $\lambda_{q,k}^{s,t}(\mathbf{K})$ for each $s \leq t \in T$ by $\lambda_{q,k}^{s,t}(\mathbf{K}) := \lambda_{q,k}^{K_s,K_t}$. We define the *$k$th up persistent eigenvalue* $\lambda_{q,\mathrm{up},k}^{s,t}(\mathbf{K})$ for each $s \leq t \in T$ to be the kth smallest eigenvalue of $\Delta_{q,\mathrm{up}}^{s,t}$. Whenever the underlying filtration $\mathbf{K}$ is clear from the context, we let $\lambda_{q,k}^{s,t} := \lambda_{q,k}^{s,t}(\mathbf{K})$ and $\lambda_{q,\mathrm{up},k}^{s,t} := \lambda_{q,\mathrm{up},k}^{s,t}(\mathbf{K})$.

In [51] the authors suggest that invariants similar to persistent eigenvalues could be useful for shape classification applications. With that in mind, we now explore both their monotonicity and stability properties, concluding with Theorem 5.10. We remark that, in the course of studying stability properties of persistent eigenvalues, we also establish the functoriality of the up persistent Laplacian and its eigenvalues.

$$\begin{tikzcd} K_{t_1} \arrow[r, "\lambda^{t_1,t_2}_{q,\mathrm{up},k}"] \arrow[rr, bend right, "\lambda^{t_1,t_3}_{q,\mathrm{up},k}"'] & K_{t_2} \arrow[r, "\lambda^{t_2,t_3}_{q,\mathrm{up},k}"] & K_{t_3} \end{tikzcd}$$

Figure 3. *All arrows represent inclusion maps and all eigenvalues $\lambda^{t_i,t_j}_{q,\mathrm{up},k}$ (for $1 \le i < j \le 3$) are shown on top of their corresponding arrows. Theorem 5.3 guarantees that $\lambda^{t_1,t_3}_{q,\mathrm{up},k}$ is the largest among these three eigenvalues.*

Theorem 5.3 (monotonicity of up persistent eigenvalues). *Let $\mathbf{K} = \{K_t\}_{t\in T}$ be a simplicial filtration and let $q \in \mathbb{N}$. Then, for any $t_1 \le t_2 \le t_3 \in T$, we have for each $k = 1,\dots,n_q^{t_1}$ that $\lambda^{t_1,t_2}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$ and $\lambda^{t_2,t_3}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$. See Figure 3 for an illustration.*

The proof of Theorem 5.3 exploits the connection of the up persistent Laplacian with Schur complements (Theorem 4.6).

Proof. By the min-max theorem (see, for example, [23, Theorem 2.1]), we have for any $s \le t \in T$ and for each $k = 1,\dots,n_q^{t_1}$ that

$$\lambda^{s,t}_{q,\mathrm{up},k} = \min_{V_k \subseteq C_q^s} \max_{g\in V_k} \frac{\left\langle \Delta^{s,t}_{q,\mathrm{up}} g, g\right\rangle_{w_q^s}}{\langle g, g\rangle_{w_q^s}},$$

where the minimum is taken over all k-dim subspaces V_k of C_q^s. Then, in order to prove that $\lambda^{t_1,t_2}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$, it suffices to verify that $\langle \Delta^{t_1,t_2}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}} \le \langle \Delta^{t_1,t_3}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}}$ for any $g \in C_q^{t_1}$.

Now, since $C^{t_2,t_1}_{q+1} \subseteq C^{t_3,t_1}_{q+1}$, we consider an orthogonal decomposition

$$C^{t_3,t_1}_{q+1} = C^{t_2,t_1}_{q+1} \bigoplus \left(C^{t_2,t_1}_{q+1}\right)^{\perp}.$$

Then, we have the decomposition $\partial^{t_3,t_1}_{q+1} = \partial^{t_2,t_1}_{q+1} \oplus \partial^{\perp}$, where $\partial^{\perp}$ maps $(C^{t_2,t_1}_{q+1})^{\perp}$ into $C_q^{t_1}$. Therefore, we have that

$$\Delta^{t_1,t_3}_{q,\mathrm{up}} = \partial^{t_3,t_1}_{q+1}\left(\partial^{t_3,t_1}_{q+1}\right)^* = \partial^{t_2,t_1}_{q+1}\left(\partial^{t_2,t_1}_{q+1}\right)^* + \partial^{\perp}\left(\partial^{\perp}\right)^* = \Delta^{t_1,t_2}_{q,\mathrm{up}} + \partial^{\perp}\left(\partial^{\perp}\right)^*. \tag{5.3}$$

This implies the following and thus $\lambda^{t_1,t_2}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$:

$$\begin{aligned} \langle \Delta^{t_1,t_3}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}} &= \langle \Delta^{t_1,t_2}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}} + \left\langle \partial^{\perp}\left(\partial^{\perp}\right)^* g, g\right\rangle_{w_q^{t_1}} \\ &= \langle \Delta^{t_1,t_2}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}} + \left\langle \left(\partial^{\perp}\right)^* g, \left(\partial^{\perp}\right)^* g\right\rangle_{w_q^{t_1}} \ge \langle \Delta^{t_1,t_2}_{q,\mathrm{up}} g, g\rangle_{w_q^{t_1}}. \end{aligned}$$

As for $\lambda^{t_2,t_3}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$, we will apply Theorem 4.6. For notational simplicity, we let $I_s^t := [n_q^t]\backslash[n_q^s]$. Since the matrix $\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}$ is positive semidefinite, both $\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_2}, I^{t_3}_{t_2})$ and $\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_1}, I^{t_3}_{t_1})$ are proper in $\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}$ (cf. Lemma 4.2). Moreover, $\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_2}, I^{t_3}_{t_2})$ is proper in

$\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_1}, I^{t_3}_{t_1})$. Then, by Lemma 4.4 and Theorem 4.6, $\mathbf{\Delta}^{t_1,t_3}_{q,\mathrm{up}} = \mathbf{\Delta}^{t_3}_{q,\mathrm{up}}/\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_1}, I^{t_3}_{t_1})$ is the Schur complement of some proper principal submatrix in $\mathbf{\Delta}^{t_2,t_3}_{q,\mathrm{up}} = \mathbf{\Delta}^{t_3}_{q,\mathrm{up}}/\mathbf{\Delta}^{t_3}_{q,\mathrm{up}}(I^{t_3}_{t_2}, I^{t_3}_{t_2})$. More precisely,

$$\mathbf{\Delta}^{t_1,t_3}_{q,\mathrm{up}} = \mathbf{\Delta}^{t_2,t_3}_{q,\mathrm{up}}/\mathbf{\Delta}^{t_2,t_3}_{q,\mathrm{up}}\left(I^{t_2}_{t_1}, I^{t_2}_{t_1}\right). \tag{5.4}$$

Then, by Lemma 4.5, we have that

$$\lambda^{t_2,t_3}_{q,\mathrm{up},k} = \lambda_k\left(\mathbf{\Delta}^{t_2,t_3}_{q,\mathrm{up}}\right) \le \lambda_k\left(\mathbf{\Delta}^{t_1,t_3}_{q,\mathrm{up}}\right) = \lambda^{t_1,t_3}_{q,\mathrm{up},k} \quad \forall k = 1,\ldots,n^s_q.$$ ∎

Note that when $q = 0$, $\Delta^{s,t}_0 = \Delta^{s,t}_{0,\mathrm{up}}$ for $s \le t$. Then, we have the following corollary.

Corollary 5.4. *Let $\mathbf{K} = \{K_t\}_{t\in T}$ be a simplicial filtration. Then for any $t_1 \le t_2 \le t_3 \in T$, we have for each $k = 1,\ldots,n^{t_1}_0$ that $\lambda^{t_1,t_2}_{0,k} \le \lambda^{t_1,t_3}_{0,k}$ and $\lambda^{t_2,t_3}_{0,k} \le \lambda^{t_1,t_3}_{0,k}$.*

We remark that the persistent Cheeger constant (cf. Definition 4.18), a quantity closely related to 0th persistent eigenvalues, is also shown to satisfy a monotonicity property (as described in Proposition 4.19). This monotonicity property is, however, weaker than the monotonicity property for 0th persistent eigenvalues established in the corollary above.

A simple adaptation of the proof of the formula $\lambda^{t_1,t_2}_{q,\mathrm{up},k} \le \lambda^{t_1,t_3}_{q,\mathrm{up},k}$ will give rise to the following monotonicity result for persistent eigenvalues.

Corollary 5.5. *Let $\mathbf{K} = \{K_t\}_{t\in T}$ be a simplicial filtration. Given $q \in \mathbb{N}$, then for any $t_1 \le t_2 \le t_3 \in T$, we have for each $k = 1,\ldots,n^{t_1}_q$ that $\lambda^{t_1,t_2}_{q,k} \le \lambda^{t_1,t_3}_{q,k}$.*

Functoriality and stability of up persistent eigenvalues. Consider the simplicial filtration $K' \hookrightarrow K \hookrightarrow L \hookrightarrow L'$. This filtration should be regarded as a morphism (in the category where objects are simplicial pairs over a fixed vertex set V) from the simplicial pair $K \hookrightarrow L$ to the simplicial pair $K' \hookrightarrow L'$:

$$\begin{tikzcd} & K \arrow[r, hook] & L \arrow[dr, hook] & \\ K' \arrow[ur, hook] \arrow[rrr, hook] & & & L' \end{tikzcd} \tag{5.5}$$

Given $\Delta^{K,L}_{q,\mathrm{up}}$ on $C_q(K)$, we induce an operator on $C_q(K')$ by considering the Schur complement $\mathbf{\Delta}^{K,L}_{q,\mathrm{up}}/\mathbf{\Delta}^{K,L}_{q,\mathrm{up}}(I^K_{K'}, I^K_{K'})$, where $I^K_{K'}$ stands for the indices corresponding to q-simplices which are not in $K' \subseteq K$. By Lemma 4.4 and Theorem 4.6, one has that $\mathbf{\Delta}^{K,L}_{q,\mathrm{up}}/\mathbf{\Delta}^{K,L}_{q,\mathrm{up}}(I^K_{K'}, I^K_{K'})$ is the matrix representation of $\Delta^{K',L}_{q,\mathrm{up}}$ (see also (5.4)). It follows from (5.3) in the proof of Theorem 5.3 that

$$\Delta^{K',L}_{q,\mathrm{up}} \preceq \Delta^{K',L'}_{q,\mathrm{up}}, \tag{5.6}$$

i.e., $\Delta^{K',L'}_{q,\mathrm{up}} - \Delta^{K',L}_{q,\mathrm{up}}$ is positive semidefinite. Hence, the operator $\Delta^{K,L}_{q,\mathrm{up}}$ on $C_q(K)$ arising from the pair $K \hookrightarrow L$ induces an operator on $C_q(K')$ which is upper bounded (in the sense of the Loewner order $\preceq$) by the operator $\Delta^{K',L'}_{q,\mathrm{up}}$ arising from the pair $K' \hookrightarrow L'$. This should be seen as expressing the *functoriality* of up persistent Laplacians. As a direct consequence of this functoriality property of up persistent Laplacians and Lemma 4.5, we establish the following functoriality (monotonicity) property of up persistent eigenvalues.

Proposition 5.6 (functoriality of up persistent eigenvalues). *For the morphism* (5.5) *between the simplicial pairs $K \hookrightarrow L$ and $K' \hookrightarrow L'$, and for any $k = 1, \ldots, n_q^{K'}$, we have that*

$$\lambda_{q,\mathrm{up},k}^{K,L} \leq \lambda_{q,\mathrm{up},k}^{K',L'}. \tag{5.7}$$

Now, based on functoriality of up persistent eigenvalues, we establish a stability result via interleaving-type distances.

Definition 5.7 (interleaving distance between simplicial filtrations over $\mathbb{R}$). *Let $\mathbf{K} = \{K_t\}_{t\in\mathbb{R}}$ and $\mathbf{L} = \{L_t\}_{t\in\mathbb{R}}$ be two simplicial filtrations over $\mathbb{R}$ with the same underlying vertex set V and the same index set $\mathbb{R}$. We define the interleaving distance between $\mathbf{K}$ and $\mathbf{L}$ by*

$$d_{\mathrm{I}}^{V}(\mathbf{K}, \mathbf{L}) := \inf \{\varepsilon \geq 0 : \forall t, K_t \subseteq L_{t+\varepsilon} \text{ and } L_t \subseteq K_{t+\varepsilon}\},$$

where when we write the inclusion $K \subseteq L$, we implicitly require that $w^K = w^L|_K$.

Definition 5.8 (interleaving distance between functions). *Let $\mathbf{Int}$ denote the set of closed intervals in $\mathbb{R}$. Let $f : \mathbf{Int} \to \mathbb{R}_{\geq 0}$ and $g : \mathbf{Int} \to \mathbb{R}_{\geq 0}$ be two nonnegative functions. We then define the interleaving distance between f and g by*

$$d_{\mathrm{I}}(f, g) := \inf \{\varepsilon \geq 0 : \forall I \in \mathbf{Int}, f(I^\varepsilon) \geq g(I) \text{ and } g(I^\varepsilon) \geq f(I)\}.$$

Above, for $\mathbf{Int} \ni I = [a, b]$ and $\varepsilon > 0$, we denoted $I^\varepsilon := [a - \varepsilon, b + \varepsilon]$.

Remark 5.9. The stability theorem given below is structurally similar to claims about stability of the rank invariant; see [42, Theorem 22] and [25, Remarks 4.10 and 4.11].

With these definitions and with (5.7) we now obtain the following stability theorem.

Theorem 5.10 (stability theorem for up persistent eigenvalues). *Let $\mathbf{K} = \{K_t\}_{t\in\mathbb{R}}$ and $\mathbf{L} = \{L_t\}_{t\in\mathbb{R}}$ be two simplicial filtrations over the same underlying vertex set V. Then,*

$$d_{\mathrm{I}}\left(\lambda_{q,\mathrm{up},k}^{\mathbf{K}}, \lambda_{q,\mathrm{up},k}^{\mathbf{L}}\right) \leq d_{\mathrm{I}}^{V}(\mathbf{K}, \mathbf{L}), \tag{5.8}$$

where $\lambda_{q,\mathrm{up},k}^{\mathbf{K}} : \mathbf{Int} \to \mathbb{R}_{\geq 0}$ is defined by $\mathbf{Int} \ni I = [a, b] \mapsto \lambda_{q,\mathrm{up},k}^{a,b}(\mathbf{K})$.

A similar but more convoluted statement, would express the stability of the persistent up Laplacians via (5.6).

Proof. If $d_{\mathrm{I}}^{V}(\mathbf{K}, \mathbf{L}) = \infty$, then (5.8) holds trivially. Otherwise we assume there exists $\varepsilon \geq 0$ such that $K_t \subseteq L_{t+\varepsilon}$ and $L_t \subseteq K_{t+\varepsilon}$ for all $t \in \mathbb{R}$. For any $I = [a, b] \in \mathbf{Int}$, then $L_{a-\varepsilon} \subseteq K_a \subseteq K_b \subseteq L_{b+\varepsilon}$ is a simplicial filtration related to the following interleaving diagram:

$$\begin{tikzcd}
& K_a \arrow[r, hook] & K_b \arrow[dr, hook] & \\
L_{a-\varepsilon} \arrow[ur, hook] \arrow[rrr, hook] & & & L_{b+\varepsilon}
\end{tikzcd}$$

By Proposition 5.6, $\lambda_{q,\mathrm{up},k}^{L_{a-\varepsilon},L_{b+\varepsilon}} \geq \lambda_{q,\mathrm{up},k}^{K_a,K_b}$. This implies that $\lambda_{q,\mathrm{up},k}^{\mathbf{L}}(I^\varepsilon) \geq \lambda_{q,\mathrm{up},k}^{\mathbf{K}}(I)$ for all $I \in \mathbf{Int}$. Similarly, $\lambda_{q,\mathrm{up},k}^{\mathbf{K}}(I^\varepsilon) \geq \lambda_{q,\mathrm{up},k}^{\mathbf{L}}(I)$ for all $I \in \mathbf{Int}$. Therefore, $d_{\mathrm{I}}(\lambda_{q,\mathrm{up},k}^{\mathbf{K}}, \lambda_{q,\mathrm{up},k}^{\mathbf{L}}) \leq \varepsilon$ and thus $d_{\mathrm{I}}(\lambda_{q,\mathrm{up},k}^{\mathbf{K}}, \lambda_{q,\mathrm{up},k}^{\mathbf{L}}) \leq d_{\mathrm{I}}^{V}(\mathbf{K}, \mathbf{L})$. ∎

6. Discussion.

As a natural progression of the ideas in this paper, where the persistent Laplacian is formulated for inclusion maps, it is of definite interest to extend it to the setting of simplicial maps—a natural extension which would enable other applications such as graph sparsification where clusters of vertices might be collapsed between consecutive levels of a filtration.

A notion of persistent Laplacian for pairs of manifolds also related by inclusion maps was developed in [6]. In the spirit of our paper, it is then natural to attempt to relate the version of the persistent Laplacian from [6] to notions of Schur complement of operators (e.g., [15]) in a suitable sense, which may also be related to Poincaré–Steklov operators [29].

The Cheeger inequality has been generalized both to higher order (eigenvalues of graph Laplacians) in [30] and to higher dimensional simplicial complexes [47, 18]. This naturally suggests to us to consider suitable extensions of our persistent Cheeger inequality to these cases which will provide interpretation of the persistent Laplacian spectrum.

It is of clear interest to elucidate stability properties of invariants associated to the persistent Laplacian which generalize the results we established in Theorem 5.10. It is conceivable that some of these developments will follow from invoking classical operator perturbation techniques—an "analytical" possibility afforded by the persistent Laplacian approach to persistent homology.

Finally, we remark that whereas the multiplicity of the zero eigenvalue recovers the rank of the corresponding homology group (i.e., Betti number), in general, both nonzero eigenvalues and (specific) eigenvectors have applications such as in partitioning [50] and shape matching [43]. This suggests the future exploration of applications of persistent spectral analysis beyond mere persistent Betti numbers.

Appendix A. Relegated proofs.

Proof of Lemma 2.4. This follows directly from the following obvious observations:

1. $C_{q-1}^K = \bigoplus_{i=1}^m C_{q-1}^{K_i}$, $C_q^K = \bigoplus_{i=1}^m C_q^{K_i}$, and $C_{q+1}^{L,K} = \bigoplus_{i=1}^m C_{q+1}^{L_i,K_i}$.
2. $\partial_q^K = \bigoplus_{i=1}^m \partial_q^{K_i}$ and $\partial_{q+1}^{L,K} = \bigoplus_{i=1}^m \partial_{q+1}^{L_i,K_i}$. ∎

Proof of Theorem 2.5. For item 1, let $c_0^K := \sum_{v\in S_0^K} w_0^K(v)[v] \in C_0^K$. We prove that $\Delta_0^{K,L} c_0^K = 0$ and thus $\lambda_{0,1}^{K,L} = 0$. Set $c_0^L := \sum_{v\in S_0^L} w_0^L(v)[v]$. Then,

$$c_0^L = \sum_{v\in S_0^L\setminus S_0^K} w_0^L(v)[v] + \sum_{v\in S_0^K} w_0^K(v)[v] = \sum_{v\in S_0^L\setminus S_0^K} w_0^L(v)[v] + c_0^K.$$

For any $c_1 \in C_1^{L,K}$, we have that

$$\left\langle \left(\partial_1^{L,K}\right)^* c_0^K, c_1\right\rangle_{w_1^{L,K}} = \left\langle c_0^K, \partial_1^{L,K} c_1\right\rangle_{w_0^K} = \left\langle c_0^L, \partial_1^{L,K} c_1\right\rangle_{w_0^L} - \left\langle \sum_{v\in S_0^L\setminus S_0^K} w_0^L(v)[v], \partial_1^{L,K} c_1\right\rangle_{w_0^L},$$

where $\langle\cdot,\cdot\rangle_{w_1^{L,K}}$ is the restriction of $\langle\cdot,\cdot\rangle_{w_1^L}$ on $C_1^{L,K}$ and we use the fact $w_0^K = w_0^L|_{S_0^K}$ in the rightmost equality.

Since $\partial_1^{L,K} c_1 \in C_0^K$, we have that $\langle \sum_{v\in S_0^L\setminus S_0^K} w_0^L(v)[v], \partial_1^{L,K} c_1\rangle_{w_0^L} = 0$. Now, assume that $c_1 = x_1[e_1] + \cdots + x_\ell[e_\ell]$ where each $e_i \in S_1^L$ and $x_i \in \mathbb{R}$. Since $\partial_1^{L,K}[e_i] = \partial_1^L[e_i] = [v_i] - [w_i]$

for some $v_i, w_i \in S_0^L$, we have that $\langle c_0^L, \partial_1^{L,K}[e_i]\rangle_{w_0^L} = 0$ for each $i = 1, \ldots, \ell$ and thus $\langle c_0^L, \partial_1^{L,K} c_1\rangle_{w_0^L} = 0$. It then follows that

$$\left\langle \left(\partial_1^{L,K}\right)^* c_0^K, c_1 \right\rangle_{w_1^{L,K}} = 0 \quad \forall c_1 \in C_1^{L,K},$$

and thus $\Delta_0^{K,L} c_0^K = \partial_1^{L,K}(\partial_1^{L,K})^* c_1 = 0$.

Now, assume that L is connected. Suppose that there exists $0 \neq c_0 \in C_0^K$ such that $\Delta_0^{L,K} c_0 = 0$. Then, $(\partial_1^{L,K})^* c_0 = 0$. For any $v, v' \in S_0^K$, since L is connected, there exists a 1-chain $c_1 \in C_1^L$ such that $\partial_1^L c_1 = [v] - [v']$ (for example, one can take a path in L connecting v and v' and let c_1 be the corresponding 1-chain). Then, $c_1 \in C_1^{L,K}$ and $\partial_1^{L,K} c_1 = [v] - [v']$. Note that

$$\langle c_0, [v] - [v']\rangle_{w_0^K} = \left\langle c_0, \partial_1^{L,K} c_1 \right\rangle_{w_0^K} = \left\langle \left(\partial_1^{L,K}\right)^* c_0, c_1 \right\rangle_{w_1^{L,K}} = 0.$$

This implies that $\langle c_0, [v]\rangle_{w_0^K} = \langle c_0, [v']\rangle_{w_0^K}$ and thus there exists $\alpha \in \mathbb{R}$ such that $\langle c_0, [v]\rangle_{w_0^K} = \alpha$ for each $v \in S_0^K$. Then, $c_0 = \alpha \cdot c_0^K$, implying that the multiplicity of 0 eigenvalue is 1.

For item 2, suppose K intersects exactly m connected components of L, denoted by $L_1, \ldots, L_m$. Then, by Lemma 2.4 we have that $\Delta_0^{K,L} = \bigoplus_{i=1}^m \Delta_0^{K_i,L_i}$. Then, the spectrum of $\Delta_0^{K,L}$ is the multiset union of the spectra of $\Delta_0^{K_i,L_i}$s. By item 1 and item 2 we have that the multiplicity of zero eigenvalue of $\Delta_0^{K,L}$ is then exactly m. ∎

Proof of Theorem 2.6. By abuse of the notation, we represent each $c^L \in C_q^L$ by a vector $c^L \in \mathbb{R}^{n_q^L}$. Then, c^K corresponds to the vector $c^K = c^L([n_q^K]) \in \mathbb{R}^{n_q^K}$. By Theorem 4.6, the matrix representation $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L}$ of $\Delta_{q,\mathrm{up}}^{K,L}$ can be computed as follows:

$$\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} = \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left([n_q^K],[n_q^K]\right) - \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left([n_q^K], I_K^L\right) \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L, I_K^L\right)^\dagger \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L,[n_q^K]\right),$$

where $I_K^L = [n_q^L] \backslash [n_q^K]$.

Suppose $\sigma_i \in S_q^K$ is an interior simplex; then the ith row of $\mathbf{\Delta}_{q,\mathrm{up}}^{L}([n_q^K], I_K^L)$ is 0 (cf. section SM1). Then,

1. the ith entry of $\mathbf{\Delta}_{q,\mathrm{up}}^{L}([n_q^K],[n_q^K])c^K$ exactly coincides with the ith entry of $\mathbf{\Delta}_{q,\mathrm{up}}^{L} c^L$;
2. the ith row of $\mathbf{\Delta}_{q,\mathrm{up}}^{L}([n_q^K], I_K^L)\mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L, I_K^L\right)^\dagger \mathbf{\Delta}_{q,\mathrm{up}}^{L}\left(I_K^L,[n_q^K]\right)$ is 0.

Therefore, the ith entry of $\mathbf{\Delta}_{q,\mathrm{up}}^{L} c^L$ $(= w_q^L(\sigma_i)\langle \Delta_{q,\mathrm{up}}^{L} c^L, [\sigma_i]\rangle_{w_q^L})$ agrees with the ith entry of $\mathbf{\Delta}_{q,\mathrm{up}}^{K,L} c^K$ $(= w_q^K(\sigma_i)\langle \Delta_{q,\mathrm{up}}^{K,L} c^K, [\sigma_i]\rangle_{w_q^K})$. Then by $w_q^K(\sigma_i) = w_q^L(\sigma_i)$, we have that

$$\left\langle \Delta_{q,\mathrm{up}}^{L} c^L, [\sigma_i]\right\rangle_{w_q^L} = \left\langle \Delta_{q,\mathrm{up}}^{K,L} c^K, [\sigma_i]\right\rangle_{w_q^K}.$$ ∎

Proof of Theorem 2.7. First, we have the following elementary linear algebra fact: The isomorphism follows from [33, Theorem 5.3] and the equality follows from [33, Theorem 5.2].

Claim A.1. *Let $A \in \mathbb{R}^{m\times n}$ and let $B \in \mathbb{R}^{n\times p}$. Suppose $AB = 0$; then we have*

$$\ker(A)/\mathrm{im}(B) \cong \ker(A) \cap \ker\left(B^{\mathrm{T}}\right) = \ker\left(BB^{\mathrm{T}} + A^{\mathrm{T}}A\right),$$

where $\cong$ denotes isomorphism between vector spaces.

The image of $H_q(K)$ under the inclusion map inside $H_q(L)$ is exactly $\ker(\partial_q^K)/\mathrm{im}(\partial_{q+1}^{L,K})$. Let B_q^K be the matrix representation of ∂_q^K. Choose an orthonormal basis of $C_{q+1}^{L,K}$ and let $B_{q+1}^{L,K}$ be the corresponding matrix representation of $\partial_{q+1}^{L,K}$ in this basis. Then, by Theorem 3.1

$$\begin{aligned}\mathbf{\Delta}_q^{K,L} &= B_{q+1}^{L,K}\left(B_{q+1}^{L,K}\right)^{\mathrm{T}}\left(W_q^K\right)^{-1} + W_q^K\left(B_q^K\right)^{\mathrm{T}}\left(W_{q-1}^K\right)^{-1}B_q^K\\ &= \left(W_q^K\right)^{\frac{1}{2}}\left(W_q^K\right)^{-\frac{1}{2}}B_{q+1}^{L,K}\left(\left(W_q^K\right)^{-\frac{1}{2}}B_{q+1}^{L,K}\right)^{\mathrm{T}}\left(W_q^K\right)^{-\frac{1}{2}}\\ &\quad+\left(W_q^K\right)^{\frac{1}{2}}\left(\left(W_{q-1}^K\right)^{-\frac{1}{2}}B_q^K\left(W_q^K\right)^{\frac{1}{2}}\right)^{\mathrm{T}}\left(W_{q-1}^K\right)^{-\frac{1}{2}}B_q^K\left(W_q^K\right)^{\frac{1}{2}}\left(W_q^K\right)^{-\frac{1}{2}}.\end{aligned}$$

Let $A := (W_{q-1}^K)^{-\frac{1}{2}}B_q^K(W_q^K)^{\frac{1}{2}}$ and $B := (W_q^K)^{-\frac{1}{2}}B_{q+1}^{L,K}$. Then,

1. $AB = (W_{q-1}^K)^{-\frac{1}{2}}B_q^K(W_q^K)^{\frac{1}{2}}(W_q^K)^{-\frac{1}{2}}B_{q+1}^{L,K} = (W_{q-1}^K)^{-\frac{1}{2}}B_q^K B_{q+1}^{L,K} = 0$,
2. $\mathbf{\Delta}_q^{K,L} = (W_q^K)^{\frac{1}{2}}(BB^{\mathrm{T}} + A^{\mathrm{T}}A)(W_q^K)^{-\frac{1}{2}}$.

Since both W_{q-1}^K and W_q^K are nonsingular, we have that $\ker(A) \cong \ker(B_q^K)$, $\mathrm{im}(B) \cong \mathrm{im}(B_{q+1}^{L,K})$, and $\ker(\mathbf{\Delta}_q^{K,L}) \cong \ker(BB^{\mathrm{T}} + A^{\mathrm{T}}A)$. It then follows from Claim A.1 that

$$\beta_q^{K,L} = \dim\left(\ker\left(B_q^K\right)/\mathrm{im}\left(B_{q+1}^{L,K}\right)\right) = \dim\left(\ker\left(\mathbf{\Delta}_q^{K,L}\right)\right) = \mathrm{nullity}\left(\Delta_q^{K,L}\right). \qquad \blacksquare$$

Proof of Lemma 3.4. Consider $\pi^\perp \circ \partial_{q+1}^L : C_{q+1}^L \to (C_q^K)^\perp$ where $\pi^\perp : C_q^L \to (C_q^K)^\perp$ is the orthogonal projection. Then, D_{q+1}^L is the matrix representation of $\pi^\perp \circ \partial_{q+1}^L$ and $C_{q+1}^{L,K} = \ker(\pi^\perp \circ \partial_{q+1}^L)$. So $R_{q+1}^L = D_{q+1}^L Y$ is the matrix representation of $\pi^\perp \circ \partial_{q+1}^L$ after a change of basis of C_{q+1}^L.

1. If $I = \emptyset$, then since R_{q+1}^L is column reduced, R_{q+1}^L has full column rank. This implies that $\pi^\perp \circ \partial_{q+1}^L : C_{q+1}^L \to (C_q^K)^\perp$ is injective and thus $C_{q+1}^{L,K} = \ker(\pi^\perp \circ \partial_{q+1}^L) = \{0\}$.
2. If $I \neq \emptyset$, then the column space of $Z = Y(:, I)$ coincides with $\ker(\pi^\perp \circ \partial_{q+1}^L) = C_{q+1}^{L,K}$. Since Y is nonsingular, Z has full column rank. Therefore, the columns of Z constitute a basis of $C_{q+1}^{L,K}$.

Obviously, $B_{q+1}^L([n_q^K], :)$ is the matrix representation of $\pi \circ \partial_{q+1}^L : C_{q+1}^L \to C_q^K$ where $\pi : C_q^L \to C_q^K$ is the orthogonal projection. Therefore, $(B_{q+1}^L Y)([n_q^K], :) = B_{q+1}^L([n_q^K], :)Y$ is the matrix representation of $\pi \circ \partial_{q+1}^L$ under the new basis Y of C_{q+1}^L. Now, assume that $I \neq \emptyset$. Since the column space of $Z = Y(:, I)$ is $C_{q+1}^{L,K}$, we have that $B_{q+1}^{L,K} = (B_{q+1}^L Y)([n_q^K], I)$ is the matrix representation of $\pi \circ \partial_{q+1}^L|_{C_{q+1}^{L,K}} = \partial_{q+1}^{L,K}$. $\blacksquare$

Note added in proof. The notion of persistent Laplacian also arose in the work of André Lieutier in 2014 [32].

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