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Finite Dimensional Functional Observer Design for Parabolic Systems

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Abstract— **This paper combines two control design aspects for a class of infinite dimensional systems, and each of the designs aims at significantly reducing the implementation complexity and computational load. A functional observer, and its extension of an unknown input functional observer, aims to reconstruct a functional of the infinite dimensional state. The resulting compensator only requires the solution to an operator Sylvester equation plus one differential equation for each dimension of the control signal, as opposed to an infinite dimensional filter evolution equation and an associated operator Riccati equation for the filter operator covariance. When the functional to be estimated coincides with the expression of a full state feedback control signal, then the functional observer becomes the minimum order compensator. When the parabolic system admits a decomposition whereby the system is decomposed into a lower finite dimensional subspace comprising the unstable eigenspectrum and an infinite stable subspace, then the functional observer-based compensator design becomes the minimum order compensator for the finite dimensional subsystem. This approach dramatically reduces the computation for solving the ARE needed for the full state controller and the associated Sylvester equation needed for the functional observer. Numerical results for a parabolic PDE in one and two spatial dimensions are included.**

I. Introduction

This work incorporates two design methods used for the efficient implementation of controllers for a class of infinite dimensional systems. The first one considers infinite dimensional systems that can be decomposed into two subsystems with the following property: the first one is a finite dimensional (slow) and possibly unstable subsystem and the other one is an infinite dimensional stable (fast) subsystem. One of the earliest works [1] examined the theoretical framework and the requisite conditions for the infinite dimensional system to admit such a decomposition and subsequently proposed a "modal" compensator by utilizing controller design methods from finite dimensional theories. Essential to this decomposition was the property of the state operator, called the spectrum determined growth assumption that relates the bound of the point spectrum to the bound of the semigroup [2].

The other design method is concerned with estimates of functionals of the state, termed *functional observers*. While most of the work on functional observers has been done on the finite dimensional setting, see the early work by Murdoch [3], the series of works by Darouach [4], [5], [6] and the book [7]. Few works extended aspects of the functional observer to the infinite dimensional case, see [8], [9], [10], [11], [12], [13]. When the functional to be estimated coincides with the full state feedback control term, where the operator is the feedback operator, the functional observer essentially estimates the control signal. This design requires the solution to an operator Sylvester equation and a number of observer states that are equal to the rank of the input operator. Such design results in the minimum-order compensator and this was first considered in [14].

To further simplify the design complexity and computational load, this paper combines the above two methods and proposes a minimum-order finite dimensional compensator for a class of infinite dimensional systems. The theoretical underpinnings for the existence and well-posedness of such a functional observer-based compensator for the finite dimensional subspace of a class of infinite dimensional systems are provided and a detailed numerical study for a parabolic PDE in one and two spatial dimensions is included.

A. Contributions

The contributions of this paper are as follows

1) Construct a functional observer based on the finite dimensional unstable subspace of the system operator.
2) Reduce the computation for solving the ARE and Sylvester equation for $A|_{D(A)}$ to its restriction on the finite dimensional, $A|_{D(A^u)}$.
3) Extend the design to an Unknown Input Functional Observer by imposing an additional operator identity condition and relaxing the conditions on the unknown temporal component of the disturbance signal.
4) Demonstrate on a diffusion PDE in 1D and 2D.

II. Mathematical Framework

Let H be a real Hilbert space equipped with the scalar product $(\cdot,\cdot)$ and norm $||\cdot||_H$. Let U be a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||_U$. Consider the controlled evolution equation

$$\dot{x}(t) = Ax + Bu, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

with the following hypothesis.

Hypothesis 1: 1) the state operator A generates an analytic C_0-semigroup e^{At} of type ω_0 (e.g. [15, p. 108]), and λ_0 is a real number in $\rho(A)$ such that $\omega_0 < \lambda_0$. The resolvent $(\lambda_0 I - A)^{-1}$ of A is compact in the Hilbert space H.

2) the input operator $B \in \mathcal{L}\big(U, [D(A^*)]'\big)$.

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In fact, 2) of Hypothesis 1 can be stated in the following equivalent way (see [15, $(HP)_1$, p. 432]):

1) B is of the form $B=(\lambda_0 I-A)\mathcal{D}$, and there exists $\gamma\in(0,1)$ such that $\mathcal{D}\in\mathcal{L}\big(U,D((\lambda_0 I-A)^{\gamma})\big)$. In other words,

$$(\lambda_0 I-A)^{\gamma-1}B\in L(U,H).$$

The operator B is called *admissible* if all *mild* solutions of the system (1)–(2) are continuous H-valued functions. Let us first recall the well-known Riesz-Schauder-Fredholm theorem (see [16, Theorem 2, p. 284] or [17, Theorem 6.29, Chapter III, p. 187]) concerning the spectral properties of an operator with compact resolvent. This result provides the essential structure for us to construct finite dimensional functional observer in Section III.

In the sequel, the symbol C denotes a generic positive constant, which is allowed to depend on the indicated parameters.

Theorem 2: Let H be a Hilbert space and let $A:D(A)\to H$ be a closed linear operator with compact resolvent $(\lambda I-A)^{-1}$ for some $\lambda\in\rho(A)$. Then

1) the spectrum of A consists of an at most countable set of points of the complex plane which has no point of accumulation except possibly $\lambda=\infty$,
2) every number in the spectrum of A is an eigenvalue of A of finite multiplicity, and
3) $\lambda\neq 0$ is an eigenvalue of A if and only if it is an eigenvalue of A^*.

Based on Theorem 2, an operator A satisfying Hypothesis 1 (H1) has a countable set of eigenvalues λ_j and corresponding eigenvectors φ_j, so that

$$A\varphi_j=\lambda_j\varphi_j,\quad j=1,2,\ldots.$$

Since A is an infinitesimal generator of an analytic C_0-semigroup, the resolvent set of A contains a sector (e.g. [15]). Therefore, there are only a finite number of eigenvalues of A in the right complex half-plane $\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda\geq 0\}$. We denote by $\Sigma_N=\{\lambda_1,\lambda_2,\ldots,\lambda_N\}$ the set of (unstable) eigenvalues repeated according to their algebraic multiplicity m_{a_j} so that

$$\cdots\mathrm{Re}\lambda_{N+1}<0\leq\mathrm{Re}\lambda_N\leq\cdots\leq\mathrm{Re}\lambda_1.$$

Let M denote the number of distinct unstable eigenvalues and let Γ_j be a positively oriented curve enclosing λ_j, but no other point of $\sigma(A)$. Define the eigenprojection $P_{N,j}$ (see [17], p. 178)

$$P_{N,j}=-\frac{1}{2\pi i}\int_{\Gamma_j}(\lambda I-A)^{-1}d\lambda,\quad j=1,2,\cdots,M.$$

The space $X_{N,j}=\mathrm{Ran}(P_{N,j})$, the range of P_N, is called the *algebraic eigenspace* for the eigenvalue λ_j, and $m_{a_j}=\dim X_{N,j}$ is the *algebraic multiplicity* of λ_j so that

$$m_{a_1}+m_{a_2}+\cdots+m_{a_M}=N.$$

Any nonzero element of $X_{N,j}$ is called a *generalized eigenfunction* for λ_j. Recall that (see [17], p. 181)

$$X_{N,j}=\mathrm{Ker}(\lambda_j-A)^{m_{a_j}},\quad j=1,2,\cdots,M.$$

We denote by $\{\psi_{j,k}\}_{k=1}^{m_{a_j}}$ the (normalized) linearly independent generalized eigenfunctions corresponding to each unstable distinct eigenvalue λ_j of A. Denote by P_N the projection, explicitly given as a contour integral and similarly its adjoint P_N^* as

$$P_N=-\frac{1}{2\pi i}\int_{\Gamma}(\lambda I-A)^{-1}d\lambda;\quad P_N^*=-\frac{1}{2\pi i}\int_{\overline{\Gamma}}(\lambda I-A^*)^{-1}d\lambda,$$

where Γ and its conjugate $\overline{\Gamma}$ are the positively oriented curves enclosing Σ_N and separate the unstable spectrum of operator A and its adjoint operator A^* from the stable spectrum, respectively.

Since some of the eigenvalues λ_j might be complex, it will be convenient in the sequel to view A as a linear operator (again denoted by A) in the complex space $\tilde{H}=H\oplus iH$. Therefore $\tilde{H}$ can be decomposed (see [17, p. 178]) as

$$\tilde{H}=\tilde{X}_u\oplus\tilde{X}_s,$$

where

$$\tilde{X}_u=P_N\tilde{H}=\bigoplus_{j=1}^{M}\tilde{X}_j,\quad \dim\tilde{X}_u=N$$

and

$$\tilde{X}_s=(I-P_N)\tilde{H}.$$

The subspaces $\tilde{X}_u\subset D(A)$ and $\tilde{X}_s\cap D(A)$ are invariant under A. $\tilde{X}_u$ and $\tilde{X}_s$ are also invariant under the C_0-semigroup e^{At} generated by A. Moreover, it is well known that the spectrum of A^* is exactly the complex conjugate of the eigenvalues of A (see [17], Chapter 3, Theorem 6.22, p. 184). Also, $\tilde{H}$ can be decomposed as a direct sum of two invariant subspaces of A^* given by

$$\tilde{H}=\tilde{X}_u^*\oplus\tilde{X}_s^*,$$

where $\tilde{X}_u^*=P_N^*\tilde{H}=\bigoplus_{j=1}^{M}\tilde{X}_j^*, \dim\tilde{X}_u^*=N$ and $\tilde{X}_s^*=(I-P_N^*)\tilde{H}$. Here, $\tilde{X}_j^*=\mathrm{Ker}(\overline{\lambda_j}-A^*)^{m_{a_j}}, j=1,2,\cdots,M$.

Let $\tilde{X}_u^{\perp}$ and $\tilde{X}_u^{*\perp}$ denote the orthogonal spaces of $\tilde{X}_u$ and $\tilde{X}_u^*$, respectively. It follows that

$$\tilde{X}_u^{\perp}=\tilde{X}_s^*\quad\text{and}\quad\tilde{X}_u^{*\perp}=\tilde{X}_s.$$

Since $\mathrm{Re}(\tilde{H})=H$, we introduce the subspace $X^s=\mathrm{Re}(\tilde{X}^s)$ and $X^u=\mathrm{Re}(\tilde{X}^u)$. Then

$$H=X_s\oplus X_u.$$

Moreover, $X_s\cap D(A)$ is invariant under A so is $X_u=X_u\cap D(A)$.

Note that the duals of X_u and X_s can be identified with X_u^* and X_s^*, respectively, (see [18, Corollary 3.14]). This result can be used to decompose our infinite dimensional problem into a finite dimensional unstable system and a stable infinite dimensional system. This decomposition is the key to properly selecting the control input functions.

Now we set

$$A_u=P_NA=A|_{X_u}:X_u\to X_u,$$

and

$$A_s=(I-P_N)A=A|_{D(A)\cap X_s}:D(A)\cap X_s\to X_s$$

to be the restrictions of A to X_u and X_s, respectively. The projections P_N and $I-P_N$ commute with A. The spectra of A

on X_u and X_s coincide with $\{\lambda_j\}_{j=1}^N$ and $\{\lambda_j\}_{j=N+1}^\infty$, so that

$$\sigma(A_u) = \{\lambda_j\}_{j=1}^N \quad \text{and} \quad \sigma(A_s) = \{\lambda_j\}_{j=N+1}^\infty.$$

Thus, A_u is bounded and of finite dimensional rank. Furthermore, since A generates an analytic C_0-semigroup on H, its restriction A_s to X_s also generates an analytic C_0-semigroup on X_s. This implies that the fractional powers of $-A_s$ are well defined. Moreover, A_s satisfies the spectrum growth condition on X_s (see [19]), and hence

$$||e^{A_s t}||_{\mathcal{L}(H)} \leq C_\omega e^{-\omega t}, \quad t \geq 0,$$

and

$$||(-A_s)^\theta e^{A_s t}||_{\mathcal{L}(H)} \leq \frac{C_\omega e^{-\omega t}}{t^\theta}, \quad t > 0, \quad 0 < \theta \leq 1, \quad (3)$$

for any $\omega \geq |\mathrm{Re}\lambda_{N+1}|$. Here, C_ω is a constant that depends on ω.

In order to apply P_N and $I - P_N$ to (1) we first need to extend their definitions. Notice that P_N^* is bounded from H to $D(A^*)$ and hence P_N can be uniquely extended as bounded operator from $[D(A^*)]'$ to H by:

$$(P_N\varphi,\phi)_H = (\varphi, P_N^*\phi)_{[D(A^*)]',D(A^*)}, \ \forall(\varphi,\phi) \in [D(A^*)]' \times D(A^*).$$

In addition, since $P_N X_s = 0$, it follows that $P_N \in L([D(A^*)]', X_u)$. Moreover,

$$\begin{aligned} D(A^*) &= D(A^*) \cap H \\ &= [D(A^*) \cap X_u^*] \oplus [D(A^*) \cap X_s^*] \\ &= X_u^* \oplus [D(A^*) \cap X_s^*], \end{aligned}$$

thus

$$\begin{aligned} [D(A^*)]' &= (X_u^*)' \oplus [D(A^*) \cap X_s^*]' \\ &= X_u \oplus [D(A^*) \cap X_s^*]'. \end{aligned} \quad (4)$$

The second equality in (4) holds because $(X_u^*)'$ can be identified with $(X_u^*)^*$, where $(X_u^*)^* = X_u$. A proof of equality (4) is also given in [18]. Consequently, $I - P_N \in \mathcal{L}([D(A^*)]', [D(A^*) \cap X_s^*]')$ and the system (1)–(2) can be decomposed as

$$x = x_u + x_s, \quad x_u = P_N x, \quad x_s = (I - P_N)x. \quad (5)$$

If applying P_N and $I - P_N$ to system (1)–(2), respectively, and letting

$$B_u = P_N B \quad \text{and} \quad B_s = (I - P_N)B,$$

we obtain

$$\dot{x}_u = A_u x_u + B_u u \in X_u, \quad (6)$$

$$x_u(0) = P_N x_0 \in X_u, \quad (7)$$

and

$$\dot{x}_s = A_s x_s + B_s u \in [D(A^*) \cap X_s^*]', \quad (8)$$

$$x_s(0) = (I - P_N)x_0 \in X_s. \quad (9)$$

Correspondingly, the control spaces are given by

$$U_u = \{B^*\varphi : \varphi \in X_u\} \quad \text{and} \quad (10)$$

$$U_s = \{B^*\psi : \psi \in [D(A^*) \cap X_s^*]\}. \quad (11)$$

System (6)–(9) is the decomposition of the abstract system (1)–(2). Recall that A generates an analytic C_0-semigroup on H and $\sigma(A_s) \subset \{\lambda\colon \mathrm{Re}\lambda \leq \mathrm{Re}\lambda_{N+1} < 0\}$, thus its restriction A_s to X^s also generates an analytic C_0-semigroup on X_s, and hence

$$||e^{A_s t}||_{\mathcal{L}(H)} \leq C_{\omega_s} e^{-\omega_s t}, \quad t \geq 0,$$

where $\omega_s = -\sup\{\mathrm{Re}\lambda\colon \lambda \in \sigma(A_s)\} = -\mathrm{Re}\lambda_{N+1} > 0$. In addition, one can always make ω_s arbitrarily large by setting Γ enclosing eigenvalues with real components smaller than $-\omega_s$.

As a result, to stabilize system (1)–(2), it suffices to stabilize the subsystem (6)–(7) by constructing $u \in U_u$ (see [20]). Assume that (A_u, B_u) is feedback stabilizable, then the feedback operator K^u can be solved from a feedback Algebraic Riccati equation (ARE) such that $A^u + B^u K^u$ generates an exponentially stable C_0-semigroup on X^u. In fact, $K^u = -B^{u*}\Pi^u$, where Π^u can be solved from the following ARE

$$A_u^*\Pi_u + \Pi_u A_u - \Pi_u B_u B_u^* \Pi_u + I_{N\times N} = 0 \text{ on } X_u, \quad (12)$$

In this case $K_u \in \mathcal{L}(X_u, X_u^*)$, where X_u^* has been identified as its dual. Moreover, $A_u + B_u K_u$ generates an exponentially stable C_0-semigroup on X^u such that

$$\|x_u(t)\|_H \leq C_u e^{-\alpha t}\|P_N x_0\|_H, \quad (13)$$

for some $\alpha > 0$ and $C_u > 0$. Now set the feedback operator $K = (K_u, 0|_{[D(A^*)\cap X_s^*]'})$. Then

$$\|u(t)\|_U = \|Kx(t)\|_U = \|K_u x_u(t)\|_U \leq C_1 e^{-\alpha t}\|x_0\|_X. \quad (14)$$

One can show that $A_s - B_s B_u^* \Pi_u$ generates an exponentially stable C_0-semigroup on X. Applying the variation of parameters formula to (8)–(9) follows

$$x_s(t) = e^{A_s t}(I - P_N)x_0 - \int_0^t e^{A_s(t-\tau)} B_s u(\tau)\, d\tau. \quad (15)$$

Recall by Hypothesis 1. 1) we have $(\lambda_0 I - A)^{\gamma-1}B \in L(U,H)$. Moreover, $(\lambda_0 I - A_s)^{1-\gamma} = (I - P_N)(\lambda_0 I - A)^{1-\gamma}$ for any $\gamma \in (0,1)$, hence

$$B_s = (I - P_N)B = (\lambda_0 I - A_s)^{1-\gamma}(I - P_N)(\lambda_0 I - A)^{\gamma-1}B$$

and

$$\begin{aligned} \|e^{A_s t}B_s u(t)\|_H &= \|(\lambda_0 I - A_s)^{1-\gamma} e^{A_s t}(I - P_N)(\lambda_0 I - A)^{\gamma-1}Bu(t)\|_H \\ &\leq C_2 \frac{e^{-\omega_s t}}{t^{1-\gamma}}\|u\|_U. \end{aligned} \quad (16)$$

Therefore, combining (15) with (14) and (16) yields

$$\begin{aligned} \|x_s(t)\|_H &\leq C_{\omega_s} e^{-\omega_s t}\|(I - P_N)x_0\|_H \\ &\quad + C_3 \int_0^t \frac{e^{-\omega_s(t-\tau)}}{(t-\tau)^{1-\gamma}} e^{-\alpha\tau}\|x_0\|_H\, d\tau \\ &\leq \max\{C_\omega, C_3\}(e^{-\omega_s t} + e^{-\alpha t}\int_0^t \frac{e^{-(\omega_s-\alpha)(t-\tau)}}{(t-\tau)^{1-\gamma}}\, d\tau)\|x_0\|_H. \end{aligned}$$

One can always adjust α and ω_s such that $0 < \alpha < \omega_s$, and hence $\int_0^t \frac{e^{-(\omega_s-\alpha)(t-\tau)}}{(t-\tau)^{1-\gamma}}\, d\tau < \infty$ for any $t > 0$. As a result, we have $\|x_s(t)\|_H \leq C_4 e^{-\alpha t}\|x_0\|_X$ for some constant $C_4 > 0$. Combining this with (13) gives

$$\|x(t)\|_H \leq C_5 e^{-\alpha t}\|x_0\|_X. \quad (17)$$

for some constant $C_5 > 0$.

Next consider that u is a linear combination of m vectors:

$$u(x,t) = \sum_{i=1}^m b_i(x)\mathrm{u}_i(t), \quad (18)$$

for $m \le N$, where $\vec{\mathrm{u}} = (\mathrm{u}_1, \mathrm{u}_2, \dots, \mathrm{u}_m)^{\mathrm{T}} \in L^2(\mathbb{R}^m)$, and $\vec{b} = (b_1, b_2, \dots, b_m)^{\mathrm{T}} \in U_u$, is chosen such that (A_u, B_u) is controllable. The necessary and sufficient condition for such $\vec{b}$ has been addressed in [20] based on the Popov-Belevitch-Hautus (PBH) controllability test. Define the operator $\mathbf{B}_m : \mathbb{R}^m \to [D(A^*)]'$ by

$$\mathbf{B}_m = B(b_1, b_2, \dots, b_m).$$

Let $\mathbf{B}_{m_u} = P_N \mathbf{B}_m$ and $\mathbf{B}_{m_s} = (I - P_N)\mathbf{B}_m$. Then we can rewrite (6)–(9) as

$$\dot{x}_u(t) = A_u x_u(t) + \mathbf{B}_{m_u} \vec{\mathrm{u}} \in X_u, \tag{19}$$

$$x_u(0) = P_N x_0 \in X_u, \tag{20}$$

and

$$\dot{x}_s(t) = A_s x_s(t) + \mathbf{B}_{m_s} \vec{\mathrm{u}}(t) \in [D(A^*) \cap X_s^*]', \tag{21}$$

$$x_s(0) = (I - P_N) x_0 \in X_s. \tag{22}$$

The detailed discussion can be found in [20].

III. Main Results: FO for infinite dimensional systems

Now consider the functional observer design for system (6)–(9) with disturbance $d(t)$

$$\begin{aligned}
&\dot{x}_u(t) = A_u x_u(t) + \mathbf{B}_{m_u} \vec{\mathrm{u}}(t) + F_u d(t) \in X_u, \\
&x_u(0) = P_N x_0 \in X_u, \\
&\dot{x}_s(t) = A_s x_s(t) + \mathbf{B}_{m_s} \vec{\mathrm{u}}(t) + F_s d(t) \in [D(A^*) \cap X_s^*]', \\
&x_s(0) = (I - P_N) x_0 \in X_s, \\
&y(t) = Cx(t), \\
&z(t) = Kx.
\end{aligned} \tag{23}$$

where $F_u = P_N F$, $F_s = (I - P_N)F$, $C = (C_u, 0\big|_{[D(A^*)\cap X_s^*]'})$, and $K = (K_u, 0|_{[D(A^*)\cap X_s^*]'})$. In this case,

$$y(t) = C_u x_u(t) \quad \text{and} \quad z(t) = K_u x_u(t). \tag{24}$$

If the operator F is not known, one may still design a functional observer that will ensure the error $e(t) = z(t) - \widehat{z}(t)$ converging to zero in the appropriate norm. The functional observer that estimates $z(t) = Kx(t)$ in (23) is given by

$$\begin{aligned}
\dot{w}(t) &= \mathbf{N} w(t) + J y(t) + H \vec{\mathrm{u}}(t), \\
\widehat{z}(t) &= w(t) + E y(t).
\end{aligned} \tag{25}$$

Assume that the derivative of the output satisfies

$$\dot{y}(t) = C_u \dot{x}_u(t). \tag{26}$$

Combining (23),(25) and using (26) one arrives at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_u(t) \\ x_s(t) \\ e(t) \end{bmatrix} &= \begin{bmatrix} A_u & 0 & 0 \\ 0 & A_s & 0 \\ 0 & 0 & \mathbf{N} \end{bmatrix} \begin{bmatrix} x_u(t) \\ x_s(t) \\ e(t) \end{bmatrix} \\
&+ \begin{bmatrix} \mathbf{B}_{m_u} \\ \mathbf{B}_{m_s} \\ \mathbf{0}_{r\times m} \end{bmatrix} \vec{\mathrm{u}}(t) + \begin{bmatrix} F_u \\ F_s \\ PF_u \end{bmatrix} d(t).
\end{aligned} \tag{27}$$

When the control is taken as $\vec{\mathrm{u}}(t) = \widehat{z}(t)$, then (27) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_u(t) \\ x_s(t) \\ e(t) \end{bmatrix} = \mathbb{A} \begin{bmatrix} x_u(t) \\ x_s(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} F_u \\ F_s \\ PF_u \end{bmatrix} d(t), \tag{28}$$

where the state operator $\mathbb{A}$ is given by

$$\mathbb{A} = \begin{bmatrix} A_u + \mathbf{B}_{m_u} K_u & 0 & -\mathbf{B}_{m_u} \\ 0 & A_s + \mathbf{B}_{m_s} K_u & -\mathbf{B}_{m_s} \\ 0 & 0 & \mathbf{N} \end{bmatrix}. \tag{29}$$

Both the open loop (27) and closed-loop (28) systems require the solution to the operator equalities

$$PA_u - \mathbf{N}P = JC_u, \qquad P\mathbf{B}_{m_u} = H, \tag{30}$$

where the solution to the Sylvester operator equation $P = K_u - EC_u : X_u \to \mathbb{R}^r$, J is an $r \times q$ matrix, H is an $r \times m$ matrix and E is an $r \times q$ matrix. If the control is selected as the estimated functional $u(t) = \widehat{z}(t)$, then $r = m$.

Lemma 1: If the pair $(A_u, \mathbf{B}_{m_u})$ is feedback stabilizable, the output $y(t)$ satisfies (26), $d \in L^2(0,\infty;X)$ and the Sylvester equation (29) is satisfied, then the FO in (27) is well-posed and for $u \in L^2(0,\infty;U_u)$ the state x is bounded with

$$\lim_{t\to\infty} |e(t)|_{\mathbb{R}^m} = 0.$$

Further, if the controller is selected as $u(t) = \widehat{z}(t)$, then the closed-loop system (28) is well-posed and

$$\lim_{t\to\infty} \|x(t)\| = 0, \quad \lim_{t\to\infty} |e(t)|_{\mathbb{R}^m} = 0. \tag{31}$$

Proof The state operator $\mathbb{A}$ in (29) generates an exponentially stable C_0-semigroup on $X \times \mathbb{R}^m$ since $\mathbf{N}$ is Hurwitz and $A + \mathbf{B}_{m_u} K$ generates an exponentially stable C_0 semigroup shown in (17) (see [21]). Using the fact that $d \in L^2(0,\infty;X)$, then the perturbed system (28) is stable leading to (31). □

Remark 1: If F_u is known, then one can implement an unknown input functional observer (UIFO) by supplementing (25), (30) with the following condition

$$PF_u = 0 \tag{32}$$

and which ensures that despite the presence of the unknown input in (23), the observer (25) can still estimate $Kx(t)$. The conditions on $d(t)$ are relaxed to the minimum needed to ensure the existence of solutions to the plant equations.

IV. Numerical Results

Two different examples that fit into the framework proposed, are considered.

A. Advection-reaction in 1D

First, we consider the 1D diffusion-reaction PDE over the interval $[0,\ell] = [0,1]$, given by

$$\begin{aligned}
&\frac{\partial}{\partial t} x(t,\xi) = a \frac{\partial^2}{\partial \xi^2} x(t,\xi) + c x(t,\xi) + b(\xi) u(t) + f(\xi) d(t) \\
&x(t,0) = x(t,1) = 0, \\
&x(0,\xi) = \sin(\pi\xi) + 3\sin(2\pi\xi) + 6\sin(4\pi\xi) \\
&y(t) = \int_0^\ell \delta(\xi - 0.625) x(t,\xi)\, \mathrm{d}\xi.
\end{aligned}$$

The eigenvalues and eigenvectors are given by

$$\lambda_i = -a\left(\frac{i\pi}{\ell}\right)^2 + c, \;\; \phi_i(\xi) = \sqrt{2}\sin(i\pi\xi), \;\; i = 1, 2, \dots, \infty.$$

The reaction constant is selected so that the first two eigenvalues are positive $c = 5a\left(\frac{\pi}{\ell}\right)^2$. The input and disturbance

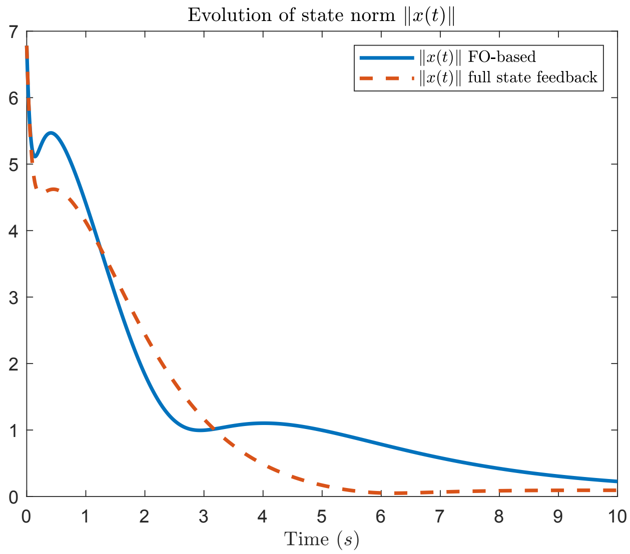

Fig. 1: 1D case: evolution of state norms.

spatial functions were selected as

$$b(\xi) = \begin{cases} \frac{1}{2\varepsilon_b} & \text{if } \xi_a - \varepsilon_b \leq \xi \leq \xi_a + \varepsilon_b \\ 0 & \text{otherwise} \end{cases}, \quad \varepsilon_b = 0.05\ell,$$

$$f(\xi) = \begin{cases} \frac{1}{2\varepsilon_f} & \text{if } \xi_d - \varepsilon_f \leq \xi \leq \xi_d + \varepsilon_f \\ 0 & \text{otherwise} \end{cases}, \quad \varepsilon_f = 0.1\ell,$$

with $\xi_a = 0.413\ell$, $\xi_d = 0.6\ell$. The resulting matrices associated with x^u, namely A_u, B_u, F_u, C_u are given by

$$A_u = \begin{bmatrix} 1.97 & 0 \\ 0 & 0.49 \end{bmatrix}, B_u = \begin{bmatrix} 1.49 \\ 0.81 \end{bmatrix}, F_u = \begin{bmatrix} 1.33 \\ -0.78 \end{bmatrix}$$

The stabilizing feedback gain $K_u = \begin{bmatrix} 5.6240 & -3.2542 \end{bmatrix}$ shifting the two unstable eigenvalues to $-2.531, -0.7943$. The solution to the Sylvester equation was $P = \begin{bmatrix} -0.7096 & -1.2079 \end{bmatrix}$ and which ensured that $PF_u = 0$. The functional observer error pole was $\mathbf{N} = -0.5$ and the initial condition for $\hat{z}(t)$ was $\hat{z}(0) = 0$, resulting in $e(0) = -K_u X_u(0) = -4.1386$.

case	norm
full-state controller	6.057
UIFO-based feedback	6.536

Table 1. 1D case: L_2 state norms.

To simulate the system, a total of 100 spectral elements were used to discretize the PDE in space. The resulting differential equations were integrated in the time interval $[0, 10]s$ using the ODE solver from the Matlab® ODE library, routine `ode45`, a 4th order Runge-Kutta scheme.

Figure 1 depicts the evolution of the L_2 state norm for the proposed case using the UIFO-based compensator, and also the case of a full-state feedback controller $u = Kx$. As expected, the full-state feedback controller exhibits a better performance over the UIFO-based compensator. The cumulative $L^2(0, 10, L^2([0, \ell]))$ norm given by

$$\|x\|_{L^2}^2 = \int_0^t \int_0^\ell x^2(\tau, \xi)\, d\xi\, d\tau$$

is summarized in Table I and which also points to comparable performance. Both controllers exhibit comparable performance, but the full-state feedback one cannot be implemented since it requires the full state, which is not available. The alternative is to implement a state observer, which as mentioned in the introduction, significantly increases the computational load.

B. Advection-reaction in 2D

The PDE is given by

$$\frac{\partial}{\partial t} x(t, \xi, \psi) = a \left(\frac{\partial^2}{\partial \xi^2} x(t, \xi, \psi) + \frac{\partial^2}{\partial psi^2} x(t, \xi, \psi) + c x(t, \xi, \psi) \right)$$
$$+ b(\xi, \psi) u(t) + f(\xi, \psi) d(t)$$
$$x(t, 0, \psi) = x(t, 1, \psi) = 0 = x(t, \xi, 0) = x(t, \xi, 1),$$
$$y(t) = \int_0^\ell \delta(\xi - 0.625) x(t, \xi)\, d\xi.$$

where

$$a = 2 \times 10^{-2}, \quad c = 1.2 \left(\left(\frac{\pi}{L_\xi} \right)^2 + \left(\frac{\pi}{L_\psi} \right)^2 \right)$$

The eigenvalues are given by

$$\begin{aligned} \lambda_{ij} &= -a \left(\left(\frac{i\pi}{L_\xi} \right)^2 + \left(\frac{j\pi}{L_\psi} \right)^2 \right) + ac \\ &= -a \left(\left(\frac{i\pi}{L_\xi} \right)^2 + \left(\frac{j\pi}{L_\psi} \right)^2 - 1.2 \left(\left(\frac{\pi}{L_\xi} \right)^2 + \left(\frac{\pi}{L_\psi} \right)^2 \right) \right) \end{aligned}$$

for $i, j = 1, \ldots, \infty$. It is seen that for $L_\xi = L_\psi$, one has that the first eigenvalue λ_{11} is positive with all other negative. The associated eigenfunctions are

$$\phi_{ij} = 2 \sin(\frac{i\pi\xi}{L_\xi}) \sin(\frac{j\pi\psi}{L_\psi})$$

The input function $b(\xi, \psi) = b_1(\xi) b_2(\psi)$ and the disturbance function $f(\xi, \psi) = f_1(\xi) f_2(\psi)$ are given by the 2D boxcar functions centered at $(0.251L_\xi, 0.361L_\psi)$ for the actuator and at $(0.563L_\xi, 0.283L_\psi)$ for the disturbance. The support for these two functions is the same as the 1D case. Here $A_u = 0.0790$, $B_u = 1.2893$ and $F_u = 1.4707$ and the controller is $K_u = 1.0631$ which shifts the unstable eigenvalue from $+0.0790$ to -1.2917. The functional observer pole was $\mathbf{N} = -0.25$ and the initial condition for $\hat{z}(0) = 0$, resulting in $e(0) = 2.6578$. The disturbance signal was selected as $w(t) = 5 \times 10^{-2} e^{-t}$ and the initial condition was

$$x(0, \xi, \psi) = 5 \left(\sin(\pi\xi/L_\xi) + \sin(2\pi\xi/L_\xi) \right) \times$$
$$\left(\sin(\pi\psi/L_\psi) + \sin(2\pi\psi/L_\psi) + \sin(3\pi\psi/L_\psi) \right)$$

To simulate the system, a total of 40×40 spectral elements were used to discretize the PDE in space. The resulting differential equations were integrated in the time interval $[0, 20]s$ using the Matlab® ODE solver `ode45`.

Figure 2 depicts the evolution of the state L_2 norm for the full state controller and the FO-based feedback. Similar to the 1D case, the performance of the full-state feedback surpasses that of the FO-based feedback but at a considerable expense in state information. Table II summarizes the results for the two cases. Similar to the 1D case, a full-state feedback controller cannot be realized and instead an

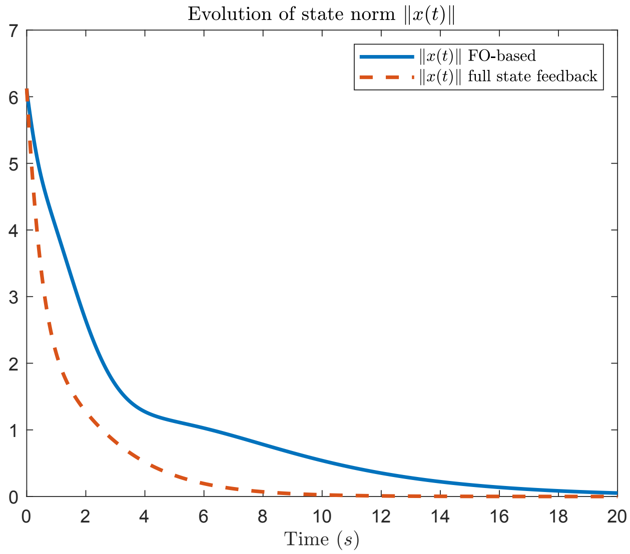

Fig. 2: 2D case: evolution of state norms.

observer-based feedback must be implemented, requiring the real-time integration of an infinite dimensional state observer $\widehat{x}(t)$ in order to realize the controller $u(t) = -K\widehat{x}(t)$. On the contrary, the UIFO-based controller, for this particular case requires the real-time integration of the scalar $w(t)$ in (25).

case	norm
full-state controller	4.34853
UIFO-based compensator	6.90367

Table 2. 2D case: L_2 state norms.

V. Conclusions

A functional observer was proposed to reconstruct a functional of the state of a parabolic PDE as a means of reducing the computational load associate with the implementation of an observer-based feedback. The functional reconstructed by the observer coincided with the full state feedback signal and this the functional observer produced an estimate of the control signal. This resulted in a minimum-order compensator. Reducing further the computational costs, the functional observer was applied a a class of parabolic PDEs that can be decomposed into a finite dimensional unstable subsystem and a stable infinite dimensional subsystem. The resulting functional observer was then requiring the solution to a finite dimensional Sylvester equation, as opposed to an operator Sylvester equation required for a functional observer applied to a general parabolic PDE system.

An extension was also considered in which the unknown input signal has its input operator known and thus, by including an additional condition on the matrix identities imposed on the functional observer, the resulting UIFO was able to reconstruct the functional to be used for control design. Such a condition enlarged the class of systems with unknown disturbances that can use a functional observer-based feedback in place of a full state feedback.

References

[1] R. Curtain, "Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input," *IEEE Trans. on Automatic Control*, vol. 27(1), pp. 98–104, February 1982.

[2] A. J. Pritchard and J. Zabczyk, "Stability and stabilizability of infinite-dimensional systems," *SIAM Rev.*, vol. 23(1), pp. 25–52, 1981.

[3] P. Murdoch, "Observer design for a linear functional of the state vector," *IEEE Trans. on Automatic Control*, vol. 18(3), pp. 308–310, 1973.

[4] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Trans. on Automatic Control*, vol. 39(3), pp. 606–609, 1994.

[5] M. Zasadzinski, M. Darouach, and M. Hayar, "Loop transfer recovery designs with an unknown input reduced-order observer-based controller," *Internat. J. Robust Nonlinear Control*, vol. 5(7), pp. 627–648, 1995.

[6] M. Darouach, "Existence and Design of Functional Observers for Linear Systems," *IEEE Trans. Automatic Control*, vol. 45(5), pp. 940–943, 2000.

[7] H. Trinh and T. Fernando, *Functional observers for dynamical systems*, ser. Lecture Notes in Control and Information Sciences. Springer, Berlin, 2012, vol. 420.

[8] Z. Emirsajł ow, "Infinite-dimensional Sylvester equations: basic theory and application to observer design," *Int. J. Appl. Math. Comput. Sci.*, vol. 22(2), pp. 245–257, 2012.

[9] Z. Emirsajłow, "Remarks on functional observers for distributed parameter systems," in *Proc. of the 23rd International Conference on Methods Models in Automation Robotics*, Aug 2018, pp. 144–147.

[10] M. A. Demetriou, "Optimization of sensor locations in transport systems using functional observers," in *Proc. of the 9th IEEE International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, August 25-28 2003.

[11] M. A. Demetriou, "Robust sensor location optimization in distributed parameter systems using functional observers," in *Proc. of the 44th IEEE Conference on Decision and Control*, Dec 2005, pp. 7187–7192.

[12] M. A. Demetriou and I. G. Rosen, "Unknown input observers for a class of distributed parameter systems," in *Proc. of the 44th IEEE Conference on Decision and Control*, Dec 2005, pp. 3874–3879.

[13] J. Deutscher, "Fault detection for linear distributed-parameter systems using finite-dimensional functional observers," *Internat. J. Control*, vol. 89(3), pp. 550–563, 2016.

[14] M. A. Demetriou and W. Hu, "Design and optimization of minimum-order compensators of distributed parameter systems via functional observers and unknown input functional observers," in *Proc. of the European Control Conference*, June 29–July 2 2021.

[15] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems*, 2nd ed., ser. Systems & Control: Foundations & Applications. Birkhäuser, Boston, 2007.

[16] K. Yosida, *Functional Analysis*, ser. Classics in Mathematics. Berlin: Springer-Verlag, 1995.

[17] T. Kato, *Perturbation Theory for Linear Operators, Vol. 132*, ser. Classics in Mathematics. Berlin: Springer-Verlag, 1995.

[18] J.-P. Raymond and L. Thevenet, "Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers," *Discrete and Continuous Dynamical Systems (DCDS-A)*, vol. 27(3), pp. 1159–1187, 2010.

[19] R. Triggiani, "On the stabilizability problem in Banach space," *J. of Mathematical Analysis and Applications*, vol. 52(3), pp. 383–403, 1975.

[20] M. Badra and T. Takahashi, "Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers: Application to the navier–stokes system," *SIAM J. on Control and Optimization*, vol. 49(2), pp. 420–463, 2011.

[21] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*. Berlin: Springer-Verlag, 1995.