Robust control of PDEs with disturbances using mobile actuators constrained over time-varying reachability sets

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Abstract— We design a practical mobile actuator guidance policy for linear parabolic equations in 2D: the guidance is chosen so that $\mathbb{H}^2$-measure of uncertainty is minimized provided the system is subject to a distributed disturbance. We first present a guidance policy where the mobile actuator location to be selected will be fixed over a certain time interval of interest. Further we add extra complexity by taking into account the dynamics of the mobile actuator over the 2D domain of interest under reachability constraints. The proposed approach is illustrated through numerical studies.

I. INTRODUCTION

Generic Advection-Diffusion-Reaction (ADR) equations are widely used to model and forecast a wide range of geophysical, biophysical and economic processes [1]: a good example is that of forecasting ozone concentrations at ground level [2]. An important class of ADR equations is represented by linear Advection-Diffusion equations used as models of heat or mass transfer in which chemically reactive effects are absent or negligible: e.g. dynamics of a concentration of a chemically non-reactive pollutant in a fluid, or the change in heat of a non-reactive flowing substance to name just a few.

In many applications, and especially in those mentioned above, it is practically impossible to have many fixed actuator locations covering the computational domain densely. A possible solution is to have mobile sensors "*where the uncertainty is the highest currently*" given that the disturbances are not known but their energy is bounded and, mathematically speaking, belong to a given bounded convex set, e.g. L^2-ellipsoid (e.g. [3]). We stress that using of mobile actuators for the control of spatially distributed systems governed by PDEs results in both implementational and computational challenges: indeed it requires the backward-in-time solution to the actuator guidance and the backward-in-time solution to the control operator Riccati equation. A way to address this computational challenge for the disturbance free-case was suggested in [4]) where the mobile actuator is repositioned at discrete time instances and resides in some spatial location for a certain time interval. In order to find optimal paths for a given time interval, a set of feasible locations is derived using the reachability set. These reachability sets are further constrained to take into account the time it takes to travel to

any spatial position with a prescribed maximum velocity. A number of different guidances, based on Lyapunov approach, for the disturbance-free case were also suggested in [5],[6]. We refer the reader to [7] for an extensive overview of the sensor and actuator placements for PDEs.

In this note, we build on the integrated control policy and mobile actuator guidance for a 2D advection-diffusion PDE system proposed in [4], and extended that result assuming that the system is subject to an unknown separable disturbance with a given spatial profile and unknown but bounded temporal profile. The idea is to reposition, within the spatial domain, the mobile platform that carries the actuating device, and dispense the appropriate control signal from this actuating device in order to attain certain performance characteristics, e.g. similarly to [8] minimal $\mathbb{H}^2$-cost. We also demonstrate the ε sub-optimality of the proposed algorithm for the disturbance-free case.

This contribution is our first step towards considering the general case of unknown but bounded set of spatially distributed disturbances and designing a computationally feasible controller minimizing the effect of the worst-case realization of the disturbance by building on ideas of [3] where a control law steering the state as close as possible (in the minimax sense) to the selected sliding surface was suggested for generic L^2-disturbances.

Assuming zero activation time and minimum residence time we demonstrate that the proposed hybrid continuous-discrete control and actuator guidance is minimizing the influence of the disturbance in $\mathbb{H}^2$-sense, and is feasible. The efficacy of the proposed mobile actuator in the presence of a disturbance given by a characteristic function with switching support is demonstrated for 2D heat equation for different geometric estimates of the reachability set.

II. PROBLEM FORMULATION

To examine the various actuator location optimization that form the basis for the proposed actuator guidance with location-dependent reachability sets, we must view the PDE as an evolution equation in Hilbert space. Assuming a 2D spatial domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$, the advection-diffusion PDE system is given by

$$\frac{\partial x(t,\xi,\psi)}{\partial t} = \mathcal{A}x(t,\xi,\psi) + b_1(\xi,\psi;t)w(t)$$
$$+ b_2(\xi,\psi;\xi_a(t),\psi_a(t))u(t), \quad (1)$$
$$z(t) = \int_\Omega c(\xi,\psi)x(t,\xi,\psi)\,d\chi + du(t),$$

where

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- $x(t, \xi, \psi)$ denotes the state at time $t \in \mathbb{R}^+$ and spatial coordinates $\chi = (\xi, \psi) \in \Omega$
- $\mathcal{A}$ is the spatial operator given by

$$\mathcal{A}\phi = \sum_{i,j}^{n} \alpha_{ij}(t,\chi) \frac{\partial^2 \phi}{\partial \chi_i \partial \chi_j} + \sum_{i=1}^{n} \beta(t,\chi) \frac{\partial \phi}{\partial \chi_i} + \gamma(t,\chi)\phi,$$

and is assumed to be uniformly elliptic in $(0,T) \times \Omega$
- $b_1(\xi, \psi; t)$ is the disturbance spatial distribution
- $w(t)$ is the disturbance input
- $b_2(\xi, \psi; \xi_a(t), \psi_a(t))$ is the actuator spatial distribution
- $\chi_a(t) = (\xi_a(t), \psi_a(t)) \in \Omega$ is the actuator centroid
- $u(t)$ is the control signal
- $z(t)$ is the output to be controlled
- $c(\xi, \psi)$ is a spatial function representing the state weight in the controlled output z.

Associated with the above PDE are the initial conditions and mixed conditions at the boundary $\partial \Omega_d \cup \partial \Omega_n = \partial \Omega$

$$x(t_0, \xi, \psi) = x_0(\xi, \psi), \qquad x\big|_{\partial \Omega_d} = 0, \quad \frac{\partial x}{\partial n}\Big|_{\partial \Omega_n} = 0,$$

with $\partial \Omega$ a smooth boundary and $\frac{\partial}{\partial n}$ the outward normal.

The state operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L_2(\Omega) \to L_2(\Omega)$ generates a strongly continuous semigroup $\mathbb{T}(t)$ on $L_2(\Omega)$. At every time $t \in \mathbb{R}^+$, the spatial distribution of the actuating device is assumed to be represented by a "shaping function" $b_2(\xi, \psi; \xi_a(t), \psi_a(t)) \in L_2(\Omega)$ centred at the control point $\chi_a(t) = (\xi_a(t), \psi_a(t))$ (see [9]) such that it approximates $\delta(\xi - \xi_a(t))\delta(\psi - \psi_a(t))$ in the sense of distributions.

Take a fixed time t_i and $\Delta t > 0$. Then $B_2(t_i) : \mathbb{R}^1 \to L_2(\Omega)$ given by

$$B_2(\chi_a(t_i))u(t) = b_2(\xi, \psi; \xi_a, \psi_a)u(t) \qquad (2)$$

denotes the location-dependent input operator on the interval $t \in [t_i, t_i + \Delta t]$. Similarly, $B_1(t_i) : \mathbb{R}^1 \to L_2(\Omega)$ defined by

$$B_1(t_i)w(t) = b_1(\xi, \psi; t_i)w(t). \qquad (3)$$

is the disturbance operator on the time interval $t \in [t_i, t_i + \Delta t]$.

The control objective is to design a control signal u and the associated guidance of the mobile platform carrying the mobile actuator in order to minimize the effects of the unknown input $w(t)$ on the system response. The actuator guidance can be expressed in terms of the coordinates of the actuator centroid $\chi_a(t) = (\xi_a(t), \psi_a(t))$. In the more complex case, one can express the guidance in terms of inputs to the equations of motion of the mobile platform carrying the actuator and whose barycenter is the actuator centroid.

III. SUMMARY OF OPTIMAL ACTUATOR LOCATION PROBLEMS

We first assume that the optimal actuator location to be selected will be fixed (immobile) throughout the time interval of interest. In this case the actuator location is given by $(\xi_a(t), \psi_a(t)) = (\xi_a, \psi_a)$ for all $t > 0$. The associated PDE in state space form is given by the evolution equation

$$\dot{x}(t) = \mathcal{A}x(t) + B_1 w(t) + B_2(\chi_a)u(t), \quad x(t_0) = x_0$$
$$z(t) = Cx(t) + Du(t) \qquad (4)$$

First, we consider the case where the control signal to be selected minimizes a prescribed performance index and for

now it is assumed that $w(t) = 0$. The LQR problem is to select the optimal actuator location from the set of candidate locations that minimizes the infinite horizon index

$$J(u, x_0, t_0) = \int_{t_0}^{\infty} \langle Cx(\tau), Cx(\tau) \rangle + \langle u(\tau), Ru(\tau) \rangle \, d\tau \quad (5)$$

where $R = R^\top > 0$. The location optimization must be restricted to the set of admissible locations which consists of all locations such that the system will have a form of controllability; for the case considered here it requires that the pair $(\mathcal{A}, B_2)$ be stabilizable. Thus the admissible set of locations is defined below.

Definition 1: The admissible set of actuator locations Θ_{ad} is given by the set of all locations in the spatial domain Ω that render the pairs $(\mathcal{A}, B_2)$ stabilizable

$$\Theta_{ad} = \{\chi_a \in \Omega : (\mathcal{A}, B_2(\chi_a)) \text{ is stabilizable}\}. \qquad (6)$$

A consequence of stabilizability is that there exist a feedback gain operator $K \in \mathcal{L}(L_2(\Omega), \mathbb{R})$ such that $A - B_2 K$ generates an exponentially stable C_0 semigroup.

When the control input operator is parameterized by the locations in Θ_{ad}, then the feedback operators are also parameterized by these locations and thus we have that the family of operators $\mathcal{A} - B_2(\chi_a)K(\chi_a)$ generate exponentially stable C_0 semigroups for all $\chi_a \in \Theta_{ad}$.

Essential to the solvability of the LQR problem, needed to ensure that an associated Operator Algebraic Riccati Equation (OARE) has a positive definite solution, is the additional condition of the detectability of the pair $(C, \mathcal{A})$.

For each candidate location in Θ_{ad}, the solution to the LQR control problem is enabled by the χ_a−dependent OARE

$$\mathcal{A}^* P(\chi_a) + P(\chi_a)\mathcal{A} - P(\chi_a)B_2(\chi_a)R^{-1}B_2^*(\chi_a)P(\chi_a) +$$
$$+ C^*C = 0, \quad \text{on } \mathcal{D}(\mathcal{A}) \subset L_2(\Omega), \quad \forall \chi_a \in \Theta_{ad}. \qquad (7)$$

For each $\chi_a \in \Theta_{ad}$ the associated optimal control is given by

$$u(t; \chi_a) = -R^{-1}B_2^*(\chi_a)P(\chi_a)x(t) \qquad (8)$$

leading to $K(\chi_a) = R^{-1}B_2^*(\chi_a)P(\chi_a), \forall \chi_a \in \Theta_{ad}$. The optimal cost (optimal value of $J(u, x_0, t_0)$) is given by

$$J(u^{opt}, x_0, t_0) = \langle x(t_0), P(\chi_a)x(t_0) \rangle, \quad \forall \chi_a \in \Theta_{ad}. \qquad (9)$$

It immediately follows that the optimal actuator location to be selected from the set of admissible locations is

$$\chi_a^{opt} = \arg \inf_{\chi_a \in \Theta_{ad}} \langle x(t_0), P(\chi_a)x(t_0) \rangle \qquad (10)$$

Next, we consider the case of $w(t) \neq 0$ and thus one is led to the following $\mathbb{H}^2$ problem of minimizing

$$\int_{t_0}^{\infty} \|z(\tau)\|_2^2 \qquad (11)$$

We assume the LQ-problem with stability is solvable, see [10]. In the $\mathbb{H}^2$ control optimization one minimizes the effects of the disturbance $w(t)$ on the closed-loop response.

Thus, we have that the $\mathbb{H}^2$ cost is given by

$$\text{trace} \left[B_1^* P(\chi_a)B_1 \right], \quad \forall \chi_a \in \Theta_{ad}. \qquad (12)$$

One has that the $\mathbb{H}^2$ optimal actuator location is given by

$$\chi_a^{opt} = \arg \inf_{\chi_a \in \Theta_{ad}} \text{trace} \left[B_1^* P(\chi_a)B_1 \right] \qquad (13)$$

Restricting the optimization to a finite time interval, one

may consider the LQR over a finite horizon

$$J(u,x_0,t_0) = \int_{t_0}^{T} \langle Cx(\tau), Cx(\tau) \rangle + \langle u(\tau), Ru(\tau) \rangle \, d\tau. \quad (14)$$

For a fixed actuator location, the solution leads to an Operator Differential Riccati Equation (ODRE). When the input operator is parameterized by the candidate actuator locations, one arrives at the location-parameterized ODRE

$$-\dot{P}(t;\chi_a) = \mathcal{A}^* P(t;\chi_a) + P(t;\chi_a)\mathcal{A}$$
$$-P(t;\chi_a)B_2(\chi_a)R^{-1}B_2^*(\chi_a)P(t;\chi_a) \quad (15)$$
$$+C^*C, \quad \text{on } \mathcal{D}(\mathcal{A}) \subset L_2(\Omega), \quad \forall \chi_a \in \Theta_{ad},$$

with terminal condition $P(T;\chi_a) = 0$ and with the optimal actuator location given by

$$\chi_a^{opt} = \arg \min_{\chi_a \in \Theta_{ad}} \langle x(t_0), P(t_0;\chi_a)x(t_0) \rangle \quad (16)$$

IV. SUBOPTIMAL GUIDANCE OF MOBILE ACTUATOR

Following the earlier work [4], one may consider a time-varying system (1) and/or repositioning the actuator in the spatial domain Ω. In this case, one is faced with the insurmountable task of having to integrate backwards-in-time both the ODRE (15) and the guidance of the mobile platform carrying the actuating device, [4].

To avoid the computational challenges with the optimal control and guidance over a finite time interval, a suboptimal policy is proposed. One considers the optimal control problem over the infinite horizon with variable lower limit t_i

$$J(u,x(t_i),t_i) = \int_{t_i}^{\infty} \langle Cx(\tau), Cx(\tau) \rangle + \langle u(\tau), Ru(\tau) \rangle \, d\tau. \quad (17)$$

Using the LQR metric, the optimal actuator location and the associated control signal for the interval $[t_i, \infty)$ are given by

$$\begin{cases} \chi_a^{opt,[t_i,\infty)} = \arg \inf_{\chi_a \in \Theta_{ad}} \langle x(t_i), P(\chi_a)x(t_i) \rangle \\ u(t;\chi_a^{opt,[t_i,\infty)}) = -R^{-1}B_2^*(\chi_a^{opt})P(\chi_a^{opt})x(t). \end{cases} \quad (18)$$

Similarly, using the $\mathbb{H}^2$ metric, the optimal actuator location and the associated control signal for the interval $[t_i, \infty)$ are

$$\begin{cases} \chi_a^{opt,[t_i,\infty)} = \arg \inf_{\chi_a \in \Theta_{ad}} \text{trace } \left[B_1^* P(\chi_a)B_1 \right] \\ u(t;\chi_a^{opt,[t_i,\infty)}) = -R^{-1}B_2^*(\chi_a^{opt})P(\chi_a^{opt})x(t). \end{cases} \quad (19)$$

At a later time $t_{i+1} = t_i + \Delta t$, the above optimization is re-examined and in this case, one considers

$$J(u,x(t_{i+1}),t_{i+1}) = \int_{t_{i+1}}^{\infty} \langle Cx(\tau), Cx(\tau) \rangle + \langle u(\tau), Ru(\tau) \rangle \, d\tau \quad (20)$$

Excluding the cost of switching, the new actuator location and control signal for the time interval $[t_{i+1}, \infty)$ is also expressed in terms of OAREs.

The above can be summarized as follows: *Starting with t_0 and setting $t_i = t_0 + i\Delta t$, $i = 1, 2, \ldots$, consider the joint actuator location and control optimization in the time subintervals $[t_0, T] = [t_0, t_1) \cup [t_1, t_2) \cup \ldots \cup [t_{n-1}, T]$. One changes the lower limit in (17) at the beginning of a new time interval $[t_i, t_{i+1})$. The optimal actuator location and control signal for a given $[t_i, t_{i+1})$ are given by (18) or (19). This is repeated for subsequent intervals.*

Two assumptions are considered here:

1) *Zero Activation Time*: In the first one, the actuator(s) change their position at the beginning of a new subinterval $[t_i, t_i + \Delta t)$. This assumes that one has either a large number of actuators placed in the spatial domain Ω and that only a subset of them can be active over a given interval of duration Δt, or that the actuators can "hop" to a new position in infinitesimal time.

2) *Minimum Residence Time*: One has vehicle dynamics for the mobile platform carrying the actuator and at the beginning of a new interval, the actuator moves to a new location. This of course would require that the time it takes to traverse (i.e. the travel time) is significantly smaller than the residence time Δt. In this case the optimizations (18) or (19) will be restricted to the intersection of Θ_{ad} and the candidate location that the mobile platform can traverse in a finite time interval in the vicinity of its current location.

A. Actuator activation in $[t_i, t_{i+1})$ with discrete time updates on the cost-to-go

To introduce another level of complexity, it is assumed that *not only* does the actuator location changes every Δt time units, *but also* the disturbance spatial distribution $b_1(\xi, \psi; t)$. Using the fact that the spatial distribution of the unknown disturbance will be changing at the beginning of a new interval, then the joint optimization departs form the one highlighted in [4], as it must consider the effects of b_1.

The algorithm summarizing the steps is provided in Algorithm 1. Notice that the actuator location is chosen to minimize the influence of the disturbance. For a disturbance free system (the disturbances are set to zero) the actuator locations in part 5 of Algorithm 1 are selected using

$$\chi_a^{opt,t_i} = \arg \inf_{\chi_a \in \Theta_{ad}} J(u(t,\chi_a(t_i)),x(t_i);t_i). \quad (21)$$

One has the following result.

Proposition 1: Consider the system (1) together with the admissible set Θ_{ad}. If the zero activation time and minimum residence time assumptions are satisfied then Algorithm 1 is feasible. Moreover, if the system (1) is disturbance free and the system guided by mobile actuator has a closed-loop trajectory is asymptotically stabilizable, then Algorithm 1 with the locations selected by (21) provides an ε-optimal mobile actuator guidance policy on a finite-time interval.

Proof: From the admissibility of the actuator locations it follows that one has optimizability satisfied. Then, the location-parameterized cost-to-go (defined at part 4 of the algorithm) is finite. Consequently, Algorithm 1 is feasible.

Consider now a disturbance free system (1) such that the closed-loop trajectory of the system guided by mobile actuator is asymptotically stabilizable. The actuator locations in the modified Algorithm 1 are selected using (21). The closed-loop trajectory of the resulting system using the modified Algorithm 1 can be written as $x_{Alg1}(T) = \Pi_{i=0}^{n-1} U_{cl,\chi_a(t_i)}(t_i, t_{i+1})x_0$, where $U_{cl,\chi_a(t_i)}(t,\tau)$ are the (mild) evolution operators corresponding to the closed loop operators on the finite time intervals. Let now $a = \sup_{s \in [0,\Delta t]} \|\mathbb{T}(t)\|$. Us-

Algorithm 1 Actuator switching in $[t_i, t_{i+1}]$

1: **initialize:** Determine Θ_{ad} consisting of all locations that render $(A, B_2(\chi_a))$ stabilizable. Divide $[t_0, T]$ into n subintervals $[t_i, t_{i+1})$ with $t_i = t_0 + i\Delta t$ and $\Delta t = \frac{(T-t_0)}{n}$.

2: **iterate:** $i = 0$

3: **loop**

4: For every $\chi_a \in \Theta_{ad}$ minimize the location-parameterized cost-to-go

$$J(u(t, \chi_a(t_i)), x(t_i); t_i) = \int_{t_i}^{\infty} \langle Cx(\tau), Cx(\tau) \rangle + \langle u(\tau), Ru(\tau) \rangle \, d\tau,$$

5: select the actuator location for $[t_i, t_{i+1}]$ using

$$\chi_a^{opt, t_i} = \arg \inf_{\chi_a \in \Theta_{ad}} \text{trace} \left[B_1^*(t_i) P(\chi_a) B_1(t_i) \right]$$

where $P(\chi_a)$ is the solution to the OARE

$$\mathcal{A}^* P(\chi_a) + P(\chi_a)\mathcal{A} - P(\chi_a) B_2(\chi_a) R^{-1} B_2^*(\chi_a) P(\chi_a) +$$
$$+ C^* C = 0, \quad \text{on } \mathcal{D}(\mathcal{A}) \subset L_2(\Omega), \quad \forall \chi_a \in \Theta_{ad}.$$

6: for $t \in [t_i, t_{i+1}]$, switch to actuator with location χ_a^{opt, t_i} and implement controller

$$u(t) = -B_2^*(\chi_a^{opt, t_i}) P(\chi_a^{opt, t_i}) x(t)$$

7: propagate (4) in the interval $[t_i, t_{i+1}[$

8: **if** $i \leq n - 2$ **then**

9: $i \leftarrow i + 1$

10: **goto** 3

11: **else**

12: terminate

13: **end if**

14: **end loop**

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ing Gronwall's lemma (as in the proof of [9, Theorem 3.2.5]) one has $\Pi_{i=0}^{n-1} \left(a + a^2 + \frac{1}{2} a^3 \| D(\chi_a(t_i)) \|^2 (\Delta t)^2 \right)$ an uniform upper bound for the trajectory, where $D(t_i) := B_2(\chi_a(t_i)) R^{-1} B_2^*(\chi_a(t_i)) P(\chi_a(t_i))(t)$. Using further the asymptotic stabilizability on the infinite time of the closed system guided by the mobile actuator, it follows that the algorithm finds an ε-optimal solution, i.e. for any $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\langle x_{Alg1}(T), P(\chi_a) x_{Alg1}(T) \rangle < \varepsilon$. ∎

B. Mobile actuator guidance in $[t_i, t_{i+1}]$ with discrete time updates on cost-to-go

A general dynamical model describing the dynamics of the mobile platform with a time-varying centroid $\chi_a(t)$ is

$$\dot{\phi}(t) = F(\phi(t), \upsilon(t)), \quad \chi_a(t) = H(\phi(t), \upsilon(t)) \quad (22)$$

where $\upsilon(t)$ is the control input to the mobile platform carrying, e.g. forces, torques. The vector field F captures the dynamics of the mobile platform, ϕ is the state of motion dynamics and H is the nonlinear output vector field that provides the Cartesian coordinates $\chi_a(t)$ of the mobile platform barycenter. While a fully dynamical model of the mobile platform will include both translational and rotational states with four translational states (positions and velocities in the ξ and ψ directions) and angular position and velocity, along with the acceleration forces in the ξ and ψ directions and torques in the vertical direction, in low to medium speeds one may consider a lower-dimensional representation of (22).

For the case under consideration, the dynamic model for the 2D case is described by the kinematic equations

$$\dot{\xi}_a(t) = v_a(t)\cos(\theta_a(t)), \quad \dot{\psi}_a(t) = v_a(t)\sin(\theta_a(t)),$$
$$\dot{\theta}_a(t) = \omega_a(t), \quad (23)$$

where the control signals are the speed $v_a(t)$ and turning rate $\omega_a(t)$; Even the kinematic model (23) can be represented in the format given in (22) with

$$\phi = \begin{bmatrix} \xi_a & \psi_a & \theta_a \end{bmatrix}^T, \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \phi.$$
$$\upsilon = \begin{bmatrix} v_a & \omega_a \end{bmatrix}^T,$$

Now, it is expected that the mobile platform repositions itself within the spatial domain Ω at the beginning of a new time subinterval $[t_i, t_{i+1}]$ while respecting the motion dynamics (23). As expected, this spatial repositioning cannot occur instantaneously. The problems associated with this case are (a) the time it takes to travel from the current optimal location $\chi_a^{opt}(t_i)$ for the interval $[t_i, t_{i+1}]$ to the next optimal location $\chi_a^{opt}(t_{i+1})$ for the interval $[t_{i+1}, t_{i+2}]$ and (b) the search for the optimal actuator location as commanded by (19). This search cannot be over the set Θ_{ad} due to the computational requirements and also due to the fact that the mobile platform cannot move to any spatial location within Θ_{ad} over a prescribed time interval. Following [4], the above two are addressed by assuming that the optimization search be performed by a set of candidate locations that render the pair $(\mathcal{A}, B_2)$ stabilizable and are within a prescribed distance from the current actuator location.

Using the assumptions made in [4], it is assumed that the time it takes the mobile platform to travel from the current position to its next position is denoted by t_{travel} and obeys

$$t_{travel} \ll \Delta t. \quad (24)$$

Further assuming that the mobile platform has a constant speed v_a leaving only the turning rate ω_a as the control input, then the distance travelled is given by vt_{travel} and thus, the mobile platform loci from the current location $\chi_a(t_i) = (\xi_a(t_i), \psi_a(t_i))$ are given by

$$\xi_a(t) = \xi_a(t_i) + (v_a t_{travel})\cos(\theta_a(t))$$
$$\psi_a(t) = \psi_a(t_i) + (v_a t_{travel})\sin(\theta_a(t)) \quad (25)$$

for $t \in [t_i + t_{travel}, t_i + \Delta t)$, and $\theta_a(t)$ satisfies

$$\theta_a(t_i) - \pi \leq \theta_a(t) \leq \theta_a(t_i) + \pi. \quad (26)$$

This essentially states that the spatial locations the mobile platform can reach in t_{travel} time units is given by a circle with center $(\xi_a(t_i), \psi_a(t_i))$ and radius $v_a t_{travel}$. This time-varying reachability set is defined by

$$\mathcal{R}_1(t_i) = \Theta_{ad} \cap \{(\xi, \psi, \theta) \text{ obey } (25), (26)\} \quad (27)$$

When the mobile platform obeying (25) has an angular constraint $\pm\Delta\theta$ with $\Delta\theta \ll \pi$, meaning that it can only rotate within a sector of the current value $\theta_a(t_i)$, then (26) is replaced by

$$\theta_a(t_i) - \Delta\theta \leq \theta_a(t) \leq \theta_a(t_i) + \Delta\theta, \quad (28)$$

and the associated reachability set is now given by

$$\mathcal{R}_2(t_i) = \Theta_{ad} \cap \{(\xi, \psi, \theta) \text{ obey } (25), (28)\}. \quad (29)$$

A third constraint in the form of angular displacement and angular rate constrains was considered in [4], but due to its similarities to $\mathcal{R}_2(t_i)$, it will not be considered here.

Algorithm 2 summarizes the integrated actuator guidance and control policies when the optimization searches are constrained in either of the time-varying reachability sets.

Algorithm 2 Actuator guidance in $[t_i, t_{i+1})$ over $\mathcal{R}(t_i)$

1: **initialize:** Determine Θ_{ad} consisting of all locations that render $(A, B_2(\chi_a))$ stabilizable. Divide $[t_0, T]$ into n subintervals $[t_i, t_{i+1})$ with $t_i = t_0 + i\Delta t$ and $\Delta t = \frac{(T-t_0)}{n}$.
2: **iterate:** $i = 0$
3: **loop**
4:     select the actuator location for $[t_i, t_{i+1})$ that minimizes the location-parameterized cost-to-go

$$J(u, x(t_i); t_i) = \int_{t_i}^{\infty} \langle Cx(\tau), Cx(\tau)\rangle + \langle u(\tau), Ru(\tau)\rangle \, d\tau,$$

    using

$$\chi_a^{opt, t_i} = \arg \inf_{\chi_a \in \mathcal{R}(t_i)} \text{trace} \left[B_1^*(t_i) P(\chi_a) B_1(t_i) \right]$$

    where $P(\chi_a)$ is the solution to the OARE

$$\mathcal{A}^* P(\chi_a) + P(\chi_a)\mathcal{A} - P(\chi_a)B_2(\chi_a)R^{-1}B_2^*(\chi_a)P(\chi_a)$$

$$+ C^*C = 0, \quad \text{on } \mathcal{D}(\mathcal{A}) \quad \forall \chi_a \in \mathcal{R}(t_i).$$

5:     for $t \in [t_i, t_{i+1})$, move to actuator location χ_a^{opt, t_i} within the reachability set $\mathcal{R}(t_i)$ and implement controller

$$u(t) = -B_2^*(\chi_a^{opt, t_i})P(\chi_a^{opt, t_i})x(t)$$

6:     propagate (1) in the interval $[t_i, t_{i+1})$
7:     **if** $i \leq n - 2$ **then**
8:         $i \leftarrow i + 1$
9:         **goto** 3
10:     **else**
11:         terminate
12:     **end if**
13: **end loop**

V. Finite dimensional approximations

Algorithms 1 and 2 require a finite dimensional approximation in order to realize the actuator location optimization. Since at the beginning of a subinterval $[t_i, t_i + \Delta t)$ one is considering an infinite horizon performance index (17), then the approximations of the OARE and the actuator location optimization will be presented for only one such subinterval.

To examine the finite dimensional representation of (4), we consider the approximation framework similar to [3, Section 5] (see also the references therein). We consider a family of finite dimensional subspaces $\mathcal{X}_n$, $n \in \mathbb{N}$, of the state space $\mathcal{X}$. We denote the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_n$ by Π_n such that $\lim_{n \to \infty} \|\Pi_n x - x\| = 0$ for all $x \in X$. Then A_n is family of state operators $A_n : \mathcal{X}_n \to \mathcal{X}_n$. For each $\chi_a \in \Omega_{ad}$, we also define the parameterized input operator approximations via $B_{2n}(\chi_a) = \Pi_n B_2(\chi_a)$. Similarly, we define the finite dimensional approximation of the unknown input operator $B_{1n} = \Pi_n B_1$. Finally, C_n is the restriction of C on $\mathcal{X}_n$.
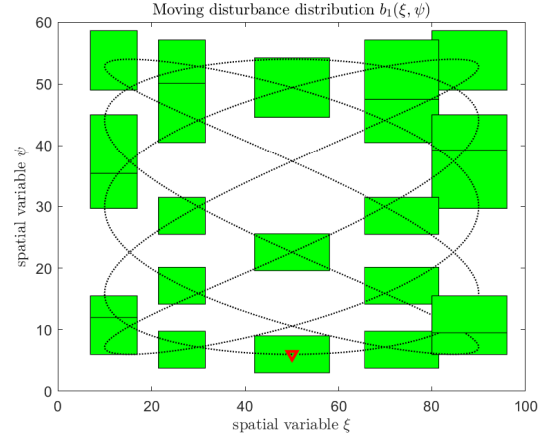

Fig. 1: Evolution of the spatial distribution of disturbances $b_1(\xi, \psi)$ given by a 2D characteristic function.

We subsequently arrive at a sequence of finite dimensional approximations to the evolution system (4)

$$\dot{x}_n(t) = A_n x_n(t) + B_{1n}w(t) + B_{2n}(\chi_a)u(t), \quad x_n(t_0) = x_{n0}$$
$$z_n(t) = C_n x_n(t) + D_n u_n(t) \tag{30}$$

The associated cost-to-go representing the approximation of (17) is given by

$$J_n(u_n, x_n(t_i), t_i) = \int_{t_i}^{\infty} \langle C_n x_n(\tau), C_n x_n(\tau)\rangle + u_n^2(\tau) \, d\tau. \tag{31}$$

In addition to requiring that the solution to the finite dimensional approximation of the OARE $P(\chi_a)$ converge to $P(\chi_a)$, we also need to require that the optimal actuator location corresponding to the finite dimensional approximation will converge to the true optimal actuator location of (4).

VI. Numerical studies

A rectangular domain $\Omega = (0, L_X) \times (0, L_Y) = (0, 100) \times (0, 60) \subset \mathbb{R}^2$ m was assumed for the PDE in (1). Similar to the study in the earlier work [4], the parameters of the elliptic operator were selected as $\alpha = 0.1, \beta = \gamma = 0$. The initial condition for the process state was set to $x_0(\xi, \psi) = 10^4 \xi^3 \psi^3 (L_X - \xi)^3 (L_Y - \psi)^3$. To realize the controller in Algorithm 2, a finite dimensional approximation scheme based on Galerkin methods was implemented with $n_x = 26, n_y = 16$ linear elements in the ξ and ψ directions.

The residence time at a particular actuator position within Ω was set to $t_{res} = 4$ s and the time to travel from one position to the next one after t_{res} time units was set at $t_{travel} = 0.4$s. A constant speed for the kinematic model (23) was assumed with a value of $v = 25$ m/s. Initially, the actuator was placed at $\chi_a(t_0) = (\xi_a(t_0), \psi_a(t_0)) = (0.312L_X, 0.123L_Y)$. The algebraic Riccati equation was solved with $C^*C = 10$ and $R_1 = 0.01$. The spatial distribution of disturbances was given by the 2D characteristic function that was switching its support every $t_{res} = 4$ s and its strength (temporal component) was given by $w(t) = 0.1$. Figure 1 demonstrates the variable support of $b_1(\xi, \psi)$ in the time interval $[0, 100]$ s.

Both reachability sets (circle (black) and sector (red)) were considered in the simulations. In fact they are depicted in Figure 2 for an actuator centered at $(10, 5)$. The angle
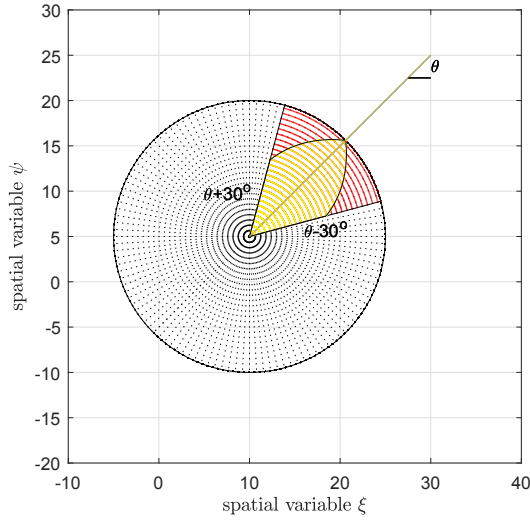
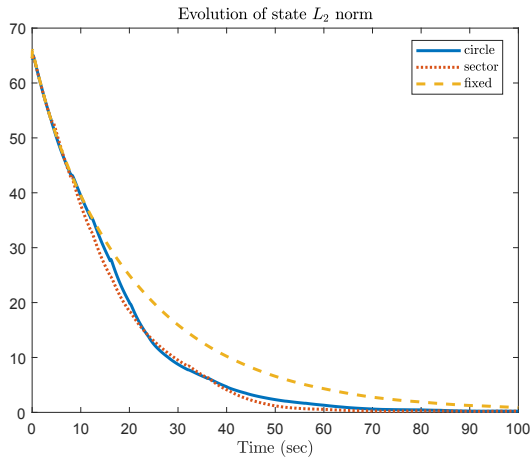Fig. 2: Reach. sets: circle, sector and segment.

Fig. 3: Evolution of L_2 state norm using the proposed actuator guidances and the fixed actuator.

constraint for the sector set uses $\Delta\theta = \pm30^o$ from the current actuator position to define the sector. In addition to the two reachability sets, another one, the segment, enforcing both angle and angular rate constraints is depicted (yellow).

The L_2 norm of the state $x(t,\xi,\psi)$ is depicted in Figure 3 for both reachability sets and also the case of a fixed-in-space actuator (i.e. an actuator with a residence time equal to the simulation time of 100s).

The actuator trajectories corresponding to the two reachability sets are depicted in Figure 4. Due to the different reachability sets, the trajectories are completely different.

The results are also summarized in Table I. The mobile actuator outperforms the fixed actuator, as expected. Comparison of the performance of the mobile actuator with the two reachability sets does not lead to any significant differences. However, the time it takes to simulate the mobile actuator with a circle as the reachability set, is six times larger than the time it takes to simulate the mobile actuator with a sector as the reachability set. Since the performance is comparable, then one may conclude that the sector reachability set is
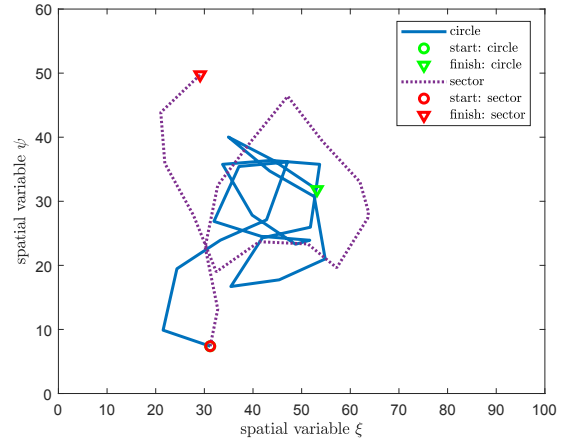

Fig. 4: Actuator trajectories using two different reach. sets.

preferable than the circle.

reachability set	circle	sector	none-fixed
state norm	195.58	191.74	209.61

TABLE I: $L_2(0,t;L_2(\Omega))$ norm; fixed and moving actuator.

VII. CONCLUSIONS

An extension of an actuator placement algorithm of [4] for the case of unknown but bounded time input is proposed and justified. It relies upon $\mathbb{H}^2$-uncertainty minimization, which numerically makes use of discretization of an AROE (required just once!) and proves efficient in simple numerical experiment for 2D heat equation. This contribution represents our first step towards considering the general case of unknown but bounded set of spatially distributed disturbances.

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