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# Modal consensus observers for distributed parameter systems

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**Abstract**

This article proposes a new type of a consensus protocol for the synchronization of distributed observers in systems governed by parabolic partial differential equations. Addressing the goal of sharing useful information among distributed observers, it delves into the details governing the modal decompositions of distributed parameter systems. Assuming that two different groups of sensors are available to provide process information to the two distributed observers, the proposed modal consensus design ensures that only useful information is transmitted to the requisite modal components of each of the observers. Without any consensus protocol, the observers capture different frequency content of the spatial process in differing degrees, as it relates to the concept of modal observability. Their modal components exhibit different learning behavior toward the process state. In the extreme case, it turns out that certain modal components of the distributed observers occasionally behave as naïve observers. To ensure that, both collectively and modal componentwise, the observers agree both with the process state and with each other, a modal component consensus protocol is proposed. Such a consensus protocol is mono-directional and provides only useful information necessary to the appropriate modal component of the distributed filters that behaves as a naïve modal observer. This protocol, when abstracted and applied to different state decompositions can be viewed as mono-directional projections of information transmitted and received by the participating distributed observers. Detailed numerical studies of advection PDE in one and two spatial dimensions are included to elucidate the details of the proposed modal consensus observers.

**KEYWORDS**

distributed parameter systems, modal consensus, modal decompositions, synchronization

## 1 | INTRODUCTION

The alternative to centralized filters for state reconstruction of dynamical systems, namely, distributed filters, brought forth significant computational savings and design complexity simplifications. Distributed filters can collaborate, via an appropriate information exchange, to reach agreement among their state estimates. The first paper to establish the link between consensus and PDEs by modeling the average consensus algorithm as an advection-diffusion process governing the homogenization of fluid mixtures was Reference 1. Migrating to a different involvement of PDEs and consensus protocols,[2] extended the consensus observers for networked agents, each of which was governed by an infinite dimensional system.

In the case of spatially distributed systems, the distributed filters become literally distributed in space, wherein sensors positioned within the spatial domain, can obtain information about the state of the spatially distributed process. Earlier work consider different aspects of consensus filters for spatially distributed systems. One of the earliest works that extended the concept of consensus in distributed filters was in Reference 2, which required the mapping from the consensus protocol of the distributed systems to the corresponding output to have a strictly positive real transfer function. The architecture of the distributed filters followed the one proposed for the finite dimensional case.[3-6] While requiring what is essentially an all-to-all connectivity, it did however paved the way for optimization of the communication among the agents by modifying the consensus operators so that only output signals (finite dimensional) are shared by the agents. Another one was the use of adaptive consensus weights whereby the consensus weights were adapted in accordance to the agreement of the state estimates of the distributed filters. The application of output feedback with consensus observers was developed in Reference 7 in order to synchronize and control distributed systems governed by partial differential equations. As a means to provide a spatial convergence, a special type of consensus protocol was proposed in Reference 8 where a spatial penalty was explicitly imposed on the disagreement of the distributed filters. An aspect closer to the proposed work was the zonal consensus observers where the consensus of the state estimates was enforced only on a portion of the spatial domain. The consensus weight was essentially a proportional weight defined on a part of the spatial domain.

The idea of consensus observers was extended to second-order distributed parameter systems in Reference 9 and the combination of the adaptation of consensus weights in Reference 2 with the natural setting of observers for second-order systems was examined in Reference 10. A version of consensus observers that introduced a concept of robustness of the finite dimensional representation of dissipative PDEs and which took advantage of the state decomposition was presented in Reference 11. In a similar vein, working on the finite dimensional decomposition of infinite dimensional dissipative systems, an $H_\infty$ design was presented in Reference 12. Combining consensus controllers with iterative learning for second-order distributed parameter systems was considered in Reference 13. Introducing the concept of a leader-follower tracking in networked systems, each governed by a diffusion PDE was considered in Reference 14 and which utilized boundary sensing in each agent. This was subsequently extended to account for boundary disturbances in Reference 15 and semilinear PDEs in Reference 16. A computational geometry concept was incorporated into the consensus problem wherein the problem of information sharing and consensus when using centroidal Voronoi tessellations algorithm to control a diffusion process was presented in Reference 17.

However, a characteristic of spatially distributed processes not found in lumped parameter systems, is the spatially dependent nature and effects of sensors; sensors can obtain information that enable the associated filter to reconstruct a portion of the state that is limited to a spatial region surrounding the sensor. A series of papers consider this aspect of regional observability[18-22] and examined the design of observers and the characterization of sensors to attain such a regional observability. Another property is that sensors in different regions of the spatial domain, can obtain information and reconstruct different frequency components (waveforms) of the spatially distributed process.

For the former case, the reconstructed state from each of the distributed filters can form a piece of the estimation puzzle and when the individual state estimates are stitched together they can produce an estimate of the spatially distributed state over the *entire* spatial domain. In the latter case, sensors and their associated filters can reconstruct a portion, in the sense of the frequency spectrum of the process state, precisely due to their spatial location. The spatially distributed filters can only tell a "piece of the story" and only when they are combined together will they provide the entire picture of the spatially distributed state.

As expected, these component filters, can only *collectively* provide the estimate of the process state; alternatively, they can also yield such a comprehensive estimate by the *judicious* sharing of information. A distributed filter that is successful in reconstructing a specific frequency band of the distributed process can broadcast such an information to all other sensors/filters, but would not need to receive any information on the process estimate from other filters if it relates to the state estimate over precisely this specific frequency band. Similarly, if a sensor/filter can efficiently reconstruct the process state over a specific subregion of the spatial domain, it does not need to receive any state estimate corresponding to that spatial subregion; instead, it needs to be supplemented with a reconstruct of the spatial state over spatial regions at which it cannot effectively reconstruct.

However, the type of information a sensor can receive and, the type and portion of the state its associated filter can reconstruct, is not absolute; rather a sensor/filter can reconstruct a given frequency band *better* than it can reconstruct other frequency bands. Thus one comes across the *relative level* of observability a given sensor/filter can attain. In particular, a given sensor may be able, through its associated filter, to observe a frequency band $A$ at a level of 100% and a frequency band $B$ at a level of 40%. Such sensor/filter need only receive information that pertains to the frequency band $B$ from other sensors/filters, and does not need any information that pertains to the frequency band $A$. Similarly, it need only transmit its state estimate that pertains to frequency band $A$ to all other sensors/filters. It does not need to transmit its state estimate as it pertains to the frequency band $B$ to all other sensors/filters since it is not doing a great job at reconstructing the process state over this frequency band. Similar arguments can be made for sensors/filters capable of reconstructing the process state over the spatial region $\Omega_A$ with a regional observability level of 100% and process state over the spatial region $\Omega \setminus \Omega_A$ with a spatial observability level of 20%. This sensor/filter will be transmitting its state estimate that pertains to the spatial region $\Omega_A$ to all other sensors/filters. It will need to receive information on the state estimate that pertains to the spatial region $\Omega \setminus \Omega_A$ from some/all other sensors/filters.

Therefore, a judicious sharing of information among the distributed observers is warranted as a means to avoid transmitting redundant and duplicate information; only useful information about the state reconstruct must be received by a given sensor/filter and in turn, only useful information should this particular sensor/filter must transmit to the other communicating sensor/filters. In a nutshell, this article answers to the following request between distributed observers: *don't send me just any information, just send me the information I need!*

*Contribution:* This article considers a spatially distributed process whose time evolution is described by a diffusion, and in general parabolic, PDE. The premise is that two sensor groups are employed; each sensor group can effectively reconstruct the process state, either in terms of frequency content or representing specific region of the spatial domain. Instead of a general consensus protocol for information sharing between the sensor groups and their associated filters, here a different information sharing protocol is proposed. This information sharing protocol termed the *modal consensus protocol*, ensures that only the appropriate portion of the state estimate is transmitted to the other distributed filter. Thus, the contribution of this article is fourfold:

- Proposes two distributed filters associated with each of the two sensor groups and summarizes their well-posedness.
- Computes the exact and useful information that a given modal observer must receive in order to agree with the other observers and provides the useful information that a given modal observer must transmit to the other modal observer. The result is that the consensus protocol is *monodirectional* whereby only useful and needed information is transmitted and received by the distributed filters.
- Introduces modification to the consensus protocol in order to minimize communications costs defrayed by the modal consensus protocols.
- Enables the selection of convergence rate of the disagreement error between the two modal observers, thereby ensuring that the disagreement error can converge to zero faster than any of the two state estimation errors.

*Contents:* A demonstration of the effects of sensor location on the amount and type of information about the spatially distributed process is demonstrated in Section 2 as a means of familiarizing the reader with the effects of sensor locations on the *level of observability* imparted by a specific sensor. Subsequently, a simple example of a spatially distributed process which admits a solution that includes only a specific frequency content, also known as the modes, is presented in a tutorial style. A sensor placed at specific spatial location, with this particular case being the zeros of certain eigenfunctions associated with the spatial operator, results in certain frequencies being absent in the output. When a combination of the input signal and initial conditions is used, the process state exhibits a nonzero spatiotemporally varying solution while the sensor output provides a zero reading! Continuing, the use of two sensors is examined whereby one sensor can provide state information over a particular modal range, while the other sensor can obtain process state information representing the orthogonal complement of the modal content of the process state. Each sensor and its associated filter cannot individually and single-handedly provide process state information. When the appropriate information sharing, via the proposed modal consensus protocol, is enacted then both distributed filters are synchronized and provide a high-fidelity estimate of the process state. This is generalized for a class of Riesz-spectral systems in Section 3. Instead of providing a generic consensus protocol, the main result in Section 3 provides an efficient and relevant information sharing among the distributed sensors/filters. Each filter receives information about the frequency or spatial region of the process state that it cannot have, and in turn, transmits information that the other spatially distributed sensor/filter does not have. Such a modal consensus protocol can be abstractly thought of as a projection of valuable information into the appropriate subspace. An extension of the modal consensus protocol regarding the information sharing is presented as a means to minimize the communication load. A detailed numerical example is provided in Section 4 as a means to highlight the particulars of the modal consensus filters for diffusion PDEs in one and two spatial dimensions, with conclusions following in Section 5.

## 2 | MOTIVATING EXAMPLE AND PROBLEM FORMULATION

Consider the 1D diffusion PDE with state $x(t,\xi)$

$$\frac{\partial x}{\partial t} = \kappa \frac{\partial^2 x}{\partial \xi^2} + \beta(\xi)u(t) \tag{1}$$

having Dirichlet boundary conditions $x(t,0) = x(t,\ell) = 0$, and initial condition $x(0,\xi) = x_0(\xi)$. The spatial domain is $\Omega = [0,\ell]$, $\kappa$ is the thermal diffusivity and $\beta(\xi)u(t)$ denotes the input with $\beta(\xi)$ the spatial distribution and $u(t)$ its scalar temporal component. The homogeneous PDE (i.e., $u = 0$) has a solution

$$x(t,\xi) = \sum_{i=1}^{\infty} \alpha_i(t)\phi_i(\xi), \tag{2}$$

where the spatial trial functions $\phi_i(\xi)$ are termed the *modes* (or modal functions). It is easy to compute the modes for (1) with Dirichlet boundary conditions,[23] given here by

$$\phi_i(\xi) = \sqrt{\frac{2}{\ell}}\ \sin\left(\frac{i\pi\xi}{\ell}\right), \quad i = 1,\dots,\infty. \tag{3}$$

For the particular case of $\beta(\xi) = \phi_{k_1}(\xi)$ with $k_1$ any integer, the solution (2) substituted into (1) results in

$$\sum_{i=1}^{\infty} \dot{\alpha}_i(t)\phi_i(\xi) = \sum_{i=1}^{\infty} \alpha_i(t)\left(-\kappa\left(\frac{i\pi}{\ell}\right)^2\right)\phi_i(\xi) + \phi_{k_1}(\xi)u(t).$$

When viewed in weak form and defining $\lambda_i = -\kappa\left(\frac{i\pi}{\ell}\right)^2$

$$\sum_{i=1}^{\infty} \dot{\alpha}_i(t)\int_0^{\ell} \phi_i(\xi)\phi_j(x)\ \mathrm{d}\xi = \sum_{i=1}^{\infty} \alpha_i(t)\lambda_i \int_0^{\ell} \phi_i(\xi)\phi_j(\xi)\ \mathrm{d}\xi + \sum_{i=1}^{\infty}\int_0^{\ell} \phi_{k_1}(\xi)\phi_j(\xi)\ \mathrm{d}\xi u(t),$$

for all test functions $\phi_j(\xi), j = 1,\dots,\infty$. When $j = k_1$, then due to the orthogonality of the eigenfunctions, we arrive at

$$\dot{\alpha}_{k_1}(t) = \lambda_{k_1}\alpha_{k_1}(t) + u(t), \tag{4}$$

and for all other values of $j$, one has the homogeneous equations for $\alpha_i(t)$. Further assuming that the initial condition $x_0(\xi) = \phi_{k_1}(\xi)$, the homogeneous equation yields the trivial solution and (4) has $\alpha_{k_1}(0) = 1$. Finally, with $u(t) = u_0$, (4) has solution

$$\alpha_{k_1}(t) = e^{\lambda_{k_1} t} + \frac{e^{\lambda_{k_1} t} - 1}{\lambda_{k_1}} u_0,$$

and the solution to (1) becomes

$$x(t,\xi) = \left(e^{\lambda_{k_1} t} + \frac{e^{\lambda_{k_1} t} - 1}{\lambda_{k_1}} u_0\right)\sqrt{\frac{2}{\ell}}\ \sin\left(\frac{k_1\pi\xi}{\ell}\right). \tag{5}$$

Assuming that a pointwise-in-space sensor is available to obtain measurements of (1)

$$y(t) = \int_0^{\ell} \delta(\xi - \xi_s)x(t,\xi)\ \mathrm{d}\xi = x(t,\xi_s), \tag{6}$$

where $\xi_s \in (0,\ell)$ is the sensor location. Using (5) in (6), the scalar measurement is now given by

$$y(t) = \left(e^{\lambda_{k_1} t} + \frac{e^{\lambda_{k_1} t} - 1}{\lambda_{k_1}} u_0\right)\sqrt{\frac{2}{\ell}}\ \sin\left(\frac{k_1\pi\xi_s}{\ell}\right).$$

Notice that if $\xi_s$ is equal to any of the zeros of the $k_1$th eigenfunction $\phi_{k_1}(\xi)$ (i.e., $\xi_s = m\ell/k_1$, $m \in \mathbb{Z}^+$, $m \geq k_1$), then

$$y(t) \equiv 0, \quad \forall t \geq 0.$$

In other words, the state $x(t,\xi)$ in (5) is nonzero, yet the measurement $y(t)$ is identical to zero for all times! Any observer for (1) with output (6) is rendered a naïve observer.

The above demonstration points to the realization that under certain conditions (inputs and initial conditions), it is possible that a single sensor placed at particular spatial locations can be ineffective in obtaining process information and any observer designed for such a process is useless.

A single sensor can be rendered ineffective in reconstructing the state of a PDE like (2). Consider now *two* sensors being available; such sensors can be placed in different regions within the spatial domain Ω. At this stage, one can consider either a single centralized observer using both sensors, *or* two collaborating observers using a sensor each. For the latter case, one does not expect that each of the two distributed observers will produce an identical estimate of the state $x(t,\xi)$. In fact, it will be demonstrated below, that each observer will be able to reconstruct "part" of the state. This portion of the state can be quantified in terms of the modes, or frequency content, present in the state estimates. Subsequent comparison via an appropriate consensus protocol will enable one to arrive at a state estimate which contains *all* modal information.

Toward this, consider the following case: the disturbance input excites two natural frequencies of (1) with the input term $\beta(\xi)u(t)$ given by

$$\beta(\xi)u(t) = \phi_{k_1}(\xi)u_1 + \phi_{k_2}(\xi)u_2, \quad k_1, k_2 \in \mathbb{Z}^+,$$

where $u_1, u_2$ are the constant amplitudes of the disturbance input terms. Using an initial condition $x_0(\xi) = x_1\phi_{k_1}(\xi) + x_2\phi_{k_2}(\xi)$, the solution to (1) is given by the truncated expansion (cf. (2))

$$x(t,\xi) = \alpha_{k_1}(t)\phi_{k_1}(\xi) + \alpha_{k_2}(t)\phi_{k_2}(\xi), \tag{7}$$

where the two modal states $\alpha_{k_1}, \alpha_{k_2}$ satisfy

$$\begin{aligned} \dot{\alpha}_{k_1}(t) &= \lambda_{k_1}\alpha_{k_1}(t) + u_1, \quad \alpha_{k_1}(0) = x_1, \\ \dot{\alpha}_{k_2}(t) &= \lambda_{k_2}\alpha_{k_2}(t) + u_2, \quad \alpha_{k_2}(0) = x_2. \end{aligned} \tag{8}$$

Their solution is given by

$$\alpha_{k_1}(t) = e^{\lambda_{k_1}t}x_1 + \frac{e^{\lambda_{k_1}t} - 1}{\lambda_{k_1}}u_1, \quad \alpha_{k_2}(t) = e^{\lambda_{k_2}t}x_2 + \frac{e^{\lambda_{k_2}t} - 1}{\lambda_{k_2}}u_2.$$

Two pointwise sensors are assumed available to provide process information, and using (7), we have

$$\begin{aligned} y_1(t) &= \int_0^\ell \delta(\xi - \xi_1)x(t,x)\,\mathrm{d}\xi = \alpha_{k_1}(t)\phi_{k_1}(\xi_1) + \alpha_{k_2}(t)\phi_{k_2}(\xi_1), \\ y_2(t) &= \int_0^\ell \delta(\xi - \xi_2)x(t,x)\,\mathrm{d}\xi = \alpha_{k_1}(t)\phi_{k_1}(\xi_2) + \alpha_{k_2}(t)\phi_{k_2}(\xi_2). \end{aligned}$$

If the first sensor is placed at a location $\xi_1$ which is a zero of $\phi_{k_2}(\xi)$ (implying $\phi_{k_2}(\xi_1) \equiv 0$), then

$$y_1(t) \equiv \alpha_{k_1}(t)\phi_{k_1}(\xi_1).$$

Similarly, if the second sensor is placed at a location $\xi_2$ which is a zero of $\phi_{k_1}(\xi)$ (with $\phi_{k_1}(\xi_2) \equiv 0$), then

$$y_2(t) \equiv \alpha_{k_2}(t)\phi_{k_2}(\xi_2).$$

In other words, each sensor obtains state information representing a particular mode. Neither of the two sensors can single-handedly obtain process information containing both modes. Collectively though, either in a centralized filter or via two collaborating distributed filters, they can reconstruct the process state. If in a centralized architecture, they result in a modal observer as was first presented in Reference 24.

If we associate a state estimate $\widehat{x}_1(t,x)$ to $y_1(t)$, then

$$\frac{\partial \widehat{x}_1}{\partial t} = \kappa \frac{\partial^2 \widehat{x}_1}{\partial \xi^2} + \beta(\xi)u(t) + L_1(\xi)\left( y_1(t) - \int_0^{\ell} \delta(\xi-\xi_1)\widehat{x}_1(t,\xi)\, \mathrm{d}\xi \right), \tag{9}$$

where $L_1(\xi)$ is the kernel of the adjoint of the observer gain for the first observer. The observer state admits a similar expansion

$$\widehat{x}_1(t,\xi) = \widehat{\alpha}^1_{k_1}(t)\phi_{k_1}(\xi) + \widehat{\alpha}^1_{k_2}(t)\phi_{k_2}(\xi). \tag{10}$$

Closer examination of the innovation term $y_1(t) - \int_0^{\ell} \delta(\xi-\xi_1)\widehat{x}_1(t,\xi)\, \mathrm{d}\xi$, reveals that

$$\varepsilon_1(t) = y_1(t) - \int_0^{\ell} \delta(\xi-\xi_1)\widehat{x}_1(t,\xi)\, \mathrm{d}\xi = \alpha_{k_1}(t)\phi_{k_1}(\xi_1) - \widehat{\alpha}^1_{k_1}(t)\phi_{k_1}(\xi_1). \tag{11}$$

Using (9), (10), and (11) and selecting the test function first as $\phi_{k_1}(\xi)$ and then as $\phi_{k_2}(\xi)$, the state observer associated with $y_1(t)$ can be decomposed into

$$\dot{\widehat{\alpha}}^1_{k_1}(t) = \lambda_{k_1}\widehat{\alpha}^1_{k_1}(t) + \varepsilon_1(t)\int_0^{\ell} L_1(\xi)\phi_{k_1}(\xi)\, \mathrm{d}\xi, \tag{12a}$$

$$\dot{\widehat{\alpha}}^1_{k_2}(t) = \lambda_{k_2}\widehat{\alpha}^1_{k_2}(t) + \varepsilon_1(t)\int_0^{\ell} L_1(\xi)\phi_{k_2}(\xi)\, \mathrm{d}\xi. \tag{12b}$$

At this stage, one can consider a simplification in the design of the filter gain kernel $L_1(\xi)$. If one sets

$$\int_0^{\ell} L_1(\xi)\phi_{k_2}(\xi)\, \mathrm{d}\xi = 0,$$

that is, $L_1(\xi)$ is designed to be orthogonal to $\phi_{k_2}(\xi)$, then (12) simplifies to

$$\begin{aligned} \dot{\widehat{\alpha}}^1_{k_1}(t) &= \lambda_{k_1}\widehat{\alpha}^1_{k_1}(t) + \varepsilon_1(t)\int_0^{\ell} L_1(\xi)\phi_{k_1}(\xi)\, \mathrm{d}\xi, \\ \dot{\widehat{\alpha}}^1_{k_2}(t) &= \lambda_{k_2}\widehat{\alpha}^1_{k_2}(t). \end{aligned} \tag{13}$$

The interpretation is that the observer associated with $y_1(t)$ can only learn about the frequency content of the state $x(t,\xi)$ associated with the mode $\phi_{k_1}(\xi)$ and thus is running a naïve observer for mode $\phi_{k_2}(\xi)$. If one imposes a similar condition for the observer associated with $y_2(t)$, then each of the two observers will be able to learn only of the mode associated with its own sensor. The result is to have a number of independent modal observers.

In a similar fashion, we associate a state estimate $\widehat{x}_2(t,\xi)$ with $y_2(t)$, then

$$\frac{\partial \widehat{x}_2}{\partial t} = \kappa \frac{\partial^2 \widehat{x}_2}{\partial \xi^2} + \beta(\xi)u(t) + L_2(\xi)\left( y_2(t) - \int_0^{\ell} \delta(\xi-\xi_2)\widehat{x}_2(t,\xi)\, \mathrm{d}\xi \right), \tag{14}$$

where $L_2(\xi)$ is the kernel of the adjoint of the observer gain for the second observer. The observer state $\widehat{x}_2(t,\xi)$ associated with (14) admits the expansion

$$\widehat{x}_2(t,\xi) = \widehat{\alpha}^2_{k_1}(t)\phi_{k_1}(\xi) + \widehat{\alpha}^2_{k_2}(t)\phi_{k_2}(\xi). \tag{15}$$

The innovation term in (14) is denoted as

$$\varepsilon_2(t) = \alpha_{k_2}(t)\phi_{k_2}(\xi_2) - \widehat{\alpha}^2_{k_2}(t)\phi_{k_2}(\xi_2), \tag{16}$$

and the observer associated with $y_2(t)$ has the decomposition

$$\dot{\hat{\alpha}}_{k_1}^2(t) = \lambda_{k_1}\hat{\alpha}_{k_1}^2(t) + \varepsilon_2(t)\int_0^{\ell} L_2(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi, \tag{17a}$$

$$\dot{\hat{\alpha}}_{k_2}^2(t) = \lambda_{k_2}\hat{\alpha}_{k_2}^2(t) + \varepsilon_2(t)\int_0^{\ell} L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi. \tag{17b}$$

Similar to the first observer, a further simplification via the assumption that $L_2(\xi)$ is orthogonal to $\phi_{k_1}(\xi)$, leads to

$$\begin{aligned}\dot{\hat{\alpha}}_{k_1}^2(t) &= \lambda_{k_1}\hat{\alpha}_{k_1}^2(t),\\ \dot{\hat{\alpha}}_{k_2}^2(t) &= \lambda_{k_2}\hat{\alpha}_{k_2}^2(t) + \varepsilon_2(t)\int_0^{\ell} L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{18}$$

and which shows that an observer associated with $y_2(t)$ can only learn about the frequency content of the state $x(t,\xi)$ associated with the mode $\phi_{k_2}(\xi)$ and thus is running a naïve observer for the mode $\phi_{k_1}(\xi)$.

The two observers (13) and (18) can collectively learn about the process state (7), but *neither* of them can individually learn completely the process state (7). These observers are also heterogeneous, in the sense that only the first component of the observer associated with sensor group #1 is learning about the $k_1$th mode of the process (7), (8) while its second component is running a naïve observer. Similarly, the observer associated with sensor group #2 has its second component learning about the $k_2$th mode of the process (7), (8) while its first component is running a naïve observer.

The synchronization task at hand is to modify (13) so that $\hat{\alpha}_{k_2}^1$ can follow $\hat{\alpha}_{k_2}^2$ in (18). Similarly, (18) must be modified so that $\hat{\alpha}_{k_1}^2$ can follow $\hat{\alpha}_{k_1}^1$ in (13).

## 2.1 Error analysis

Subtracting the first component (12a) of (12) from the first component of (8), one arrives at the $k_1$th *modal estimation error*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_{k_1} - \hat{\alpha}_{k_1}^1\right) = \left(\lambda_{k_1} - \phi_{k_1}(\xi_1)\int_0^{\ell} L_1(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi\right) \times \left(\alpha_{k_1} - \hat{\alpha}_{k_1}^1\right). \tag{19}$$

If one selects the filter operator gain (operator associated with the kernel $L_1(\xi)$) to be equal to a constant multiple of the adjoint of the output operator for $y_1(t)$, then

$$\int_0^{\ell} L_1(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi = \int_0^{\ell} m_1\delta(\xi-\xi_1)\phi_{k_1}(\xi)\,\mathrm{d}\xi = m_1\phi_{k_1}(\xi_1),$$

simplifying (19) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_{k_1} - \hat{\alpha}_{k_1}^1\right) = \left(\lambda_{k_1} - m_1\phi_{k_1}^2(\xi_1)\right)\left(\alpha_{k_1} - \hat{\alpha}_{k_1}^1\right). \tag{20}$$

In the absence of inputs, one can establish the exponential convergence of the modal error $\alpha_{k_1} - \hat{\alpha}_{k_1}^1$ to zero. Similarly, when the kernel $L_2(\xi)$ is equal to a constant multiple of the adjoint of the output operator for $y_2(t)$, then

$$\int_0^{\ell} L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi = m_2\phi_{k_2}(\xi_2).$$

The above when used with the second component (12b) of (12) and the second component of (8), produces the evolution of the $k_2$th modal estimation error

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_{k_2} - \hat{\alpha}_{k_2}^2\right) = \left(\lambda_{k_2} - m_2\phi_{k_2}^2(\xi_2)\right)\left(\alpha_{k_2} - \hat{\alpha}_{k_2}^2\right). \tag{21}$$

It is easily seen that the modal estimation errors (20), (21) converge to zero. The first observer (9), or its modal decomposition (12), partially reconstructs the state $x(t,x)$ of (1). In fact, it captures all the state characteristics associated with the first mode $\lambda_{k_1}$ *only*. Similarly, the second observer (14), or its modal decomposition (17), captures all the state characteristics associated with the second mode $\lambda_{k_2}$ *only*. Each observer captures half of the state information, but not the same half. The second half of the first observer, given by (12b), and the first half of the second observer given by (17a), cannot provide any estimates of their respective modal component.

To address this, a modification is proposed in (12b) and (17a), and which takes the form of a consensus protocol. Equation (12b) is now modified to include a penalty on the difference $\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}$ and whose goal is to ensure that (12b) learns about the second modal component from the successful observer (17b). It now becomes

$$\dot{\hat{\alpha}}^1_{k_2}(t) = \lambda_{k_2}\hat{\alpha}^1_{k_2}(t) + \varepsilon_1(t)\int_0^\ell L_1(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi + f_{21}, \tag{22}$$

where the term $f_{21}$ is to be selected in order for the difference $\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}$ to converge to zero. Subtracting (22) from (17b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right) &= \lambda_{k_2}\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right) - f_{21} + \varepsilon_2(t)\int_0^\ell L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi - \varepsilon_1(t)\int_0^\ell L_1(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi \\ &= \left(\lambda_{k_2} - m_2(\phi_{k_2}(\xi_2))^2\right)\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right) - f_{21} + \left(\alpha_{k_2} - \hat{\alpha}^1_{k_2}\right)\phi_{k_2}(\xi_2)\int_0^\ell L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi - \varepsilon_1(t)\int_0^\ell L_1(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Let us examine the terms above in order to select $f_{21}$ accordingly. The term $(\alpha_{k_2} - \hat{\alpha}^1_{k_2})\phi_{k_2}(\xi_2)\int_0^\ell L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi$ is simplified

$$\left(\alpha_{k_2} - \hat{\alpha}^1_{k_2}\right)\phi_{k_2}(\xi_2)\int_0^\ell L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi = \left(\alpha_{k_2} - \hat{\alpha}^1_{k_2}\right)\phi_{k_2}(\xi_2)m_2\phi_{k_2}(\xi_2).$$

The term $\alpha_{k_2}\phi_{k_2}(\xi_2) = y_2(t)$ and the term $\hat{\alpha}^1_{k_2}\phi_{k_2}(\xi_2) = C_2\hat{x}_1$. If $L_1(\xi)$ is normal to $\phi_{k_2}(\xi)$, then the obvious choice for $f_{21}$ is

$$f_{21} = \left(y_2(t) - C_2\hat{x}_1(t,\xi)\right) + q_{21}\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right), \qquad \int_0^\ell L_1(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi = 0, \tag{23}$$

and which results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right) = \left(\lambda_{k_2} - m_2(\phi_{k_2}(\xi_2))^2 - q_{21}\right)\left(\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}\right).$$

This provides the desired convergence of $\hat{\alpha}^1_{k_2}$ to $\hat{\alpha}^2_{k_2}$ and subsequently to $\alpha_{k_2}$ since $|\alpha_{k_2} - \hat{\alpha}^1_{k_2}| \le |\alpha_{k_2} - \hat{\alpha}^2_{k_2}| + |\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}|$. It should be noted that the convergence rate of the disagreement error $\hat{\alpha}^2_{k_2} - \hat{\alpha}^1_{k_2}$, via the choice of $q_{21}$ is larger than the convergence rate of the modal estimation error $\alpha_{k_2} - \hat{\alpha}^2_{k_2}$ in (21).

In a similar fashion, the modal consensus observer is implemented for (17a). This observer is modified to

$$\dot{\hat{\alpha}}^2_{k_1}(t) = \lambda_{k_1}\hat{\alpha}^2_{k_1}(t) + \varepsilon_2(t)\int_0^\ell L_2(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi + f_{12}, \tag{24}$$

where

$$f_{12} = \left(y_1(t) - C_1\hat{x}_2(t,x)\right) + q_{12}\left(\hat{\alpha}^1_{k_1} - \hat{\alpha}^2_{k_1}\right), \qquad \int_0^\ell L_2(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi = 0. \tag{25}$$

As was similarly noted above, the convergence rate of the disagreement error $\hat{\alpha}^2_{k_1} - \hat{\alpha}^1_{k_1}$, via the choice of $q_{12}$ is larger than the convergence rate of the modal estimation error $\alpha_{k_1} - \hat{\alpha}^1_{k_1}$ in (17).

Summarizing the above by updating (12) and (17) using (23), (25) we have

$$\begin{cases} \dot{\widehat{\alpha}}^1_{k_1}(t) = \lambda_{k_1}\widehat{\alpha}^1_{k_1}(t) + \varepsilon_1(t)\int_0^{\ell} L_1(\xi)\phi_{k_1}(\xi)\,\mathrm{d}\xi, \\ \dot{\widehat{\alpha}}^1_{k_2}(t) = \lambda_{k_2}\widehat{\alpha}^1_{k_2}(t) + \big(y_2(t) - C_2\widehat{x}_1(t,x)\big) + q_{21}\left(\widehat{\alpha}^2_{k_2} - \widehat{\alpha}^1_{k_2}\right), \end{cases} \tag{26}$$

and

$$\begin{cases} \dot{\widehat{\alpha}}^2_{k_1}(t) = \lambda_{k_1}\widehat{\alpha}^2_{k_1}(t) + \big(y_1(t) - C_1\widehat{x}_2(t,x)\big) + q_{12}\left(\widehat{\alpha}^1_{k_1} - \widehat{\alpha}^2_{k_1}\right), \\ \dot{\widehat{\alpha}}^2_{k_2}(t) = \lambda_{k_2}\widehat{\alpha}^2_{k_2}(t) + \varepsilon_2(t)\int_0^{\ell} L_2(\xi)\phi_{k_2}(\xi)\,\mathrm{d}\xi. \end{cases} \tag{27}$$

Both (26) and (27) must be written in terms of the PDEs (9) and (14) and not their modal components. This is presented in the next section for the more general class of PDEs described by Riesz-spectral systems.

## 3 | GENERALIZATION TO RIESZ-SPECTRAL SYSTEMS

To consider a family of PDEs that exhibit similar behavior to the above diffusion PDE, we view these PDEs as evolution equations in an appropriate Hilbert space. The interplay and relationships between the time derivative of the state and the various spatial derivatives will be absorbed into the definition of the state operator which will absorb the boundary conditions through the definition of its domain. The premise in this case is that two clusters, or groups, of sensors are available to provide process information. The first sensor group can collectively obtain process information over a particular modal content; similarly, the second sensor group can collectively obtain process information over a different modal content. Such a modal content may or may not be the complement of the frequency content of the first sensor group. These sensor groups may be selected to contain frequency content or frequency range, for example, the first sensor group may contain frequencies below a threshold and the second sensor group may contain frequencies above the frequency threshold. Or, the first sensor group may contain the frequencies of the even-numbered eigenfunctions and the second sensor group may contain the frequencies of the odd-numbered eigenfunctions, and so forth. Additionally, the first group may be able to observe "more" one frequency band and observe "less" another frequency band. The reverse can be assumed for the second sensor group and the difference on their process state information is on the relative observability. Neither of the two sensor groups will solely have a 100% observability over one frequency content and 0% over a different frequency content. The difference of the sensor groups is that one sensor group can attain higher observability levels over certain frequency content compared with the other sensor group. This concept of the level of *modal observability* will be clarified below. It essentially follows the definitions of modal observability first appeared in References 25,26 for flexible structures and then applied to parabolic PDEs in References 27.

We consider PDEs, like the one in (1), written as evolution equations in a Hilbert space $X$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1x(t) \\ C_2x(t) \end{bmatrix}, \end{aligned} \tag{28}$$

where the output operators $C_1, C_2$ represent multidimensional measurements associated with group #1 and group #2 of the eigenfunctions of the operator $A$. To allow for a more general class of input and primarily output operators, we consider the Gelfand space triple $V \hookrightarrow X \hookrightarrow V^*$ with the state operator $A \in \mathcal{L}(V, V^*)$ and $C_1 \in \mathcal{L}(V, Y_1)$, $C_2 \in \mathcal{L}(V, Y_2)$, where $Y_1, Y_2$ are the Euclidean output spaces of dimensions $n_1$ and $n_2$, respectively. These dimensions may or may not be the same, but both $n_1, n_2 \geq 1$.

The PDE in (1), as written as an evolution equation (28), can be solved independently in terms of the modes by taking advantage of the orthogonality property of the eigenfunctions (i.e., modes) of the spatial operator $A$ in (28).[23] The Hilbert space $X$, equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{\Omega} \phi_1^*(\xi)\phi_2(\xi)\,\mathrm{d}\xi,$$

yields

$$\langle \phi_i, A\phi_j \rangle = \int_\Omega \phi_i^*(\xi) A\phi_j(\xi)\, \mathrm{d}\xi = \lambda_j \delta_{ij},$$

where $\delta_{ij}$ denotes the Kronecker delta, and $\lambda_j$ the $j$th eigenvalue, cf. $\lambda_i$ in (3), (4). When the eigenfunctions are used as a basis function set for the Hilbert space $X$, the spatially varying solution to an equation of the form (28) can be written as

$$x(t,\xi) = \sum_{i=1}^{\infty} x_i(t)\phi_i(\xi).$$

These two sensor groups represent two bands of frequencies. The first group of frequencies is associated with the output operator $C_1$ and is such that the level of *modal observability* of the sensors in group #1 is significantly above a certain modal observability threshold. Similarly, the second group of frequencies is associated with the output operator $C_2$ and is such that the level of *modal observability* of the sensors in group #2 is significantly above a certain modal observability threshold. However, it should be emphasized again that both groups may observe (almost) all frequencies but their level of modal observability may not necessarily exceed the threshold; each sensor group may observe certain frequencies "more" than other frequencies and this is quantified in terms of the level of modal observability defined below.

To better understand the effects of sensor locations on the ability of a sensor to measure a given mode and the amount at which it can measure a given mode, we first visit certain definitions from Reference 27. The two sensor groups in (28) are given in detail

$$y_i(t) = \begin{bmatrix} \int_\Omega c_{i,1}(\xi) x(t,\xi)\, \mathrm{d}\xi \\ \vdots \\ \int_\Omega c_{i,n_i}(\xi) x(t,\xi)\, \mathrm{d}\xi \end{bmatrix}, \quad i = 1, 2,$$

where $c_{i,j}(\xi)$ is the spatial distribution of the $j$th sensor of the $i$th sensor group, $j = 1, \ldots, n_i$, $i = 1, 2$. With regard to (6), we identify $c_{1,1}(\xi) = \delta(\xi - \xi_s)$. To quantify how much a given sensor, described by $c_{i,j}(\xi)$, can measure a specific mode, we borrow some definitions on modal observability from Reference 27. However, the concept of $\mathbb{H}_2$ spatial observability is not considered and instead a slightly different definition of modal observability is presented.

To distinguish between the collective notion of observability of a sensor group and the observability of a given sensor within the sensor group, the spatial distribution of the sensing devices is further parameterized by the sensor locations. Thus the spatial functions $c_{i,j}(\xi)$ are further parameterized by the sensor locations and the process measurements are written as

$$y_{ij}(t;\xi_{sj}) = \int_\Omega c_{i,j}(\xi;\xi_{sj}) x(t,\xi)\, \mathrm{d}\xi, \quad j = 1, \ldots, n_i, i = 1, 2.$$

The vector of sensor locations is denoted by $\boldsymbol{\xi}_{si} = \{\xi_{s1}, \ldots, \xi_{sn_i}\} \in \Omega_{si}$, $i = 1, 2$, where the domain of admissible sensor locations is $\Omega_{si} \subset \prod^{n_i}\Omega$.

The first definition quantifies the ability of a sensor, described by $c_{i,j}(\xi;\xi_s)$ to effectively measure a specific mode of (28). For the $i$th sensor group and the $k$th mode, define

$$f_k^i(\xi_s) = \left\| \begin{pmatrix} \int_\Omega c_{i,1}(\xi;\xi_{s1})\phi_k(\xi)\, \mathrm{d}\xi \\ \vdots \\ \int_\Omega c_{i,n_i}(\xi;\xi_{sn_i})\phi_k(\xi)\, \mathrm{d}\xi \end{pmatrix} \right\|, \quad i = 1, 2.$$

The $j$th component of $f_k^i(\boldsymbol{\xi}_s)$ is given by

$$f_{k,j}^i(\xi_{sj}) = \left| \int_\Omega c_{i,j}(\xi;\xi_{sj})\phi_k(\xi)\, \mathrm{d}\xi \right|, \quad i = 1, 2,$$

and it is easily established that for the $k$th mode we have

$$f_k^i(\boldsymbol{\xi}_s) = \sqrt{\sum_{j=1}^{n_i} (f_{k,j}^i(\xi_{sj}))^2}, \quad i = 1, 2.$$

Now, to have an assessment of the measuring ability of a given sensor group over a specific mode, we consider the *modal observability*.

**Definition 1** ($k$ th modal observability). The $k$ th modal observability of the $i$ th sensor group at the sensor location $\boldsymbol{\xi}_s$ is defined as

$$\mathcal{O}_k^i(\boldsymbol{\xi}_s) = \frac{f_k^i(\boldsymbol{\xi}_s)}{\max\limits_{\xi \in \Omega_{si}} f_k^i(\boldsymbol{\xi})}, \quad i = 1, 2, \quad k = 1, 2, \dots .$$

The modal observability describes the ability of the $i$th sensor group that is placed at the sensor location $\boldsymbol{\xi}_s$, to measure the $k$th mode.

A value of 1, or 100%, for a given $\mathcal{O}_k^i(\boldsymbol{\xi}_s)$ indicates that the $i$th sensor group, placed at location $\boldsymbol{\xi}_s$, can measure the $k$th mode completely. Such a measurement of the $k$th mode is collectively and is not necessarily attributed to a particular sensor within the sensor group.

A related definition considers the component of the sensor group, that is, a particular sensor, and its ability to measure a specific mode.

**Definition 2** ($k$th modal observability of $j$th sensor). The $k$ th modal observability of the $j$th sensor at location $\xi_{sk}$ within the $i$ th sensor group is defined as

$$\mathcal{O}_{k,j}^i(\xi_{sj}) = \frac{f_{k,j}^i(\xi_{sj})}{\max\limits_{\xi \in \Omega} f_{k,j}^i(\xi)}, \quad i = 1, 2, \quad k = 1, 2, \dots .$$

It is easily established by the definition of the Euclidean norm in $\mathbb{R}^{n_i}$ that

$$f_k^i(\xi_s) = \sqrt{\sum_{j=1}^{n_i} \left(f_{k,j}^i(\xi_{sj})\right)^2}, \quad i = 1, 2.$$

One can view the definition of $k$th Modal Observability as the summation of the $k$th modal observability of $j$th sensor over all sensors within the sensor group. One can subsequently consider a summation of any of the above two quantities over all modes.

Since we will be predominantly working with a finite set of modes, as they represent the salient dynamics of the process, then a measure of how a sensor within a sensor group or, a how sensor group can measure these modes is necessitated.

**Definition 3** (cumulative spatial observability). The cumulative spatial observability of the $i$th sensor group is defined for the first $N$ modes and is given by

$$\mathcal{S}_N^i(\boldsymbol{\xi}_s) = \frac{\sqrt{\sum_{k=1}^N \left(f_k^i(\boldsymbol{\xi}_s)\right)^2}}{\max\limits_{\xi \in \Omega_{si}} \sqrt{\sum_{k=1}^N \left(f_k^i(\boldsymbol{\xi})\right)^2}}, \quad i = 1, 2.$$

In a similar fashion, one can define the *component spatial observability* which corresponds to the observability of a single sensor at a spatial location $\xi_{sj}$ over the first $N$ modes. It is the normalized sum of squares of the $k$th modal observabilities of $j$th sensor over the first $N$ modes.

*Remark* 1. In the above definition of cumulative spatial observability, one can define it over a specific set of modes $\mathcal{N}_i$. In this case, the associated cumulative spatial observability is given by

$$\mathcal{S}_{\mathcal{N}_i}(\xi_s) = \frac{\sqrt{\sum_{k\in\mathcal{N}_i}\left(f_k^i(\xi_s)\right)^2}}{\max\limits_{\xi\in\Omega_{si}}\sqrt{\sum_{k\in\mathcal{N}_i}\left(f_k^i(\xi)\right)^2}}, \quad i = 1, 2.$$

**Example 1.** Consider the diffusion equation (1) in $[0, \ell] = [0, 1]$ with eigenfunctions (3). If a single ($j = 1 = n_1 = n_2$) pointwise sensor is placed at the location $\xi_s = 0.5$, then

$$C\phi_k = \int_0^1 \delta(\xi - 0.5)\phi_k(\xi)\,\mathrm{d}\xi = \phi_k(0.5).$$

In this case $f_{k,1}(\xi_{s,1}) = \left|\phi_k(0.5)\right|$, and

$$O^1_{k,1}(\xi_{s,1}) = \frac{f_{k,1}(\xi_{s,1})}{\max\limits_{\xi_s\in[0,1]} f_{k,1}(\xi_s)} = \frac{\left|\phi_k(0.5)\right|}{\max\limits_{\xi_s\in[0,1]}\left|\phi_k(\xi_s)\right|}.$$

When only the first $N = 4$ modes are of interest, then the cumulative spatial observability is

$$\mathcal{S}^1_4(\xi_s) = \frac{\sqrt{\sum_{k=1}^4\left|\phi_k(\xi_s)\right|^2}}{\max\limits_{\xi\in\Omega}\sqrt{\sum_{k=1}^4\left|\phi_k(\xi)\right|^2}} = \frac{\sqrt{\left|\phi_1(\xi_s)\right|^2 + \left|\phi_2(\xi_s)\right|^2 + \left|\phi_3(\xi_s)\right|^2 + \left|\phi_4(\xi_s)\right|^2}}{\max\limits_{\xi\in[0,1]}\sqrt{\left|\phi_1(\xi)\right|^2 + \left|\phi_2(\xi)\right|^2 + \left|\phi_3(\xi)\right|^2 + \left|\phi_4(\xi)\right|^2}}.$$

When the sensor location is $\xi_s = 0.25$, then $\mathcal{S}^1_4(0.25) = 0.8669$, whereas $\mathcal{S}^1_4(0.50) = 0.8513$. This is depicted graphically in Figure 1, where the cumulative spatial observability $\mathcal{S}_{\mathcal{N}}(\xi_s)$ $(=\mathcal{S}^1_4(\xi_s))$, is plotted versus the sensor location $\xi_s \in (0, 1)$. The highest value occurs at $\xi_s = 0.16$ and $\xi_s = 0.84$, meaning that a sensor placed in either location will have the highest value of the cumulative spatial observability for *all* four modes. If there is only a single sensor group having a single sensing device with $\mathcal{N} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, then the best location for $\xi_s$ will be either of the two values.

Next, two sensor groups are considered containing different modes. First, group #1 having a single sensor is associated with the first two modes $\mathcal{N}_1 = \{\phi_1, \phi_2\}$ and group #2 also having a single sensor is associated with the modes $\mathcal{N}_2 = \{\phi_3, \phi_4\}$. The associated cumulative spatial observability for these two sets $\mathcal{S}_{\mathcal{N}_i}(\xi_s)$ is plotted against the sensor location $\xi_s$ and presented in Figure 2 along with a desired threshold of 85%; that is, consider sensor locations that ensure $\mathcal{S}_{\mathcal{N}_i}(\xi_s) \geq 0.85$. For the first sensor group, the best sensor location is at $\xi_s = 0.29$ (and also $\xi_s = 0.71$), since $\mathcal{S}_{\mathcal{N}_1}(0.29) = 1$. This means that if one wants to observe modes $\phi_1$ and $\phi_2$ by a single sensor, the best possible location is $\xi_s = 0.29$. For group #2, the best sensor location is $\xi_s = 0.14$ (and also $\xi_s = 0.86$) since $\mathcal{S}_{\mathcal{N}_2}(0.14) = 1$. A sensor placed at $\xi_s = 0.14$ will have the highest modal observability of modes $\phi_3$ and $\phi_4$; if these two modes are dominant, then the associated filter gain will be the smallest possible of all sensor locations. Notice that such a sensor will also be able to observe the other two modes, $\phi_1, \phi_2$ since $\mathcal{S}_{\mathcal{N}_1}(0.14) = 0.65$.

When the sets $\mathcal{N}_i$ are changed, then different spatial locations appear to be better suited for each new $\mathcal{N}_i$, $i = 1, 2$. Figure 3 depicts the cumulative spatial observability for $\mathcal{N}_1 = \{\phi_1, \phi_3\}$ and $\mathcal{N}_2 = \{\phi_2, \phi_4\}$ versus the sensor location $\xi_s$.

The highest cumulative spatial observability for $\mathcal{N}_1$ is at $\xi_s = 0.5$ having $\mathcal{S}_{\mathcal{N}_1}(0.5) = 1$. This means that an observer for a system with $\phi_1, \phi_3$ being the dominant modes, will be able to reconstruct them *completely* with the smallest possible filter gain. Similarly, the highest cumulative spatial observability for $\mathcal{N}_2$ is at $\xi_s = 0.15$ (also at $\xi_s = 0.35$, or $\xi_s = 0.65$ and $\xi_s = 0.85$) having $\mathcal{S}_{\mathcal{N}_2}(0.15) = 1$. If the system is exited by the even-numbered modes, then this sensor location is the best ever to reconstruct these two modes.

When each mode is considered individually, as depicted in Figure 4, one considers the case of having four sensor groups with a single sensor in each with $\mathcal{N}_1 = \{\phi_1\}$, $\mathcal{N}_2 = \{\phi_2\}$, $\mathcal{N}_3 = \{\phi_3\}$, and $\mathcal{N}_4 = \{\phi_4\}$. An optimal sensor location for one group may be the worst for another group; for example, at $\xi_s = 0.5$, the modes $\phi_1$ and $\phi_3$ have the best modal observability, but at the same time represents a zero observability for the modes $\phi_2$ and $\phi_4$.

As it turns out, one can group the modes in different sets (e.g., the first two and last two, or odd-numbered and even-numbered modes) and select the sensor location, or locations, that have a desired level of modal observability for that modal group. In this particular example, it appears that grouping them into the odd-numbered and even-numbered

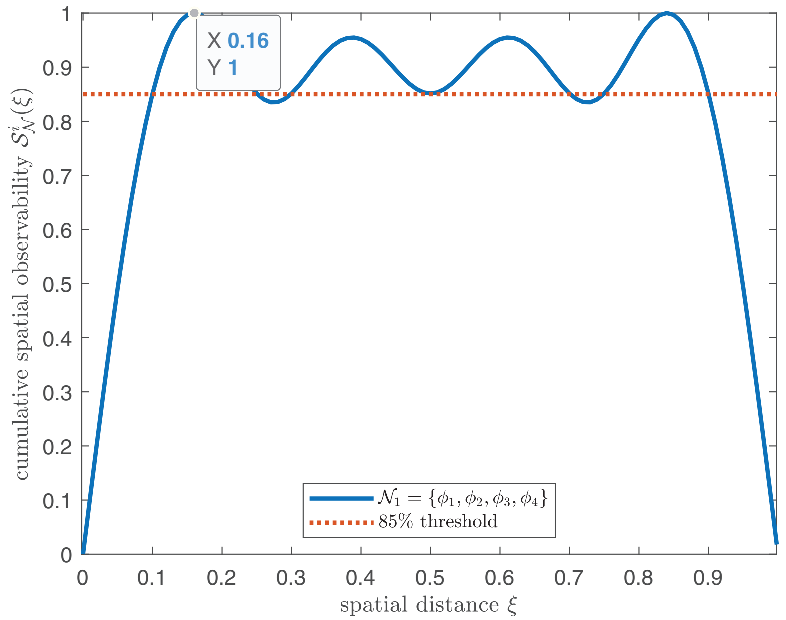


**FIGURE 1** Cumulative spatial observability for $N = 4$ modes with $\mathcal{N}_1 = \{\phi_1, \phi_2, \phi_3, \phi_4\}$

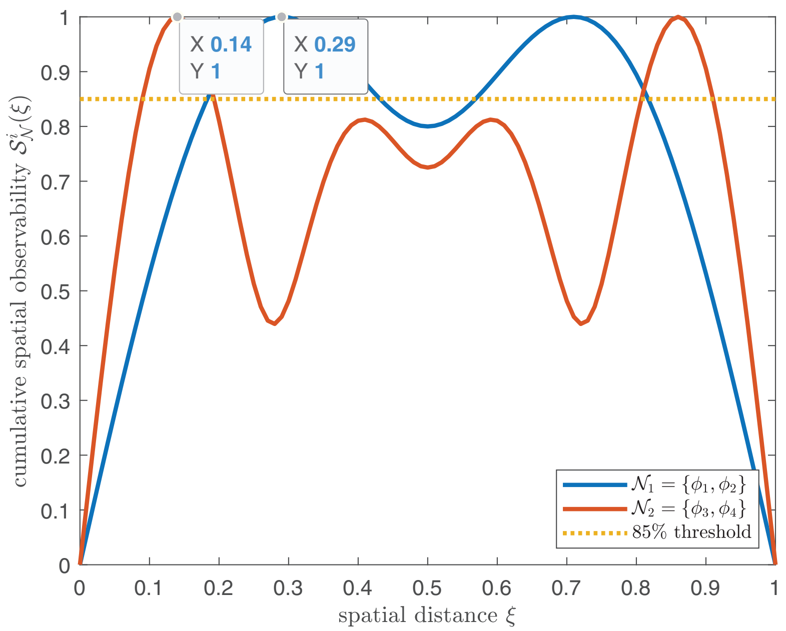


**FIGURE 2** Cumulative spatial observability for $N = 4$ modes with $\mathcal{N}_1 = \{\phi_1, \phi_2\}$ and $\mathcal{N}_2 = \{\phi_3, \phi_4\}$

modes is the better policy since the even-numbered and odd-numbered modes share their zeros. Following Figure 3, a sensor placed at $\xi_s = 0.5$ can handle the odd-numbered modes and another sensor placed at $\xi_s = 0.15$ (or at 0.35, 0.65, 0.85) can handle the even-numbered modes. Examining Figure 4, a sensor placed at $\xi_s = 0.5$ will be able to reconstruct all odd-numbered modes for this process. If the appropriate conditions are present, for example, initial conditions and spatial distribution of input/disturbance, results in a response containing only the odd-numbered modes, then this sensor can reconstruct all of them. However, if the conditions are such that only the even-numbered modes appear in the state response, then the sensor placed at $\xi_s = 0.5$ will be ineffective.

We let $P_1 : X \to X_1$ be the orthogonal projection operator associated with the first sensor group, represented by $C_1$. Similarly, we define $P_2 : X \to X_2$ be the orthogonal projection operator associated with the second sensor group,

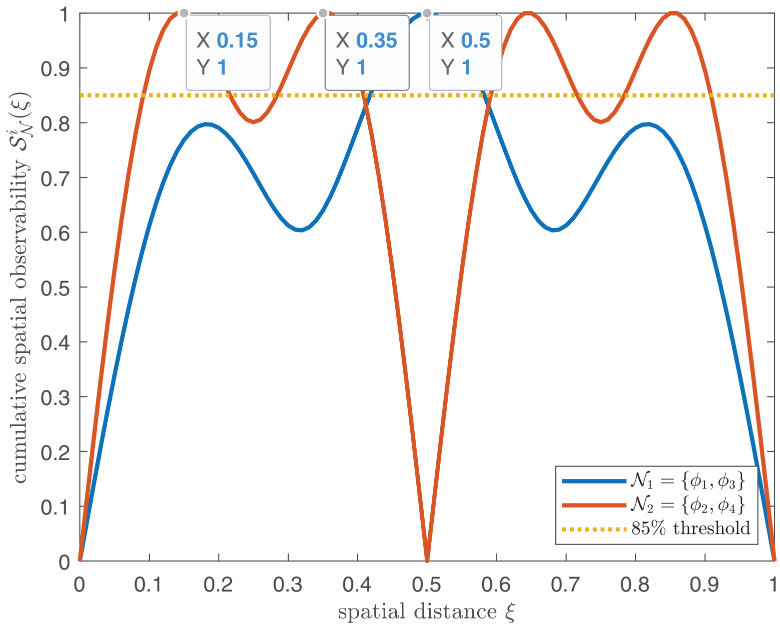


**FIGURE 3** Cumulative spatial observability for $N = 4$ modes with $\mathcal{N}_1 = \{\phi_1, \phi_3\}$ and $\mathcal{N}_2 = \{\phi_2, \phi_4\}$

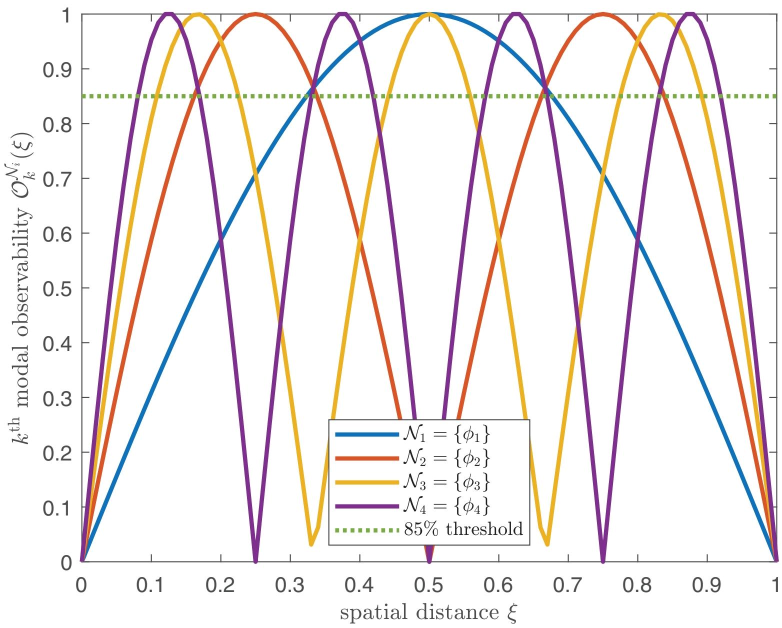


**FIGURE 4** Modal observability for $N = 4$ modes with $\mathcal{N}_i = \{\phi_i\}, i = 1, 2, 3, 4$

represented by $C_2$. The state $x \in X$ admits the following decomposition $x = P_1x + P_2x = x_1 + x_2$. The projection operators have the following properties

- $C_1(P_1x) = C_1x_1$
- $\langle P_1x, \Phi \rangle = \langle x_1\Phi_1 \rangle = \langle x, \Phi_1 \rangle$

  and

- $C_2(P_2x) = C_2x_2$
- $\langle P_2x, \Phi \rangle = \langle x_2, \Phi_2 \rangle = \langle x, \Phi_2 \rangle$

*Remark* 2. Please note that the above do not imply that $C_1x_2 = 0$ and $C_2x_1 = 0$. Through the relevant definitions of modal observability, it will be shown that $C_1x_1$ is "larger" than $C_2x_1$ in an appropriate sense; similarly $C_2x_2$ is "larger" than $C_1x_2$.

Examining the above system in weak form, with the test functions $\Phi = \Phi_1 + \Phi_2$, we have that (28) is equivalently written as

$$\langle \dot{x}_1, \Phi_1 \rangle = \langle Ax_1, \Phi_1 \rangle + \langle Bu, \Phi_1 \rangle , \tag{29a}$$

$$y_1 = C_1 x, \tag{29b}$$

$$\langle \dot{x}_2, \Phi_2 \rangle = \langle Ax_2, \Phi_2 \rangle + \langle Bu, \Phi_2 \rangle , \tag{29c}$$

$$y_2 = C_2 x. \tag{29d}$$

The first observer, associated with the state $x_1(t)$ in (29a) and output $y_1(t)$ in (29b), is

$$\dot{\hat{x}}_1 = A\hat{x}_1 + L_1 \left( y_1 - C_1\hat{x}_1 \right) + Bu, \tag{30}$$

which is subsequently decomposed into the two components

$$\left\langle \dot{\hat{x}}_{1,1}, \Phi_1 \right\rangle = \left\langle A\hat{x}_{1,1}, \Phi_1 \right\rangle + \left\langle L_1 \left( y_1 - C_1\hat{x}_{1,1} \right), \Phi_1 \right\rangle + \langle Bu, \Phi_1 \rangle , \tag{31a}$$

$$\left\langle \dot{\hat{x}}_{1,2}, \Phi_2 \right\rangle = \left\langle A\hat{x}_{1,2}, \Phi_2 \right\rangle + \left\langle L_1 \left( y_1 - C_1\hat{x}_{1,2} \right), \Phi_2 \right\rangle + \langle Bu, \Phi_2 \rangle . \tag{31b}$$

Similarly, the second observer, associated with the state $x_2(t)$ in (29c) and output $y_2(t)$ in (29d), is given by

$$\dot{\hat{x}}_2 = A\hat{x}_2 + L_2 \left( y_2 - C_2\hat{x}_2 \right) + Bu, \tag{32}$$

which is subsequently decomposed into the two components

$$\left\langle \dot{\hat{x}}_{2,1}, \Phi_1 \right\rangle = \left\langle A\hat{x}_{2,1}, \Phi_1 \right\rangle + \left\langle L_2 \left( y_2 - C_2\hat{x}_{2,1} \right), \Phi_1 \right\rangle + \langle Bu, \Phi_1 \rangle , \tag{33a}$$

$$\left\langle \dot{\hat{x}}_{2,2}, \Phi_2 \right\rangle = \left\langle A\hat{x}_{2,2}, \Phi_2 \right\rangle + \left\langle L_2 \left( y_2 - C_2\hat{x}_{2,2} \right), \Phi_2 \right\rangle + \langle Bu, \Phi_2 \rangle . \tag{33b}$$

To better understand the selection of the operator filter gains $L_1, L_2$ and the subsequent modifications for modal consensus terms, consider the first part of the first observer, given by (31a). Comparing it with (29a), we have

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left( x_1 - \hat{x}_{1,1} \right), \Phi_1 \right\rangle = \left\langle (A - L_1C_1) \left( x_1 - \hat{x}_{1,1} \right), \Phi_1 \right\rangle .$$

This immediately provides the first condition imposed on the filter operator $L_1$: it should be such that $A - L_1C_1$ generates an exponentially stable $C_0$ semigroup on $X$. This is attained by requiring the pair $(A, C_1)$ be *approximately observable*.[28] Thus, the first part of the first observer (30) is able to reconstruct the first part of the state, as described by (29a). The second part of the observer (30), as given by (31b) will not be able to reconstruct the second part $x_2$ of the process state in (29c). In a similar fashion to the example in Section 2, it will be augmented with a consensus protocol in order to synchronize with $\hat{x}_{2,2}$.

Now, we move to the second observer (32). Comparing (33b) with (29c), we have

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left( x_2 - \hat{x}_{2,2} \right), \Phi_2 \right\rangle = \left\langle (A - L_2C_2) \left( x_2 - \hat{x}_{2,2} \right), \Phi_2 \right\rangle .$$

It is obvious that imposing $(A, C_2)$ be approximately observable, the operator $A - L_2C_2$ will generate an exponentially stable $C_0$ semigroup on $X$ and thus the difference $(x_2 - \hat{x}_{2,2})$ will converge (in norm) to zero asymptotically. The first part

**TABLE 1** Convergence properties of modal observers (31) and (33)

| Observer (31) | Observer (33) |
|---|---|
| $\hat{x}_{1,1} \to x_1$ | $\hat{x}_{2,1} \not\to x_1$ |
| $\hat{x}_{1,2} \not\to x_2$ | $\hat{x}_{2,2} \to x_2$ |

of the observer (32), as given by (33a) will not be able to reconstruct the first part $x_1$ of the process state in (29a). However, it will be augmented with a consensus protocol in order to synchronize with $\hat{x}_{1,1}$.

Summarizing, we have Table 1 which lead to the particular consensus design requirements:

- the design must ensure $\hat{x}_{1,2}$ is synchronized with $\hat{x}_{2,2}$,
- the design must ensure $\hat{x}_{2,1}$ is synchronized with $\hat{x}_{1,1}$.

These requirements are *one-directional*: $\hat{x}_{2,2}$ does not need to follow $\hat{x}_{1,2}$, and $\hat{x}_{1,1}$ does not need to follow $\hat{x}_{2,1}$. Such modal consensus couplings are viewed as carefully projected consensus protocols that ensure the monodirectional synchronization, and are presented next.

## 3.1 | Deriving modal consensus protocols

To derive the additional terms needed in (31b) to attain the necessary synchronization ($\hat{x}_{1,2} \to \hat{x}_{2,2}$), we compare (33b) to (31b) where we also include the unknown *modal consensus term* $f_{12}$ in the right-hand side of (31b)

$$\begin{aligned}\left\langle \dot{\hat{x}}_{2,2} - \dot{\hat{x}}_{1,2}, \Phi_2 \right\rangle &= \left\langle A(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle + \left\langle L_2\left(y_2 - C_2\hat{x}_{2,2}\right), \Phi_2 \right\rangle - \left\langle L_1\left(y_1 - C_1\hat{x}_{1,2}\right), \Phi_2 \right\rangle - \langle f_{21}, \Phi_2 \rangle \\ &= \left\langle (A - L_2C_2)(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle + \left\langle L_2(y_2 - C_2\hat{x}_{1,2}), \Phi_2 \right\rangle - \left\langle L_1\left(y_1 - C_1\hat{x}_{1,2}\right), \Phi_2 \right\rangle - \langle f_{12}, \Phi_2 \rangle .\end{aligned}$$

In the above, we have

- the term $L_2(y_2 - C_2\hat{x}_{1,2})$ is available, since $y_2$ is the output from sensor group # 2. The term $C_2\hat{x}_{1,2}$ is artificial but available since it is equal to the second component of the first observer (30) evaluated at the sensor locations of group #2. This of course requires that the observer associated with sensor group #1 be aware of the output matrix $C_2$ of sensor group #2.
- the operator $A - L_2C_2$ generates an exponentially stable $C_0$ semigroup since the pair $(A, C_2)$ is approximately observable.
- the term $L_1\left(y_1 - C_1\hat{x}_{1,2}\right)$ can be made available since it is equal to a weighted multiple (by the operator $L_1$) of the difference (innovation) $y_1 - C_1\hat{x}_{1,2}$. Alternatively, if the design of $L_1$ permits it, it can be selected so that $\langle L_1, \Phi_2 \rangle = 0$.

The above considerations prompt the selection of the term $f_{12}$ above as

$$\langle f_{12}, \Phi_2 \rangle = \left\langle L_2(y_2 - C_2\hat{x}_{1,2}), \Phi_2 \right\rangle - \left\langle L_1\left(y_1 - C_1\hat{x}_{1,2}\right), \Phi_2 \right\rangle - \left\langle F_{12}(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle ,$$

or if the design of $L_1$ permits it, as

$$\langle f_{21}, \Phi_2 \rangle = \left\langle L_2(y_2 - C_2\hat{x}_{1,2}), \Phi_2 \right\rangle - \left\langle F_{12}(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle . \tag{34}$$

In either expression, the consensus operator $F_{12}$ is selected so that the resulting modal disagreement error

$$\left\langle \dot{\hat{x}}_{2,2} - \dot{\hat{x}}_{1,2}, \Phi_2 \right\rangle = \left\langle (A - L_2C_2 - F_{12})(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle , \tag{35}$$

not only converges to zero asymptotically, but does so with a rate higher than that of the operator $A - L_2C_2$. This will ensure that the difference $\hat{x}_{2,2} - \hat{x}_{1,2}$ converges to zero faster than the difference $x_2 - \hat{x}_{2,2}$ does.

*Remark* 3. The above consensus protocol (34) imposes a heavy communication load from the second observer (32) to the first observer (30). In particular, the observer (33b) must transmit its state $\hat{x}_{2,2}$ to the observer (31b) in order to realize the coupling $\left\langle F_{12}(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle$. An alternate is so assume the specific structure of $F_{12} = F_1C_2$ and in this case the coupling term requires only the $n_2$-dimensional measurement $C_2\hat{x}_{2,2}$ to be realized and transmitted, thus resulting in

$$\left\langle \dot{\hat{x}}_{2,2} - \dot{\hat{x}}_{1,2}, \Phi_2 \right\rangle = \left\langle (A - L_2C_2 - F_1C_2)(\hat{x}_{2,2} - \hat{x}_{1,2}), \Phi_2 \right\rangle . \tag{36}$$

The updated observer (31), which now includes a consensus protocol needed to ensure that only $\hat{x}_{1,2}$ follows $\hat{x}_{2,2}$, is given by

$$\left\langle \dot{\hat{x}}_{1,1}, \Phi_1 \right\rangle = \left\langle A\hat{x}_{1,1}, \Phi_1 \right\rangle + \left\langle L_1 \left(y_1 - C_1\hat{x}_{1,1}\right), \Phi_1 \right\rangle + \langle Bu, \Phi_1 \rangle \tag{37a}$$

$$\begin{aligned}\left\langle \dot{\hat{x}}_{1,2}, \Phi_2 \right\rangle = {} & \left\langle A\hat{x}_{1,2}, \Phi_2 \right\rangle + \left\langle L_1 \left(y_1 - C_1\hat{x}_{1,2}\right), \Phi_2 \right\rangle + \left\langle L_2(y_2 - C_2\hat{x}_{1,2}), \Phi_2 \right\rangle - \left\langle L_1 \left(y_1 - C_1\hat{x}_{1,2}\right), \Phi_2 \right\rangle \\ & + \langle Bu, \Phi_2 \rangle - \left\langle F_1(C_2\hat{x}_{1,2} - C_2\hat{x}_{2,2}), \Phi_2 \right\rangle .\end{aligned} \tag{37b}$$

The above modification in (37b) ensures that $\hat{x}_{1,1} \to x_1$ and $\hat{x}_{1,2} \to \hat{x}_{2,2} \to x_2$. However, the entire expression (37) must be written in the single-observer form of (30). With the aid of the projection operators, this is given by

$$\boxed{\dot{\hat{x}}_1 = A\hat{x}_1 + L_1 \left(y_1 - C_1\hat{x}_1\right) + Bu + P_2r_1}, \tag{38}$$

where the modal consensus signal is given by

$$r_1 = L_2(y_2 - C_2\hat{x}_{1,2}) - L_1 \left(y_1 - C_1\hat{x}_{1,2}\right) - F_{12}(\hat{x}_{1,2} - \hat{x}_{2,2}). \tag{39}$$

Please note that $P_2r_1$ only contributes to the second component of $\hat{x}_1$ as given by (37b). It has no contribution to the first part given by (37a).

In a similar fashion, one can update the observer (33) associated with sensor group #2 to

$$\begin{aligned}\left\langle \dot{\hat{x}}_{2,1}, \Phi_1 \right\rangle = {} & \left\langle A\hat{x}_{2,1}, \Phi_1 \right\rangle + \left\langle L_2 \left(y_2 - C_2\hat{x}_{2,1}\right), \Phi_1 \right\rangle + \left\langle L_1(y_1 - C_1\hat{x}_{2,1}), \Phi_1 \right\rangle - \left\langle L_2 \left(y_2 - C_2\hat{x}_{2,1}\right), \Phi_1 \right\rangle , \\ & + \langle Bu, \Phi_1 \rangle - \left\langle F_2(C_1\hat{x}_{2,1} - C_1\hat{x}_{1,1}), \Phi_1 \right\rangle ,\end{aligned} \tag{40a}$$

$$\left\langle \dot{\hat{x}}_{2,2}, \Phi_2 \right\rangle = \left\langle A\hat{x}_{2,2}, \Phi_2 \right\rangle + \left\langle L_2 \left(y_2 - C_2\hat{x}_{2,2}\right), \Phi_2 \right\rangle + \langle Bu, \Phi_2 \rangle . \tag{40b}$$

In a single-observer form, it is given by

$$\boxed{\dot{\hat{x}}_2 = A\hat{x}_2 + L_2 \left(y_2 - C_2\hat{x}_2\right) + Bu + P_1r_2}, \tag{41}$$

where the modal consensus signal is given by

$$r_2 = L_1 \left(y_1 - C_1\hat{x}_{2,1}\right) - L_2 \left(y_2 - C_2\hat{x}_{2,1}\right) - F_{21} \left(\hat{x}_{2,1} - \hat{x}_{1,1}\right) . \tag{42}$$

Once again, we note that $P_1r_2$ only contributes to the first component of $\hat{x}_2$ as given by (40a). It has no contribution to the second part given by (40b).

## 3.2 | Design procedure

The modal observers (37) and (40) ensure that with the appropriate design of the filter gains $L_1$ and $L_2$, one has $\hat{x}_{1,1} \to x_1$ and $\hat{x}_{2,2} \to x_2$ as $t \to \infty$. Through the additional modal consensus protocols given in (38), (39) and (41), (42), one has $\hat{x}_{1,2} \to \hat{x}_{2,2}$ and $\hat{x}_{2,1} \to \hat{x}_{1,1}$ as $t \to \infty$.

This means that through the selective consensus protocols given by the modal consensus terms (39) and (42) one has that the two observers (38) and (41) can reconstruct the process state. Now it remains to provide the details leading to the design of (38) and (41). We denote by $\mathcal{N}_1$ to be the set of modes associated with group #1 and by $\mathcal{N}_2$ to be the set of modes associated with group #2; in other words $\mathcal{N}_1 = \{\phi_i : P_1\phi_i = \phi_i,\ i = 1, 2, \dots, \}$ with $\mathcal{N}_2$ similarly defined. The vector of sensor locations associated with sensor group #1 is denoted by $\xi_{s1} = \left[\xi_{s11}, \xi_{s12}, \dots, \xi_{s1n_1}\right]$ and the vector of sensor locations associated with sensor group #2 is denoted by $\xi_{s2} = \left[\xi_{s21}, \xi_{s22}, \dots, \xi_{s2n_2}\right]$.

1. Given (31), decompose the modes into group #1 and group #2. Any element $\Phi$ of the state space $X = X_1 \bigoplus X_2$ is written as $\Phi = \Phi_1 + \Phi_2$, with $\Phi_1 \in X_1$ and $\Phi_2 \in X_2$. Group #1 of modes ensures that the output operator $C_1$ has a cumulative spatial observability with a prescribed threshold $O_1$, that is,

$$\mathcal{S}_{\mathcal{N}_1}(\xi_{s1}) \geq O_1, \quad \mathcal{S}_{\mathcal{N}_2}(\xi_{s1}) \ll O_1. \tag{43}$$

   Equivalently stated, one selects the sensor locations for group #1 so that the output operator $C_1$ is such that the pair $(A, C_1)$ is approximately observable and (43) is satisfied. Similarly, group #2 of modes ensures that the output operator $C_2$ has a modal observability with a prescribed threshold $O_2$, that is,

$$\mathcal{S}_{\mathcal{N}_2}(\xi_{s2}) \geq O_2, \quad \mathcal{S}_{\mathcal{N}_1}(\xi_{s2}) \ll O_2. \tag{44}$$

2. Given the pairs $(A, C_1)$ and $(A, C_2)$, design the filter operator gains $L_1$ and $L_2$ such that $A - L_1C_1$ and $A - L_2C_2$ are the generators of exponentially stable $C_0$ semigroups on $X$.
3. Set up the modal consensus observers (38) and (41) using (39) and (42) to compute the consensus coupling. To realize the term $C_2\hat{x}_{1,2}$ in (37b), the output operator (equivalently the sensor locations) of group #2 must be made available to sensor group #1. Similarly, to realize the term $C_1\hat{x}_{2,1}$ in (40a), the output operator of group #1 must be made available to sensor group #2. To reduce communication costs, the consensus operator $F_{12}$ can be selected as

$$F_{12} = F_1C_2, \tag{45}$$

   with the property that the spectrum bound of the semigroup generated by $A - L_2C_2 - F_1C_2$ is larger than the spectrum bound of the semigroup generated by $A - L_2C_2$. Similarly, the consensus operator $F_{21}$ can be selected as

$$F_{21} = F_2C_1, \tag{46}$$

   with the property that the spectrum bound of the semigroup generated by $A - L_1C_1 - F_2C_1$ is larger than the spectrum bound of the semigroup generated by $A - L_1C_1$.

The modal consensus observers are summarized in Table 2.

**TABLE 2** Summary of modal consensus observers

| Sensor groups | Modal consensus observer equations |
|---|---|
| Sensor group #1 | $\dot{\hat{x}}_1 = A\hat{x}_1 + L_1\left(y_1 - C_1\hat{x}_1\right) + Bu + P_2r_1$ |
| | $r_1 = L_2(y_2 - C_2\hat{x}_{1,2}) - L_1\left(y_1 - C_1\hat{x}_{1,2}\right) - F_{12}(\hat{x}_{1,2} - \hat{x}_{2,2})$ |
| | $F_{12} = F_1C_2\ :\ A - L_2C_2 - F_1C_2$ “more” stable than $A - L_2C_2$ |
| Sensor group #2 | $\dot{\hat{x}}_2 = A\hat{x}_2 + L_2\left(y_2 - C_2\hat{x}_2\right) + Bu + P_1r_2$ |
| | $r_2 = L_1(y_1 - C_1\hat{x}_{2,1}) - L_2\left(y_2 - C_2\hat{x}_{2,1}\right) - F_{21}(\hat{x}_{2,1} - \hat{x}_{1,1})$ |
| | $F_{21} = F_2C_1\ :\ A - L_1C_1 - F_2C_1$ “more” stable than $A - L_1C_1$ |

## 3.3 | Well-posedness and convergence

One starts with the assumption that the process state (29) can be decomposed into (30) and that it is well-posed. Both the well-posedness of the modal observers (38), (39) and (41), (42) and the resulting convergence to the process state (29a)–(29d) easily follow from basic theory of evolution equations of systems governed by state operators generating exponentially stable semigroups and is thus presented without a formal theoretical statement. The theoretical results are simply stated as a lemma at the end of this section.

First, the well posedness of the two components of the modal observers (37a) and (40b) is considered. Define the following modal errors as follows

$$e_{1,1} = x_1 - \hat{x}_{1,1}, \quad e_{2,2} = x_2 - \hat{x}_{2,2}. \tag{47}$$

Using (30) and (37a), (40b), we have

$$\langle \dot{e}_{1,1}, \Phi_1 \rangle = \langle (A - L_1C_1)e_{1,1}, \Phi_1 \rangle , \tag{48}$$

and

$$\langle \dot{e}_{2,2}, \Phi_2 \rangle = \langle (A - L_2C_2)e_{2,2}, \Phi_2 \rangle . \tag{49}$$

The above two error equations are well-posed if $\hat{x}_{1,1}(0) \in X_1$ and $\hat{x}_{2,2}(0) \in X_2$ with the errors converging weakly to zero due to the exponential stability of the semigroups generated by $A - L_1C_1$ and $A - L_2C_2$ on $X_1$ and $X_2$, respectively.

Now, to consider the remaining two modal components, define the remaining modal errors as follows

$$e_{1,2} = x_2 - \hat{x}_{1,2}, \quad e_{2,1} = x_1 - \hat{x}_{2,1}. \tag{50}$$

Using (47), (50), we have

$$\begin{aligned} e_{1,2} &= x_2 - \hat{x}_{1,2} = (x_2 - \hat{x}_{2,2}) + (\hat{x}_{2,2} - \hat{x}_{1,2}) \\ &= e_{2,2} + (\hat{x}_{2,2} - \hat{x}_{1,2}). \end{aligned}$$

Using (49) and (35), (45) (or (36)), along with the well-posedness of the process state (30), we can immediately obtain the well-posedness of (37b) with $\lim_{t\to\infty} \|e_{1,2}\| = 0$. Similarly, using (47), (50), we have the identity

$$\begin{aligned} e_{2,1} &= x_1 - \hat{x}_{2,1} = (x_1 - \hat{x}_{1,1}) + (\hat{x}_{1,1} - \hat{x}_{2,1}) \\ &= e_{1,1} + (\hat{x}_{1,1} - \hat{x}_{2,1}). \end{aligned}$$

Using the remaining modal observer equations (37a) and (40a) along with (30), (46), one arrives at (cf. (36))

$$\left\langle \dot{\hat{x}}_{1,1} - \dot{\hat{x}}_{2,1}, \Phi_2 \right\rangle = \left\langle (A - L_1C_1 - F_2C_1)(\hat{x}_{1,1} - \hat{x}_{2,1}), \Phi_2 \right\rangle . \tag{51}$$

Following (48), the modal error $e_{1,1}$ converges to zero due to the observer design, and the disagreement error $\hat{x}_{1,1} - \hat{x}_{2,1}$ governed by (51) converges to zero due to the modal consensus protocol designed. Thus, the error $e_{2,1}$ also converges to zero and therefore the proposed modal consensus observers (38) and (41) converge to the process state (31).

The above results are summarized in the lemma below.

**Lemma 1.** *Given the infinite dimensional system (28) decomposed into the two subsystems (29a)–(29d). Assume that the sensor locations in each sensor group are such that inequalities (43), (44) are satisfied for some prescribed user-defined observability thresholds $O_1, O_2$. Further assume that the pairs $(A, C_1)$ and $(A, C_2)$ are approximately observable and such that there exists filter gains $L_1$ and $L_2$ such that the operators $A - L_1C_1$ and $A - L_2C_2$ generate exponentially stable $C_0$ semigroups on X. Then the modal consensus observers given by (37) and (40), or by their compact formulations (38) and (41), along with the modal consensus protocols $r_1$, $r_2$, given by (39) and (42) are well-posed with*

$$\lim_{t\to\infty} \|x(t) - \hat{x}_1(t)\| = 0, \quad \lim_{t\to\infty} \|x(t) - \hat{x}_2(t)\| = 0.$$

*Additionally, the modal pairs $\hat{x}_{1,1}, \hat{x}_{2,1}$ and $\hat{x}_{2,2}, \hat{x}_{1,2}$ attain exponential synchronization in the sense of*

$$\lim_{t\to\infty} \|\hat{x}_{1,1} - \hat{x}_{2,1}\| = 0, \quad \lim_{t\to\infty} \|\hat{x}_{2,2} - \hat{x}_{1,2}\| = 0,$$

*with rate given by the spectrum bound of the semigroup generated by the operators $A - L_1C_1 - F_2C_1$ and $A - L_2C_2 - F_1C_2$, and which is larger than the spectrum bound of the semigroups generated by the operators $A - L_1C_1$ and $A - L_2C_2$, respectively.*

*Remark* 4 (communication cost reduction). Following Table 1, the modal disagreement errors $\hat{x}_{1,1} - \hat{x}_{2,1}$ and $\hat{x}_{2,2} - \hat{x}_{1,2}$ must converge to zero in order to achieve modal synchronization. Through the modal consensus protocols in (38), (39) and (41), (42) this was made possible. The additional requirement is that the consensus gains $F_{12}$ in (45) and $F_{21}$ in (46) can be selected so that the rate given by the spectrum bound of the semigroup generated by the operators $A - L_1C_1 - F_2C_1$ and $A - L_2C_2 - F_1C_2$, be made larger than the spectrum bound of the semigroups generated by the operators $A - L_1C_1$ and $A - L_2C_2$, respectively. Such a design minimizes the communication costs since in (38) the modal component $\hat{x}_{2,2}$ from sensor group #2 must be transmitted to the modal observer associated with sensor group #1. If the decomposition (45) can be realized, then only an $n_2$ dimensional signal must be transmitted. If this decomposition is not feasible, one must still impose the requirement that the spectrum bound from $A - L_2C_2 - F_{12}$ is larger than the spectrum bound from $A - L_2C_2$. In this case, the communication cost increase since now the entire state estimate $\hat{x}_{2,2}$ must be transmitted to the modal observer (37). A similar requirement is imposed on the gain $F_{21}$ where either an $n_1$ dimensional signal is transmitted from modal observer #1 to modal observer #2 or the entire modal estimate $\hat{x}_{1,1}$ must be transmitted to modal observer #2.

# 4 | NUMERICAL STUDIES

## 4.1 | 1D PDE

We consider the PDE in (1) over the spatial domain $[0, \ell] = [0, 1]$ and use the appropriate conditions to ensure that only the first four modes are present in the system response, as considered in Example 1. A thermal conductivity was used with a value $\kappa = 10^{-3}$.

The two sensors were selected so that one can cater to the odd-numbered eigenmodes $\sin(\frac{(2j+1)\pi\xi}{\ell})$, $j = 0, 1, \ldots$ and the other one to favor the even-numbered eigenmodes $\sin(\frac{2j\pi\xi}{\ell})$, $j = 1, 2, \ldots$, that is, $n_1 = 1$, $n_2 = 1$. Following Example 1, and in particular Figure 3, the location $\xi_{s1} = 0.50\ell$ was selected for group#1 and which provided a cumulative spatial observability level of 1, and the location $\xi_{s2} = 0.15\ell$ was selected for group#2 and which provided a cumulative spatial observability level of 1.

The initial condition for the process (1) was selected as

$$x(0, \xi) = \sin\left(\frac{\pi\xi}{\ell}\right) + 3\sin\left(\frac{2\pi\xi}{\ell}\right) + 6\sin\left(\frac{3\pi\xi}{\ell}\right) + 7\sin\left(\frac{4\pi\xi}{\ell}\right),$$

and the spatial distribution of the input was

$$\beta(\xi) = \sin\left(\frac{\pi\xi}{\ell}\right) + \sin\left(\frac{2\pi\xi}{\ell}\right) + \sin\left(\frac{3\pi\xi}{\ell}\right) + \sin\left(\frac{4\pi\xi}{\ell}\right).$$

To facilitate the consensus design, the filter gains were selected as $L_1 = P_1\left(20C_1^*\right)$ and $L_2 = P_2\left(20C_2^*\right)$. This simplified the consensus terms $r_1$ and $r_2$ in (39), (42), since $P_1L_2 = 0$ and $P_2L_1 = 0$. These two modal consensus terms were selected as

$$r_1 = L_2\left(y_2 - C_2\hat{x}_{1,2}\right) - 20\left(\hat{x}_{1,2} - \hat{x}_{2,2}\right), \quad r_2 = L_1\left(y_1 - C_1\hat{x}_{2,1}\right) - 20\left(\hat{x}_{2,1} - \hat{x}_{1,1}\right),$$

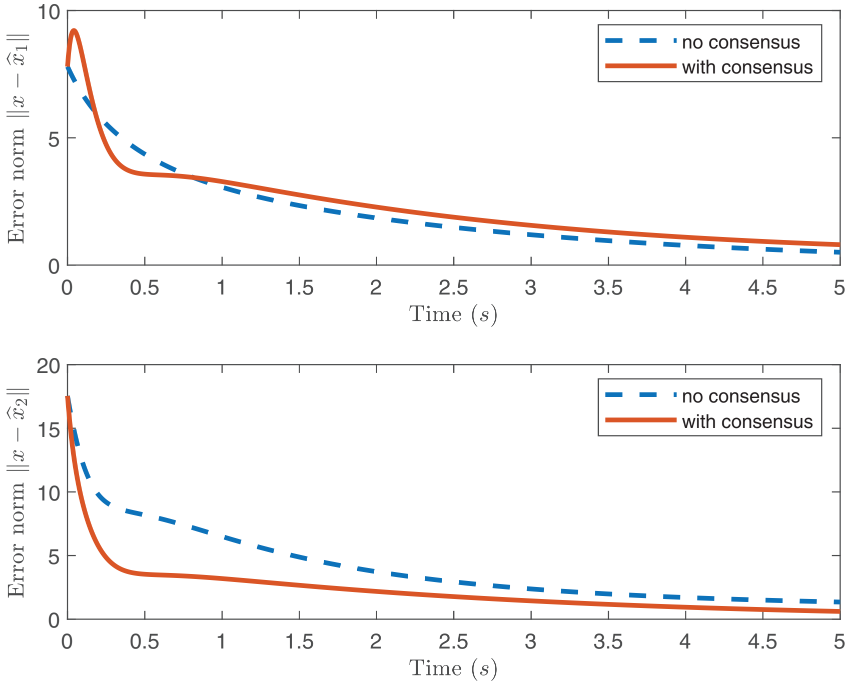


**FIGURE 5** 1D case: Evolution of estimation errors $\|x - \hat{x}_i\|$

with the consensus gains $F_{12} = F_{21} = 20\mathbf{I}$. Please note that as per Remark 4, this selection does not minimize the communication costs; however, the communication burden is not large since each modal observer must transmit a two-dimensional state to each other, as opposed to a scalar signal.

Figure 5 depicts the $L_2$ error norms $\|x - \hat{x}_i\|$, $i = 1, 2$ for the case with consensus and the case without any consensus terms (i.e., $r_1 = 0, r_2 = 0$ in (39), (42)). The inclusion of the consensus terms in (38) and (41) shows a slight improvement, especially for the second observer $\hat{x}_2$. However, the main thrust of this article is on the agreement of the two state estimates $\hat{x}_1$ and $\hat{x}_2$. Figure 6 demonstrates the success of the proposed modal consensus observers by depicting the evolution of the norm $\|\hat{x}_1 - \hat{x}_2\|$ representing the disagreement error, with and without consensus terms in (38) and (41). As expected, the use of consensus protocols, and in particular the modal consensus protocol, has a superior performance over the case of the observers without consensus. Finally, the spatial evolution of the disagreement error $\hat{x}_1(t, \xi) - \hat{x}_2(t, \xi)$ is depicted in Figure 7, where it is also observed that the disagreement has a faster pointwise convergence when a modal consensus protocol is implemented.

## 4.2 | 2D PDE

The diffusion PDE with state $x(t, \xi, \psi)$ in the 2D domain $\Omega = [0, \ell_X] \times [0, \ell_Y] = [0, 1] \times [0, 1]$ is given by

$$\frac{\partial x}{\partial t} = \kappa \left( \frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \psi^2} \right) + \beta(\xi, \psi) u(t), \quad y_i(t) = \begin{bmatrix} \int_\Omega c_{i,1}(\xi, \psi) x(t, \xi, \psi) \, \mathrm{d}\psi \mathrm{d}\xi \\ \vdots \\ \int_\Omega c_{i,n_i}(\xi, \psi) x(t, \xi, \psi) \, \mathrm{d}\psi \mathrm{d}\xi \end{bmatrix}, \quad i = 1, 2.$$

The mode shapes in this case are given by

$$\phi_{ij}(\xi, \psi) = \sqrt{\frac{2}{\ell_X}} \sin\left(\frac{i\pi\xi}{\ell_X}\right) \sqrt{\frac{2}{\ell_Y}} \sin\left(\frac{j\pi\xi}{\ell_X}\right),$$

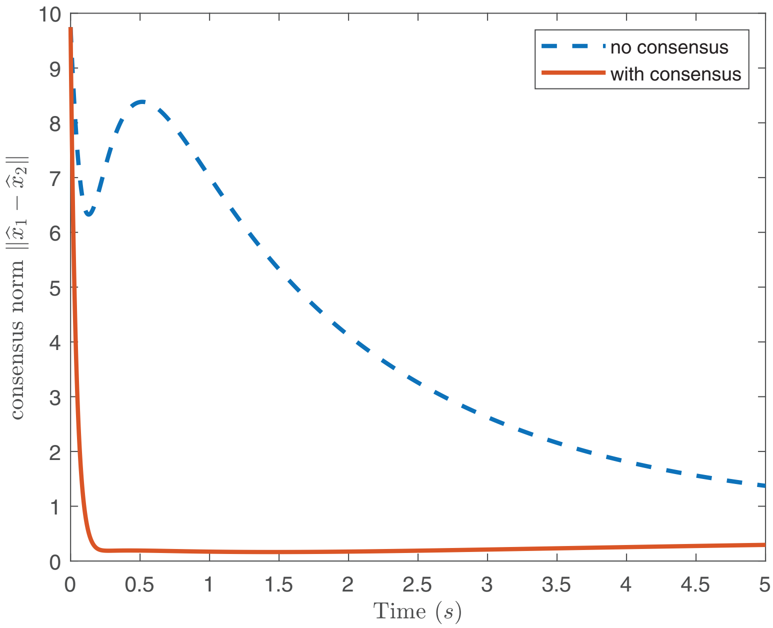


FIGURE 6 1D case: Evolution of disagreement error $\|\hat{x}_1 - \hat{x}_2\|$

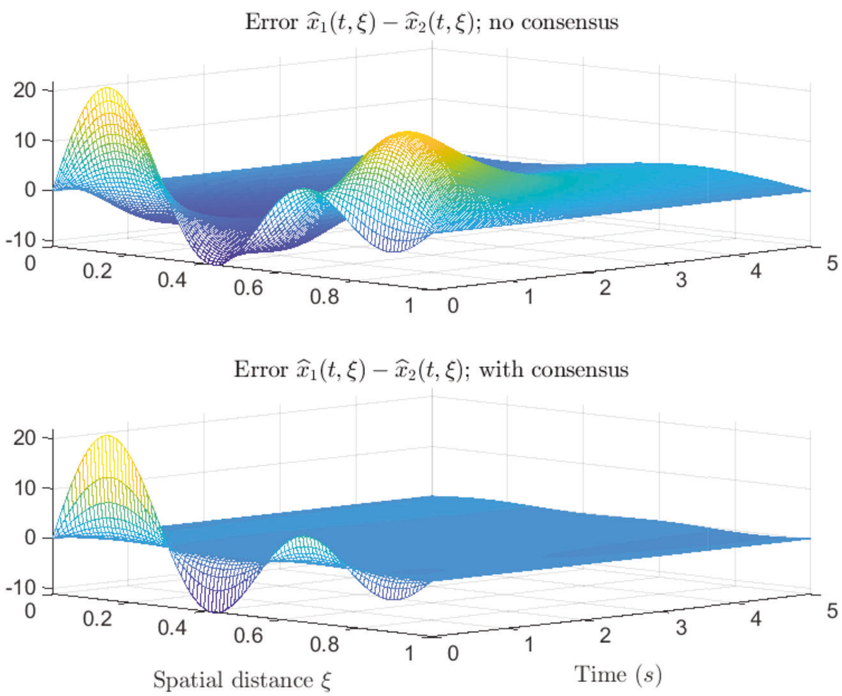


FIGURE 7 1D case: Distribution of estimation disagreement

with $i, j = 1, 2, \ldots$. For the particular study, it is assumed that the modes for $i = 1, 2$ and $j = 1, 2, 3$ are the dominant ones; that is, $\phi_{11}(\xi, \psi), \phi_{12}(\xi, \psi), \phi_{13}(\xi, \psi), \phi_{21}(\xi, \psi), \phi_{22}(\xi, \psi)$, and $\phi_{23}(\xi, \psi)$. These are depicted in Figure 8. The modal response is ensured by using the following input spatial distribution

$$\beta(\xi, \psi) = \sin(\pi\xi/\ell_X)\sin(2\pi\psi/\ell_Y) + \sin(2\pi\xi/\ell_X)\sin(\pi\psi/\ell_Y),$$

and initial condition

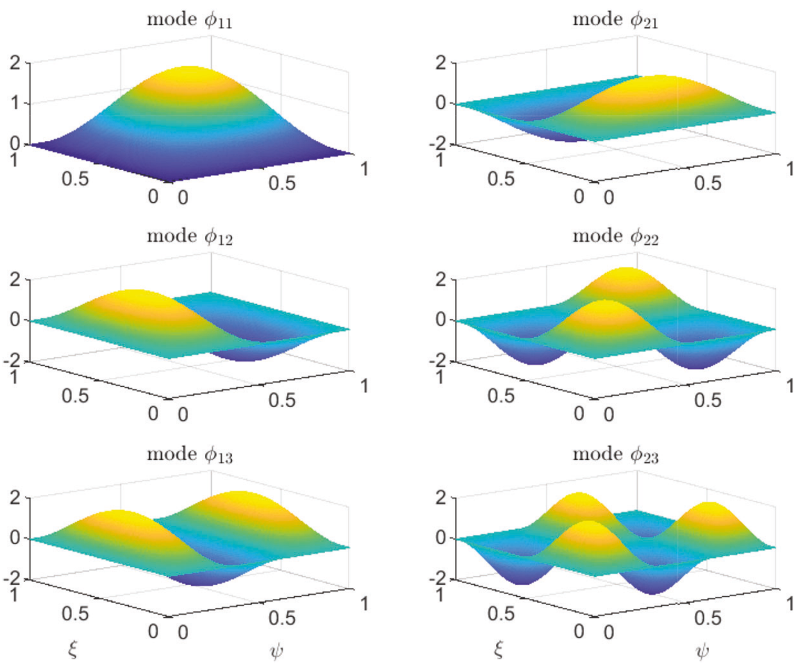


FIGURE 8 2D case: Modes for the 2D PDE

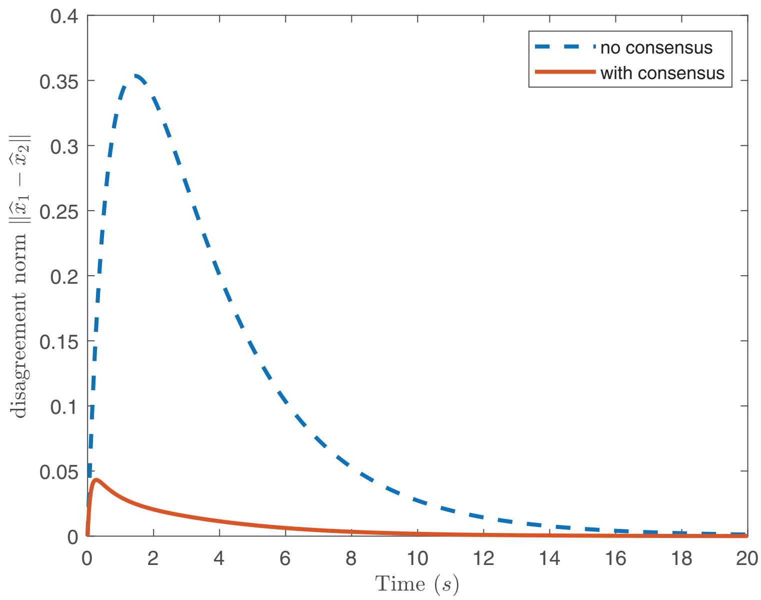


FIGURE 9 2D case: Evolution of disagreement error

$$x_0(\xi,\psi) = (\sin(\pi\xi/\ell_X) + \sin(2\pi\xi/\ell_X))\,(\sin(\pi\psi/\ell_Y) + \sin(2\pi\psi/\ell_Y) + \sin(3\pi\psi/\ell_Y))\,.$$

Using the 1D expressions for the cumulative spatial observability in each direction separately is not a suitable measure for sensor location, precisely because of the dependence of the modes on two spatial variables. Using the 2D version of the cumulative spatial observability for the two sets $\mathcal{N}_1 = \{\phi_{11}, \phi_{12}, \phi_{13}\}$ and $\mathcal{N}_2 = \{\phi_{21}, \phi_{22}, \phi_{23}\}$ with

$$\mathcal{S}_{\mathcal{N}_1}(\xi_{s1},\psi_{s1}) = \frac{\sqrt{|\phi_{11}(\xi_{s1},\psi_{s1})|^2 + |\phi_{12}(\xi_{s1},\psi_{s1})|^2 + |\phi_{13}(\xi_{s1},\psi_{s1})|^2}}{\max\limits_{\xi\in[0,1],\psi\in[0,1]} \sqrt{|\phi_{11}(\xi,\psi)|^2 + |\phi_{12}(\xi,\psi)|^2 + |\phi_{13}(\xi,\psi)|^2}},$$

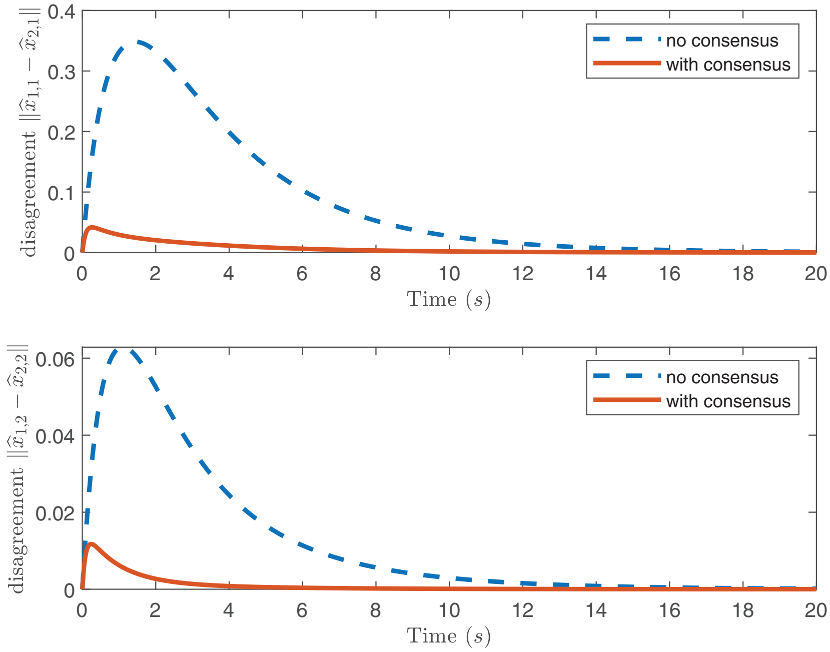


**FIGURE 10** 2D case: Evolution of disagreement error

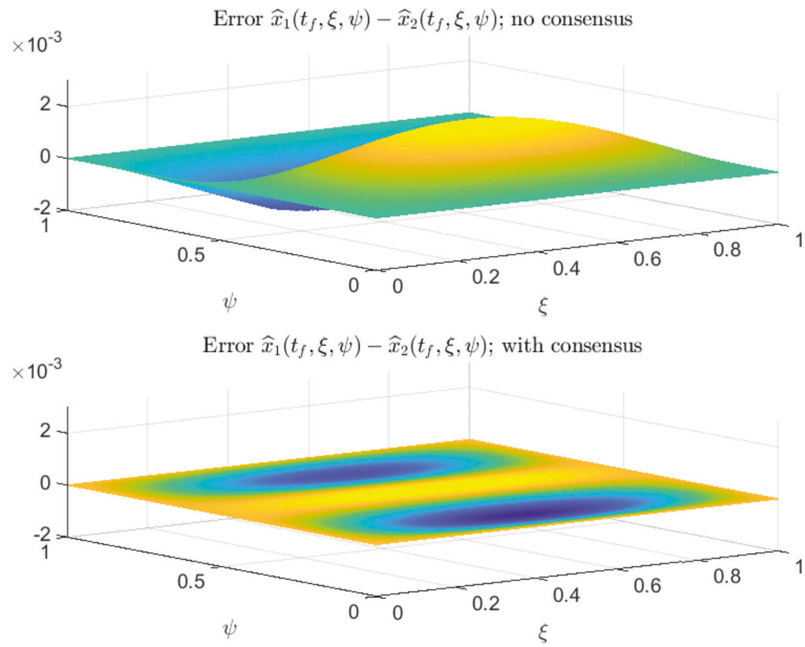


**FIGURE 11** 2D case: Spatial evolution of disagreement

$$S_{\mathcal{N}_2}(\xi_{s2},\psi_{s2}) = \frac{\sqrt{|\phi_{21}(\xi_{s2},\psi_{s2})|^2 + |\phi_{22}(\xi_{s2},\psi_{s2})|^2 + |\phi_{23}(\xi_{s2},\psi_{s2})|^2}}{\max\limits_{\xi\in[0,1],\psi\in[0,1]} \sqrt{|\phi_{21}(\xi,\psi)|^2 + |\phi_{22}(\xi,\psi)|^2 + |\phi_{23}(\xi,\psi)|^2}},$$

the sensor location for group #1 was selected at $(\xi_{s1},\psi_{s1}) = (0.5\ell_X, 5\ell_Y/12)$ and the sensor location for group #2 was selected at $(\xi_{s2},\psi_{s2}) = (0.1\ell_X, 5\ell_Y/6)$.

Figure 9 depicts the evolution of the disagreement error $\|\widehat{x}_1 - \widehat{x}_2\|$ where it is once again observed that the proposed modal consensus (39), (42) ensures agreement between the estimates $\widehat{x}_1$ and $\widehat{x}_2$.

In particular, the modal components of the disagreement errors $\hat{x}_{1,1} - \hat{x}_{2,1}$ governed by (51) and $\hat{x}_{1,2} - \hat{x}_{2,2}$ governed by (35) are depicted in Figure 10, and which provide an insight on the agreement of the modal components. Continuing, the spatial evolution at the final time $t_f = 20$ s is presented in Figure 11, where it is observed that inclusion of modal consensus significantly improves the agreement of the distributed observers.

## 5 | CONCLUSIONS AND OUTLOOK

This article addressed the problem of quality versus quantity of information exchanged among spatially distributed observers for partial differential equations. For infinite dimensional systems admitting a modal decomposition, the observers associated with different sensor groups can reconstruct the various modal components of the process state with different accuracy. This was precisely addressed in the design of the modal consensus observers whereby the appropriate information was judiciously transmitted from one modal observer to another in order for the modal components of the observer to agree with the corresponding modal components of the other distributed observer. Extensions enabled the reduction of communication costs among the modal observers by transmitting finite dimensional output signals as opposed to infinite dimensional state estimates.

The proposed modal consensus protocols were viewed as orthogonal projections of various consensus terms onto the appropriate subspaces related to the unobservable modes of the spatially distributed process. Continuing with possible extensions is to consider multiple distributed observers with a prescribed communication topology and the task at hand is to derive the appropriate modal consensus protocols for each of the modal observers within the confines of the communication topology.

### CONFLICT OF INTEREST

The authors declared that they have no conflict of interest to this work.

### DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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