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Unknown input functional observers for vector second order structural systems

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ABSTRACT

This paper considers a class of linear time invariant systems that describe the dynamics of mechanical systems. Due to their algebraic structure, the dynamics of such systems are written in their natural second order framework in order to exploit this structure with the obvious computational benefits in controller and observer design. A functional observer along with an unknown input observer are combined and are presented for this class of systems. The additional advantage of this combined observer is that when certain conditions are imposed, it reduces to the standard natural second observer. This translates to guaranteeing that the derivative of the estimated position vector coincides with the estimate of the velocity vector, a case not always ensured when such system is brought in a first order realization. An added benefit resulting from the second order formulation is the minimum order compensator whose order is dictated by the rank of the control input matrix, when the proposed functional estimate is used in place of a full state control signal.

1. Introduction

Dynamical systems, written in vector second order form describe the dynamics of flexible structures and their control and estimation has been the topic of research for quite some time. When written in the second order setting, termed as the “natural setting”, offer certain algebraic advantages and retain a direct link with the physical variables (e.g. displacement, velocity and acceleration).

When such systems are brought into a 1st order setting, in order to take advantage of system-theoretic results on estimation and control design, certain advantages are lost. One of these lost attributes has been pointed out first in [1] for single degree of freedom (SDOF) systems and in [2] for multi degree of freedom (MDOF) and infinite dimensional systems regarding the design of state estimators. The observer design for mechanical systems expressed in a 1st order setting result in a certain deficiency: *the time derivative of the estimated position component of the state vector is* not *equal to the estimated velocity component*. This issue was subsequently visited in [2,3] for distributed and lumped parameter systems; for the latter, the MDOF systems were considered with both position and velocity measurements. Consideration of acceleration measurements for the design of natural observers, along with an experimental verification was given in [4].

When a portion of the state is desired to be identified, as expressed via a linear functional of the state, one may use already established results [5] to design a *functional observer* of a mechanical system. However this entails the observer construction of a 1st order representation of the mechanical system. The algebraic advantages in such case are lost. To address this, a *natural* functional observer for mechanical systems expressed as vector second order systems is warranted.

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In the event that one may want to implement a controller and a static output feedback controller is not applicable, then a dynamic compensator must be considered. However, this would require the reconstruction of the entire state. It is possible to utilize a functional observer, where the functional is identical to a full state controller and the functional observer essentially estimates the control input. As a result, the functional observer, under mild assumptions, can be used in lieu of a compensator, providing significant computational savings in the form of a minimum order compensator.

This paper addresses the need for a natural *functional observer* (FO) and by imposing additional conditions, it also presents a natural *unknown input functional observer* (UIFO). Since the UIFO is the more general type of observer, it is examined first and via appropriate relaxations, it reduces to a FO, to an UIO or to a natural observer. Thus the reason to present an UIFO for vector 2nd order systems in the natural setting is to arrive at the most general observer that can revert to a natural observer when certain conditions are imposed/relaxed. Such a feat is not possible when one considers the 2nd system in a 1st order setting, design the UIFO, and then impose the conditions to revert to an observer; such an observer will not be a natural observer! A special case of a functional observer for vector 2nd systems which has 1st order observer dynamics is also considered. In all cases, by making appropriate assumptions, the natural UIFO is shown to reduce to either a natural FO or a natural UIO for vector 2nd systems. Extending the results to compensator design, the full state feedback takes the form of the functional to be identified, and thus the FO and UIFO are utilized for controller design, thus significantly reducing the computational burden and enabling the real time implementation of UIFO or FO-based feedback.

Contributions: The contributions of this paper are threefold: (i) it modifies earlier UIO results for vector 2nd systems in [6] to include velocity measurements; (ii) proposes a natural UIFO for vector 2nd systems and shows how to recover a natural FO, a natural UIO and a natural observer, by relaxing appropriate conditions; (iii) presents a minimum-order compensator when the functional output coincides with a full-state feedback controller. The proposed framework enables one to show that a natural UIO reverts to a natural observer, a natural FO reverts to a natural observer, and a natural UIFO reverts to a natural UIO or a natural FO. Special cases dealing with restrictive measurements and 1st order structure observers are provided.

The remainder of this paper is as follows: Section 2 summarizes prior results on UIFO as they pertain to 1st order systems. Additionally, the conditions for UIFO to become either an FO or an UIO are examined. Section 3 introduces the vector 2nd systems under consideration, and the natural UIO design for vector 2nd are summarized in Section 3.1. The proposed natural UIFO for vector 2nd systems is presented in Section 4 and is shown to revert to a natural FO, a natural UIO and a natural observer. An alternate FO for vector 2nd systems is presented in Section 5, which produces a 1st order FO. The use of the estimated functional as the full-state control input is given in Section 6. An example implementing an UIFO compensator for a 3 DOF system is presented in Section 7 and conclusions follow in Section 8.

2. Summary of UIFO's, UIO's and FO's for 1st order systems

The class of systems considered is described by the following LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu + Fd, \\ y &= Cx, \\ z &= Lx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state signal, $u \in \mathbb{R}^m$ denotes the control signal, $d \in \mathbb{R}^q$ is the unknown input, $y \in \mathbb{R}^p$ is the measured output and $z \in \mathbb{R}^r$ with $r \leqslant n$ is the vector to be estimated. The matrices A, B, F, C, L are known constant matrices of appropriate dimensions. The associated UIFO proposed for (1) by Darouach [7] with the notation adopted from the earlier work [5], is given by

$$\left\{ \dot{w} = Nw + Jy + Hu, \widehat{z} = w + Ey, \right. \tag{2}$$

where one imposes the following matrix identities

$$P \triangleq L - EC, PA - NP - JC = \mathbf{0}_{r\times n}, PF = \mathbf{0}_{r\times q}, H = PB. \tag{3}$$

The estimate of z is $\widehat{z}$ and w is the state of the observer. The above result in asymptotically stable dynamics for the r-dimensional error $e = z - \widehat{z}$, given by $\dot{e} = Ne$. The following pseudo-algorithm, which forms the basis for the design algorithms in Sections 3–5, is based on

[7], but adapted to the notation in [5].

Algorithm 1: Designing UIFO observers for 1st order systems using (2)

1: Multiply $PA - NP - JC = 0$ by $[L^\dagger \quad (\mathbf{I}_{n\times n} - L^\dagger L)]$ to obtain

$$N = PAL^\dagger + \left(NE - J\right)CL^\dagger \Rightarrow N = LAL^\dagger - [E \quad (J-NE)]\begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}[E \quad (J-NE)]\begin{bmatrix} CA(\mathbf{I}_{n\times n} - L^\dagger L) \\ C(\mathbf{I}_{n\times n} - L^\dagger L) \end{bmatrix} = LA\left(\mathbf{I}_{n\times n} - L^\dagger L\right)$$

2: Incorporate the constraint $PF = \mathbf{0}_{r\times q}$ in above, to obtain

$$[E \quad (J-NE)]\underbrace{\begin{bmatrix} CA(\mathbf{I}_{n\times n} - L^\dagger L) & CF \\ C(\mathbf{I}_{n\times n} - L^\dagger L) & \mathbf{0}_{p\times q} \end{bmatrix}}_{\Sigma} = \underbrace{[LA(\mathbf{I}_{n\times n} - L^\dagger L) \quad LF]}_{G}$$

3: Solve $[E \quad (J-NE)]\Sigma = G$ to obtain $[E \quad (J-NE)] = G\Sigma^\dagger + Z(\mathbf{I}_{2p\times 2p} - \Sigma\Sigma^\dagger)$

4: Update N to

$$N = LAL^\dagger - G\Sigma^\dagger\begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix} - Z\left(\mathbf{I}_{2p\times 2p} - \Sigma\Sigma^\dagger\right)\begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}$$

5: Use pole placement techniques to select Z in step 4 and compute N

6: Use Z from step 5 to solve for $[E \quad (J-NE)]$from step 3 and extract E

7: Using E from step 6, solve for P using $P = L - EC$, then solve for J using $J = NE + (J-NE)$ and set $H = PB$

8: Use the computed N, J, H, E to implement the UIFO in (2)

2.1. Reduction of the 1st order UIFO in (2), (3) to UIO, FO and Luenberger observer

One may impose additional conditions that reduce the UIFO in (2) to either an UIO or FO and subsequently to a Luenberger observer. In fact these conditions will be used as a criterion to design natural observers for vector 2nd order systems. Basically, these are as follows:

1. *When the vector z is the entire state x, i.e. $z = x$ with $L \equiv \mathbf{I}_{n\times n}$ and $d \neq \mathbf{0}_{q\times 1}$, then the UIFO should coincide with the UIO with $\widehat{x} \equiv \widehat{z}$.*
2. *When the vector z is the entire state x and $d = \mathbf{0}_{q\times 1}$, then the UIFO should reduce to the Luenberger observer with $\widehat{x} \equiv \widehat{z}$.*
3. *When $d = \mathbf{0}_{q\times 1}$ and L is any $r \times n$ matrix with $r \leqslant n$, then the UIFO should become the standard FO.*

We revisit the UIFO given in [7] and examine the above three conditions. In order to do so, we rewrite the proposed UIFO in Eq. 2, in terms of the $\widehat{z}$ state, which is only used for analysis purposes and is given by

$$\begin{aligned}\dot{\widehat{z}} &= \dot{w} + E\dot{y} = Nw + Jy + Hu + EC(Ax + Bu + Fd) = N\widehat{z} + (-NEC + JC + ECA)x + (H + ECB)u + ECFd \\ &= N\widehat{z} + (LA - NL)x + LBu + LFd. \end{aligned} \tag{4}$$

One can clearly see that equation Eq. 4 (cf Eq. 2) cannot be implemented as it requires knowledge of the unknown input d and process state x. However, in the ensuing analysis, it will help verify the above conditions.

1. When $z = x$ and $d \neq \mathbf{0}_{q\times 1}$, the UIFO in Eq. 2 becomes

$$\widehat{z} = \widehat{x} = w + Ey, \dot{w} = Nw + Jy + PBu,$$

with the following matrix identities (cf Eq. 3) $P = \mathbf{I}_n - EC, PA - NP - JC = 0, PF = 0, H = PB$, which is the UIO described in Chen and Patton [8]. Additionally, when one sets $F = \mathbf{0}_{n\times q}$, the UIO yields $P = \mathbf{I}_n$ with $N = A - JC$ and $H = B$. This now coincides with the standard Luenberger observer [9].

2. When $z = x$ and $d = \mathbf{0}_{q\times 1}$ (equivalently $F = \mathbf{0}_{n\times q}$), then the observer given by Eq. 4 in terms of $\widehat{z}$ is

$$\begin{aligned}\dot{\widehat{z}} &= N\widehat{z} + \left(LA - NL\right)x + LBu + LFd \\ &= N\widehat{z} + (A - N)x + Bu,\end{aligned}$$

where now $PA - NP - JC = 0$, with $P = \mathbf{I}_n, E = \mathbf{0}_{n\times p}$, yields

$$0 = PA - NP - JC = (\mathbf{I}_n - EC)A - N(\mathbf{I}_n - EC) - JC = A - N - JC \Rightarrow N \equiv A - JC.$$

In this case, the proposed observer with $\widehat{x} = \widehat{z}$ reduces to

$$\dot{\hat{z}} = N\hat{z} + \left(A - N\right)x + Bu = \left(A - JC\right)\hat{z} + JCx + Bu = \left(A - JC\right)\hat{z} + Jy + Bu,$$

which is the familiar Luenberger observer. The state estimation error $e = z - \hat{z} = x - \hat{x}$ is governed by $\dot{e} = (A - JC)e$, with $(A - JC)$ being Hurwitz.

3. When $d \equiv \mathbf{0}_{q\times 1}$, then the UIFO in (2) is now given by

$$\hat{z} = w + Ey, \dot{w} = Nw + Jy + PBu,$$

with the following matrix identities $P = L - EC, PA - NP - JC = 0, H = PB$, which is the FO proposed by Darouach in [5]. When one further chooses $L = \mathbf{I}_n$, then $P = \mathbf{I}_n$ and $N = A - JC$ with $H = B$. Further examination reveals that this is the standard Luenberger observer [9].

Remark 2.1. One may enforce the condition $PF = \mathbf{0}_{r\times q}$ in order to obtain E via $E = (LF)(CF)^{\dagger}$ and then select the matrices N and J with N Hurwitz so that the Sylvester equation $PA - NP - JC = \mathbf{0}_{r\times n}$ holds. This may not be always feasible, but if it is, it results in a minimum complexity design. In fact, this was used for the design of natural UIO for vector 2nd systems in [6].

2.2. UIFO and FO of order larger than r

In a recent work by [10], an interesting aspect of the above UIFO was set forth, namely in addressing the case where the rank conditions for existence are not satisfied, or the resulting observer is not satisfying the Sylvester equation. The latter issue is important when the FO or UIFO is used as a compensator and it can destabilize the resulting closed-loop system [11].

Conditions for the existence of a FO of order larger than r were examined and the procedure for constructing FO was provided in [11]. In summary the UIFO (2) is replaced by

$$\left\{ \dot{w} = Nw + Jy + Hu, \hat{z} = Qw + Ey, \right. \tag{5}$$

where $\dim(w) = r + \chi$, with $0 \leqslant \chi \leqslant n - r - p$. The matrix $Q \in \mathbb{R}^{r\times(r+\chi)}$ with $P \in \mathbb{R}^{(r+\chi)\times r}$ is defined via $QP = L - EC$, and now one imposes the following (modified) matrix identities (cf (3))

$$QP = L - EC, Q\left(PA - NP - JC\right) = \mathbf{0}_{r\times n}, QPF = \mathbf{0}_{r\times q}, Q\left(PB - H\right) = \mathbf{0}_{r\times m}. \tag{6}$$

As part of the construction, one augments the L matrix by additional rows so that it becomes full row rank matrix. The additional rows are needed to ensure that the two rank conditions in [7] are satisfied. The construction procedure in this case is essentially the same as the one in Algorithm 1, see also [12].

3. Problem formulation and summary of earlier works on natural UIOs

The class of systems under consideration is governed by the vector 2nd order system

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = B_0 u(t) + F_0 d(t), \tag{7a}$$

$$y\left(t\right) = \begin{bmatrix} y_p\left(t\right) \\ y_v\left(t\right) \end{bmatrix} = \begin{bmatrix} C_p x(t) \\ C_v \dot{x}\left(t\right) \end{bmatrix} = \begin{bmatrix} C_p & \mathbf{0}_{p_1\times n} \\ \mathbf{0}_{p_2\times n} & C_v \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \tag{7b}$$

$$z\left(t\right) = L_p x(t) + L_v \dot{x}(t), \tag{7c}$$

where $x(t) \in \mathbb{R}^n$ is the *position* vector, $\dot{x}(t) \in \mathbb{R}^n$ is the *velocity* vector and the $n\times n$ matrices M, D, K denote the *mass, damping* and *stiffness* matrices respectively. The mass and stiffness matrices, M and K, are assumed to be symmetric and positive definite. To allow for a larger class of mechanical systems, it is assumed that $D \geqslant 0$, i.e. possibly consider non-gyroscopic systems with $D = -D^T$, [13]. The $(p_1 + p_2)$-dimensional measurement vector $y(t)$ in (7b) has both position and velocity components $y_p(t)$ and $y_v(t)$, with the associated observation matrices C_p, C_v having dimensions $p_1 \times n$ and $p_2 \times n$, respectively. The control input vector $u(t)$ is assumed to be an m-dimensional vector with B_0 an $n\times m$ control influence matrix. The q-dimensional unknown input signal is denoted by $d(t)$ and its $n \times q$ distribution matrix is denoted by F_0.

The r-dimensional vector $z(t)$ in (7c), comprising the functional form of the state position and velocity vectors, is desired to be estimated using only input/output information. The associated $r\times n$ known constant matrices L_p, L_v are in fact the weight matrices that form the state functional to be estimated.

Problem statement: it is desired to use the measurement outputs (position y_p, or velocity y_v or both y_p and y_v) of (7a) to estimate a linear functional $z \in \mathbb{R}^r$ of the position and velocity states given by (7c).

One approach is to write the 2nd order system (7) into its 1st order form

$$\dot{X}(t)=\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{I}_n\\ -M^{-1}K & -M^{-1}D\end{bmatrix}X(t)+\begin{bmatrix}\mathbf{0}_{n\times m}\\ M^{-1}B_0\end{bmatrix}u(t)+\begin{bmatrix}\mathbf{0}_{n\times q}\\ M^{-1}F_0\end{bmatrix}d(t)=AX(t)+Bu(t)+Fd(t),y(t)=\begin{bmatrix}C_p & \mathbf{0}_{p_1\times n}\\ \mathbf{0}_{p_2\times n} & C_v\end{bmatrix}X(t)=CX(t),X(t)=\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}\in\mathbb{R}^{2n}, \tag{8}$$

and take advantage of existing results for full order, reduced order or functional observers [12]. Similar to that, is to consider (7) in first order descriptor form

$$\begin{bmatrix}K & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & M\end{bmatrix}\dot{X}(t)=\begin{bmatrix}\mathbf{0}_{n\times n} & K\\ -K & -D\end{bmatrix}X(t)+\begin{bmatrix}\mathbf{0}_{n\times m}\\ B_0\end{bmatrix}u(t)+\begin{bmatrix}\mathbf{0}_{n\times q}\\ F_0\end{bmatrix}d(t),y(t)=CX(t). \tag{9}$$

and use the results by Darouach for observer design in descriptor form in [14,15].

For the case of a full order observer, there are computational advantages in considering an observer in a natural setting, i.e. consider an observer in a vector 2nd order form [3]. Due to the sparsity of the state matrix, one should be able to see the possible reduction in computations when considering the 2nd order setting. In the specific case of the system corresponding to an approximation of an elastic structure as for example a cable, the use of finite element approximation results in tri-diagonal matrices M,D and K, [16]. Bringing it in the first order setting would destroy the tri-diagonal structure via the multiplications $M^{-1}D$ and $M^{-1}K$, unless of course one considers observer design in the descriptor form (9). In addition to the computational benefits, a natural observer ensures that the derivative of the estimated position *is indeed* the estimated velocity; in contrast, an observer for a system in the 1st order form (8), has the following form for the estimated position

$$\dot{\hat{x}}=\hat{v}+G_{1p}\left(y_p-C_p\hat{x}\right)+G_{1v}\left(y_v-C_v\hat{v}\right),$$

where $\hat{v}$ is the velocity estimate, $\hat{x}$ is the position estimate and G_{1p},G_{1v} are the position filter gains, and which *cannot* guarantee $\frac{d}{dt}\hat{x}(t)$ will be equal to $\hat{v}(t)$ for $t\geqslant 0$, unless $G_{1p}C_p=\mathbf{0}_{n\times n},G_{1v}C_v=\mathbf{0}_{n\times n}$, see [4]. However, one cannot recover the natural observer.

Let us examine an UIFO for the vector 2nd system in the form (8) using the 1st order UIFO (2), (3). Following the remarks in Section 2.1 and using $\dim(X)=2n$, by setting $d=\mathbf{0}_{q\times 1},P=\mathbf{I}_{2n},L=\mathbf{I}_{2n}$, and $E_p=\mathbf{0}_{2n\times p_1}$, $E_v=\mathbf{0}_{2n\times p_v}$ one arrives at

$$H=\begin{bmatrix}\mathbf{0}_{n\times m}\\ M^{-1}B_0\end{bmatrix},$$

and the $2n\times 2n$ state observer matrix $N=A-JC$, with $J\in\mathbb{R}^{2n\times(p_1+p_2)}$ is given by

$$N=A-JC=\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{I}_n\\ -M^{-1}K & -M^{-1}D\end{bmatrix}-\begin{bmatrix}J_{pp} & J_{pv}\\ J_{vp} & J_{vv}\end{bmatrix}\begin{bmatrix}C_p & \mathbf{0}_{p_1\times n}\\ \mathbf{0}_{p_2\times n} & C_v\end{bmatrix}=\begin{bmatrix}-J_{pp}C_p & \mathbf{I}_n-J_{pv}C_v\\ -M^{-1}K-J_{vp}C_p & -M^{-1}D-J_{vv}C_v\end{bmatrix},$$

with the matrices $J_{pp},J_{vp}\in\mathbb{R}^{n\times p_1},J_{pv},J_{vv}\in\mathbb{R}^{n\times p_2}$. Further examination of the above reveals that for the state of the observer $w=(w_1,w_2)$, one has

$$\begin{aligned}\dot{w}_1&=-J_{pp}C_pw_1+\left(\mathbf{I}_{n\times n}-J_{pv}C_v\right)w_2+J_{pp}y_p+J_{pv}y_v\\ w_2&=\left(-M^{-1}K-J_{vp}C_p\right)w_1+\left(-M^{-1}D-J_{vv}C_v\right)w_2+Bu+J_{vp}y_p+J_{vv}y_v,\end{aligned}$$

which is *not* a natural observer! Considering the estimation errors, define $e_1=x-w_1$ and $e_2=\dot{x}-w_2$; then

$$\begin{aligned}\dot{e}_1&=e_2-J_{pp}C_pe_1-J_{pv}C_ve_2\\ e_2&=\left(-M^{-1}K-J_{vp}C_p\right)e_1+\left(-M^{-1}D-J_{vv}C_v\right)e_2,\end{aligned}$$

which also demonstrates that unless one constrains $J_{pp}=\mathbf{0}_{n\times p_1},J_{pv}=\mathbf{0}_{n\times p_2}$ (or $J_{pp}C_p=\mathbf{0}_{n\times n},J_{pv}C_v=\mathbf{0}_{n\times n}$), the UIFO in (2), (3) applied to vector 2nd systems in the form (8) will *never* reduce to a natural observer.

For reduced order observers, one has to find estimates $\hat{x}$ and $\hat{\dot{x}}$ of x and $\dot{x}$ and then take $\hat{z}=L_p\hat{x}+L_v\hat{\dot{x}}$ as the estimate of the signal z. However, one must ensure that the derivative of the position estimate is indeed the velocity estimate, i.e. to ensure that $\frac{d}{dt}\hat{x}=\hat{\dot{x}}$. The best approach in this case is to consider a reduced order *natural* observer as well.

When considering functional observers, one comes across the solution of an associated Sylvester equation. Once again, to minimize unnecessary computations and take advantage of the algebraic structure of the matrices in their natural setting, we design UIOs for (7) instead of (8), [14]. The computational savings take the form of solving for $n\times n$ dimensional Sylvester equations instead of $2n\times 2n$ dimensional ones.

We will consider the more general case of designing a FO for $L_px+L_v\dot{x}$ using both position and velocity measurements y_p and y_v. Any other combination with different measurements and functional outputs can be shown to be special cases of the general case. The exception is when one has only position measurements y_p and it is desired to design a position functional observer, i.e. estimate $z=$

$L_p x$. This is a degenerate case and one must then consider a special functional observer in a 2nd order setting as well.

Prior to the description of the observer designs, we summarize a result on the solution to matrix equations.

Lemma 3.1. ([17]) *Given matrices X, A, B of appropriate dimensions, $XA = B$ has a solution if and only if $B(I - A^{\dagger}A) = 0$, where $A^{\dagger}$ denotes the Moore–Penrose inverse of A. The latter is equivalent to $\text{rank}\begin{pmatrix} B \\ A \end{pmatrix} = \text{rank}(A)$, and the corresponding solution is $X = BA^{\dagger} + Z(I - AA^{\dagger})$ where Z is an arbitrary matrix. Similarly, $AX = B$ has a solution if and only if $(I - AA^{\dagger})B = 0$, and it is given by $X = A^{\dagger}B + (I - A^{\dagger}A)Z$.*

3.1. Natural UIOs for vector 2nd order systems with position and velocity measurements

We summarize the UIO for 2nd order systems considered in [6]. Here, a modification not published elsewhere, is included in order to allow for velocity measurements. This modification removes the need to take the time derivative of position measurements y_p and uses both y_p and y_v. The proposed *natural* UIO for (7) with $z = (x, \dot{x})^T$, which requires $p_1 = p_2 = p$ (equal number of position and velocity measurements), is

$$\left\{ M\ddot{w} + D_1\dot{w} + K_1 w = TB_0 u + J_p y_p + J_v y_v, \widehat{x} = w + E_p y_p. \right. \tag{10}$$

The n-dimensional vector $\widehat{x}$ is the estimate of the position x, and, by construction of the *natural* UIO for 2nd order systems, the estimate of the velocity vector *is* equal to the derivative of $\widehat{x}$; i.e. $\widehat{\dot{x}}\left(t\right) = \frac{d}{dt}\widehat{x}\left(t\right), \forall t \geqslant 0$.

In a similar fashion as in [6], one imposes the following matrix identities

$$\begin{array}{ll} \left(I - ME_pC_pM^{-1}\right)F_0 = 0, & T = I - ME_pC_pM^{-1} = M\left(I - E_pC_p\right)M^{-1}, \\ D_1 = TD + J_{v_1}C_v, & K_1 = TK + J_{p_1}C_p, \\ J_v = J_{v_1} + J_{v_2}, & J_p = J_{p_1} + J_{p_2}, \\ J_{v_2} = -D_1E_pQ^{-1}, & J_{p_2} = -K_1E_p, \end{array} \tag{11}$$

with the additional assumption that the two output matrices are related via

$$C_v = QC_p, \tag{12}$$

where Q is any $p \times p$ nonsingular matrix. The above condition (collocated-type) eliminates the need to differentiate the position output, as required in the observer design of [6]; the difference comes at the condition $J_{v_2} = -D_1E_pQ^{-1}$. When equations Eq. 11, Eq. 12 are satisfied, then the estimation error $e \triangleq x - \widehat{x}$ is governed by

$$M\ddot{e} + D_1\dot{e} + K_1 e = 0. \tag{13}$$

Conditions that guarantee the convergence of the position and velocity errors to zero regardless of the presence of the unknown input d are given below.

Theorem 3.1. *The necessary and sufficient conditions for the proposed observer in Eq. 10, which uses both y_p and y_v, to be a natural UIO for the 2nd order system in (7) are:*

1. $\text{rank}\left(C_pM^{-1}F_0\right) = \text{rank}\begin{pmatrix} M^{-1}F_0 \\ C_pM^{-1}F_0 \end{pmatrix}$.
2. *the pairs (C_p, TK) and (C_v, TD) are detectable, in the sense that one may assign the spectrum of both K_1 and D_1 such that the following quadratic pencil [18] has a desired eigenstructure*

$$\mathscr{P}_1\left(s\right) = s^2M + s\left(TD + J_{v_1}C_v\right) + \left(TK + J_{p_1}C_p\right).$$

3. *$C_v = QC_p$ with Q nonsingular.*

Proof. The proof is similar to the one in [6] and whose main difference is in the additional arguments required for the added velocity measurements. When $C_v = C_p$, then the proof is identical to that in [6] with $\dot{y}_p = y_v$. Due to the similarity in the proof for the current case of $C_v = QC_p$, the reader is directed in [6]. □

Algorithm 2: Designing minimum complexity natural UIO observers for 2nd order systems using (10)

1: Use $M(I-E_pC_p)M^{-1}F_0=0$ to obtain the solution $E_p=\left(M^{-1}F_0\right)\left(C_pM^{-1}F_0\right)^{\dagger}$

2: Compute T using $T=M(I-E_pC_p)M^{-1}$

3: Compute TK and TD

4: Compute $K_1=TK+J_{p_1}C_p$ and $D_1=TD+J_{v_1}C_v$ by using pole placement techniques in the pairs (C_p,TK) and (C_v,TD) to select J_{p_1} and J_{v_1}

5: Compute $J_p=J_{p_1}-K_1E_p$ and $J_v=J_{v_1}-D_1E_pQ^{-1}$

6: Implement the minimum complexity natural UIO

$$M\ddot{w}+D_1\dot{w}+K_1w=TB_0u+J_py_p+J_vy_v,\widehat{x}=w+E_py_p.$$

Remark 3.1. (*Restrictions of the natural UIO extended from [6]*) While the proposed natural UIO in (10) is an improvement of the natural UIO presented in [6] as it explicitly uses velocity measurements, it is nonetheless a special case of natural UIOs. The reason is that the Sylvester equations

$$\begin{aligned}&-TK-J_pC_p+K_1M^{-1}TM=0,\\&-TD-J_vC_v+D_1M^{-1}TM=0,\end{aligned}$$

resulting from the estimation error dynamics, should have incorporated the constraint $TF_0=0$ in order to obtain the solution E_p, and the general solution for E_p should have used the matrix Z in Lemma 3.1. The solution for E_p solely relies on condition 1 of Theorem 3.1. If this cannot be satisfied, then E_p has to be solved using the above two Sylvester equations. Even if condition 1 is satisfied, there is no guarantee that the detectability conditions (condition 2) can be satisfied. However, if all three conditions of Theorem 3.1 are satisfied, then the design procedure for the natural UIO in (10), as given in Algorithm 1, will result in a natural UIO with minimal design complexity.

Remark 3.2. (*Natural UIO to natural observer*) One can observe that when $F_0=\mathbf{0}_{n\times q}$, the UIO in Eq. 10-Eq. 12 coincides with the natural 2nd order observer summarized in [2] for lumped parameter systems. Indeed, when $F_0=\mathbf{0}_{n\times q}$, then $\widehat{x}=w+E_py_p$ with $E_p=\mathbf{0}_{n\times p_1}$, gives $\widehat{x}\equiv w$ and the governing equation becomes $M\ddot{\widehat{x}}+D_1\dot{\widehat{x}}+K_1\widehat{x}=B_0u+J_{p_1}y_p+J_{v_1}y_v$, with $D_1=D+J_{v_1}C_v$, $K_1=K+J_{p_1}C_p$, $T=\mathbf{I}_n$, $J_{v_2}=J_{p_2}=\mathbf{0}_{p\times n}$, or $M\ddot{\widehat{x}}+(D+J_{v_1}C_v)\dot{\widehat{x}}+(K+J_{p_1}C_p)\widehat{x}=Bu+J_{p_1}y_p+J_{v_1}y_v$. The above framework retains the 2nd order structure of mechanical systems with the time derivative of the estimated position equal to the estimated velocity, [2].

To show that the error system in Eq. 13 converges to zero asymptotically, one employs the parameter-dependent Lyapunov function that was used in [6]. Detailed arguments leading to the convergence of $e,\dot{e}$ to zero are given in [6], and in [2] for the case of a natural 2nd order observer. Alternatively, one may use condition 2 of Theorem 3.1 to argue stability of the error Eq. (13) corresponding to (14).

In order to arrive at a natural UIO that is not restrictive, one must utilize the general framework of UIFO that rely on associated Sylvester equations to enforce a condition similar to $TF_0=0$. In that case, the general natural UIFO presented in Section 4 can be used to design natural UIO and FO for vector 2nd order systems.

4. Natural UIFOs for vector 2th order systems

The proposed *natural* UIFO for (7) is given by

$$M_0\ddot{\zeta}+D_0\dot{\zeta}+K_0\zeta=J_py_p+J_vy_v+Hu. \tag{14a}$$

$$\widehat{z}=A_1\zeta+A_2\dot{\zeta}+E_py_p+E_vy_v, \tag{14b}$$

The estimate of $z(t)$ contains a weighted sum of the "projected" position and velocity $\zeta(t)$ and $\dot{\zeta}(t)$. The matrices E_p,E_v have dimensions $r\times p_1$ and $r\times p_2$, respectively. The "projected" mass, damping and stiffness matrices M_0,D_0,K_0 have dimension $s\times s$ with $\min(n,r)\leqslant s\leqslant n$. Similarly, the matrices A_1,A_2 have dimension $r\times s$. Finally, J_p,J_v and H have dimensions $s\times p_1$, $s\times p_2$ and $s\times m$. A way to view (14), is that the observer (14a) with state $(w,\dot{w})$ is of order $2s$ and its output z is of order r. Define the matrices $P_1=L_p-E_pC_p$, $P_2=L_v-E_vC_v$. Theorem 4.1 provides conditions for (14) to be a natural UIFO for (7).

Theorem 4.1. *The signal $\widehat{z}$ in (14a), (14b) is an asymptotic estimate of the functional z in (7) for any initial position $x(0)$, initial velocity $\dot{x}(0)$, any $\widehat{z}(0)$ and any bounded input $u(t)$ in (7) if and only if the equations*

$$\begin{aligned} -P_2M^{-1}K-NP_1-A_2M_0^{-1}J_pC_p&=\mathbf{0}_{r\times n},\\ P_1-P_2M^{-1}D-NP_2-A_2M_0^{-1}J_vC_v&=\mathbf{0}_{r\times n},\\ P_2M^{-1}B_0-A_2M_0^{-1}H&=\mathbf{0}_{r\times m},\\ P_2M^{-1}F_0&=\mathbf{0}_{r\times q}, \end{aligned} \tag{15}$$

along with the following auxiliary matrix identities

$$A_2M_0^{-1}K_0+NA_1=\mathbf{0}_{r\times s}, A_2M_0^{-1}D_0+NA_2-A_1=\mathbf{0}_{r\times s}, \tag{16}$$

with N Hurwitz, hold and in that case the associated estimation error converges to zero asymptotically.

Proof. One defines the *estimation error*

$$e=z-\widehat{z}=L_px+L_v\dot{x}-E_py_p-E_vy_v-A_1\zeta-A_2\dot{\zeta}=\left(L_p-E_pC_p\right)x+\left(L_v-E_vC_v\right)\dot{x}-A_1\zeta-A_2\dot{\zeta}=P_1x+P_2\dot{x}-A_1\zeta-A_2\dot{\zeta}.$$

The evolution of the estimation error is given by

$$\begin{aligned} \dot{e}&=P_1\dot{x}+P_2\ddot{x}-A_1\dot{\zeta}-A_2\ddot{\zeta}\\ &=P_1\dot{x}+P_2\left(-M^{-1}D\dot{x}-M^{-1}Kx+M^{-1}B_0u+M^{-1}F_0d\right)-A_1\dot{\zeta}-A_2\left(-M_0^{-1}D_0\dot{\zeta}-M_0^{-1}K_0\zeta+M_0^{-1}J_py_p+M_0^{-1}J_vy_v\right.\\ &\quad\left.+M_0^{-1}Hu\right)+Ne-N(P_1x+P_2\dot{x}-A_1\zeta-A_2\dot{\zeta})\\ &=\left(P_1-P_2M^{-1}D-NP_2-A_2M_0^{-1}J_vC_v\right)\dot{x}+\left(-P_2M^{-1}K-NP_1-A_2M_0^{-1}J_pC_p\right)x+\left(P_2M^{-1}B_0\right.\\ &\quad\left.-A_2M_0^{-1}H\right)u+P_2M^{-1}F_0d+\left(A_2M_0^{-1}K_0+NA_1\right)\zeta+\left(-A_1+A_2M_0^{-1}D_0+NA_2\right)\dot{\zeta}+Ne. \end{aligned}$$

If (15), (16) hold then $\dot{e}=Ne$. Since N is Hurwitz by design, e converges to zero asymptotically. □

Lemma 4.1. *The necessary and sufficient conditions for the existence and asymptotic convergence of the UIFO observer (14a), (14b) to the functional output z of the vector 2nd order system (7) are*

1. $\operatorname{rank}\left(\begin{bmatrix} L_v\overline{K} & (L_vM^{-1}F_0)\\ C_v\overline{K} & (C_vM^{-1}F_0)\\ \overline{C}_p & \mathbf{0}_{p_1\times q}\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix} C_v\overline{K} & (C_vM^{-1}F_0)\\ \overline{C}_p & \mathbf{0}_{p_1\times q}\end{bmatrix}\right),$
2. $\operatorname{rank}\left(\begin{bmatrix} L_v\overline{D}-L_p(\mathbf{I}_n-L_v^{\dagger}L_v)\\ -C_p(\mathbf{I}_{n\times n}-L_v^{\dagger}L_v)\\ C_v\overline{D}\\ \overline{C}_v\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix} -C_p(\mathbf{I}_{n\times n}-L_v^{\dagger}L_v)\\ C_v\overline{D}\\ \overline{C}_v\end{bmatrix}\right),$

where $\overline{K}=M^{-1}K\left(\mathbf{I}_n-L_p^{\dagger}L_p\right)$, $\overline{D}=M^{-1}D(\mathbf{I}_n-L_v^{\dagger}L_v)$, $\overline{C}_p=C_p\left(\mathbf{I}_n-L_p^{\dagger}L_p\right)$, $\overline{C}_v=C_v(\mathbf{I}_n-L_v^{\dagger}L_v)$.

4.1. Design procedure for natural UIFOs and FOs

The natural UIFO and its reduction to FO and UIO, require similar matrix identities and therefore we consider one set of matrix identities; we provide the design steps to the general matrix identities below. Their solution can then be adapted to the appropriate observer. Consider the following matrix equations

$$\begin{aligned} &-P_2M^{-1}K-NP_1-A_2M_0^{-1}J_pC_p=\mathbf{0}_{r\times n},\\ &P_1-P_2M^{-1}D-NP_2-A_2M_0^{-1}J_vC_v=\mathbf{0}_{r\times n},\\ &P_2M^{-1}F_0=\mathbf{0}_{s\times q}, \end{aligned}$$

which represent the first part of the identities (15), (16). The auxiliary matrix identities (16) are written as

$$-[A_1\quad A_2]\underbrace{\begin{bmatrix} \mathbf{0}_{s\times s} & \mathbf{I}_s\\ -M_0^{-1}K_0 & -M_0^{-1}D_0\end{bmatrix}}_{A_0}+N[A_1\quad A_2]=[\mathbf{0}_{r\times s}\quad \mathbf{0}_{r\times s}], \tag{17}$$

and can thus be used to obtain A_1,A_2 once M_0,D_0,K_0,N are designed. If one defines

$$\widetilde{J}_p=A_2M_0^{-1}J_p, \widetilde{J}_v=A_2M_0^{-1}J_v,$$

then the above equations are re-written as

$$-P_2M^{-1}K - NP_1 - \widetilde{J}_pC_p = \mathbf{0}_{r\times n}, \tag{18a}$$

$$P_1 - P_2M^{-1}D - NP_2 - \widetilde{J}_vC_v = \mathbf{0}_{r\times n}, \tag{18b}$$

$$P_2M^{-1}F_0 = \mathbf{0}_{s\times q}. \tag{18c}$$

Eqs. (18) will eventually produce N,P_1,P_2,E_p,E_v,$\widetilde{J}_p$,$\widetilde{J}_v$, and M_0, D_0, K_0 needed for the auxiliary identities. When $\widetilde{J}_p, \widetilde{J}_v$ are found from the solution to (18), then knowledge of A_2, M_0 will enable the computation of J_p, J_v needed for the implementation of the UIFO (14). The solution to the auxiliary matrix identity (17) can be obtained via vectorization $\left(\mathbf{I}_{2s}\otimes N - A_0^T\otimes \mathbf{I}_r\right)\mathrm{vec}([A_1 \quad A_2]) = \mathbf{0}_{2sr\times 1}$, with the solution given by $\left[A_1 \quad A_2\right] \in \ker\left(\mathbf{I}_{2s}\otimes N - A_0^T\otimes \mathbf{I}_r\right)$. Once A_2 is available, then $J_p = \left(A_2M_0^{-1}\right)^{\dagger}\widetilde{J}_p$and $J_v = \left(A_2M_0^{-1}\right)^{\dagger}\widetilde{J}_v$.

The design for UIFO and FO differ slightly in that the former additionally enforces $P_2M^{-1}F_0 = \mathbf{0}_{s\times q}$ whereas the latter only requires the solution to (18a) and (18b).

4.1.1. Design procedure for natural UIFOs in (14)

Eq. (18a) which is equivalent to $\left(-P_2M^{-1}K - NP_1 - \widetilde{J}_pC_p\right)\left[L_p^{\dagger} \qquad \mathbf{I}_n - L_p^{\dagger}L_p\right] = \mathbf{0}_{r\times(r+n)}$, where $L_p^{\dagger}$ denotes the generalized inverse of L_p with $L_pL_p^{\dagger} = \mathbf{I}_r$, produces

$$-P_2M^{-1}KL_p^{\dagger} - NP_1L_p^{\dagger} - \widetilde{J}_pC_pL_p^{\dagger} = \mathbf{0}_{r\times r}, \tag{19a}$$

$$-P_2M^{-1}K\left(\mathbf{I}_n - L_p^{\dagger}L_p\right) - NP_1\left(\mathbf{I}_n - L_p^{\dagger}L_p\right) - \widetilde{J}_pC_p\left(\mathbf{I}_n - L_p^{\dagger}L_p\right) = \mathbf{0}_{r\times n}. \tag{19b}$$

Similarly, (18b), which is equivalent to $\left(P_1 - P_2M^{-1}D - NP_2 - \widetilde{J}_vC_v\right)\left[L_v^{\dagger} \qquad \mathbf{I}_n - L_v^{\dagger}L_v\right] = \mathbf{0}_{r\times(r+n)}$, where $L_v^{\dagger}$ denotes the generalized inverse of L_v with $L_vL_v^{\dagger} = \mathbf{I}_r$, produces

$$P_1L_v^{\dagger} - P_2M^{-1}DL_v^{\dagger} - NP_2L_v^{\dagger} - \widetilde{J}_vC_vL_v^{\dagger} = \mathbf{0}_{r\times r}, \tag{20a}$$

$$P_1\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) - P_2M^{-1}D\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) - NP_2\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) - \widetilde{J}_vC_v\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) = \mathbf{0}_{r\times n}. \tag{20b}$$

Using the fact that $P_1L_p^{\dagger} = \mathbf{I}_r - E_pC_pL_p^{\dagger}$ and the definition of P_1, (19a) produces

$$N = -L_vM^{-1}KL_p^{\dagger} + \left[E_v \qquad \left(NE_p - \widetilde{J}_p\right)\right]\begin{bmatrix} C_vM^{-1}KL_p^{\dagger} \\ C_pL_p^{\dagger} \end{bmatrix}, \tag{21}$$

and (19b) produces

$$\left[E_v \qquad \left(NE_p - \widetilde{J}_p\right)\right]\begin{bmatrix} C_vM^{-1}K\left(\mathbf{I}_n - L_p^{\dagger}L_p\right) \\ C_p\left(\mathbf{I}_n - L_p^{\dagger}L_p\right) \end{bmatrix} = L_vM^{-1}K\left(\mathbf{I}_n - L_p^{\dagger}L_p\right). \tag{22}$$

Using the definition of P_2 and $P_2L_v^{\dagger} = \mathbf{I}_r - E_vC_vL_v^{\dagger}$, Eq. (20a) produces

$$N = L_pL_v^{\dagger} - L_vM^{-1}DL_v^{\dagger} + E_vC_vM^{-1}DL_v^{\dagger} + \left[E_p \quad \left(NE_v - \widetilde{J}_v\right)\right]\begin{bmatrix} -C_pL_v^{\dagger} \\ C_vL_v^{\dagger} \end{bmatrix}. \tag{23}$$

The second expression (20b) simplifies to

$$\left[E_p \quad \left(NE_v - \widetilde{J}_v\right)\right]\begin{bmatrix} -C_p\left(\mathbf{I}_{n\times n} - L_v^{\dagger}L_v\right) \\ C_v\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) \end{bmatrix} = \left(L_v - E_vC_v\right)M^{-1}D\left(\mathbf{I}_n - L_v^{\dagger}L_v\right) - L_p\left(\mathbf{I}_{n\times n} - L_v^{\dagger}L_v\right). \tag{24}$$

Eqs. ()()()()(21)–(24) will be solved concurrently. To simplify notation, define matrices

$$\begin{aligned}
&\overline{C}_p = C_p\left(\mathbf{I}_n - L_p^\dagger L_p\right), \overline{C}_v = C_v\left(\mathbf{I}_n - L_v^\dagger L_v\right), \overline{K} = M^{-1}K\left(\mathbf{I}_n - L_p^\dagger L_p\right), \overline{D} = M^{-1}D\left(\mathbf{I}_n - L_v^\dagger L_v\right),\\
&\Sigma_1 = \begin{bmatrix} C_v\overline{K} \\ \overline{C}_p \end{bmatrix}, \Sigma_2 = \begin{bmatrix} -C_p\left(\mathbf{I}_{n\times n} - L_v^\dagger L_v\right) \\ \overline{C}_v \end{bmatrix}, \Sigma_3 = \begin{bmatrix} C_v\overline{K} & \left(C_v M^{-1}F_0\right) \\ \overline{C}_p & \mathbf{0}_{p_1\times q} \end{bmatrix},\\
&G_1 = L_v\overline{K}, G_2 = \left(L_v - E_v C_v\right)\overline{D} - L_p\left(\mathbf{I}_{n\times n} - L_v^\dagger L_v\right), G_3 = \begin{bmatrix} L_v\overline{K} & \left(L_v M^{-1}F_0\right) \end{bmatrix}.
\end{aligned} \tag{25}$$

Using $P_2 = L_v - E_v C_v$, one has from (18c)

$$\left(L_v - E_v C_v\right)M^{-1}F_0 = \mathbf{0}_{s\times q} \Rightarrow E_v\left(C_v M^{-1}F_0\right) = \left(L_v M^{-1}F_0\right).$$

Combine (22) and the above (cf step 2 of Algorithm 1)

$$\begin{bmatrix} E_v & \left(NE_p - \widetilde{J}_p\right) \end{bmatrix}\begin{bmatrix} C_v\overline{K} & \left(C_v M^{-1}F_0\right) \\ \overline{C}_p & \mathbf{0}_{p_1\times q} \end{bmatrix} = \begin{bmatrix} L_v\overline{K} & \left(L_v M^{-1}F_0\right) \end{bmatrix}. \tag{26}$$

Following (25), Eq. (26) is compactly written as $G_3 = \begin{bmatrix} E_v & (NE_p - \widetilde{J}_p) \end{bmatrix}\Sigma_3$ and, using condition 1 in Lemma 4.1, has a solution if and only if

$$\operatorname{rank}\begin{pmatrix} G_3 \\ \Sigma_3 \end{pmatrix} = \operatorname{rank}\left(\Sigma_3\right).$$

Using Lemma 3.1, the solution in this case is given by the left inverse

$$\begin{bmatrix} E_v & \left(NE_p - \widetilde{J}_p\right) \end{bmatrix} = G_3\Sigma_3^\dagger + Z_3\left(\mathbf{I}_{p_1+p_2} - \Sigma_3\Sigma_3^\dagger\right), \tag{27}$$

where Z_3 is an arbitrary matrix. Using the above in (21), we have

$$N = -L_v M^{-1}KL_p^\dagger + G_3\Sigma_3^\dagger\begin{bmatrix} C_v M^{-1}KL_p^\dagger \\ C_p L_p^\dagger \end{bmatrix} + Z_3\left(\mathbf{I}_{p_1\times p_1} - \Sigma_3\Sigma_3^\dagger\right)\begin{bmatrix} C_v M^{-1}KL_p^\dagger \\ C_p L_p^\dagger \end{bmatrix}. \tag{28}$$

Using pole placement, Z_3 can be obtained from (28), and subsequently, N can be computed. Using Z_3 in (27) provides the solution to E_v and $(NE_p - \widetilde{J}_p)$. Using E_v, compute $P_2 = L_v - E_v C_v$. Now, to compute E_p and $(NE_v - \widetilde{J}_v)$, consider Eq. (24) which simplifies to $\begin{bmatrix} E_p & (NE_v - \widetilde{J}_v) \end{bmatrix}\Sigma_2 = G_2$ and (using condition 2 in Lemma 4.1), it has a solution if and only if

$$\operatorname{rank}\begin{pmatrix} G_2 \\ \Sigma_2 \end{pmatrix} = \operatorname{rank}\left(\Sigma_2\right).$$

Using Lemma 3.1, the solution to (24) is

$$\begin{bmatrix} E_p & \left(NE_v - \widetilde{J}_v\right) \end{bmatrix} = G_2\Sigma_2^\dagger + Z_2\left(\mathbf{I}_{p_1+p_2} - \Sigma_2\Sigma_2^\dagger\right), \tag{29}$$

where Z_2 is a matrix of appropriate dimensions. Using (23), (29)

$$N = \left(L_p L_v^\dagger - P_2 M^{-1}DL_v^\dagger + G_2\Sigma_2^\dagger\begin{bmatrix} -C_p L_v^\dagger \\ C_v L_v^\dagger \end{bmatrix}\right) + Z_2\left(\left(\mathbf{I}_{p_1+p_2} - \Sigma_2\Sigma_2^\dagger\right)\begin{bmatrix} -C_p L_v^\dagger \\ C_v L_v^\dagger \end{bmatrix}\right). \tag{30}$$

Using N from (28) above, the matrix Z_2 can be obtained by pole placement. Using the solution Z_2 back in (29), one can compute E_p and $(NE_v - \widetilde{J}_v)$. From the last one, the knowledge of N and E_v from (27), (28) subsequently provides the solution to $\widetilde{J}_v$ via $\widetilde{J}_v = NE_v - (NE_v - \widetilde{J}_v)$. Similarly, the knowledge of E_p allows one to compute $\widetilde{J}_p$ via $\widetilde{J}_p = NE_p - (NE_p - \widetilde{J}_p)$.

To recover J_p, J_v, compute M_0, D_0, K_0 so that the quadratic pencil $\mathscr{P}_0(s) = s^2M_0 + sD_0 + K_0$ has all its roots in $\mathbb{C}^-$. The solution to the matrix identities (17) can be obtained and subsequently $J_p = (A_2M_0^{-1})^\dagger\widetilde{J}_p, J_v = (A_2M_0^{-1})^\dagger\widetilde{J}_v$ and $H = (A_2M_0^{-1})^\dagger P_2M^{-1}B$. Using $M_0, D_0, K_0, J_p, J_v, H, A_1, A_2, E_p$ and E_v the UIFO in (14) can be realized. Algorithm 3 summarizes the procedure for designing the proposed UIFO.

Algorithm 3: Designing natural UIFO's for vector 2nd order systems: solving (18a), (18b) and (18c)

1:	Using $L_p, L_v, C_p, C_v, M, D, K$ set up matrices $\overline{C}_p, \overline{C}_v, \overline{K}, \overline{D}$, and $\Sigma_1, \Sigma_2, \Sigma_3, G_1, G_3$ in (25)
2:	Using pole placement techniques, solve for Z_3 in (28)
3:	Solve for $[E_v\ (NE_p - \widetilde{J}_p)]$ using (27)
4:	Use E_v to solve for $P_2 = L_v - E_v C_v$ and set-up G_2 in (25)

(continued on next page)

(*continued*)

Algorithm 3: Designing natural UIFO's for vector 2nd order systems: solving (18a), (18b) and (18c)

5:	Using pole placement in N, solve for Z_2 in (30)
6:	Solve for E_p and $\widetilde{J}_v$ using Z_2 and $\widetilde{J}_v = NE_v - (NE_v - \widetilde{J}_v)$ in (29)
7:	Compute P_1 using $P_1 = L_p - E_pC_p$
8:	Solve for $\widetilde{J}_p$ using $\widetilde{J}_p = NE_p - (NE_p - \widetilde{J}_p)$
9:	Select positive definite $r \times r$ matrices M_0, D_0 and K_0 such that the quadratic pencil $\mathscr{P}_0(s) = s^2M_0 + sD_0 + K_0$ has all roots $s \in \mathbb{C}^-$
10: :	Use M_0, D_0, K_0 and N to solve for the auxiliary matrix identities for A_1, A_2
11:	Solve for J_p and J_v using $J_p = (A_2M_0^{-1})^{\dagger}\widetilde{J}_p$ and $J_v = (A_2M_0^{-1})^{\dagger}\widetilde{J}_v$
12:	Solve for H using $H = (A_2M_0^{-1})^{\dagger}P_2M^{-1}B$
13:	Using $N, E_p, E_v, J_p, J_v, P_2$ implement the natural UIFO

$$\left\{ M_0\ddot{\zeta} + D_0\dot{\zeta} + K_0\zeta = J_py_p + J_vy_v + Hu, \widehat{z} = A_1\zeta + A_2\dot{\zeta} + E_py_p + E_vy_v. \right.$$

4.1.2. Design procedure for natural FOs in (14)

The procedure is very similar. The difference comes at the removal of (18c). In this case one uses (22) to solve for $[E_v \quad (NE_p - \widetilde{J}_p)]$. The condition for solvability is now

$$\operatorname{rank}\begin{pmatrix} G_1 \\ \Sigma_1 \end{pmatrix} = \operatorname{rank}(\Sigma_1).$$

The solution to (22) is

$$\begin{bmatrix} E_v & \left(NE_p - \widetilde{J}_p\right) \end{bmatrix} = G_1\Sigma_1^{\dagger} + Z_1\left(\mathbf{I}_{p_1+p_2} - \Sigma_1\Sigma_1^{\dagger}\right) \tag{31}$$

where Z_1 is an arbitrary matrix. Substituting to (21)

$$N = -L_vM^{-1}KL_p^{\dagger} + G_1\Sigma_1^{\dagger}\begin{bmatrix} C_vM^{-1}KL_p^{\dagger} \\ C_pL_p^{\dagger} \end{bmatrix} + Z_1\left(\mathbf{I}_{p_1\times p_1} - \Sigma_1\Sigma_1^{\dagger}\right)\begin{bmatrix} C_vM^{-1}KL_p^{\dagger} \\ C_pL_p^{\dagger} \end{bmatrix}. \tag{32}$$

Using pole placement, Z_1 can be obtained from (32), and subsequently, N can be computed. Using Z_1 in (31) provides the solution to E_v and $(NE_p - \widetilde{J}_p)$. Using E_v, compute $P_2 = L_v - E_vC_v$. The rest of the steps follow the ones for UIFO in Eqs. (29), (30). Algorithm 14 summarizes the design of the natural FO (14).

Algorithm 4: Designing natural FOs for vector 2nd order systems: solving (18a), (18b)

1:	Using $L_p, L_v, C_p, C_v, M, D, K$ set up matrices $\overline{C}_p, \overline{C}_v, \overline{K}, \overline{D}$ and $\Sigma_1, \Sigma_3, G_1, G_3$ using (25)
2:	Using pole placement techniques, solve for Z_1 and then for N in (32)
3:	Using Z_1, solve for E_v and $(NE_p - \widetilde{J}_p)$ in (31)
4:	Solve for $P_2 = L_v - E_vC_v$
5:	Using pole placement in N, solve for Z_2 in (30)
6:	Solve for E_p and $\widetilde{J}_v$ using Z_2 and $\widetilde{J}_v = NE_v - (NE_v - \widetilde{J}_v)$ in (29)
7:	Solve for P_1 using $P_1 = L_p - E_pC_p$
8:	Solve for $\widetilde{J}_p$ using $\widetilde{J}_p = NE_p - (NE_p - \widetilde{J}_p)$
9:	Select positive definite $r \times r$ matrices M_0, D_0 and K_0 such that the quadratic pencil $P_0(s) = s^2M_0 + sD_0 + K_0$ has all roots $s \in \mathbb{C}^-$
10:	Use M_0, D_0, K_0 and N to solve for the auxiliary matrix identities for A_1, A_2
11:	Solve for J_p and J_v using $J_p = (A_2M_0^{-1})^{\dagger}\widetilde{J}_p$ and $J_v = (A_2M_0^{-1})^{\dagger}\widetilde{J}_v$
12:	Solve for H using $H = (A_2M_0^{-1})^{\dagger}P_2M^{-1}B$
13:	Using $N, E_p, E_v, J_p, J_v, P_2$ implement the natural FO

$$\left\{ M_0\ddot{\zeta} + D_0\dot{\zeta} + K_0\zeta = J_py_p + J_vy_v + Hu, \widehat{z} = A_1\zeta + A_2\dot{\zeta} + E_py_p + E_vy_v. \right.$$

4.2. Reducing the proposed natural UIFO to natural UIO, natural FO and natural observer

Since the criterion for any UIFO of vector 2nd order systems to be termed natural, is its property to reduce to a natural FO, to a natural UIO and to a natural observer, we consider the proposed UIFO in (14) and verify that it can be reduced to a FO, to an UIO and subsequently to a natural observers.

4.2.1. Reduction of natural UIFO to a natural UIO and then to a natural observer

We now show that when the vector to be estimated is the entire state vector, i.e. $r = 2n$ with

$$z = \begin{bmatrix} \mathbf{I}_{n\times n} \\ \mathbf{0}_{n\times n} \end{bmatrix} x + \begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathbf{I}_{n\times n} \end{bmatrix} \dot{x} = L_p x + L_v \dot{x},$$

and the dimension of (14) is now $s = n$, the proposed UIFO in (14a), (14b) reduces to a natural UIO. By inspection, one has $M_0 = M$ and $A_1 = L_p$ and $A_2 = L_v$. With this choice of A_1, A_2, the Eqs. (17) produce

$$N = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}K_0 & -M^{-1}D_0 \end{bmatrix},$$

and the matrix identities in (15) become

$$[P_1 \quad P_2]\begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix} - N[P_1 \quad P_2] + \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ -M^{-1}J_pC_p & -M^{-1}J_vC_v \end{bmatrix} = [\mathbf{0}_{2n\times n} \quad \mathbf{0}_{2n\times n}],$$

and which produce the natural UIO. Now to reduce the natural UIO to a natural observer, set $d = \mathbf{0}_q$. Then $E_p = \mathbf{0}_{2n\times p_1}, E_v = \mathbf{0}_{2n\times p_2}$ and thus

$$P_1 = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0}_{n\times n} \end{bmatrix}, P_2 = \begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathbf{I}_n \end{bmatrix}.$$

The above Sylvester equation reduces to

$$\begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}\left(K + J_pC_p\right) & -M^{-1}\left(D + J_vC_v\right) \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}K_0 & -M^{-1}D_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}, \tag{33}$$

leading to $K_0 = K + J_pC_p, D_0 = D + J_vC_v$. Using the above, the natural UIFO (14a), (14b) is now given by

$$\begin{cases} M\ddot{\zeta}(t) + \left(D + J_vC_v\right)\dot{\zeta}(t) + \left(K + J_pC_p\right)\zeta(t) = B_0u + J_py_p + J_vy_v, \widehat{z}(t) = \begin{bmatrix} \zeta(t) \\ \dot{\zeta}(t) \end{bmatrix}, \end{cases} \tag{34}$$

which coincides with the natural observer with $\zeta = \widehat{x}$ and $\dot{\zeta} = \dot{\widehat{x}}$.

4.2.2. Reduction of natural UIFO to natural FO and then to natural observer

This is a rather trivial exercise. Indeed, when $F_0 = \mathbf{0}_{n\times q}$, then the equations (15), (16) without the additional condition $P_2M^{-1}F_0 = 0$, follow the procedure in Section 4.1.2 and summarized in Algorithm 14.

To recover the natural observer from the natural FO, with z being a $2n$-dimensional vector (i.e. estimate the entire state), one has $r \equiv 2n$ with the associated matrices having dimension $2n \times n$ and given by

$$L_p = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0}_{n\times n} \end{bmatrix} \text{and} L_v = \begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathbf{I}_n \end{bmatrix}.$$

Setting $E_p = \mathbf{0}_{2n\times p_1}, E_v = \mathbf{0}_{2n\times p_2}$ produces $[P_1 \quad P_2] = \mathbf{I}_{2n}$. Following the steps similar to the previous case, one arrives at (33) and therefore the UIFO in (14) reduces to the natural observer (34).

4.3. Alternate design to (14)

Similar to the 1st order case in [11], one can have an observer whose dimension is different to the estimated functional dimension r. While this was presented in (14), wherein the dimension of ζ was not necessarily equal to the dimension of z, we consider this in the context of a re-defined Sylvester matrices P_1, P_2 in (15).

The UIFO is identical to (14a) with the estimate of z given by

$$\widehat{z} = Q_1\zeta + Q_2\dot{\zeta} + E_py_p + E_vy_v, \tag{35}$$

but the Sylvester Eqs. (15) will differ. Using the same estimation error $e = z - \widehat{z} = (L_p - E_pC_p)x + (L_v - E_vC_v)\dot{x} - Q_1\zeta - Q_2\dot{\zeta}$, we have that

$$\begin{aligned} \dot{e} &= \left(L_p - E_pC_p\right)\dot{x} + \left(L_v - E_vC_v\right)\ddot{x} - Q_1\dot{\zeta} - Q_2\ddot{\zeta} \\ &= \left(- \left(L_v - E_vC_v\right)M^{-1}K - Q_2M_0^{-1}J_pC_p - N\left(L_p - E_pC_p\right)\right)x + \left(\left(L_p - E_pC_p\right) - \left(L_v - E_vC_v\right)M^{-1}D - Q_sM_0^{-1}J_vC_v - N\left(L_v\right.\right. \\ &\quad \left.\left. - E_vC_v\right)\right)\dot{x} + \left(\left(L_v - E_vC_v\right)M^{-1}B - Q_2M_0^{-1}H\right)u + \left(L_v - E_vC_v\right)M^{-1}Fd + \left(Q_2M_0^{-1}K_0 + NQ_1\right)\zeta + \left(- Q_1 + Q_2M_0^{-1}D_0 + NQ_2\right)\dot{\zeta}. \end{aligned} \tag{36}$$

Defining $Q_1P_1 = L_p - E_pC_p, Q_2P_2 = L_v - E_vC_v$, then (36) produces the counterpart of (15)

$$-Q_2P_2M^{-1} - NQ_1P_1 - Q_2M_0^{-1}J_pC_p = 0, \tag{37a}$$

$$Q_1P_1 - Q_2P_2M^{-1}D - NQ_2P_2 - Q_2M_0^{-1}J_vC_v = 0, \tag{37b}$$

$$NQ_1 = Q_2M_0^{-1}K_0, \tag{37c}$$

$$NQ_2 = Q_1 + Q_2M_0^{-1}D_0, \tag{37d}$$

$$Q_2\left(P_2M^{-1}B - M_0^{-1}H\right) = 0, \tag{37e}$$

$$Q_2P_2M^{-1}F = 0. \tag{37f}$$

5. Special cases

5.1. Special case: dim(z) = dim(ζ) in (14)

One may simplify the proposed UIFO (14) by choosing $A_2M_0^{-1} = \mathbf{I}_{r\times r}$, i.e. the dimension s of the observer state ζ in Eq. 18b is set equal to the dimension r of the state functional z. In this case, the identities (15) become

$$\begin{aligned} -P_2M^{-1}K - NP_1 - J_pC_p &= \mathbf{0}_{r\times n},\\ P_1 - P_2M^{-1}D - NP_2 - J_vC_v &= \mathbf{0}_{r\times n},\\ P_2M^{-1}B_0 - H = \mathbf{0}_{r\times m}, P_2M^{-1}F_0 &= \mathbf{0}_{r\times q}, \end{aligned}$$

and the auxiliary identities (16) simplify to

$$K_0 + NA_1 = \mathbf{0}_{r\times r}, D_0 + NM_0 - A_1 = \mathbf{0}_{r\times r}, \Rightarrow D_0 = -\left(NM_0 + N^{-1}K_0\right), A_1 = -N^{-1}K_0.$$

In this case, the proposed natural UIFO (14), becomes

$$\left\{ M_0\ddot{\zeta} - \left(NM_0 + N^{-1}K_0\right)\dot{\zeta} + K_0\zeta = J_py_p + J_vy_v + P_2M^{-1}B_0u, \widehat{z} = -N^{-1}K_0\zeta + M_0\dot{\zeta} + E_py_p + E_py_v, \right.$$

and which provides a "closed-loop" damping matrix D_0 which is proportional to air damping (via $-NM_0$) and structural damping (via $-N^{-1}K_0$). To obtain J_p, J_v, E_p, E_v one simply follows Algorithm 3.

5.2. First order FOs for vector 2nd order systems

When both $y_p(t)$ and $y_v(t)$ are available, it is possible to have a 1st order structure for the FO. In fact it represents the components of a FO based on the 1st representation (8). The 1st order FO $\widehat{z}$ of z is given by

$$\begin{aligned} \dot{w} &= Nw + Q_vM^{-1}B_0u + J_py_p + J_vy_v,\\ \widehat{z} &= w + E_py_p + E_vy_v, \end{aligned} \tag{38}$$

where $N, Q_v, J_p, J_v, E_p, E_v$ will be defined below. The r-dimensional vector w denotes the observer state and $\widehat{z}$ denotes the estimate of z. The theorem below provides the conditions for $\widehat{z}$ to be an estimate of z.

Theorem 5.1. *The signal $\widehat{z}$ in (38) is an asymptotic estimate of the state functional z in (7) for any initial position $x(0)$, initial velocity $\dot{x}(0)$, any $\widehat{z}(0)$ and any bounded input $u(t)$ in (34) if and only if*

$$-Q_vM^{-1}K - NQ_p - J_pC_p = \mathbf{0}_{r\times n}, \tag{39a}$$

$$Q_p - Q_vM^{-1}D - NQ_v - J_vC_v = \mathbf{0}_{r\times n}, \tag{39b}$$

where $Q_p = L_p - E_pC_p$, $Q_v = L_v - E_vC_v$, and the $r\times r$ matrix N is Hurwitz.

Proof. The resulting estimation error e is

$$e = z - \widehat{z} = L_px + L_v\dot{x} - E_py_p - E_vy_v - w = Q_px + Q_v\dot{x} - w.$$

The time derivative of e is $\dot{e} = Q_p\dot{x} + Q_v\ddot{x} - \dot{w}$. Using the expression for $\ddot{x}$ in (7), the expression for $\dot{w}$ in (38) and adding and subtracting Ne, then $\dot{e}$ is given by

$$\begin{aligned}\dot{e} &= Ne + Q_p\dot{x} + Q_v\ddot{x} - Nw - Q_vM^{-1}B_0u - J_py_p - J_vy_v - N\big(Q_px + Q_v\dot{x} - w\big)\\ &= Ne + Q_p\dot{x} - Q_vM^{-1}D\dot{x} - Q_vM^{-1}Kx - J_py_p - J_vy_v - NQ_px - NQ_v\dot{x}\\ &= Ne + \big(Q_p - Q_vM^{-1}D - NQ_v - J_vC_v\big)\dot{x} + \big(-Q_vM^{-1}K - NQ_p - J_pC_p\big)x.\end{aligned}$$

Following the conditions (39a), (39b) in Theorem 5.1, the estimation error is then governed by

$$\dot{e} = Ne, e(0) = e_0 \neq \mathbf{0}_{r\times 1}.$$

Since the matrix N is Hurwitz by design, then we have $\lim_{t\to\infty}|e(t)| = 0$. □

Remark 5.1. The proposed FO in (38) cannot be termed a natural FO for the vector 2nd because it does not reduce to a natural observer. Setting $r = 2n$ with $E_p = \mathbf{0}_{2n\times p_1}$, $E_v = \mathbf{0}_{2n\times p_2}$ and

$$L_p = Q_p = \begin{bmatrix}\mathbf{I}_n\\ \mathbf{0}_{n\times n}\end{bmatrix}, L_v = Q_v = \begin{bmatrix}\mathbf{0}_{n\times n}\\ \mathbf{I}_n\end{bmatrix},$$

and which leads to a similar estimator as presented at the beginning of Section 3.

Remark 5.2. The equations in (39) can also be derived when one considers a functional observer for the system in 1st order form (8). In this case, one has

$$z = [L_p \quad L_v]X, \widehat{z} = w + [E_p \quad E_v]\begin{bmatrix}y_p\\ y_v\end{bmatrix}, P = [Q_p \quad Q_v], JC = [J_pC_p \quad J_vC_v],$$

and the associated Sylvester Eq. (3) is given by $PA - NP = JC$, or

$$[Q_p \quad Q_v]A - N[Q_p \quad Q_v] = [J_pC_p \quad J_vC_v].$$

A solution to this Sylvester equation exists when the spectra are disjoint $\sigma(A)\cap\sigma(N) = \varnothing$, [12]. Two special cases arise namely having solely position measurements ($C_v = \mathbf{0}_{p_2\times n}$) and it is desired to obtain an estimate of a functional of the position ($L_v = \mathbf{0}_{r\times n}$); similarly having only velocity measurements ($C_p = \mathbf{0}_{p_1\times n}$) and it is desired to estimate a functional of the velocity ($L_p = \mathbf{0}_{r\times n}$). These two cases cannot be dealt with using the procedure presented earlier, as both degenerate and result in naïve observers. Any other combination can be handled under the framework set forth in Theorem 4.1.

5.3. Asymptotic observer for functional position using position measurements

In this case, the output is $y = y_p = C_px = \begin{bmatrix}C_p & \mathbf{0}_{r\times n}\end{bmatrix}X$, and one must provide an estimate for $z = L_px = \begin{bmatrix}L_p & \mathbf{0}_{r\times n}\end{bmatrix}X$. Here, one cannot set $L_v = \mathbf{0}_{r\times n}$ in (7c), $C_v = \mathbf{0}_{p_2\times n}, J_v = \mathbf{0}_{r\times p_2}, E_v = \mathbf{0}_{r\times p_2}, P_2 = \mathbf{0}_{r\times n}$ in (14) and solve (18) with $P_2 = \mathbf{0}_{r\times n}, J_v = \mathbf{0}_{r\times p_2}$, since it degenerates. Eq. (18a) reduces to $NP_1 = -J_pC_p$ and (18b) gives $P_1 = \mathbf{0}_{r\times n}$. Upon substitution into (14), one arrives at a naïve observer.

We propose the following *natural* functional observer (cf (14))

$$\begin{aligned}&M_1\ddot{w} + D_1\dot{w} + K_1w = M_1P_1M^{-1}B_0u + J_py_p\\ &\widehat{z} = w + E_py_p\end{aligned} \tag{40}$$

where $\widehat{z}$ denotes the estimate of the unknown functional position z. The functional position error is given by

$$e = z - \widehat{z} = L_px - E_pC_px - w = P_1x - w, P_1 = L_p - E_pC_p.$$

We use the following expressions to relate the pairs (x,e) and (x,w) via

$$\begin{bmatrix}x\\ e\end{bmatrix} = \begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times r}\\ P_1 & -\mathbf{I}_r\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix} \text{and} \begin{bmatrix}x\\ w\end{bmatrix} = \begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times r}\\ P_1 & -\mathbf{I}_r\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}. \tag{41}$$

To facilitate the asymptotic properties of the error equation, consider the vector 2nd order systems (7), (41)

$$\begin{bmatrix}M & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & M_1\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{w}\end{bmatrix} + \begin{bmatrix}D & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & D_1\end{bmatrix}\begin{bmatrix}\dot{x}\,\dot{w}\end{bmatrix} + \begin{bmatrix}K & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & K_1\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix} = \begin{bmatrix}B_0u\\ M_1P_1M^{-1}B_0u\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n\times 1}\\ J_pC_px\end{bmatrix}.$$

Using (41) in the above expression, we have

$$\begin{bmatrix}M & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & M_1\end{bmatrix}\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times r}\\ P_1 & -\mathbf{I}_r\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{e}\end{bmatrix} + \begin{bmatrix}D & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & D_1\end{bmatrix}\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times r}\\ P_1 & -\mathbf{I}_r\end{bmatrix}\begin{bmatrix}\dot{x}\,\dot{e}\end{bmatrix} + \begin{bmatrix}K & \mathbf{0}_{n\times r}\\ \mathbf{0}_{r\times n} & K_1\end{bmatrix}\begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times r}\\ P_1 & -\mathbf{I}_r\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} = \begin{bmatrix}B_0u\\ M_1P_1M^{-1}B_0u\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n\times 1}\\ J_pC_px\end{bmatrix}$$

Simplifying the above, one arrives at

$$\begin{bmatrix} M & \mathbf{0}_{n\times r} \\ M_1P_1 & -M_1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} D & \mathbf{0}_{n\times r} \\ D_1P_1 & -D_1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} + \begin{bmatrix} K & \mathbf{0}_{n\times r} \\ K_1P_1 & -K_1 \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} B_0u \\ M_1P_1M^{-1}B_0u \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n\times 1} \\ J_pC_px \end{bmatrix}.$$

In order to diagonalize the mass matrix of the augmented system, we multiply both sides by the left inverse

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times r} \\ M_1P_1M^{-1} & -\mathbf{I}_r \end{bmatrix}$$

to arrive at

$$\begin{bmatrix} M & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & M_1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} D & \mathbf{0}_{n\times r} \\ M_1P_1M^{-1}D - D_1P_1 & D_1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} + \begin{bmatrix} K & \mathbf{0}_{n\times r} \\ M_1P_1M^{-1}K - K_1P_1 & K_1 \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} B_0u \\ \mathbf{0}_{m\times 1} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times r} \\ -J_pC_p & \mathbf{0}_{r\times r} \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix}.$$

Solving for the following matrix equations

$$\begin{cases} M_1P_1M^{-1}K - K_1P_1 + J_pC_p = \mathbf{0}_{r\times n}, \\ M_1P_1M^{-1}D - D_1P_1 = \mathbf{0}_{r\times n}, \end{cases} \tag{42}$$

one arrives at

$$\begin{bmatrix} M & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & M_1 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} D & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & D_1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} + \begin{bmatrix} K & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & K_1 \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} B_0u \\ \mathbf{0}_{r\times 1} \end{bmatrix}.$$

With the above choice one has that the functional position error satisfies

$$M_1\ddot{e} + D_1\dot{e} + K_1e = 0. \tag{43}$$

Lemma 5.1. *When only position output y_p is available and it is desired to reconstruct $z = L_px$, then the natural functional observer (40) ensures that the associated functional position error governed by (43) asymptotically converges to zero if and only if the Sylvester Eqs. (42) are satisfied.*

The proof follows from the preceding analysis and uses arguments similar to the proof of Theorem 5.1.

Remark 5.3. Similar to the general case in (18), one can obtain the solution to (42). Rewriting (42)

$$\left(P_1\widetilde{K} - \widetilde{K}_1P_1 + \widetilde{J}_pC_p\right)\left[L_p^\dagger \quad I - L_p^\dagger L_p\right] = 0, P_1\widetilde{D} - \widetilde{D}_1P_1 = 0,$$

where $\widetilde{K} = M^{-1}K, \widetilde{K}_1 = M_1^{-1}K_1$, $\widetilde{D} = M^{-1}D, \widetilde{D}_1 = M_1^{-1}D_1$, $\widetilde{J}_p = M_1^{-1}J_p$. The first one produces

$$K_1 = M_1L_pM^{-1}KL_p^\dagger + \left[-M_1E_p \quad (K_1E_p + J_p)\right]\begin{bmatrix} C_pM^{-1}KL_p^\dagger \\ C_pL_p^\dagger \end{bmatrix}.$$

From the second one

$$\left[-M_1E_p \quad (K_1E_p + J_p)\right]\begin{bmatrix} C_p\overline{K} \\ C_p \end{bmatrix} = -M_1L_pM^{-1}\overline{K}.$$

One may follow the steps presented in Section 4 for the solution to the above two matrix equations. Due to their similarities, the reader is directed to Section 4.

Table 1
natural UIFO.

Case	is $C_p = \mathbf{0}_{p_1\times n}$?	is $C_v = \mathbf{0}_{p_2\times n}$?	is $L_p = \mathbf{0}_{r\times n}$?	is $L_v = \mathbf{0}_{r\times n}$?	corresponding UIFO
(a1)	N	N	N	N	(14)
(a2)	N	N	N	Y	(14)
(a3)	N	N	Y	N	(14)
(a4)	Y	N	N	N	(14)
(a5)	N	Y	N	N	(14)
(b)	N	Y	N	Y	(40)
(c)	Y	N	Y	N	(44)

5.4. Asymptotic observer for functional velocity using velocity measurements

Similar to the previous case, one now has $L_p = \mathbf{0}_{r\times n}, C_p = \mathbf{0}_{p_1\times n}, E_p = \mathbf{0}_{r\times p_1}$ with $P_1 = \mathbf{0}_{r\times n}$. When these are used in (18), they produce a constrained solution for P_2. Eq. (18a) produces $P_2M^{-1}K = \mathbf{0}_{r\times n}$. Eq. (18b) simplifies to $P_2M^{-1}D + NP_2 + \widetilde{J}_vC = \mathbf{0}_{r\times n}$. Multiplying by $[L_v^\dagger \quad I_n - L_v^\dagger L_v]$ and incorporating $P_2M^{-1}K = \mathbf{0}_{r\times n}$ along with (18c) yields

$$N = -L_vM^{-1}DL_v^\dagger + \left[E_v \quad \left(NE_v - \widetilde{J}_v\right)\right]\left[C_vM^{-1}DLv^\dagger C_vL_v^\dagger\right], \left[E_v \quad \left(NE_v - \widetilde{J}_v\right)\right]\begin{bmatrix} C_v\overline{D} & C_vM^{-1}K & C_vM^{-1}F_0 \\ \overline{C}_v & 0 & 0 \end{bmatrix}$$
$$= \left[L_v\overline{D} \quad L_vM^{-1}K \quad L_vM^{-1}F_0\right].$$

Solving the second equation and then using pole placement for N, one can obtain the solution to $E_v, \widetilde{J}_v$. Once $N, \widetilde{J}_v$ are obtained and M_0, D_0, K_0 are selected, one can solve for A_1, A_2 in (17) and for H in (15). Then the UIFO (14) for this case becomes

$$\begin{aligned} &M_0\ddot{\zeta} + D_0\dot{\zeta} + K_0\zeta = J_vy_v + Hu, \\ &\widehat{z} = A_1\zeta + A_2\dot{\zeta} + E_vy_v. \end{aligned} \tag{44}$$

The stability and convergence arguments are similar to the earlier cases and are omitted.

Table 1 summarizes the UIFO and the special cases in terms of the matrices C_p, C_v, L_p, L_v.

6. Compensator design using natural functional observer: Functional output as the control input

In the event that one would like to use $z = L_px + L_v\dot{x}$ as a compensator in place of a full state feedback, one must set $r = m$ and find the gains L_p and L_v such that the closed loop system $M\ddot{x} + (D - B_0L_v)\dot{x} + (K - B_0L_p)x = 0$, satisfies some a priori defined stability and optimality criteria. For example one may use (8) to design an optimal feedback control law. Using the 1st order setting (8), a linear quadratic regulator with cost

$$J = \int_0^\infty X^T(\tau)QX(\tau) + u^T(\tau)Ru(\tau)\,d\tau,$$

will produce an optimal feedback controller $u(t) = -\overline{G}X(t)$, where $\overline{G} \in \mathbb{R}^{r\times 2n}$ is the feedback gain obtained via the solution to an associated Algebraic Riccati Equation. Closer examination of u reveals that

$$u(t) = -\overline{G}X(t) = -\left[\overline{G}_p \quad \overline{G}_v\right]\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = -\overline{G}_px(t) - \overline{G}_v\dot{x}(t), \tag{45}$$

where $\overline{G}_p, \overline{G}_v\mathbb{R}^{r\times n}$ are the decompositions of $\overline{G}$. The above controller results in the closed-loop system

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}X(t) + \begin{bmatrix} \mathbf{0}_{n\times q} \\ M^{-1}B_0 \end{bmatrix}u(t) = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}X(t) - \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ M^{-1}B_0\overline{G}_p & M^{-1}B_0\overline{G}_v \end{bmatrix}X(t) \\ &= \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}\left(K + B_0\overline{G}_p\right) & -M^{-1}\left(D + B_0\overline{G}_v\right) \end{bmatrix}X(t) = A_{cl}X(t), \end{aligned} \tag{46}$$

with the closed-loop state matrix A_{cl} given by

$$A_{cl} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ -M^{-1}\left(K + B_0\overline{G}_p\right) & -M^{-1}\left(D + B_0\overline{G}_v\right) \end{bmatrix}.$$

In its natural setting, the closed-loop system is given by $M\ddot{x} + (D + B_0\overline{G}_v)\dot{x} + (K + B_0\overline{G}_p)x = 0$. The stability of the closed loop system (46) directly follows from the optimal control formulation.

In the absence of full state information, the state feedback $u(t) = -\overline{G}X(t)$, cannot be implemented. In its place, one uses the state estimate of $X(t)$ given by

$$u(t) = -G\widehat{X}(t). \tag{47}$$

The controller (47) requires the estimate $\widehat{X}(t)$ to be realized. A state estimator, based on a Luenberger observer or a Kalman filter, would require the implementation of an $2n$ dimensional compensator which becomes computationally expensive. A computationally inexpensive alternate is to use a functional observer for the r-dimensional signal $-GX(t)$. In other words, one implements the functional observer for

$$z(t) = -\overline{G}_px(t) - \overline{G}_v\dot{x}(t), \tag{48}$$

which requires the implementation of a $r \ll 2n$ dimensional dynamical system, and uses its estimate $\widehat{z}(t)$ in place of the control signal; i.e. use $u = \widehat{z}$. The question that arises is whether the controller $u(t) = \widehat{z}(t)$ with $z(t)$ given by $z = -\overline{G}X$ can result in a stable closed-loop system.

The use of the estimate of (48) as a control signal in the UIO (10), in the UIFO and FO (14), in the first order FO (38), in the functional position observer (40), and the functional velocity observer (44), results in a stable closed-lop system. The stability arguments are similar and thus only one case will be detailed.

Lemma 6.1. *Assume that $m = r$ and consider the vector 2nd order system (7) with the linear functional (41) where the gains $\overline{G}_p, \overline{G}_v \in \mathbb{R}^{q\times n}$ are designed so that the corresponding closed-loop system (46) is stable. The minimum order controller $u(t) = \widehat{z}(t)$ where $\widehat{z}(t)$ is given by (14) and the requisite matrices in (18), are solved using $L_p = -\overline{G}_p$ and $L_v = -\overline{G}_v$, results in an exponentially stable closed-loop system, whose convergence rate is dictated by the spectra of A_{cl} and N.*

Proof. Setting $z = -\overline{G}X$ and using $u = \widehat{z} = z - e$, the closed loop system (7), (14), (45) with $d = 0$ becomes

$$\left\{ M\ddot{x} + \left(D + B_0\overline{G}_v\right)\dot{x} + \left(K + B_0\overline{G}_p\right)x = -B_0 e, \dot{e} = Ne. \right. \tag{49}$$

Using the first order formulation (46), then the augmented system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} X(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{cl} & \begin{bmatrix} \mathbf{0}_{n\times q} \\ M^{-1}B_0 \end{bmatrix} \\ \mathbf{0}_{r\times 2n} & N \end{bmatrix}\begin{bmatrix} X(t) \\ e(t) \end{bmatrix}. \tag{50}$$

The spectrum of the augmented state matrix, an upper diagonal matrix, consists of the spectra $\sigma(A_{cl})$ and $\sigma(N)$. Since both are Hurwitz matrices, the exponential stability of (49) immediately follows, [19,12]. □

Remark 6.1. The same conclusions can be drawn for the special case of functional position observer in (40). In this case one must require that the control input $u = z = L_p x$ would result in a stable nominal system $M\ddot{x} + D\dot{x} + (K + B_0\overline{G}_p)x = 0$. Use of the control input $u = \widehat{z} = z - e$ results in

$$\begin{aligned} &M\ddot{x} + D\dot{x} + \left(K + B_0\overline{G}_p\right)x = -B_0 e \\ &M_1\ddot{e} + D_1\dot{e} + K_1 e = 0. \end{aligned}$$

Similar results hold for the UIO (10), the 1st order FO (38), and the functional velocity observer (44). However, the presence of a nonzero d in (7) requires additional conditions. When the control signal $u = \widehat{z} - e$ is applied in (7), it results in

$$\left\{ M\ddot{x} + \left(D + B_0\overline{G}_v\right)\dot{x} + \left(K + B_0\overline{G}_p\right)x = -B_0 e + F_0 d, \dot{e} = Ne. \right.$$

Asymptotic stability can be invoked in this case when the disturbance signal $d \in L_2(0,\infty;\mathbb{R}^q)$, [19].

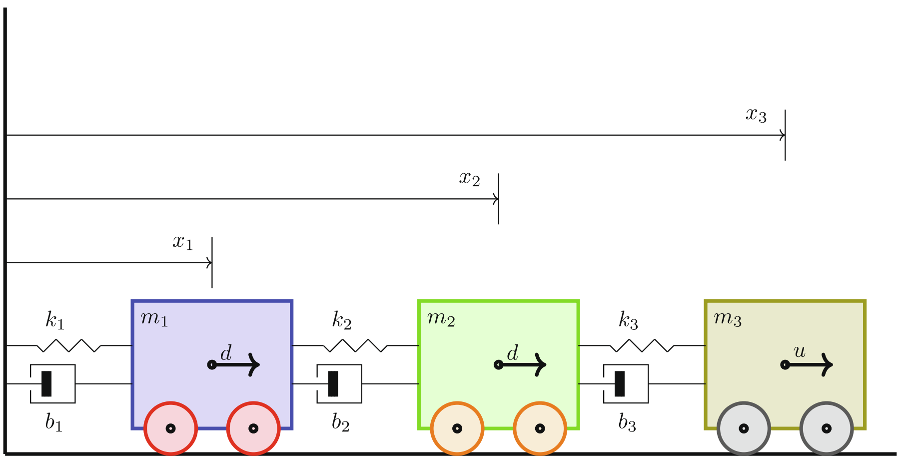

Fig. 1. Mechanical system: three masses connected via springs and dampers.

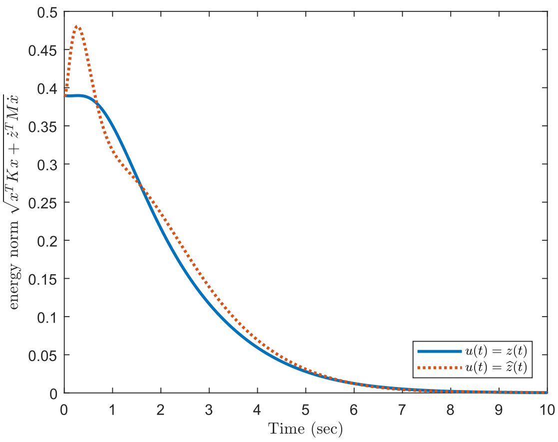

Fig. 2. Evolution of energy norm using $u = -\overline{G}X$ and using an UIFO with $u = \hat{z}$.

Table 2
$L_2(0, 10; \mathbb{R})$ norm for different controllers.

Control policy	Control norm	State norm
full state	0.269	0.5136
UIFO-based	1.737	0.5438

7. Numerical example

Consider the system (7) with

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, K = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, F_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_v = [0 \quad 1 \quad 0].$$

representing three masses connected through springs and dampers with only the last one having a control force and depicted in Fig. 1. The unknown input $d(t)$ enters in the first two masses and position measurements are available for the first and third mass, while velocity measurements are available for the second mass. The numerical values of the mass, damper and stiffness elements are $m_1 = 2, m_2 = 3, m_3 = 5, b_1 = 2, b_2 = 3, b_3 = 5, k_1 = 4, k_2 = 2$ and $k_3 = 1$. Using a pole placement scheme [20,21] to place the open loop poles $(-2.5, -0.9193 \pm 0.9193j, -0.5664, -0.0976 \pm 0.3206j)$ to the desired locations $(-2.5, -2, -1.75, -1.5, -1.25, -1)$, the resulting feedback gains for $u(t) = -\overline{G}X(t)$, are $L_p = [5.1376 \quad 10.9089 \quad 29.1756]$, $L_v = [11.3586 \quad 16.2619 \quad 24.5000]$.

We consider the special case with $\dim(z) = \dim(\zeta) = 1$ in Section 5.1. This yields $A_2 = M_0$. Selecting $M_0 = 1$ and $K_0 = 4$ with $N = -1.4$, then $D_0 = -(NM_0 + N^{-1}K_0) = 4.2571$ and $A_1 = 2.8571$. The associated UIFO in Section 5.1 was implemented with (7a) and $d(t) = 10^{-4}\sin(2t)$ using $u = -z$ and $u = -\hat{z}$ resulting in the closed-loop system (49). Fig. 2 depicts the evolution of the closed-loop norm with the full-state feedback and the UIFO-based compensator. As expected, the UIFO-based compensator has a comparable performance to the full state feedback with the difference of only implementing the reduced-order UIFO as opposed to an observer-based feedback. Table 2 summarizes the results for the two cases and which also shows the control effort when an UIFO-based compensator is used. While an increased control effort is warranted in the UIFO-based compensator, the computational savings can be used to counterbalance the increased control effort, especially when one has to deal with large mechanical systems which would require a $2n$ dimensional observer to be simulated in real time versus a $2m$ dimensional UIFO with $m \ll n$.

8. Conclusions

A unknown input functional observer, representing the most general case of a functional observer, has been proposed for a class of mechanical systems described by vector 2nd order systems. A computational benefit in considering UIFO's in a 2nd order setting was the solution of two n dimensional Sylvester equations as opposed to $2n$ dimensional Sylvester equations when using a 1st order setting. Associated with this 2nd order setting for the design of UIFOs is the ability to recover natural observers when the entire state is desired to be estimated. Such property cannot be guaranteed when designing UIFO or UIO of mechanical systems in a 1st order setting. An additional and important benefit when considering compensator design, when the UIFO is used as the estimate of a full state feedback, it requires a minimum order compensator given by $2m$, where $m \ll n$ is the rank of the input distribution matrix. The alternative of an observer-based feedback controller requires a $2n$ dimensional observer to be realized.

An immediate extension involves mechanical systems with nonlinear dynamics whereby certain parameterizations are required to produce an adaptive natural observer for the on-line estimation of both system parameters and functionals of the state. A related extension concerns the extension to time-varying systems which produce differential Sylvester equations. Both such extensions are considered by the author and will appear in a forthcoming publication.

CRediT authorship contribution statement

Michael A. Demetriou: Conceptualization, Methodology, Validation, Investigation, Writing - original draft, Project administration, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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