

Inducing persistence of excitation through sensor motion in the adaptive estimation of spatial fields

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Abstract—An alternative to regression-based estimation of spatial fields is adaptive-based estimation. Harnessing a widely used assumption on the series expansion of an unknown spatial field, the on-line estimation of the spatial field enables the integration of the real-time estimation of the field with any other tasks required of sensing agents. Parameter convergence in the adaptive estimation case requires the property of persistence of excitation. This condition reduces to imposing the integral of the outer product of a regressor vector be uniformly positive definite. With a single sensor measurement this is impossible to achieve unless the measurements are mobile. In this work, it is shown that in the adaptive estimation of a spatial field, a single mobile sensor is capable of inducing persistence of excitation and hence provide the sought after parameter convergence. Thus, the motion of a single sensor is a necessary condition for parameter convergence. It is shown with the appropriate control design for the platform carrying onboard the sensor, it also is a sufficient condition for persistence of excitation. Numerical results examining the time-variation of the regressor vector to induce a persistence of excitation along with user-defined guidance for the adaptive estimation of spatial fields are included to demonstrate the effects of mobile sensors in inducing persistence of excitation.

I. INTRODUCTION

The estimation of spatial fields for exploration and coverage [1] used regression-based estimation schemes wherein measurement(s) from static or mobile sensor(s) deployed in the spatial field were processed using a particular algorithm. In [2] multi-robot coverage considered the density function and relied on Bayesian optimization with gaussian processes. An approach with a flavor towards quantized and imperfect channels communication constraints in wireless sensor networks was presented in [3]. A review of all existing methods for spatial field estimation are summarized in [4], [5], [6], [7], [8].

A different approach is to obtain spatial measurements, either from a network of static (immobile) sensors or from a smaller set of mobile sensors and process the information to provide an estimate of the spatial field using adaptive techniques. Instead of utilizing regression techniques, the adaptive techniques for parameter estimation rely on the iterative updating of the estimates of the unknown parameters in real time. This essentially embeds a static process (true field) into the dynamic process of its estimate and attempts to select the update rules so that an associated output error converges to zero asymptotically. Parameter convergence, and hence function estimate convergence, is only insured

when a persistence of excitation (PE) condition is satisfied. As it turns out, this PE condition imposes the uniform positive definiteness of the integral of the output product of a regressor vector. In the case of static sensors, the regressor vector, and also its outer product, are time invariant and subsequently cannot yield a uniformly positive definite integral, unless the number of sensors is greater than or equal to the number of unknown parameters and the sensors are placed in locations that do not coincide with the zeros of the interpolating functions.

Thus, when a single immobile sensor is used, parameter convergence cannot be established. However, when the sensor is mobile, it may ensure that this rank-one time varying matrix resulting from the outer product of the regressor vector, to have a uniformly positive definite integral. *Thus, persistence of excitation required for parameter convergence can be achieved by a single mobile sensor!*

The motion of the sensor is *necessary condition* to establish parameter convergence. When platform dynamics for the onboard sensor are accounted for, then the appropriate guidance control for the platform may provide the *sufficient condition* for parameter convergence.

A related and relevant work in [4] considered the kinematics of the mobile sensor and its effects on the robot motion, reminiscent of Zermelo's navigation problem with position-dependent vector velocity [9]. These field effects on the kinematics were additive terms of the velocity vector of the spatial field weighted by two unknown field parameters to be identified adaptively. Through the construction of an observer of the vehicle motion, a persistence of excitation condition was discovered to provide convergence of the unknown values of the field present in the vehicle kinematic equations. The difference of [4] with this work is that here the estimation scheme is field-centric wherein an adaptive scheme for the entire spatial field reconstruction is proposed and via the requirements for spatial field error convergence, the PE condition is expressed in terms of the time variation of the outer product of the regressor vector. A way to induce time variation of the regressor vector and possibly lead to PE is via the motion of the sensor. The resulting outcome is the complete reconstruction of the entire spatial field. Inducing the PE can be done off-line prior to any experiments by carefully designing a mobile sensor path within the spatial domain via the time-variation of the interpolating functions.

The remainder of the paper is as follows. The assumptions made for the modeling of the spatial fields are presented in Section II along with the definition of static and mobile sensor measurement. A summary of the adaptive estimation

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of time invariant spatial fields, assumed to admit a series expansion, are given in Section III. A Lyapunov-based guidance along with a user-defined guidance are also provided in Section III along with the results on the necessity of mobile sensor for PE. Showing the sufficiency part for PE via sensor motion is also given in this section which is formulated as a control problem and which only operates on the sensor-parameterized regressor vector. This *a priori* selects the mobile sensor path to achieve PE. Extensive numerical results on achieving a PE for a selected regressor vector along with the reconstruction of a spatial field in 1D and 2D are presented in Section IV. Conclusions and future work follow in Section V.

II. MODELING OF SPATIAL FIELDS

Consider the spatial field denoted by $f(x)$, where $x \in \Omega$ is the spatial variable and $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is the spatial domain. The unknown spatial field is assumed to be bounded $f \in L^\infty(\Omega)$, differentiable $f \in C^\infty(\Omega)$ and further it assumes an expansion of the form

$$f(x) = \sum_{i=1}^n \theta_i \phi_i(x), \quad x \in \Omega, \quad (1)$$

where θ_i are the unknown parameters and $\phi_i(x)$ are known interpolating functions with the appropriate regularity.

Measurement from a pointwise-in-space sensor is assumed available. The sensor is placed at the spatial location $x_s \in \Omega$, termed the *sensor centroid*, and is given by the value of the unknown function $f(x)$ evaluated at this point via

$$y = \int_{\Omega} \delta(x - x_s) f(x) dx = f(x_s). \quad (2)$$

To account for a mobile sensor, the centroid is assumed time-varying and (2) is re-written to account for the sensor motion

$$y(t) = \int_{\Omega} \delta(x - x_s(t)) f(x) dx = f(x_s(t)). \quad (3)$$

Note that the sensor measurement in (3) is time varying due to the sensor motion. Additionally, (3) includes (2) as a special case when $\dot{x}_s = 0$.

Using (1), the output $y(t)$ in (3) is explicitly expressed in terms of the unknown parameters

$$y(t) = \sum_{i=1}^n \theta_i \phi_i(x_s(t)). \quad (4)$$

The measurement (4) can be written in compact form using

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}, \quad \Phi(x_s(t)) = \begin{bmatrix} \phi_1(x_s(t)) \\ \vdots \\ \phi_n(x_s(t)) \end{bmatrix},$$

as follows

$$y(t) = \Phi^T(x_s(t)) \theta. \quad (5)$$

Using the assumption $f \in L^\infty(\Omega)$, the unknown parameter θ belongs to the parameter space $\Theta = \{p \in \mathbb{R}^n : \|p\|_{\mathbb{R}^n} < \infty\}$.

Problem statement: Using the mobile sensor measurement (3), identify the unknown parameters $\theta \in \Theta$ in real-time.

III. ADAPTIVE ESTIMATION OF SPATIAL FIELDS

If the expansion (1) is assumed, then the *adaptive estimate* of $f(x)$ is given by

$$\hat{f}(t, x) = \sum_{i=1}^n \hat{\theta}_i(t) \phi_i(x), \quad x \in \Omega, \quad t \in \mathbb{R}^+, \quad (6)$$

where $\hat{\theta}_i(t) \in \mathbb{R}^1$, $i = 1, \dots, n$ are the *adaptive estimates* of the unknown constant coefficients θ_i . Evaluating the above expression at the sensor centroid $x_s(t)$, we have

$$\hat{y}(t) = \int_0^L \delta(x - x_s(t)) \hat{f}(t, x) dx = \hat{f}(t, x_s(t)) = \sum_{i=1}^n \hat{\theta}_i(t) \phi_i(x_s(t)).$$

Similar to (5), the *estimated output* can be written as

$$\hat{y}(t) = \Phi^T(x_s(t)) \hat{\theta}(t), \quad (7)$$

where the $n \times 1$ vector $\Phi(x_s(t))$, was defined earlier. The *output estimation error* $\epsilon(t) \triangleq \hat{y}(t) - y(t)$ is

$$\epsilon(t) = \Phi^T(x_s(t)) \tilde{\theta}(t), \quad (8)$$

where $\tilde{\theta}(t)$ is the *parameter error* vector. This is also given by the spatially varying *state error*

$$e(t, x) = \sum_{i=1}^n \tilde{\theta}_i(t) \phi_i(x), \quad x \in \Omega, \quad t \in \mathbb{R}^+,$$

evaluated pointwise at the sensor centroid x_s . The expression in (8) is in a form suitable for extracting the adaptive laws for $\tilde{\theta}(t)$, [10]. The mobile sensor guidance is represented by the sensor velocity $\dot{x}_s(t)$ and is also desired to be identified.

A. Lyapunov-based adaptation and guidance

To extract the adaptation of $\tilde{\theta}(t)$ and the sensor guidance $\dot{x}_s(t)$, consider the Lyapunov function

$$V(\epsilon) = \frac{1}{2\gamma} \epsilon^2(t), \quad \gamma > 0.$$

Its derivative is

$$\begin{aligned} \dot{V} &= \frac{1}{\gamma} \epsilon(t) \frac{d}{dt} \left(\Phi^T(x_s(t)) \tilde{\theta}(t) \right) \\ &= \frac{1}{\gamma} \epsilon(t) \left[\frac{\partial \Phi^T(x)}{\partial t} \Big|_{x=x_s(t)} \tilde{\theta}(t) + \Phi^T(x_s(t)) \dot{\tilde{\theta}}(t) \right], \end{aligned}$$

The adaptation

$$\dot{\tilde{\theta}}(t) = \hat{\tilde{\theta}}(t) = -\gamma \Phi(x_s(t)) \epsilon(t), \quad \hat{\tilde{\theta}}(0) \in \Theta, \quad (9)$$

changes the Lyapunov derivative to

$$\dot{V} = -\epsilon^2(t) \Phi^T(x_s(t)) \Phi(x_s(t)) + \frac{1}{\gamma} \epsilon(t) \left[\frac{\partial \Phi^T(x)}{\partial t} \Big|_{x=x_s(t)} \right] \tilde{\theta}(t)$$

The second term is simplified to

$$\frac{1}{\gamma} \epsilon(t) \left[\frac{\partial \Phi^T(x)}{\partial t} \Big|_{x=x_s(t)} \right] \tilde{\theta}(t) = \frac{1}{\gamma} \epsilon(t) \dot{x}_s(t) \epsilon_x(t). \quad (10)$$

The proposed guidance is

$$\dot{x}_s(t) = -v \text{sign}(\epsilon(t)) \text{sign}(\epsilon_x(t)), \quad x_s(0) = x_{s0}, \quad (11)$$

where v is the assumed sensor speed and ϵ_x the spatial gradient or ϵ , defined in (13). The guidance (11) results in the further simplification of the Lyapunov derivative

$$\begin{aligned} \dot{V} &= -\epsilon^2(t) \Phi^T(x_s(t)) \Phi(x_s(t)) - \frac{v}{\gamma} |\epsilon(t)| |\epsilon_x(t)| \\ &\leq -\epsilon^2(t) \Phi^T(x_s(t)) \Phi(x_s(t)). \end{aligned} \quad (12)$$

A more straightforward way to extract the guidance (11) is to consider the individual terms in (10)

$$\begin{aligned}\varepsilon(t) \left[\frac{\partial \Phi^T(x)}{\partial t} \Big|_{x=x_s(t)} \right] \tilde{\theta}(t) &= \varepsilon(t) \sum_{i=1}^n \frac{\partial}{\partial t} (\varphi_i(x_s(t))) \tilde{\theta}_i(t) \\ &= \varepsilon(t) \sum_{i=1}^n \tilde{\theta}_i(t) \left(\frac{\partial \varphi_i(x)}{\partial x} \Big|_{x=x_s(t)} \right) \dot{x}_s(t) \\ &= \varepsilon(t) \left(\frac{\partial \varepsilon(t)}{\partial x} \Big|_{x=x_s(t)} \right) \dot{x}_s(t),\end{aligned}$$

where

$$\varepsilon_x(t) \triangleq \frac{\partial \varepsilon(t)}{\partial x} \Big|_{x=x_s(t)} = \frac{\partial \varepsilon(t, x)}{\partial x} \Big|_{x=x_s(t)}. \quad (13)$$

Lemma 1: Consider the unknown function $f(x)$ admitting the expansion (1). Assume that the mobile sensor can measure both the value of the function at its current location and the associated spatial gradient. The adaptations of the unknown parameters in (9) along with the guidance (11) ensure that all signals are bounded if

$$\Phi^T(x_s(t))\Phi(x_s(t)) = \sum_{i=1}^n \varphi_i^2(x_s(t)) \geq \beta, \quad \forall t \in \mathbb{R}^+,$$

is uniformly positive definite with lower bound $\beta > 0$. Additionally, one has

$$\lim_{t \rightarrow \infty} |\varepsilon(t)| = 0 \quad (14)$$

with $\hat{\theta}, \dot{\hat{\theta}} \in L^\infty(0, \infty; \mathbb{R}^n)$ (i.e. $\hat{\theta}, \dot{\hat{\theta}} \in \Theta$). If additionally, one has that the $n \times n$ rank-one matrix $\Phi(x_s(t))\Phi^T(x_s(t))$ has a uniformly positive definite integral via

$$\alpha_1 \mathbf{I}_n \geq \frac{1}{T_0} \int_t^{t+T_0} \Phi(x_s(\tau))\Phi^T(x_s(\tau)) d\tau \geq \alpha_0 \mathbf{I}_n, \quad (15)$$

then it guarantees parameter convergence

$$\lim_{t \rightarrow \infty} \|\hat{\theta}(t) - \theta\|_{\mathbb{R}^n} = 0 \quad (16)$$

Proof: The uniform boundedness assumption of $\sum_{i=1}^n \varphi_i^2(x_s(t))$, updates (12) to

$$\dot{V} \leq -\beta \varepsilon^2(t) < 0.$$

Using Lyapunov stability theory one has convergence of ε to zero and $\varepsilon \in L^\infty$. Using (8) we have $\tilde{\theta} \in L^\infty(0, \infty; \mathbb{R}^n)$ and using (9), we have $\dot{\tilde{\theta}} \in L^\infty(0, \infty; \mathbb{R}^n)$. If (15) is satisfied, then using [10, ch. 4] we have parameter convergence. ■

Remark 1: The condition (15) is called *persistence of excitation* (PE). The matrix $\Phi(x_s(t))\Phi^T(x_s(t))$, formed by the outer product of the regressor vector $\Phi(x_s(t))$, gives an $n \times n$ matrix of rank one. If the sensor is fixed with $\dot{x}_s = 0$, then this matrix is constant and can *never* satisfy (15) since

$$\begin{aligned}\frac{1}{T_0} \int_t^{t+T_0} \Phi(x_s)\Phi^T(x_s) d\tau &= \Phi(x_s)\Phi^T(x_s) \frac{1}{T_0} \int_t^{t+T_0} 1 d\tau \\ &= \Phi(x_s)\Phi^T(x_s) \not\geq \alpha_0 \mathbf{I}_n.\end{aligned}$$

This leads to the next result.

Corollary 1: The motion of the single sensor is a necessary condition for parameter convergence (16).

Indeed, for a single sensor one must have φ_i be time varying and this is only achieved by the sensor motion.

One can also obtain the sufficiency if the guidance ensures condition (15). This is considered in the next section.

B. User-defined guidance

The adaptive law (9) can also be obtained using a gradient scheme to extract the adaptive laws. From [10] for finite dimensional systems or from [11] for spatially distributed systems, using

$$J(\tilde{\theta}) = \frac{1}{2} (\tilde{\theta}^T(t)\Phi(x_s(t)) - y(t))^2, \quad (17)$$

the gradient-based law $\dot{\hat{\theta}} = -\gamma \nabla J(\hat{\theta})$, from [10] results in

$$\dot{\hat{\theta}}(t) = -\gamma \Phi(x_s(t))\varepsilon(t),$$

which is identical to (9). To analyze the stability of the gradient-based adaptive law, one uses the Lyapunov function

$$V = \frac{1}{2\gamma} \tilde{\theta}^T(t)\tilde{\theta}(t),$$

see [10]. However, the gradient-based adaptation provides the necessary condition for persistence of excitation.

Lemma 2: If the sensor guidance, defined by $\dot{x}_s(t)$, is selected so that (15) is satisfied, then the prescribed motion of the sensor is a necessary and sufficient condition for parameter convergence (16).

Proof: When the sensor is mobile, rendering the centroid $x_s(t)$ time-varying, then it is possible for the $n \times n$ rank-one matrix $\Phi(x_s(t))\Phi^T(x_s(t))$ to have a uniformly positive definite integral over a time interval $[t, t+T_0]$. If the sensor obeys the integrator dynamics

$$\dot{x}_s(t) = u(t) \quad (18)$$

where u denotes the control signal, then $u(t)$ can likewise be selected to ensure (15) is satisfied. ■

Remark 2: If the integrator dynamics (18) are not given, then the guidance given by $\dot{x}_s(t)$ can be selected to ensure that (15) is satisfied. In this case we have a *user-defined guidance*.

The user-defined guidance selects the sensor centroid $x_s(t)$ to ensure (15) is satisfied. This selection operates on the interpolating functions $\varphi_i(x)$ and can be made prior to any spatial field identification experiments. Once the interpolating functions are given, then the appropriate choice of $x_s(t)$ (path planning) for (15) provides the sensor guidance.

The question that arises is how does one select $\dot{x}_s(t)$, or $u(t)$ in (18) such that (15) is satisfied. This will be considered in the numerical examples.

C. Mobile sensor dynamics

If the mobile sensor has dynamics given by an equation of the form (18), then it is possible to select the control signal $u(t)$ so that a user-defined guidance ensures that parameter convergence is attained. Denote the function $\xi(t)$ that ensures $\Phi(\xi(\tau))\Phi^T(\xi(\tau))$ has uniformly positive definite integral. Then a tracking control of the form

$$u(t) = \dot{\xi}(t) - k_p(x_s(t) - \xi(t)), \quad k_p > 0, \quad (19)$$

ensures that the mobile sensor platform attains the desired sensor guidance guaranteeing the PE condition (15).

The above law cannot be used for the Lyapunov-based guidance (11) since $\xi(t)$ cannot be generated in closed form and is not differentiable.

When second integrator dynamics are assumed

$$\ddot{x}_s(t) = u(t), \quad (20)$$

then a law similar to (19) can be derived to ensure that the PE condition is satisfied. Using $\xi(t)$ to denote the function (the desired sensor trajectory) that satisfies (15), then the control law for (20) is given by

$$u(t) = \ddot{\xi}(t) - k_d(\dot{x}_s(t) - \dot{\xi}(t)) - k_p(x_s(t) - \xi(t)), \quad (21)$$

with $k_p, k_d > 0$. The algorithm below summarizes the control design that ensures the PE condition is satisfied.

Algorithm 1 Inducing PE via sensor motion

- 1: **initialize:** Determine the function $\xi(t)$ so that the rank-one $n \times n$ matrix $\Phi(\xi(t))\Phi^T(\xi(t))$ has a uniformly positive definite integral
 - 2: **if** mobile sensor has no dynamics **then**
 - 3: select the sensor centroid as $x_s(t) = \xi(t)$
 - 4: **end if**
 - 5: **if** mobile sensor obeys (18) **then**
 - 6: implement controller (19)
 - 7: **end if**
 - 8: **if** mobile sensor obeys (20) **then**
 - 9: implement controller (21)
 - 10: **end if**
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IV. EXAMPLES

A. Example 1: Testing the PE condition (15)

It is assumed that the unknown function has the expansion

$$f(x) = \sum_{i=1}^3 a_i \varphi_i(x), \text{ in } \Omega = [0, L] = [0, 1] \subset \mathbb{R}^1 \text{ where}$$

$$\varphi_i(x) = (\sqrt{2}/\sqrt{L}) \sin(i\pi x/L), \quad i = 1, 2, 3.$$

A way to check (15) is to consider the matrix $\Phi(\xi(t))\Phi^T(\xi(t))$ for a prescribed sensor trajectory $\xi(t)$

$$\begin{bmatrix} \varphi_1^2(\xi(t)) & \varphi_1(\xi(t))\varphi_2(\xi(t)) & \varphi_1(\xi(t))\varphi_3(\xi(t)) \\ \varphi_2(\xi(t))\varphi_1(\xi(t)) & \varphi_2^2(\xi(t)) & \varphi_2(\xi(t))\varphi_3(\xi(t)) \\ \varphi_3(\xi(t))\varphi_1(\xi(t)) & \varphi_3(\xi(t))\varphi_2(\xi(t)) & \varphi_3^2(\xi(t)) \end{bmatrix}.$$

Following the analysis from a regression-based estimation, one could try to avoid the sensor residing at any of the zeros of the three functions φ_i . For the above selection of φ_i , the zeros are at the spatial locations $L/3, L/2, 2L/3$. However, as mentioned earlier, the 3×3 matrix is rank one regardless where the sensor resides. Such a matrix can be made to have uniformly positive definite integral with a moving sensor.

We consider different sensor trajectories and evaluate the integral of the above matrix over an interval $[t, t + T_0]$ and examine its positive definiteness

$$\xi_1(t) = (0.5 + 0.45 \sin(2\pi t))L$$

$$\xi_2(t) = (0.413 + 0.1v(t))L$$

$$\xi_3(t) = (2 + \cos(\pi i))L/5, \quad i \in \mathbb{Z}.$$

trajectory	time interval	α_0	α_1
ξ_1	[0.1, 0.6]	0.6976	1.3437
ξ_1	[5.09, 5.1]	34.8781	67.1825
ξ_1	[8, 9]	0.3488	0.6718
ξ_2	[0.1, 0.6]	0.0566	2.5730
ξ_2	[5.09, 5.1]	2.8276	128.6520
ξ_2	[8, 9]	0.0283	1.2865
ξ_3	[0.1, 0.6]	1.2500	1.2500
ξ_3	[5.09, 5.1]	62.5000	62.5000
ξ_3	[8, 9]	0.6250	0.6250

Table 1. Effects of sensor trajectory $\xi(t)$ on the PE bounds in (15).

The last one switches sequentially between the two values $x_s = 0.2L$ and $x_s = 0.6L$ emulating a sensor that “jumps” between these two locations. The function $v(t)$ in $\xi_2(t)$ is a random number with zero mean and variance 1, generated from the Matlab function `randn`.

Using the above three sensor trajectories, the matrix $\Phi\Phi^T$ was integrated in the intervals [0.1, 0.6], [5.09, 5.1] and [8, 9]. In all three cases, the integral was uniformly positive definite with the bounds α_0, α_1 in (15) given in Table I.

The most surprising observation is that of $\xi_2(t)$ which switches sequentially between two locations. This means that there are *only two* pointwise spatial field measurements that are cycled through the adaptation via the sensor jumping (i.e. hopping) and can provide a complete parameter convergence.

B. Example 2: Estimating $f(x)$ using minimum number of measurements

The unknown function is

$$f(x) = \varphi_1(x) + 0.3\varphi_2(x) - 0.7\varphi_3(x), \quad x \in [0, 1],$$

with the φ_i defined in Example 1. The user assumes that the unknown function has the expansion

$$f(x) = \sum_{i=1}^3 \theta_i \varphi_i(x).$$

In other words, it is assumed that there are 3 unknown coefficients with true values $\theta_1 = 1, \theta_2 = 0.3, \theta_3 = -0.7$. The adaptive estimation scheme must identify the coefficients and must achieve

$$\lim_{t \rightarrow \infty} \hat{\theta}_1(t) = 1, \quad \lim_{t \rightarrow \infty} \hat{\theta}_2(t) = 0.3, \quad \lim_{t \rightarrow \infty} \hat{\theta}_3(t) = -0.7,$$

We consider a moving sensor with measurements and a user-defined trajectory

$$y(t) = f(x_s(t)), \quad x_s(t) = 0.5 + 0.05 \sin(50\pi t).$$

The adaptation (9) was used with $\gamma = 20$ and $\hat{\theta}_i(0) = 0.1, i = 1, 2, 3$. The moving sensor is able to induce the PE condition (15) and subsequently provide the desired parameter convergence. This is depicted in Figure 1, where it is seen that all three errors $\hat{\theta}_i(t)$ asymptotically converge to zero.

The unknown $f(x)$ is successfully reconstructed as observed in Figure 2; the function estimate $\hat{f}(x, t)$ at the final time $t_f = 30s$ is indistinguishable from the true function.

The evolution of the relative error, given by the norm

$$100 \frac{\|e(t, x)\|_{L_2(\Omega)}}{\|x\|_{L_2(\Omega)}} = 100 \frac{\sqrt{\int_0^1 e^2(t, x) dx}}{\sqrt{\int_0^1 f^2(x) dx}},$$

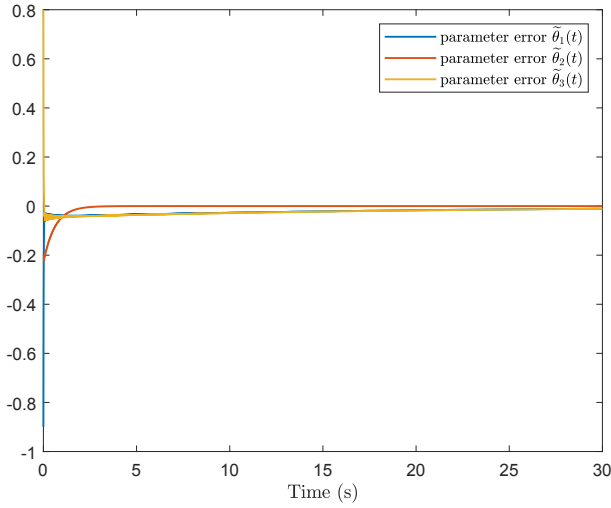


Fig. 1: Example 2: Evolution of parameter errors $\tilde{\theta}_i(t)$.

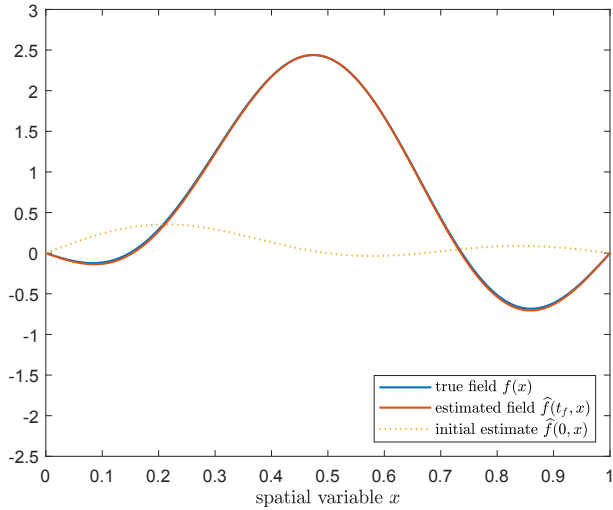


Fig. 2: Example 2: Spatial evolution of true unknown $f(x)$, and its adaptive estimate $\hat{f}(t, x)$ at $t = 0$ and at $t_f = 30$.

is depicted in Figure 3. It starts with a value of 97.11% and finishes with a value of 1.214% at the final time.

When the unknown spatial field is known to admit a series expansion, then a user-defined guidance can provide PE and hence convergence of $\hat{f}(t, x)$ to $f(x)$. This enables agent-based monitoring and control systems to deploy mobile agents in a spatial field and *concurrently* perform field estimation along with any other tasks required.

C. Example 3: Estimating a 2D field $f(x, y)$

The unknown 2D function is selected as

$$f(x, y) = 1.6\phi_2(x)\phi_1(y) - 0.2\phi_3(x)\phi_1(y), \quad x \in [0, L_x], y \in [0, L_y],$$

where ϕ_i are the same interpolating functions used in Example 1. It is assumed to admit the expansion

$$f(x, y) = \sum_{i=1}^3 \theta_i \phi_i(x) \phi_1(y).$$

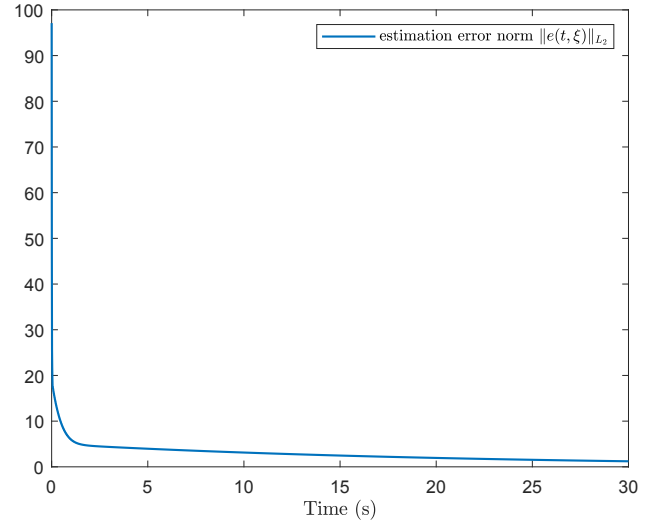


Fig. 3: Example 2: Evolution of the relative $L_2(\Omega)$ norm of the estimation error $e(t, x)$. At initial time the relative error is 97.11% and at the final time it is 1.214%.

A user-defined guidance is selected with

$$x_s(t) = (0.5 - 0.1 \sin(10\pi t))L_x, y_s(t) = (0.5 - 0.1 \cos(6\pi t))L_y.$$

The mobile sensor measurement is given by

$$\begin{aligned} z(t) &= \int_0^{L_x} \int_0^{L_y} \delta(x - x_s(t)) \delta(y - y_s(t)) f(x, y) dy dx \\ &= f(x_s(t), y_s(t)). \end{aligned}$$

The mobile sensor output can be written in the form (5) with

$$\Phi(x_s(t), y_s(t)) = \begin{bmatrix} \phi_1(x_s(t))\phi_1(y_s(t)) \\ \phi_2(x_s(t))\phi_1(y_s(t)) \\ \phi_3(x_s(t))\phi_1(y_s(t)) \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}.$$

Using the adaptive law (9) with $\gamma = 2$, the parameter errors $\tilde{\theta}_i(t)$ are depicted in Figure 4, where it is observed that all three converge to zero. The relative $L_2(\Omega)$ error norm

$$100 \sqrt{\int_0^{L_x} \int_0^{L_y} e^2(t, x, y) dy dx} / \sqrt{\int_0^{L_x} \int_0^{L_y} f^2(x, y) dy dx}$$

is depicted in Figure 5, where it is also observed that the relative state error converges to zero in the $L_2(\Omega)$ norm. Finally, the spatial distribution of $e(t, x, y)$ at the final time $t_f = 40$ s is depicted in Figure 6. For comparison, the distribution of the unknown function $f(x, y)$ is also included. The pointwise convergence of the error $e(t, x, y)$ is observed in Figure 6.

V. CONCLUSIONS AND POSSIBLE EXTENSIONS

In estimating spatial fields, the use of adaptive methods with mobile sensors provides an alternative to the use of regression-based field reconstruction. The advantage of the adaptive techniques is that the mission of a mobile agent within a spatial field can be integrated with the learning scheme and the mobile sensor will be able to learn “on-the-fly” without separating the tasks of learning and exploration/coverage. In doing so, it turns out that a single sensor due to its motion can induce a PE condition and thus ensure

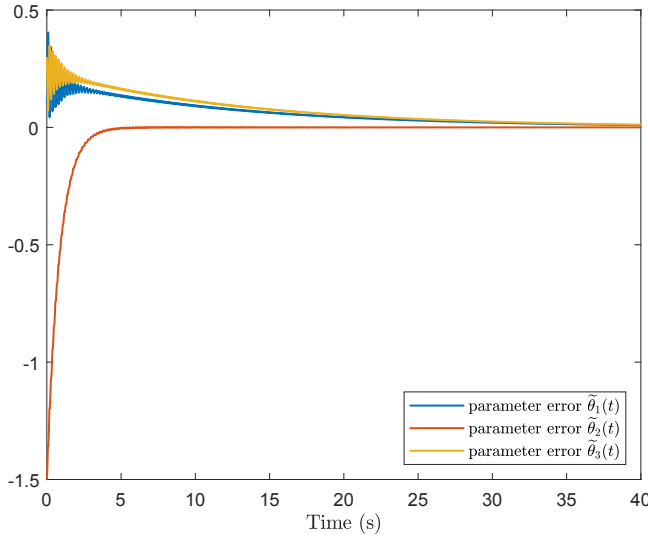


Fig. 4: Example 3: Evolution of parameter errors $\tilde{\theta}_i(t)$.

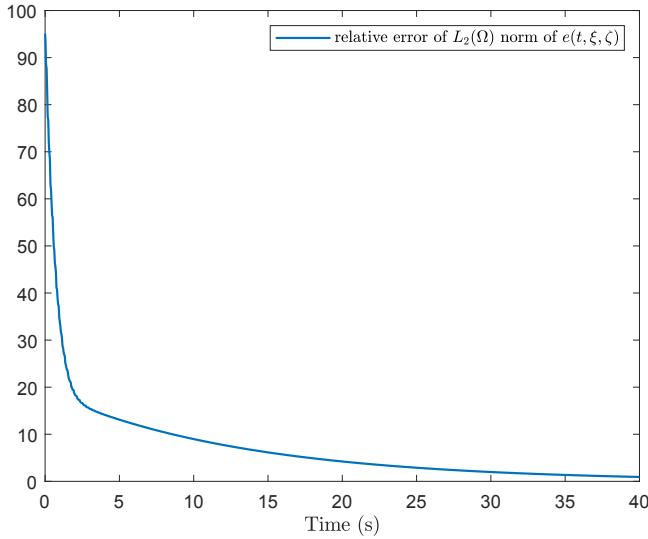


Fig. 5: Example 3: Evolution of relative error norm.

estimated spatial field convergence.

The extension to 2D and possibly 3D spatial fields may require more regularity assumptions on the field and its assumed expansion. Another interesting extension is when the assumed series expansion is not guaranteed and instead

$$f(x) = \sum_{i=1}^n \theta_i \phi_i(x) + \Delta f$$

where Δf is the approximation error. The above expansion is artificial with the “known” parameters θ_i being the artificial parameters that minimize the L^2 norm between $f(x)$ and $\sum_{i=1}^n \theta_i \phi_i(x)$. The “true” parameters are not known, but are assumed to exist for analysis purposes. In other words, there exist unknown artificial parameters θ_i that minimize the L_2 norm of Δf . Use of the adaptive techniques presented in this paper may guarantee state error convergence to zero. However, this will not be the true state error $\hat{f}(t, x) - f(x)$, but the error between $\hat{f}(t, x)$ and the assumed expansion

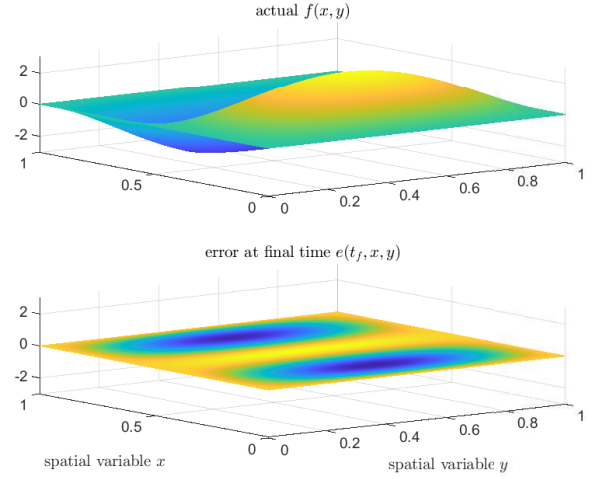


Fig. 6: Example 3: Evolution of relative error norm.

$\sum_{i=1}^n \theta_i \phi_i(x)$. The convergence of $\hat{f}(t, x)$ is offset by the unknown approximation error Δf . However, such an error can be reduced when larger n is used. Alas, the mathematical framework may not be suitable for this and instead one may consider kernel spaces as considered in [12]. This work is currently being undertaken by the author and will appear in a forthcoming publication.

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