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Fault detection and accommodation of positive real infinite dimensional systems using adaptive RKHS-based functional estimation

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Abstract— **This paper presents an adaptive functional estimation scheme for the fault detection and diagnosis of nonlinear faults in positive real infinite dimensional systems. The system is assumed to satisfy a positive realness condition and the fault, taking the form of a nonlinear function of the output, is assumed to enter the system at an unknown time. The proposed detection and diagnostic observer utilizes a Reproducing Kernel Hilbert Space as the parameter space and via a Lyapunov redesign approach, the learning scheme for the unknown functional is used for the detection of the fault occurrence, the diagnosis of the fault and finally its accommodation via an adaptive control reconfiguration. Results on parabolic PDEs with either boundary or in-domain actuation and sensing are included.**

I. Introduction

In the adaptive parameter identification of dynamical systems, provided the said systems admit a certain parametrization, a necessary condition for the extraction of the adaptive laws using input and output information, is that the system satisfies a positive realness condition [1]. The transfer matrix from the input to the output must have certain properties which in the time domain are equivalent to the system satisfying a Lyapunov equation and the input and output matrices are coupled via the solution to the Lyapunov equation.

This property found its way in the use of adaptive techniques for fault detection, diagnosis and accommodation of dynamical systems. When faults, be they component, actuator or sensor, are modelled as nonlinear functions of the available signals (input or output), then one may assume a series expansion parametrization for the nonlinear function and adaptively estimate the weights in the expansion. This approach is used for both parameter estimation in structurally perturbed systems and systems where the fault function enters as an input and admits the series expansion parametrization [2]. Similar approaches were applied in [3] using flatness-based fault diagnosis utilizing input and output signals and in [4] using backstepping methods. When the function does not admit such a series expansion, then neural networks are utilized to aid with the adaptive estimation. However, one a priori selects the dimension of the parameter space and proceeds with the adaptive estimation design. This of course is restrictive since one must fix a priori the parameter space dimension.

Another way of modeling the adaptive functional estimation is via the use of Reproducing Kernel Hilbert Spaces (RKHS), which provides a *natural* setting for the parameter space. The paper [5], which itself followed the fundamental work in [6], formulated the problem of adaptive functional estimation of strictly positive real infinite dimensional systems via the use of a RKHS as the parameter space.

The idea of using RKHS as the natural parameter space is used in this work to model faults in positive real infinite dimensional systems. The fault functions are assumed to be nonlinear functions of available signals, such as inputs or outputs, and this paper proposes an adaptive fault detection observer to detect the presence of the fault. The diagnosis of the fault, namely the adaptive functional estimation, is abstractly viewed in the Hilbert space (the state space) and the RKHS (the functional parameter space). The last component of the fault management policy is the fault accommodation stage which is achieved via the appropriate control reconfiguration. This reconfiguration uses the adaptive estimates of the functional estimation to cancel the effects of the fault function, thereby attempting to bring a faulty system, into its pre-fault performance.

II. Motivation and Abstract framework

A representative PDE that falls under the abstract framework considered here is the diffusion PDE with boundary actuation and sensing

$$\begin{aligned}&\frac{\partial x(t,\xi)}{\partial t} = a_1\frac{\partial^2 x(t,\xi)}{\partial \xi^2},\;\; x(0,\xi)=x_0(\xi),\;\; 0\le\xi\le\ell,\\ &x(t,0)=0,\quad x_\xi(t,\ell)=u(t)+\beta(t-\tau_f)f(y(t)),\\ &y(t)=x(t,\ell),\end{aligned} \tag{1}$$

The state is $x(t,\xi)$ with $t\in\mathbb{R}^+$, $\xi\in\Omega=[0,\ell]$, $u(t)$ is the control signal applied at the right boundary, $\beta(t-\tau_f)$ is the *time profile* of the fault which describes the time evolution of the fault and $f(y(t))$ is the *fault function*. The process measurement $y(t)$ is given by the state at the right boundary.

The fault profile function can represent an abrupt or an incipient fault and is given by

$$\beta(t-\tau_f)=\begin{cases}0 & \text{if } t<\tau_f\\ 1-e^{-\lambda(t-\tau_f)} & \text{if } t\ge\tau_f\end{cases} \tag{2}$$

When $\lambda=\infty$, the above profile becomes the Heaviside step function representing abrupt fault and for $0<\lambda<\infty$ it represents an incipient fault (slowly developing), [2].

The fault detection, diagnosis and accommodation objective is to (i) detect the presence of the fault, i.e. estimate when $\beta\neq 0$, (ii) diagnose the fault, i.e. estimate $f(y)$, and (iii) change the control signal, in an automated manner using the estimate of $f(y)$, in order to retain the performance of the control for the healthy system. For the system in (1), detection is declared when the monitoring scheme "senses"

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the presence of the term $\beta(t-\tau_f)f(y(t))$ in the boundary. Such a detection may not occur at exactly $t=\tau_f$, but at a time thereafter $t>\tau_f$. The time instance the presence of the fault is declared is termed the *detection time* t_d and the difference of the two is termed the *detection delay* $\tau_d = t_d - \tau_f \geq 0$. Obviously one wants τ_d as small as possible.

The above PDE can be written as an evolution system in a Hilbert space X as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \beta(t-\tau_f)Bf(y(t)) \\ y(t) &= B^*x(t), \quad x(0) \in D(A). \end{aligned} \tag{3}$$

Accommodating for unbounded input and output operators, a Gelfand triple is considered $V \hookrightarrow X \hookrightarrow V^*$ with the embeddings dense and continuous. The state space X serves as the pivot space and V is a reflexive Banach space with V^* denoting its conjugate dual, [7].

The output space is denoted by $\mathcal{Y} \in \mathbb{R}^m$ and represents process measurement that is an m-dimensional vector. Assuming a square system (equal number of controls and measurements) the input space $\mathcal{U}$ coincides with $\mathcal{Y}$. Further, with collocated inputs and outputs, we have $B \in \mathcal{L}(\mathcal{Y}, V^*)$ and $B^* \in \mathcal{L}(V, \mathcal{Y})$. The state operator $A \in \mathcal{L}(V, V^*)$. For the system in (1), we have that the input and output operators are rank 1; i.e. $m=1$ with $\mathcal{U}=\mathcal{Y}=\mathbb{R}^1$.

Prior to the occurrence of the unknown fault, i.e. for $t<\tau_f$, the *healthy* system is described by the linear system

$$\text{healthy system:} \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = B^*x(t), \quad x(0) \in D(A), \end{cases} \tag{4}$$

and in terms of the PDE in (1) is

$$\begin{aligned} &\frac{\partial x(t,\xi)}{\partial t} = a_1 \frac{\partial^2 x(t,\xi)}{\partial \xi^2}, \; x(0,\xi) = x_0(\xi), \; 0 \leq \xi \leq \ell, \\ &x(t,0) = 0, \quad x_\xi(t,\ell) = u(t), \\ &y(t) = x(t,\ell). \end{aligned} \tag{5}$$

III. Adaptive fault detection, diagnosis and accommodation

The adaptive fault detection is easily accomplished with an adaptive detection observer. The adaptive fault diagnosis requires additional assumptions on the operators (A, B, B^*) and a particular parametrization of the fault function $f(y)$. For a functional parametrization based on a RKHS, additional assumptions and formulations are required.

A. Adaptive Fault Detection Observer

The proposed adaptive detection observer is given by

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + L\left(y(t) - B^*\widehat{x}(t)\right) + Bu(t), \; \widehat{x}(0) \neq x(0), \tag{6}$$

where L is the filter operator such that $A-LB^*$ generates an exponentially stable C_0 semigroup. The latter is achieved by imposing that the pair (B^*, A) be approximately observable [8]. The detection observer (6) essentially is unaware of the presence of the term $\beta(t-\tau_f)Bf(y(t))$ in (3) and thinks that the system is instead described by the healthy system (4).

To analyze the ability of the detection observer to "sense" a change in the system, define the *estimation error* $e(t) = x(t) - \widehat{x}(t)$. Using (3) and (6) we arrive at the error system

$$\dot{e}(t) = (A-LB^*)e(t) + \beta(t-\tau_f)Bf(y(t)), \; e(0) \neq 0. \tag{7}$$

A suitable signal to monitor the detection system is the *residual* signal $\varepsilon(t)$ given by

$$\varepsilon(t) = B^*e(t). \tag{8}$$

The residual, which coincides with the output error, serves as a means to detect the presence of a fault. Prior to the fault, i.e. for $t<\tau_f$, the residual is norm-bounded by $r_0(t)$

$$|\varepsilon(t)| \leq r_0(t) \triangleq M\|B^*\|e^{-\lambda_{A-LB^*}t}\|e(0)\|, \; \forall t<\tau_f, \tag{9}$$

where T_{A-LB^*} is the semigroup generated by the filter operator $A-LB^*$. Such a signal is exponentially converging to zero and one can find a bound, which will serve as a time-varying *threshold*. Thus, the threshold is given by $r_0(t)$. At the onset of the abrupt fault, the residual is given by

$$\begin{aligned} \varepsilon(t) &= B^*T_{A-LB^*}(t)e(0) \\ &\quad + \int_0^t B^*T_{A-LB^*}(t-s)B\beta(s-\tau_f)f(y(s))\,\mathrm{d}s. \end{aligned} \tag{10}$$

It is obvious that the instance the residual (10) exceeds the time varying threshold (9) the presence of a fault is declared. In fact, when $t>\tau_f$, the residual satisfies

$$\begin{aligned} r(t) = |\varepsilon(t)| &\leq r_0(t) \\ &\quad + \left| \int_0^t B^*T_{A-LB^*}(t-s)B\beta(s-\tau_f)f(y(s))\,\mathrm{d}s \right| \\ &\leq r_0(t) + \left| \int_0^t B^*T_{A-LB^*}(t-s)Bf(y(s))\,\mathrm{d}s \right|. \end{aligned} \tag{11}$$

Lemma 1: The detection observer (6) prior to the occurrence of the fault, i.e. for all $t<\tau_f$, is such that $\varepsilon(t)$ satisfies

$$|\varepsilon(t)| \leq r_0(t), \quad \forall t<\tau_f.$$

The presence of the fault in the system (3) is declared the instance the residual exceeds the time varying threshold $r_0(t)$. The proof of Lemma 1 follows from [2], [9].

B. Adaptive fault diagnosis using RKHS

The fault detection observer (6) can be deactivated once the fault is declared. However, one can easily modify (6) to become an *adaptive detection and diagnostic observer*. Finding the appropriate parametrization of the unknown term $f(y)$ would enable one (a) to ensure that no parameter adaptation takes place prior to t_d and (b) to activate the parameter updating scheme once the fault is declared.

If the unknown function $f(y)$ admits the expansion

$$f(y(t)) = \sum_{i=1}^{N} \alpha_i \phi_i(y(t)), \tag{12}$$

where $\phi_i(y(t))$ are known functions of y and α_i the unknown constant weights, with its on-line estimate taking the form

$$\widehat{f}(t, y(t)) = \sum_{i=1}^{N} \widehat{\alpha}_i(t)\phi_i(y(t)), \tag{13}$$

then the adaptive estimation scheme in [5] with the dead-zone modification presented in [2] would ensure that no parameter adaptation takes place prior to the fault declaration and it is activated after the fault declaration. The extraction of the adaptive laws for the weights $\widehat{\alpha}_i(t)$ is based on Lyapunov-redesign methods. To examine the well-posedness and convergence properties one must consider the parameter space $\Theta \in \mathbb{R}^N$ which is the space of the N-dimensional vectors with inner product $\langle\theta,\chi\rangle_\Theta = \theta^T\chi,\ \theta,\chi \in \Theta$. The resulting evolution system is viewed in $X \times \Theta$. The well-posedness of such an adaptive scheme was presented in [2].

However, the above parametrization and subsequently the adaptive detection and diagnostic observer cannot be implemented if the unknown fault function $f(y)$ does not have a parametrization given by (12).

To address this, a Hilbert space is used as the parameter space for the estimation of $f(y)$. We denote this Hilbert space of functions by Q and defined on the output space $\mathcal{Y}$ via

$$f : \mathcal{Y} \to \mathbb{R}^1, \tag{14}$$

with the *evaluation functional* over Q; this evaluation functional evaluates each function at a point $y \in \mathcal{Y}$ via

$$\lambda_y : f \to f(y), \quad \forall f \in Q. \tag{15}$$

In other words, one has

$$f(y) = \lambda_y(f). \tag{16}$$

Following the earlier work [5], when the kernels are appropriately constructed, then the evaluation functional λ_y is bounded. Then the parameter space Q becomes a Reproducing Kernel Hilbert Space (RKHS). The Riesz representation theorem has that for all $y \in \mathcal{Y}$, there exists an element $\kappa(\cdot,\cdot) : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^1$ so that the reproducing kernel $\kappa_y = \kappa(y,\cdot)$ has the *reproducing property*, in other words

$$f(y) = \lambda_y(f) = \langle f, \kappa(y,\cdot)\rangle_Q = \langle f, \kappa_y\rangle_Q, \tag{17}$$

for all $f \in Q$ and for all $y \in \mathcal{Y}$. As mentioned in [5], the inner product representation allows one to evaluate the kernel function at points in the data space, i.e. the output space $\mathcal{Y}$. By considering two different outputs y_i, y_j in the data space $\mathcal{Y}$ and the corresponding elements $f(y_i), f(y_j)$ in the feature space, i.e. the parameter space Q, one has

$$\langle f(y_i), f(y_j)\rangle_Q = \langle \kappa(y_i,\cdot), \kappa(y_j,\cdot)\rangle_Q = \kappa(y_i,y_j). \tag{18}$$

Finally, to arrive at the skew-adjoint structure of the closed-loop operator representing the state and parameter error, one must define the adjoint of the evaluation functional λ_y, denoted by $\lambda_y^* : \mathcal{Y} \to Q$ and given by

$$\langle \eta, \lambda_y(f)\rangle_{\mathcal{Y}} = \langle \eta\kappa_y, f\rangle_Q = \langle \lambda_y^*(\eta), f\rangle_Q, \quad \eta \in \mathcal{Y}. \tag{19}$$

Using the above, the system (3) is now written as

$$\dot{x}(t) = Ax(t) + Bu(t) + \beta(t-\tau_f)B\lambda_{y(t)}(f), \quad \text{in } V^*. \tag{20}$$

The structure of the detection and diagnostic observer for (20) follows the one presented in [2], [9], but with the main difference of the adaptive estimate $\widehat{f}$ of f. This adaptive detection and diagnostic observer is given by

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + L(y(t) - B^*\widehat{x}(t)) + Bu(t) + B\lambda_{y(t)}(\widehat{f}). \tag{21}$$

To demonstrate the learning of the unknown fault function $f(y)$, we first consider the case where the fault is present from the beginning, i.e. $\tau_f = 0$ and then modify the adaptive law to ensure that no adaptation takes place prior to the declaration of the fault. With $\beta(t-\tau_f)$ set to 1, (20) and (21) yield the error system

$$\dot{e}(t) = (A - LB^*)e(t) + B\lambda_{y(t)}(\widetilde{f}), \tag{22}$$

where $\widetilde{f}$ is the parameter error given by $\widetilde{f} = \widehat{f} - f$. To extract the on-line learning rules for $\widehat{f}$, we state the assumptions on the system operators and formally introduce them in the lemma statement. The operator $A - LB^*$ is denoted by A_o and the assumption on the triple (A_o, B, B^*) is that it satisfies a special case of operator Lure's equation. In this case, there exists a nonnegative constant μ and an operator $S \in \mathcal{L}(D(A_o), X)$ such that for $\varphi \in D(A_o)$

$$(A_o + \mu I)^*\varphi + (A_o + \mu I)\varphi = -S^*S\varphi. \tag{23}$$

Using Lyapunov-redesign methods to extract the on-line learning rules for $\widehat{f}$, we consider the Lyapunov functional

$$V(e,\widetilde{f}) = |e(t)|_X^2 + \langle \mathcal{G}^{-1}\widetilde{f}, \widetilde{f}\rangle_Q, \quad \mathcal{G} \in \mathcal{L}(Q,Q). \tag{24}$$

The derivative of (24) along (22) is

$$\dot{V} = \langle e, A_o e\rangle + \langle A_o e, e\rangle + 2\langle e, B\lambda_{y(t)}(\widetilde{f})\rangle_{V,V^*} + 2\langle \mathcal{G}^{-1}\dot{\widetilde{f}}, \widetilde{f}\rangle_Q.$$

The third term is the one that would provide the update laws for the learning scheme. However, it must be brought into a more suitable form. Use

$$\begin{aligned} \langle e, B\lambda_{y(t)}(\widetilde{f})\rangle_{V,V^*} &= \langle B^*e, \lambda_{y(t)}(\widetilde{f})\rangle_{\mathcal{Y}} \\ &= \langle \lambda_y^*(B^*e), \widetilde{f}\rangle_Q = \langle \lambda_y^*(\varepsilon), \widetilde{f}\rangle_Q. \end{aligned}$$

Using the above equivalent expression and the last term of the Lyapunov derivative we have

$$\begin{aligned} &2\langle e, B\lambda_{y(t)}(\widetilde{f})\rangle_{V,V^*} + 2\langle \mathcal{G}^{-1}\dot{\widetilde{f}}, \widetilde{f}\rangle_Q \\ &\qquad = 2\langle \lambda_y^*(\varepsilon) + \mathcal{G}^{-1}\dot{\widetilde{f}}, \widetilde{f}\rangle_Q. \end{aligned}$$

Forcing the above terms to zero yields the adaptive law in weak form in terms of the adjoint of the evaluation functional

$$\langle \dot{\widetilde{f}}, p\rangle_Q = \langle \dot{\widehat{f}}, p\rangle_Q = -\langle \mathcal{G}\lambda_{y(t)}^*(\varepsilon(t)), p\rangle_Q, \qquad p \in Q, \tag{25}$$

or in terms of the evaluation functional as

$$\langle \dot{\widetilde{f}}, p\rangle_Q = -\langle \varepsilon(t), \lambda_{y(t)}(\mathcal{G}p)\rangle_{\mathcal{Y}}, \qquad p \in Q. \tag{26}$$

Equations (22) and (25) are considered in the space $X \times Q$ with the augmented state $z = (e, \widetilde{f})$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \mathcal{A}(t)z(t) \tag{27}$$

where the operator $\mathcal{A}(t) : V \times Q \to V^* \times Q$ is given by

$$\mathcal{A}(t) = \begin{bmatrix} A_o & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B\lambda_{y(t)}([\cdot]) \\ -\mathcal{G}\lambda_{y(t)}^*(B^*[\cdot]) & 0 \end{bmatrix}.$$

To remove the assumption that the fault occurs at the beginning, the learning law (25) must be made to ensure that no adaptation takes place prior to the declaration of the fault. Following the *dead-zone* modification presented in [2] and adjusted for the adaptive functional estimation considered here, the adaptive law for the adaptive detection and diagnostic observer is

$$\langle \dot{\widetilde{f}}, p\rangle_Q = -\langle D_{r_0}[\varepsilon(t)], \lambda_{y(t)}(\mathcal{G}p)\rangle_{\mathcal{Y}}, \qquad p \in Q, \tag{28}$$

where the dead-zone operator is

$$D_{r_0}[\varepsilon] = \begin{cases} 0 & \text{if } |\varepsilon(t)| < r_0(t) \\ 0 & \text{if } |\varepsilon(t)| \geq r_0(t) \end{cases}.$$

We summarize the above results for the adaptive detection and diagnostic observer below.

Lemma 2: Assume that the class of PDEs under consideration is represented by (3) where the input operator $B \in \mathcal{L}(\mathcal{Y}, V^*)$ and the output operator $B^* \in \mathcal{L}(V, \mathcal{Y})$ are rank 1 operators with $\mathcal{Y} = \mathbb{R}^1$. Also assume that the control input and the fault function are not catastrophic, in the sense that the post-fault plant is bounded with $\|x(t)\| \leq c$, a.e. $t > 0$ for some positive constant c. Further assume that there exists an operator L such that $A - LB^*$ generates an exponentially stable C_0 semigroup on X and satisfying the weak version of positive realness (23). Then the proposed adaptive detection and diagnostic observer (21) with the learning laws (28) ensure that prior to the declaration of the fault no adaptation takes place. A fault is declared when the residual signal $r(t) = |\varepsilon(t)|$ exceeds the time-varying threshold $r_0(t)$ and the adaptation is activated with

$$\langle \dot{\widehat{f}}, p \rangle_Q = -\langle \varepsilon(t), \lambda_{y(t)}(\mathcal{G}p) \rangle_{\mathcal{Y}}, \qquad p \in Q, \quad t \geq t_d.$$

This ensures that after the fault occurrence, the system is well-posed and bounded, and the state estimation error converges to zero

$$\lim_{t_d < t \to \infty} |e(t)|_X \to 0.$$

Functional convergence in the sense of

$$\lim_{t \to \infty} \|\widehat{f}(y) - f(y)\|_Q = 0,$$

can be concluded provided that the system (21), (28) is persistently excited in the sense that there exists T_0, δ_0 and ε_0 such that for each $q \in Q$ with $|q|_Q = 1$ and sufficiently large $t > t_d$, there exists a $t' \in [t, t + T_0]$ such that

$$\left\| \int_{t'}^{t'+\delta_0} B\lambda_{y(\tau)}(q) \, \mathrm{d}\tau \right\|_{V^*} \geq \varepsilon_0.$$

Note that norm convergence of $\widetilde{f}$ to 0, i.e. $\|\widetilde{f}(y)\|_Q$, implies pointwise (in time) convergence of $|\widetilde{f}(y(t))|_{\mathcal{Y}}$ to 0.

Proof: A sketch is provided since it relies on arguments made in earlier works on adaptive fault detection schemes in strictly positive real infinite dimensional systems [2]. Using the positive real condition (23) for the collocated infinite dimensional system, one has that the derivative of the Lyapunov functional (24) for the error system given by (22) and (26) yields $\dot{V} \leq -2\mu |e(t)|_X^2$ for $t \geq t_d$. For such a system, the operator $\mathcal{A}(t)$ in the evolution of the augmented system (27) satisfies all the conditions laid for in [10] and therefore one has well-posedness of (27) with norm convergence of the state estimation error $e(t)$ to zero. If the persistence of excitation condition is also satisfied, then one can immediately claim functional convergence. For the interval $[\tau_f, \tau_d)$, the adaptation is not activated due to dead-zone modification and thus the error system is governed by (22) with $\widetilde{f}$ replaced by $-f$. Using the assumption that the presence of that fault function ensures a bounded $f(y)$ (via a Lipschitz condition) and via the assumption of A_o being the generator of an exponentially stable semigroup, one can argue well-posedness and boundedness of $e(t)$. ■

C. Adaptive fault accommodation using RKHS

The nominal control signal for the healthy system (4) is denoted by $u_0(t)$ and can be chosen as static feedback (function of y) or as dynamic feedback (function of $\widehat{x}$). For simplicity, it is assumed that this nominal controller is

$$u_0(t) = -K\widehat{x}(t), \tag{29}$$

where the feedback operator $K \in \mathcal{L}(V^*, \mathbb{R}^1)$ is such that the closed-loop operator $A - BK$ generates an exponentially stable semigroup on X. The closed-loop system consisting of the healthy system (4), the observer (6) with the control $u(t) = u_0(t)$ can equivalently be expressed in terms of the healthy system (4), the control $u(t) = u_0(t)$ and the error system $\dot{e}(t) = A_o e(t)$ via

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A_o \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \tag{30}$$

It is easily seen that the system (30) is exponentially stable.

When a fault is present in the system, the control (29) may not be able to guarantee the same performance as for the healthy system. To accommodate for the presence of the fault function $f(y)$ the control signal must be reconfigured so that it "cancels" the effects of $f(y)$. This is done by subtracting the estimate of the fault function

$$u(t) = u_0(t) - \lambda_{y(t)}(\widehat{f}), \quad t \geq t_d. \tag{31}$$

For $t \geq t_d$, the closed-loop system with this controller is

$$\begin{aligned} \dot{e} &= A_o e + B\lambda_y(\widetilde{f}) \\ \dot{x} &= (A - BK)x + BKe + B\lambda_y(\widetilde{f}) \\ \dot{\widetilde{f}} &= -\mathcal{G}\lambda_y^*(D_{r_0}[B^*e]) \end{aligned}$$

Expressed in terms of the aggregate state $\zeta(t) = (e(t), x(t), \widetilde{f}(t))$ for $t \geq t_d$, the closed-loop system (3), (21), (28) and (31) is

$$\dot{\zeta} = \begin{bmatrix} A_o & 0 & B\lambda_y([\cdot]) \\ BK & A - BK & B\lambda_y([\cdot]) \\ -\mathcal{G}\lambda_y^*(D_{r_0}[B^*[\cdot]]) & 0 & 0 \end{bmatrix} \zeta. \tag{32}$$

The following lemma summarizes the well-posedness of the fault accommodated closed-loop system.

Lemma 3: The fault accommodating controller (31) ensures that the closed-loop system (32) is well-posed and

$$\lim_{t \to \infty} |e(t)|_X^2 = 0.$$

The convergence rate of the state x to a residual set dictated by $\lambda_y(\widetilde{f})$ is the same as the convergence rate of the healthy system to zero. Convergence of the actual state to zero can be achieved either by imposing persistence of excitation or an L_2 bound on the functional error $\widetilde{f}$, see [11].

IV. Special Case: Finite Dimensional Systems

In the finite dimensional case the Hilbert/Sobolev spaces collapse and are equal to the state space with $V = X = V^* = \mathbb{R}^n$. The linear system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \beta(t - \tau_f)B\lambda_{y(t)}(f), \quad \text{in } \mathbb{R}^n, \\ y(t) &= Cx(t) \end{aligned} \tag{33}$$

where the state matrix $A \in \mathbb{R}^{n \times n}$, the input matrix $B \in \mathbb{R}^{n \times 1}$ and the output matrix $C \in \mathbb{R}^{1 \times n}$. In this case, we do not require collocation and therefore we can impose the general condition for SPR systems [1], whereby the triple

$\Sigma = (A, B, C)$ satisfies the matrix Lur'e equations

$$A_o^T P + P A_o = -W^T W, \qquad B^T P = C. \tag{34}$$

The associated adaptive observer is symbolically identical to the infinite dimensional counterpart in (21) and given by

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + L(y(t) - C\widehat{x}(t)) + Bu(t) + B\lambda_{y(t)}(\widehat{f}), \tag{35}$$

where $\widehat{x}(t) \in \mathbb{R}^n$ is the adaptive estimate of $x(t)$ and $\widehat{f}$ is still the adaptive estimate of f. The error equation is

$$\dot{e}(t) = (A - LC)e(t) + B\lambda_{y(t)}(\widetilde{f}). \tag{36}$$

The finite dimensional version of the Lyapunov functional (24) is given by

$$V(e, \widetilde{f}) = e^T(t)Pe(t) + \langle \mathcal{G}^{-1}\widetilde{f}(t), \widetilde{f}(t)\rangle_Q \tag{37}$$

where $\mathcal{G} \in \mathcal{L}(Q, Q)$ is the same adaptive gain operator as in (24). The derivative of this V along (36) is

$$\begin{aligned}\dot{V} &= e^T P A_o e + e^T A_o^T P e + 2e^T P B \lambda_{y(t)}(\widetilde{f}) + 2\langle \mathcal{G}^{-1}\dot{\widetilde{f}}, \widetilde{f}\rangle_Q \\ &= -e^T W^T W e + 2\varepsilon^T \lambda_{y(t)}(\widetilde{f}) + 2\langle \mathcal{G}^{-1}\dot{\widetilde{f}}, \widetilde{f}\rangle_Q\end{aligned} \tag{38}$$

Using the same manipulations as in the infinite dimensional case, the third term is

$$\varepsilon^T \lambda_{y(t)}(\widetilde{f}) = \langle \varepsilon, \lambda_{y(t)}(\widetilde{f})\rangle_{\mathcal{Y}} = \langle \varepsilon\kappa_{y(t)}, \widetilde{f}\rangle_Q = \langle \lambda^*_{y(t)}(\varepsilon), \widetilde{f}\rangle_Q$$

which provides an adaptive law identical to (25).

Lemma 4: Assume that the finite dimensional system in (33) satisfies the more general form of the strictly positive real system via Lur'e equations (34). Then the adaptive detection and diagnostic observer (35), along with the robust learning scheme that uses a dead-zone (28) ensures that no adaptation of $\widehat{f}$ takes place prior to the presence of the fault and detects the presence of the fault the instance the residual signal $\varepsilon(t) = Ce(t)$ exceeds the time-varying threshold

$$r_0(t) = |C|e^{-\lambda_{(A-LC)}t}|e(0)|.$$

Then the accommodating controller $u = -K\widehat{x} - \lambda_y(\widehat{f})$ ensures that the closed-loop system will have a performance close to that of the healthy system and that the system with its adaptive detection observer is a well-posed system in $\mathbb{R}^n \times \mathbb{R}^n \times Q$ with boundedness of all signals and asymptotic convergence of the plant state $x(t)$ and estimation error $e(t)$.

V. Numerical Example and Conclusions

Example 1: PDE with in-domain actuation and sensing. The following PDE is considered in $[0, \ell] = [0, 1]$

$$\begin{aligned}x_t(t,\xi) &= a_1 x_{\xi\xi}(t,\xi) + a_2 x_\xi(t,\xi) \\ &\quad + 1.5a_1\delta(\xi - \xi_s)\int_0^\ell \delta(\xi - \xi_s)x(t,\xi)\,\mathrm{d}\xi \\ &\quad + \delta(\xi - \xi_s)\left(u(t) + \beta(t - \tau_f)f(y(t))\right) \\ y(t) &= \int_0^\ell \delta(\xi - \xi_s)x(t,\xi)\,\mathrm{d}\xi\end{aligned}$$

with $a_1 = 0.01, a_2 = 0.05$ and the actuator location $\xi_s = 0.251\ell$. The fault term was set as $f(y) = 0.1y$ with an abrupt fault profile given by $\beta(t - \tau_f) = H(t - 1.5)$. The state feedback controller gain K was on LQR design with cost functional described by $\|x(t)\|^2$ and $100u^2(t)$. The filter gain was $L(\xi) = 2a_1\delta(\xi - \xi_s)$.

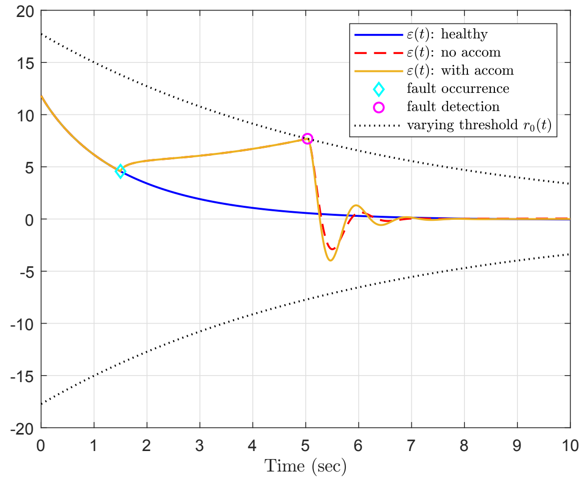

Fig. 1: Example 1: Evolution of $\varepsilon(t)$ and its threshold.

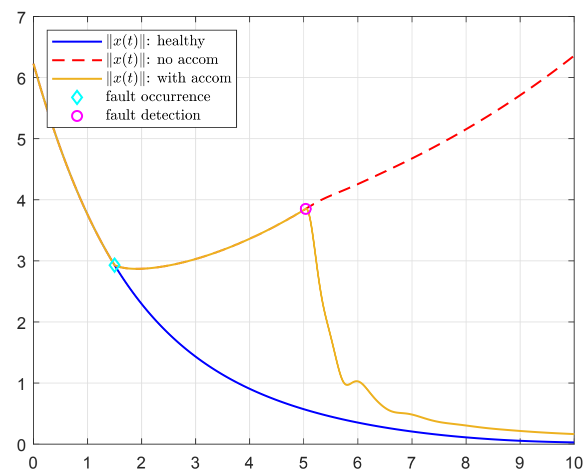

Fig. 2: Example 1: Evolution of $\|x(t)\|$: healthy case, faulty with no accommodation and faulty with accommodation.

Radial basis functions (RBFs) were selected for the functional estimation, with $\kappa_y(q) = \exp\{-\frac{|y-q|^2}{2\sigma^2}\}$. The standard deviation was $\sigma = \frac{100}{2\sqrt{\log(2)}}$ with the means evenly distributed in the interval $[-10, 10]$. In the approximation of $f(y)$, via finite dimensional subspaces $Q^N \subset Q$, a total number $N = 41$ of RBFs were used $f(y) \approx \sum_{i=1}^N \theta_i \kappa_{y_i}(\cdot)$. For the approximation of the PDE a Galerkin scheme was used with a total of 50 elements.

Figure 1 depicts the evolution of the residual signal and its time varying threshold. The presence of the fault was detected at $t_d = 5.034$s resulting in a fault delay of $\tau_d = 3.534$s. The state norm with and without the proposed fault accommodation are depicted in Figure 2, where it is observed that when the fault is accommodated, the performance approaches that of the healthy system.

Example 2: PDE with boundary actuation and sensing. We consider

$$\begin{aligned}&x_t(t,\xi) = a_1 x_{\xi\xi}(t,\xi), \;\; x(0,\xi) = x_0(\xi), \;\; 0 \le \xi \le \ell, \\ &x(t,0) = 0, \quad x_\xi(t,\ell) = u(t) + \beta(t - \tau_f)g(y), \\ &y(t) = x(t,\ell).\end{aligned}$$

where $a_1 = 0.05$, $f(y) = 0.02y^3$, $\beta(t - \tau_f) = H(t - 2)$. In

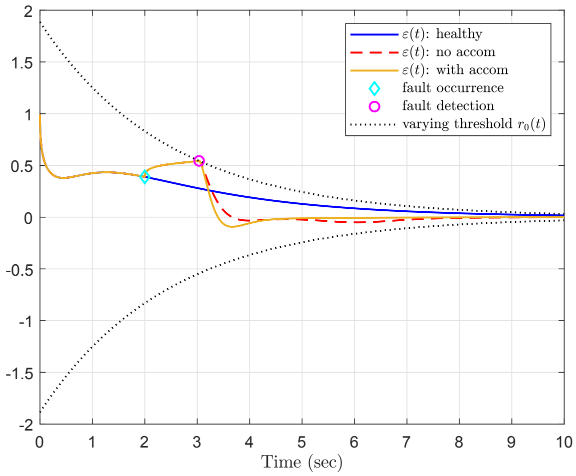

Fig. 3: Example 2: Evolution of $\varepsilon(t)$ and its threshold.

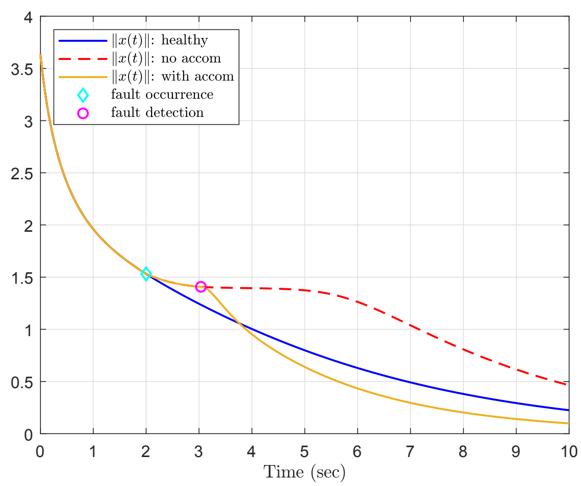

Fig. 4: Example 2: Evolution of $\|x(t)\|$: healthy case, faulty with no accommodation and faulty with accommodation.

this example, the state feedback controller gain K was based on LQR design with cost functional defined by $\|x(t)\|^2$ and $25u^2(t)$. The observer used a gain with kernel given by $L(\xi) = 10a_1\delta(\xi - \ell)$.

Figure 3 depicts the evolution of the residual signal and its time varying threshold. The presence of the fault was detected at $t_d = 3.036$ a resulting in a fault delay of 1.036s. Similarly, the state norm with and without the proposed fault accommodation are depicted in Figure 4, where once more it is observed that when the fault is accommodated, the performance approaches that of the healthy system. The functional error $f(y(t)) - \widehat{f}(t,y(t))$ is depicted in Figure 5, where it is observed that when the dead-zone modification is implemented, no adaptation takes place prior to the fault declaration ($t_d = 3.036$) and converges to zero thereafter.

VI. Conclusions

An adaptive fault detection and diagnosis scheme for a class of positive real infinite dimensional systems was proposed. The fault function, given by a nonlinear function of the output, was diagnosed via an adaptive functional estimation scheme utilizing a reproducing kernel Hilbert

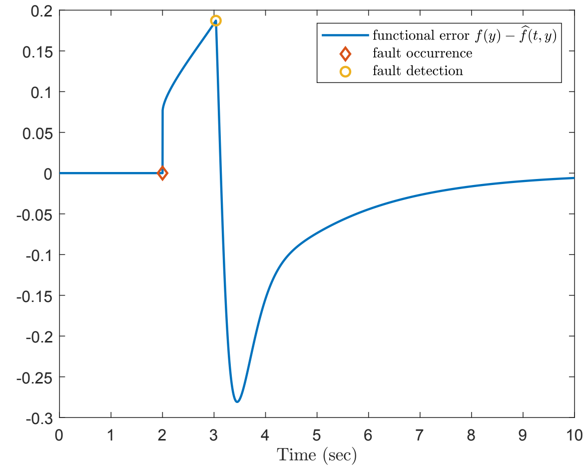

Fig. 5: Example 2: Evolution of error $f(y(t)) - \widehat{f}(t,y(t))$.

space as the parameter space. This resulted in both the state and parameter spaces be Hilbert spaces. Well-posedness and convergence were summarized. The finite dimensional case was treated as a special case of the infinite dimensional case and which allowed for more general output matrices. Extensive simulation studies on parabolic PDEs with in-domain and boundary actuation and sensing were presented.

Immediate extensions involve strictly positive real infinite dimensional systems where the input-output collocation condition is no longer assumed and a more general operator Lur'e equation is used along with the effects of partial persistence of excitation on the functional convergence.

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