2022 American Control Conference (ACC)
Atlanta, USA, June 8-10, 2022

Multi-band modal consensus filters for parabolic partial differential equations

Michael A. Demetriou

Abstract— **This paper presents a new formulation of consensus filters for parabolic PDEs. Using modal decompositions, the information a given distributed filter transmits to and receives from the remaining networked filters depends on the modal information needed. If a given distributed filter can completely reconstruct a specific mode or modes of the PDE, then it does not need any information from any of the networked filters. Similarly, if a distributed filter cannot adequately reconstruct a given mode, then it receives information from the filter that can completely reconstruct that specific mode. This then presents a connectivity which is based on the information needed. This consensus protocol which is dictated by the information a filter does not have but needs, is essentially a projection of information needed onto the unobservable space. This is demonstrated for a diffusion PDE in 1D and subsequently its abstraction is formulated for Riesz-spectral systems. Numerical studies demonstrate the proposed modal consensus filters.**

I. Introduction

A new paradigm of consensus protocols for distributed filters of PDEs is examined. Earlier works on distributed filters for PDEs were given a prescribed communication topology and had to design the consensus gains in order for each filter to synchronize [1], [2], [3], [4], [5], [6], [7], [8]. Here, the information transmitted is on a *need-basis* and constitutes the main contriobution of this work. A filter transmits its state estimate information only to the distributed filters that need this specific information. And in that case, it only transmits the specific information customised for each filter. Similarly, a filter only receives very particular information from filters that have done a better job at reconstructing specific modes of the PDE. Thus, the bi-directional information sharing is on a need-basis and only the experts transmit the needed information. This modal consensus filter architecture avoids superfluous information transmitted to the networked filters.

The problem is formulated in Section II for the 1D diffusion PDE which is excited by specific modes and whose response is given by a finite sum of the modal expansion. In order to link sensor locations to the distributed filters, it is assumed that a given sensor, or sensor group, can have a larger modal observability of specific modes over the other sensors. The modal consensus filters are presented in Section III. The abstraction and generalization of the proposed modal consensus filters is summarized in Section IV for PDEs that can be represented by Riesz-spectral systems. Numerical results with conclusions follow in Section V.

M. A. Demetriou is with WPI, Aerospace Engineering Department, Worcester, MA 01609, USA, mdemetri@wpi.edu. The author gratefully acknowledges financial support from NSF-CMMI grant # 1825546.

II. Problem formulation

The key aspects of the proposed multi-band consensus filters will be demonstrated via the 1D diffusion PDE and in a gradual manner the design of such consensus filters will be extended to a general class of PDEs. The PDE is

$$\frac{\partial x(t,\xi)}{\partial t} = \kappa \frac{\partial^2 x(t,\xi)}{\partial \xi^2} + \beta(\xi)u(t), \tag{1}$$

where $x(t,\xi)$ denotes the state at time $t \in \mathbb{R}^+$ and spatial location $\xi \in (0,\ell) = \Omega$. The parameter κ denotes the thermal diffusivity and the spatial function $\beta(\xi)$ denotes the spatial distribution of the actuating device. The associated control signal is denoted by $u(t)$. It is assumed that Dirichlet boundary conditions hold with $x(t,0) = x(t,\ell) = 0$.

A. Modal decomposition and observability

Starting with the homogeneous case, i.e., $u = 0$, one can write the solution to (1) in a series expansion

$$x(t,\xi) = \sum_{i=1}^{\infty} \alpha_i(t)\phi_i(\xi). \tag{2}$$

The functions $\phi_i(\xi)$, $i = 1,\dots,\infty$ are the modes and using established results on the solution of the eigenvalue problem for (1) with Dirichlet boundary conditions [9], one obtains

$$\phi_i(\xi) = \sqrt{(2/\ell)}\, \sin(i\pi\xi/\ell), \qquad i = 1,\dots,\infty, \tag{3}$$

with the associated eigenvalues given by

$$\lambda_i = -\kappa (i\pi/\ell)^2, \qquad i = 1,\dots,\infty. \tag{4}$$

To appreciate the effects of the sensor location, assume that a pointwise sensor with centroid at location $\xi_s \in \Omega$ provides process measurements

$$y(t) = \int_0^{\ell} \delta(\xi - \xi_s)x(t,\xi)\,\mathrm{d}\xi = x(t,\xi_s). \tag{5}$$

For argument's sake, assume that the initial condition $x(0,\xi) = x_0(\xi)$ is given by $x_0(\xi) = x_{0k}\phi_k(\xi)$, i.e. the initial condition is equal to the k^{th} mode. Using (2) in (1), one has

$$\sum_{i=1}^{\infty} \dot{\alpha}_i(t)\phi_i(\xi) = \sum_{i=1}^{\infty} \lambda_i \alpha_i(t)\phi_i(\xi).$$

Multiplying with a test function equal to the j^{th} eigenfunction and integrating over the spatial domain, one arrives at

$$\sum_{i=1}^{\infty} \dot{\alpha}_i(t) \int_0^{\ell} \phi_i(\xi)\phi_j(x)\,\mathrm{d}\xi = \sum_{i=1}^{\infty} \lambda_i \alpha_i(t) \int_0^{\ell} \phi_i(\xi)\phi_j(\xi)\,\mathrm{d}\xi. \tag{6}$$

Using the initial condition $x_0(\xi) = x_{0k}\phi_k(\xi)$ and the orthogonality of the eigenfunctions, (4) reduces to

$$\dot{\alpha}_k(t) = \lambda_k \alpha_k(t), \quad \alpha_k(0) = x_{0k}, \tag{7}$$

with a solution $\alpha_k(t) = x_{0k}e^{\lambda_k t}$ and subsequently $x(t,\xi) =$

$x_{0k}e^{\lambda_k t}\phi_k(\xi)$. The output in (5) is now given by

$$y(t) = \int_0^{\ell} \delta(\xi-\xi_s) x_{0k} e^{\lambda_k t}\phi_k(\xi)\,\mathrm{d}\xi = x_{0k}e^{\lambda_k t}\phi_k(\xi_s).$$

In terms of the eigenfunctions (3), the state and output are

$$\begin{aligned} x(t,\xi) &= x_{0k}\sqrt{2/\ell}\,e^{\lambda_k t}\sin(k\pi\xi/\ell), \\ y(t) &= x_{0k}\sqrt{2/\ell}\,e^{\lambda_k t}\sin(k\pi\xi_s/\ell). \end{aligned} \tag{8}$$

Examination of (8) has that for $x_{0k} \neq 0$, the state $x(t,\xi)$ is nonzero and time-varying and that it eventually converges to zero exponentially due to (4). However, the output signal $y(t)$ may be zero for certain sensor locations. Indeed, when ξ_s is a zero of $\phi_k(\xi) = \sqrt{2/\ell}\sin(k\pi\xi/\ell)$, i.e. $\xi_s = n\ell/k$, $n \in \mathbb{Z}^+, n \geq k$, then the output is $y(t) = 0$ for all $t \geq 0$.

As it will be examined further, one can easily see that when the input $u(t)$ is selected in a specific way in terms of the weighted sum of few eigenfunctions and the initial condition is similarly chosen as a weighted sum of the same few eigenfunctions, then the state will have a similar finite series expansion. When a sensor is placed at a spatial location that coincides with the zeros of those few eigenfunctions, the output once again will be equal to zero. While the state will be non-zero, the process output will be identically zero for all times. An attempt to design an observer for such a system will simply result in a naïve observer.

To demonstrate the effects of the sensor locations, it is assumed that the input term $\beta(\xi)u(t)$ and initial condition $x_0(\xi)$ are expressed in terms of the first three modes

$$\beta(\xi)u(t) = \sum_{i=1}^{3} u_i\phi_i(\xi), \;\; x_0(\xi) = \sum_{i=1}^{3} x_{0i}\phi_i(\xi). \tag{9}$$

Remark 1: Please note that one seldom has an actuating device with spatial distribution $\beta(\xi)$ equipped with such a modal discrimination. Similarly, the initial condition in (9) also is artificial and one almost never has such a modal content in $x_0(\xi)$. However, the use of such a specific and artificial expansion is made to accentuate the proposed work. In the abstract representation in Section IV, such assumptions are removed since input terms have no effect in filter design.

In a manner similar to (6), (7) the modal components of the solution to (1) are described by

$$\dot{\alpha}_i(t) = \lambda_i\alpha_i(t) + u_i, \;\; \alpha_i(0) = x_{0i}, \;\; i = 1,2,3, \tag{10}$$

yielding the modal solutions

$$\alpha_i(t) = e^{\lambda_i t}x_{0i} + \frac{e^{\lambda_i t}-1}{\lambda_i}u_i, \;\; i = 1,2,3, \tag{11}$$

and the state solution

$$\begin{aligned} x(t,\xi) &= \left(e^{\lambda_1 t}x_{01} + \tfrac{e^{\lambda_1 t}-1}{\lambda_1}u_1\right)\sqrt{(2/\ell)}\,\sin(1\pi\xi/\ell) \\ &+ \left(e^{\lambda_2 t}x_{02} + \tfrac{e^{\lambda_2 t}-1}{\lambda_2}u_2\right)\sqrt{(2/\ell)}\,\sin(2\pi\xi/\ell) \\ &+ \left(e^{\lambda_3 t}x_{03} + \tfrac{e^{\lambda_3 t}-1}{\lambda_3}u_3\right)\sqrt{(2/\ell)}\,\sin(3\pi\xi/\ell). \end{aligned} \tag{12}$$

The question that arises now, is where to put the sensor? To obtain an insight, consider the three modes weighted by $\sqrt{\ell/2}$ (i.e. plot the sinusoids). Examining Figure 1, if a sensor is placed at the location $\xi_s = 0.5$ that maximizes $\phi_1(\xi)$, one has that $\phi_2(0.5) = 0$ and $\phi_3(0.5) = -1$.

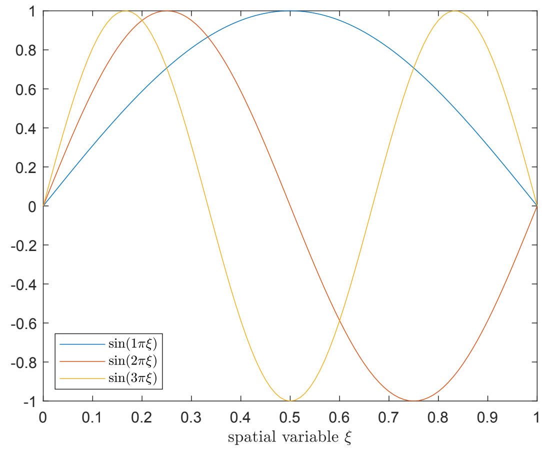

Fig. 1: Plot of modes $\sin(i\pi\xi/\ell)$, $i = 1,2,3$ in $[0,1]$.

This means that a sensor placed at $\xi_s = 0.5$ will not be able to provide any information about the second mode of (12). Using for now the absolute value of the mode at the selected sensor location as a measure of "modal observability" consider the sensor locations $\xi_s = 1/2, 1/3, 1/6$. Table I summarizes the results

mode	$\xi_s = 1/2$	$\xi_s = 1/3$	$\xi_s = 1/6$
1	$\lvert\phi_1(\xi_s)\rvert = 1$	$\lvert\phi_1(\xi_s)\rvert = \sqrt{3}/2$	$\lvert\phi_1(\xi_s)\rvert = 1/2$
2	$\lvert\phi_2(\xi_s)\rvert = 0$	$\lvert\phi_2(\xi_s)\rvert = \sqrt{3}/2$	$\lvert\phi_2(\xi_s)\rvert = \sqrt{3}/2$
3	$\lvert\phi_3(\xi_s)\rvert = 1$	$\lvert\phi_3(\xi_s)\rvert = 0$	$\lvert\phi_3(\xi_s)\rvert = 1$

Table I. Level of "modal observability" expressed as $\lvert\sin(i\pi\xi_s)\rvert$.

- A sensor placed at $\xi_s = 1/2$ can completely observe modes 1 and 3 and cannot observe mode 2.
- A sensor placed at $\xi_s = 1/3$ can partially observe modes 1 and 2 and cannot observe mode 3.
- A sensor placed at $\xi_s = 1/6$ can partially observe modes 1 and 2 and can observe mode 3 completely.

Several conclusions follow from the above:

- A (centralized) filter with a sensor at $\xi_s = 1/6$, which yields observability of all three modes, can efficiently reconstruct the process state whose solution is given by (12). While it completely observes mode 3, it partially observes modes 1 and 2.
- A filter with two sensors at $\xi_s = 1/6$ and $\xi_s = 1/2$ can efficiently reconstruct the process state; it completely observes modes 1 and 2 and partially observes mode 2.
- A filter with three sensors at $\xi_s = 1/2$, $\xi_s = 1/3$ and at $\xi_s = 1/6$ can efficiently reconstruct the process state; it completely observes modes 1 and 2 and partially observes mode 2.
- A filter with three sensors at $\xi_s = 1/2$, $\xi_s = 1/4$ and at $\xi_s = 1/6$ can completely reconstruct the process state; it completely observes all three modes *collectively*.

When different modes other than the first three modes are present in (12), then a single sensor location will never be able to provide complete observability for all modes. In fact, it may have zero modal observability for certain modes. A network of sensors placed at different locations may collectively have observabilty of all modes. However

filter	estimate mode 1?	estimate mode 2?	info needed
$\widehat{x}_1$	Y	N	mode 2 estimate
$\widehat{x}_2$	N	Y	mode 1 estimate

Distributed filters and specialized information needed.

such an obsevability may not be complete for each mode, i.e. the value of the modal observability will not be unity for all modes. One can argue that with an adequate number of sensors placed at specific locations will produce a system that has partial modal observability of all modes and can subsequently reconstruct the process state adequately.

B. Distributed modal filters and modal consensus protocol

Against the centralized filter architecture is the decentralized filter architecture that employs multiple filters that may share information. One may consider the same number of filters as the number of modes present in a response like (12). For each sensor that is placed in a location that has a complete modal observability over a single mode and perhaps a partial modal observability over few other modes, one can associate a single filter. Thus each filter will be able to completely reconstruct a specific mode and can "partially' reconstruct few other modes.

The question is how to share the information so that all such distributed filters agree with each other. A consensus protocol can be implemented by all such distributed filters and the learning abilities will then be dictated by the communication topology and the appropriate selection of consensus weights. However, one aspect of this information sharing will be explored here: *each filter must share what it knows best to the remaining filters it communicates with. Not only that, but each filter should receive the necessary information it needs from the remaining filters*. As an example consider two filters denoted by $\widehat{x}_1(t,\xi)$ and $\widehat{x}_2(t,\xi)$, with a single sensor each. If each filter can only reconstruct completely one mode and not a second mode, one will have the following: $\widehat{x}_1(t,\xi)$ can reconstruct mode 1 and not reconstruct mode 2. Similarly, $\widehat{x}_2(t,\xi)$ can reconstruct mode 2 and not reconstruct mode 1. The first filter needs only information about mode 2 and filter $\widehat{x}_2(t,\xi)$ needs only information about mode 1. A standard consensus protocol will send all modal information to each distributed filter. Thus we have that $\widehat{x}_1(t,\xi)$ will receive information from $\widehat{x}_2(t,\xi)$ that contains estimates of modes 1 and 2; however $\widehat{x}_1(t,\xi)$ does not need any information on mode 1, since it does a superb job in reconstructing it. The reverse is observed when filter $\widehat{x}_2(t,\xi)$ receives information from $\widehat{x}_1(t,\xi)$ that contains estimates on modes 1 and 2; however $\widehat{x}_2(t,\xi)$ does not need any information on the estimate of mode 2 since it is doing a better job to reconstruct mode 2 than $\widehat{x}_1(t,\xi)$ does. Table II summarizes the relevant information needed for each filter to each consensus. A generic consensus protocol will share unnecessary information between the distributed filters.

Therefore, a modification to a consensus protocol is warranted to ensure that only useful information is transmitted from one filter and that superfluous information is not shared. Similarly, a given filter will receive only useful information from the remaining filters that have a better ability to reconstruct specific modes that this filter cannot reconstruct and must hence receive. In other words, each filter will get the modal information it does not have from the "experts".

This in fact constitutes the main contribution of this paper and is presented in the next section for a system that has a response similar to (12). Subsequently an attempt will be made to carefully extend this to a general class of PDEs with response given by a more general expression than (12).

III. Multi-band modal consensus filters

Assume that the system has a response given by (12) and that three sensors placed at locations ξ_{s1}, ξ_{s2}, and ξ_{s3} provide process information

$$y_i(t) = C_i x(t) = \int_0^\ell \delta(\xi - \xi_{si}) x(t,\xi)\,\mathrm{d}\xi = x(t,\xi_{si}), \qquad (13)$$

for $i = 1,2,3$, where C_i are the output operators associated with the sensor spatial distributions $\delta(\xi - \xi_{si})$. To each sensor we associate a single filter and thus we have the following filters used to reconstruct (1) that has the response (12)

$$\begin{aligned}
\frac{\partial \widehat{x}_1}{\partial t} &= \kappa \frac{\partial^2 \widehat{x}_1}{\partial \xi^2} + \beta u + L_1 \left(y_1 - \int_0^\ell \delta(\xi - \xi_{s1}) \widehat{x}_1 \,\mathrm{d}\xi \right), \\
\frac{\partial \widehat{x}_2}{\partial t} &= \kappa \frac{\partial^2 \widehat{x}_2}{\partial \xi^2} + \beta u + L_2 \left(y_2 - \int_0^\ell \delta(\xi - \xi_{s2}) \widehat{x}_2 \,\mathrm{d}\xi \right), \qquad (14) \\
\frac{\partial \widehat{x}_3}{\partial t} &= \kappa \frac{\partial^2 \widehat{x}_3}{\partial \xi^2} + \beta u + L_3 \left(y_3 - \int_0^\ell \delta(\xi - \xi_{s3}) \widehat{x}_3 \,\mathrm{d}\xi \right),
\end{aligned}$$

where we have suppressed the dependence on t and ξ for brevity. The spatial functions $L_i(\xi)$, $i = 1,2,3$ are the filter kernels and correspond to the adjoints of the observer operator gains. They are selected either using a Luenberger observer design or Kalman filter design with the property that an associated estimation error will converge to zero. Each filter in (14) admits a truncated series expansion as the process state $x(t,\xi)$ does, and so one has

$$\widehat{x}_i(t,\xi) = \sum_{i=1}^{3} \widehat{\alpha}_{ij}(t) \phi_i(\xi) \qquad (15)$$

where $\widehat{\alpha}_{ij}(t)$ denotes the estimate of the j^{th} modal weight $\alpha_j(t)$ in (12) and the first superscript denotes the i^{th} filter; e.g. $\widehat{\alpha}_{23}(t)$ is the 3rd modal component of filter $\widehat{x}_2(t,\xi)$.

Let us examine the innovation terms in (14). Define

$$\begin{aligned}
\varepsilon_i(t) &= y_i(t) - \int_0^\ell \delta(\xi - \xi_{si}) \widehat{x}_i(t,\xi)\,\mathrm{d}\xi \\
&= \sum_{j=1}^{3} \left(\alpha_j(t) - \widehat{\alpha}_{ij}(t)\right) \phi_j(\xi_{si}), \;\; i = 1,2,3.
\end{aligned} \qquad (16)$$

Then the modal components of the i^{th} filter associated with $y_i(t)$ are given by

$$\begin{aligned}
\dot{\widehat{\alpha}}_{i1} &= \lambda_1 \widehat{\alpha}_{i1} + u_1 + \varepsilon_i(t) \int_0^\ell L_i(\xi) \phi_1(\xi)\,\mathrm{d}\xi \\
\dot{\widehat{\alpha}}_{i2} &= \lambda_2 \widehat{\alpha}_{i2} + u_2 + \varepsilon_i(t) \int_0^\ell L_i(\xi) \phi_2(\xi)\,\mathrm{d}\xi \qquad (17) \\
\dot{\widehat{\alpha}}_{i3} &= \lambda_3 \widehat{\alpha}_{i3} + u_3 + \varepsilon_i(t) \int_0^\ell L_i(\xi) \phi_3(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

To demonstrate the need for "usefull and necessary" information to be shared amongst the filters in (17), for now let us make the rather stringent assumption

$$\int_0^\ell L_i(\xi)\phi_i(\xi)\,\mathrm{d}\xi \neq 0,\quad \int_0^\ell L_i(\xi)\phi_j(\xi)\,\mathrm{d}\xi = 0,\quad i \neq j. \tag{18}$$

Then the three filters are given by

$$\text{filter 1}\begin{cases} \dot{\widehat{\alpha}}_{11} = \lambda_1\widehat{\alpha}_{11} + u_1 + \varepsilon_1 \int_0^\ell L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi, \\ \dot{\widehat{\alpha}}_{12} = \lambda_2\widehat{\alpha}_{12} + u_2, \\ \dot{\widehat{\alpha}}_{13} = \lambda_3\widehat{\alpha}_{13} + u_3, \end{cases} \tag{19}$$

$$\text{filter 2}\begin{cases} \dot{\widehat{\alpha}}_{21} = \lambda_1\widehat{\alpha}_{21} + u_1, \\ \dot{\widehat{\alpha}}_{22} = \lambda_2\widehat{\alpha}_{22} + u_2 + \varepsilon_2 \int_0^\ell L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi, \\ \dot{\widehat{\alpha}}_{23} = \lambda_3\widehat{\alpha}_{23} + u_3, \end{cases} \tag{20}$$

$$\text{filter 3}\begin{cases} \dot{\widehat{\alpha}}_{31} = \lambda_1\widehat{\alpha}_{31} + u_1, \\ \dot{\widehat{\alpha}}_{32} = \lambda_2\widehat{\alpha}_{32} + u_2, \\ \dot{\widehat{\alpha}}_{33} = \lambda_3\widehat{\alpha}_{33} + u_3 + \varepsilon_3 \int_0^\ell L_3(\xi)\phi_3(\xi)\,\mathrm{d}\xi. \end{cases} \tag{21}$$

Equations (19)–(21) will be used in the error analysis which will provide the necessary modifications to (14) in order to include the appropriate consensus terms. It is observed that filter 1 can only reconstruct mode 1, filter 2 can reconstruct mode 2 and filter 3 can reconstruct mode 3. Thus we have

- filter 1 needs information from filter 2 regarding mode 2 and from filter 3 regarding mode 3,
- filter 2 needs information from filter 1 regarding mode 1 and from filter 3 regarding mode 3,
- filter 3 needs information from filter 1 regarding mode 1 and from filter 2 regarding mode 2.

Prior to utilizing the above in the construction of the consensus terms in (19)–(21), we consider the filter kernels $L_i(\xi)$ and impose additional conditions to (18). Consider for example the estimation error between mode 1 of $x(t,\xi)$ and of mode 1 of $\widehat{x}(t,\xi)$. Using (10), (16) and (19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1 - \widehat{\alpha}_{11}) = \left(\lambda_1 - \phi_1(\xi_{s1})\int_0^\ell L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi\right)(\alpha_1 - \widehat{\alpha}_{11}),$$

where we have made the implicit assumption from (18) that the innovation terms in (16) simplify to

$$\varepsilon_i(t) = (\alpha_i(t) - \widehat{\alpha}_{ii}(t))\,\phi_i(\xi_{si}), \tag{22}$$

i.e. $\phi_i(\xi_{sj}) = 0$. By selecting now the filter kernels $L_i(\xi)$ as

$$\begin{aligned} \int_0^\ell L_i(\xi)\phi_i(\xi)\,\mathrm{d}\xi &= m_i \int_0^\ell \delta(\xi - \xi_{si})\phi_i(\xi)\,\mathrm{d}\xi \\ &= m_i\phi_i(\xi_{si}), \end{aligned} \tag{23}$$

for some scalar gains $m_i > 0$, we have that the modal estimation errors are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1 - \widehat{\alpha}_{11}) = \left(\lambda_1 - m_1\phi_1^2(\xi_{s1})\right)(\alpha_1 - \widehat{\alpha}_{11}).$$

Similar results can be obtained for the other cases and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_i - \widehat{\alpha}_{ii}) = \left(\lambda_i - m_i\phi_i^2(\xi_{si})\right)(\alpha_i - \widehat{\alpha}_{ii}). \tag{24}$$

The exponential convergence of the above modal estimation errors immediately follows with the aid of (4). What (24) reveals is that each of the three filters can completely reconstruct the mode associated with its modal observability. This strengthens the above observations on what modal information is needed by each filter.

We now present the design for the modal consensus filters. The first modal component of filter 1 does not need to receive any information. The second and third components of filter 1 in (19) are now modified

$$\begin{aligned} \dot{\widehat{\alpha}}_{11} &= \lambda_1\widehat{\alpha}_{11} + u_1 + \varepsilon_1(t)\int_0^\ell L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi, \\ \dot{\widehat{\alpha}}_{12} &= \lambda_2\widehat{\alpha}_{12} + u_2 + f_{12} \\ \dot{\widehat{\alpha}}_{13} &= \lambda_3\widehat{\alpha}_{13} + u_3 + f_{13} \end{aligned} \tag{25}$$

The functions f_{12}, f_{13} denote the *consensus terms* that filter 1 requires and they are interpreted as follows: f_{ij} represents the i^{th} filter and the information it receives is from the j^{th} filter. Comparing the error between the second modal component of filter 2 and the second modal component of filter 1 in (25), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) &= \lambda_2(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) \\ &\quad + \varepsilon_2(t)\int_0^\ell L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi - f_{12} \\ &= \lambda_2(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) - f_{12} \\ &\quad + m_2\phi_2^2(\xi_{s2})(\alpha_2 - \widehat{\alpha}_{22} + \widehat{\alpha}_{12} - \widehat{\alpha}_{12}) \\ &= (\lambda_2 - m_2\phi_2^2(\xi_{s2}))(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) \\ &\quad + m_2\phi_2^2(\xi_{s2})(\alpha_2 - \widehat{\alpha}_{12}) - f_{12}. \end{aligned}$$

Using (13), the term $m_2\phi_2^2(\xi_{s2})(\alpha_2 - \widehat{\alpha}_{12})$ is

$$\begin{aligned} &m_2\phi_2^2(\xi_{s2})(\alpha_2 - \widehat{\alpha}_{12}) \\ &\qquad = (y_2(t) - C_2\widehat{x}_1(t))\int_0^\ell L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{26}$$

where y_2 denotes the measurement signal from the second sensor and the signal $C_2\widehat{x}_1(t)$ is the value of the state of filter 1 evaluated at the second sensor. Using (26), a logical choice of f_{12} in (25) is

$$f_{12} = (y_2 - C_2\widehat{x}_1)\int_0^\ell L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi + q_{12}(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) \tag{27}$$

where $q_{12} > 0$ is a scalar *consensus weight*. Substitution of (27) in (25) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}) = \left(\lambda_2 - m_2\phi_2^2(\xi_{s2}) - q_{12}\right)(\widehat{\alpha}_{22} - \widehat{\alpha}_{12}). \tag{28}$$

Equation (28) reveals that the consensus error $\widehat{\alpha}_{22} - \widehat{\alpha}_{12}$ exponentially converges to zero with a rate faster than $a_2 - \widehat{\alpha}_{22}$ does. The remaining term f_{13} in (25) can be determined in a similar fashion. The same applies with the additional terms in filter 2 of (20) and in filter 3 of (21). These are stated in the lemma below.

Lemma 1: Consider the system (1) which admits a solution similar to (12). Assume that the filter kernels are selected using (18) and (23). Then the proposed multi-band modal consensus filters given by

$$\text{filter 1}\begin{cases}\dot{\widehat{\alpha}}_{11}=\lambda_1\widehat{\alpha}_{11}+u_1+\varepsilon_1\int_0^{\ell}L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi,\\ \dot{\widehat{\alpha}}_{12}=\lambda_2\widehat{\alpha}_{12}+u_2+f_{12},\\ \dot{\widehat{\alpha}}_{13}=\lambda_3\widehat{\alpha}_{13}+u_3+f_{13},\end{cases}\tag{29}$$

$$\text{filter 2}\begin{cases}\dot{\widehat{\alpha}}_{21}=\lambda_1\widehat{\alpha}_{21}+u_1+f_{21},\\ \dot{\widehat{\alpha}}_{22}=\lambda_2\widehat{\alpha}_{22}+u_2+\varepsilon_2\int_0^{\ell}L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi,\\ \dot{\widehat{\alpha}}_{23}=\lambda_3\widehat{\alpha}_{23}+u_3+f_{23},\end{cases}\tag{30}$$

$$\text{filter 3}\begin{cases}\dot{\widehat{\alpha}}_{31}=\lambda_1\widehat{\alpha}_{31}+u_1+f_{31},\\ \dot{\widehat{\alpha}}_{32}=\lambda_2\widehat{\alpha}_{32}+u_2+f_{32},\\ \dot{\widehat{\alpha}}_{33}=\lambda_3\widehat{\alpha}_{33}+u_3+\varepsilon_3\int_0^{\ell}L_3(\xi)\phi_3(\xi)\,\mathrm{d}\xi.\end{cases}\tag{31}$$

with the consensus terms given by

$$\begin{aligned}
f_{12}&=(y_2-C_2\widehat{x}_1)\int_0^{\ell}L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi+q_{12}(\widehat{\alpha}_{22}-\widehat{\alpha}_{12})\\
f_{13}&=(y_3-C_3\widehat{x}_1)\int_0^{\ell}L_3(\xi)\phi_3(\xi)\,\mathrm{d}\xi+q_{13}(\widehat{\alpha}_{33}-\widehat{\alpha}_{13})\\
f_{21}&=(y_1-C_1\widehat{x}_2)\int_0^{\ell}L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi+q_{21}(\widehat{\alpha}_{11}-\widehat{\alpha}_{21})\\
f_{23}&=(y_3-C_3\widehat{x}_2)\int_0^{\ell}L_3(\xi)\phi_3(\xi)\,\mathrm{d}\xi+q_{23}(\widehat{\alpha}_{33}-\widehat{\alpha}_{23})\\
f_{31}&=(y_1-C_1\widehat{x}_3)\int_0^{\ell}L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi+q_{31}(\widehat{\alpha}_{11}-\widehat{\alpha}_{31})\\
f_{32}&=(y_2-C_2\widehat{x}_3)\int_0^{\ell}L_2(\xi)\phi_2(\xi)\,\mathrm{d}\xi+q_{32}(\widehat{\alpha}_{22}-\widehat{\alpha}_{32})
\end{aligned}\tag{32}$$

ensure that the filters (29)–(31) reconstruct the process state of (1) with the modal errors satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_i-\widehat{\alpha}_{ii})=\left(\lambda_i-m_i\phi_i^2(\xi_{si})\right)(\alpha_i-\widehat{\alpha}_{ii})\tag{33}$$

for $i=1,2,3$ and the consensus errors satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji})=\left(\lambda_i-m_i\phi_i^2(\xi_{si})-q_{ji}\right)(\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji})\tag{34}$$

for $i,j=1,2,3$. Furthermore, as a consequence of the triangle inequality one has

$$\lim_{t\to\infty}|\alpha_i-\widehat{\alpha}_{ji}|=0\tag{35}$$

for $i,j=1,2,3$, $i\neq j$, with exponential convergence having a rate at least $\lambda_i-m_i\phi_i^2(\xi_{si})$, $i=1,2,3$.

Proof: Consider (10), the first equation of (29), the second equation of (30) and the third equation of (31)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_i-\widehat{\alpha}_{ii})=\lambda_i(\alpha_i-\widehat{\alpha}_{ii})-\varepsilon_1\int_0^{\ell}L_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi.$$

Using (22) and (23), the above produces (33). Using the $\widehat{\alpha}_{ji}$ components from (29)–(31) along with the definitions of the consensus terms in (32), one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji})&=\lambda_i(\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji})+\varepsilon_i\int_0^{\ell}L_i(\xi)\phi_i(\xi)\,\mathrm{d}\xi-f_{ji}\\&=\left(\lambda_i-m_i\phi_i^2(\xi_{si})-q_{ji}\right)(\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji}).\end{aligned}$$

Finally, using the triangle inequality

$$\begin{aligned}|\alpha_i-\widehat{\alpha}_{ji}|&=|\alpha_i-\widehat{\alpha}_{ii}+\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji}|\\&\leq|\alpha_i-\widehat{\alpha}_{ii}|+|\widehat{\alpha}_{ii}-\widehat{\alpha}_{ji}|\end{aligned}$$

and the convergence from (33), (34), one obtains (35). ∎

As a first step in considering the modal consensus filters for a more general class of PDEs with a response that has more than three modes present, the modal consensus filters (29)–(31) must be written in the form (14). Additionally, the assumption (18), (23) can be relaxed. In such a case, each filter in (29)–(31) will have additional residual terms that will not cancel out and must be absorbed by the consensus terms f_{ij}. Please note that when $L_i(\xi)=m_i\delta(\xi-\xi_{si})$, one may not ensure that each sensor location ξ_{si} will coincide with the zeros of all eigenfunctions $\phi_j(\xi)$ except $\phi_i(\xi)$. Referring to Table I, when $\xi_s=1/6$, one had $\phi_i(1/6)\neq 0$ for $=1,2,3$.

IV. Abstraction of Modal Consensus Filters (29)–(31)

We consider PDEs expressed as a Riesz-spectral system

$$\dot{x}(t)=Ax(t)+Bu(t)\tag{36}$$

over the state space $\mathcal{X}$, having three different outputs

$$y_1(t)=C_1x(t),\;\; y_2(t)=C_2x(t),\;\; y_3(t)=C_3x(t),\tag{37}$$

produced by three different *sensor groups*. It is assumed that each sensor group is represented by the output operator C_i, $i=1,2,3$ and is associated with a distinct set of eigenfunctions. Each sensor group may not have the same number of sensors as another group, but all three sensor groups have more than one sensor each.

Define the projection operators $P_i:\mathcal{X}\to\mathcal{X}$, $i=1,2,3$, associated with each of the three sensor groups, as represented by their output operators C_i, $i=1,2,3$. We need to obtain the three components of each filter $\widehat{x}_i(t)$ associated with the process (36). Then the three filters are abstractly written as

$$\begin{aligned}\dot{\widehat{x}}_1&=A\widehat{x}_1+Bu+L_1(y_1-C_1\widehat{x}_1)+P_2f_{12}+P_3f_{13},\\ \dot{\widehat{x}}_2&=A\widehat{x}_2+Bu+L_2(y_2-C_2\widehat{x}_2)+P_1f_{21}+P_3f_{23},\\ \dot{\widehat{x}}_3&=A\widehat{x}_3+Bu+L_3(y_3-C_3\widehat{x}_3)+P_1f_{31}+P_2f_{32}.\end{aligned}\tag{38}$$

The consensus terms f_{ij} are given by

$$\begin{aligned}f_{ij}=&\,L_j(y_j-C_j(P_j\widehat{x}_i)-L_i(y_i-C_i(P_j\widehat{x}_j))\\&-q_{ij}(P_j\widehat{x}_i-P_j\widehat{x}_j),\quad i,j=1,2,3,\;\; i\neq j.\end{aligned}\tag{39}$$

Please notice that the term $L_i(y_i-C_i(P_j\widehat{x}_j))$ corresponds to (18) when one selects the filter gains as $L_i=\mu_iP_iC_i^*$, $i=1,2,3$, for some scalar weights $\mu_i>0$, $i=1,2,3$. In other words, when the filter gains are selected as a scalar multiple of the adjoint of the output operator associated with a sensor group and projected onto the appropriate subspace, then the consensus terms in (39) simplify to

$$f_{ij}=L_j(y_j-C_j(P_j\widehat{x}_i)-q_{ij}(P_j\widehat{x}_i-P_j\widehat{x}_j),\tag{40}$$

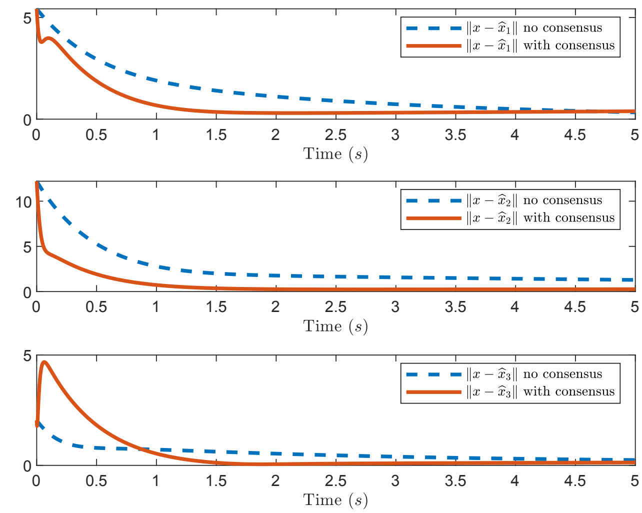

Fig. 2: Evolution of estimation error norms $\|x - \widehat{x}_i\|$.

for $i, j = 1, 2, 3$, $i \neq j$. The scalar gains q_{ij} are the *consensus weights* that were similarly given in (32) for the simpler case of each sensor group containing only one sensor. The abstractions of (33)–(35) are given by

$$\lim_{t\to\infty} \|P_i(x - \widehat{x}_i)\| = 0, \quad \lim_{t\to\infty} \|P_i(\widehat{x}_i - \widehat{x}_j)\| = 0,$$

and $\lim_{t\to\infty} \|P_i(x - \widehat{x}_j)\| = 0$.

V. Numerical results and conclusion

For simplicity, consider (1) with $\kappa = 0.01$ having each sensor group consisting of a single sensing device. In other words, we consider sensors associated with three different modes $\{\phi_{k_1}(\xi)\}$, $\{\phi_{k_2}(\xi)\}$, $\{\phi_{k_3}(\xi)\}$. For the specific case, we selected $k_1 =$, $k_2 = 2$ and $k_3 = 4$. Due to the fact that $\phi_{k3}(\xi) = \sqrt{2/\ell}\sin(4\pi\xi/\ell)$, the sensor locations were different from the ones in Table I and were selected as $\xi_{s1} = \ell/2$, $\xi_{s2} = \ell/4$ and $\xi_{s3} = \ell/8$. For simplicity, $L_i = P_i C_i^*$, $i = 1, 2, 3$. The consensus weights were selected as $q_{12} = q_{13} = q_{21} = q_{23} = 20$ and $q_{31} = q_{32} = 50$. The input in (9) was selected as $\beta(\xi)u(t) = \phi_1(\xi) + \phi_2(\xi) + \phi_3(\xi)$ and $x_0(\xi) = \phi_1(\xi) + 3\phi_2(\xi) + 6\phi_3(\xi)$. To simulate (1), a total of $N = 14$ modes were used in the truncated expansion (2) given by $x(t,\xi) = \sum_{i=1}^{N} \alpha_i(t)\phi_i(\xi)$. This of course still produces a solution similar to (12) due to the choice of u and x_0 and is

$$\begin{aligned} x(t,\xi) &= \left(e^{\lambda_1 t} + (e^{\lambda_1 t} - 1)/\lambda_1\right)\sqrt{2/\ell}\,\sin(1\pi\xi/\ell) \\ &+ \left(3e^{\lambda_2 t} + (e^{\lambda_2 t} - 1)/\lambda_2\right)\sqrt{2/\ell}\,\sin(2\pi\xi/\ell) \\ &+ \left(6e^{\lambda_3 t} + (e^{\lambda_3 t} - 1)/\lambda_3\right)\sqrt{2/\ell}\,\sin(4\pi\xi/\ell). \end{aligned}$$

with $\lambda_1 = -\kappa(1\pi)^2$, $\lambda_2 = -\kappa(2\pi)^2$ and $\lambda_3 = -\kappa(4\pi)^2$. The evolution of the norm of the estimation errors $x(t) - \widehat{x}_i(t)$ is depicted in Figure 2 for both the case of using the proposed modal consensus filters in (38) and the case of distributed noninteracting filters ($f_{1j} = 0$).

To quantify the amount of disagreement of the three filters, an appropriate metric is the disagreement potential $\Psi_G(\widehat{x}) = \frac{1}{2}\sum_{(i,j)\in E} \|\widehat{x}_i - \widehat{x}_j\|^2$ where G is the communication graph and E denotes the set of edges of the graph G. For the infinite dimensional case this potential is given by

$$\Psi_G(\widehat{x}) = \frac{1}{2}\left(\|\widehat{x}_1 - \widehat{x}_2\|^2 + \|\widehat{x}_1 - \widehat{x}_3\|^2 + \|\widehat{x}_2 - \widehat{x}_3\|^2\right)$$
$$= \frac{1}{2}\Big(\int_0^\ell (\widehat{x}_1(t,\xi) - \widehat{x}_2(t,\xi))^2\,\mathrm{d}\xi + \int_0^\ell (\widehat{x}_1(t,\xi) - \widehat{x}_3(t,\xi))^2\,\mathrm{d}\xi$$
$$+ \int_0^\ell (\widehat{x}_2(t,\xi) - \widehat{x}_3(t,\xi))^2\,\mathrm{d}\xi\Big)$$

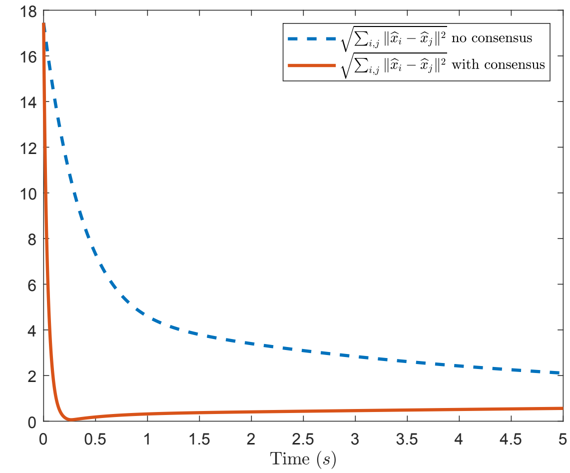

Fig. 3: Evolution of disagreement $\sqrt{\sum_{i,j} \|\widehat{x}_i - \widehat{x}_j\|^2}$.

Figure 3 depicts $\sqrt{2\Psi_G(\widehat{x})}$ which shows the positive effects of consensus in (38) in ensuring filter synchronization.

The proposed multi-band modal consensus filters dictated the communication topology based on the value of information generated and needed. An immediate extension would require to modify this to an integrated sensor grouping and subsequent sensor location together with consensus protocols constrained by parameterized communication topologies.

References

[1] M. A. Demetriou, "Design of consensus and adaptive consensus filters for distributed parameter systems," *Automatica J. IFAC*, vol. 46(2), pp. 300–311, 2010.

[2] Y. Miyasato, "Adaptive H_∞ consensus control for distributed parameter systems of hyperbolic type." Proceedings of the IEEE International Conference on Control Applications (CCA), Aug 2013, pp. 1018–1023.

[3] Y. Orlov, A. Pilloni, A. Pisano, and E. Usai, "Consensus-based leader-follower tracking for a network of perturbed diffusion pdes via local boundary interaction," *IFAC-PapersOnLine*, vol. 49(8), pp. 228 – 233, 2016.

[4] A. Pilloni, A. Pisano, Y. Orlov, and E. Usai, "Consensus-based control for a network of diffusion PDEs with boundary local interaction," *IEEE Trans. Automat. Control*, vol. 61, no. 9, pp. 2708–2713, 2016.

[5] H.-D. Wang and H.-N. Wu, "Design of finite dimensional robust H_∞ distributed consensus filters for dissipative PDE systems with sensor networks," *Internat. J. Robust Nonlinear Control*, vol. 25(1), pp. 1454–1471, 2015.

[6] P. Li, X. Liu, and Y. Lan, "PI-type iterative learning consensus control for second-order hyperbolic distributed parameter models multi-agent systems," *IEEE Access*, vol. 8, pp. 16 049–16 056, 2020.

[7] J. Xiong and J. Li, "Non-fragile consensus algorithms for a network of diffusion PDEs with boundary local interaction," *Internat. J. Systems Sci.*, vol. 48, no. 9, pp. 1829–1835, 2017.

[8] C. Yang, H. He, T. Huang, A. Zhang, J. Qiu, J. Cao, and X. Li, "Consensus for non-linear multi-agent systems modelled by PDEs based on spatial boundary communication," *IET Control Theory Appl.*, vol. 11, no. 17, pp. 3196–3200, 2017.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*. Wiley-Interscience, 1989, vol. 2.