A SIMPLE TEST FOR THE SEPARABILITY OF SYMMETRIC QUANTUM STATES

Domenico D'Alessandro

Department of Mathematics, Iowa State University,
440 Carver Hall, Ames, IA 50011, U.S.A.
(e-mail: daless@iastate.edu)

(*Received November 1, 2021 — Revised May 28, 2022*)

We present a direct test to verify separability of symmetric pure states of n quantum bits, that is, n-qubit states that are invariant under any element of the permutation group.

Keywords: multipartite quantum pure states, separability and entanglement of quantum states, symmetric states.

1. Introduction

The study of quantum mechanical states is an important part of quantum mechanics and quantum information theory [4]. One relevant feature of such states is *separability* and *entanglement* [5]. These properties refer to the situation where the quantum system is a composition of n subsystems. Separable states are states that can be expressed as (statistical mixtures of) tensor products of states of the single subsystems. Entangled states are states that cannot. Entanglement is considered one of the main resources in quantum computation [7], and therefore the study of entanglement has generated a large interest in the last few decades. Types of results that are important in this theory are *criteria* to test if a state is entangled (separable) and the formulation of meaningful *measures* of entanglement. Results are available for *pure* and *mixed* states, *bipartite* and *multipartite* systems and systems whose subsystems are two-dimensional (*qubits*) or have general dimensions (*qudits*). A review is given in [5]. One may also consider special classes of states and see whether new results can be generated or results which are known for general states can be simplified.

In this paper, we are interested in *pure states* of *multipartite quantum systems* where each component subsystem is a two-dimensional system, that is, a *quantum bit (qubit)*. In other words, the states we are interested in are vectors in the 2^n-dimensional complex vector space $\mathcal{H} := \mathcal{H}_2^{\otimes n}$, where $\mathcal{H}_2$ is spanned by the basis of two elements $\{|0\rangle, |1\rangle\}$. As it is customary in quantum mechanics (see, e.g. [10]) vectors which differ by a multiplicative factor are identified. If $|\psi\rangle$ is a state in $\mathcal{H}$ a simple separability-entanglement test is as follows: One considers the *density* *matrix* $\rho := |\psi\rangle\langle\psi|$ and, for each one of the qubits, takes the *partial trace* (see, e.g. [7]) with respect to the remaining ($(n-1)$-qubit) system. If and only if the resulting reduced density matrices represent pure states, the state $|\psi\rangle$ is separable. In this paper we present an even more direct test valid for quantum *symmetric* states which is in terms of the coefficients of the expansion of $|\psi\rangle$ in a given (natural) basis and does not require passing through the density matrix description of the state.

Symmetric states $|\psi\rangle$ are states that are invariant under any element of the *permutation group* S_n, that is, $P|\psi\rangle = |\psi\rangle$ for every $P \in S_n$[1]. Such states have recently received much attention [2, 3, 6, 8, 9] both at the theoretical and at the experimental level. They span the so-called *symmetric sector* and are the largest ($(n+1)$-dimensional) representation of SU(2) in the Clebsch-Gordan decomposition of the tensor product $(\mathrm{SU}(2))^{\otimes n}$ (see, e.g. [12]). The symmetric sector is spanned by the basis

$$\mathcal{B} := \{\phi_0, \phi_1, \ldots, \phi_n\},$$

with

$$\phi_j := \sum |a_1 \cdots a_n\rangle, \tag{1}$$

where the a_k's are 0 or 1 and the sum runs over all the combinations which have j 1's (and $n-j$ 0's). Therefore, for example, the symmetric sector of three qubits is spanned by

$$\begin{aligned}
\phi_0 &:= |000\rangle, \\
\phi_1 &:= |100\rangle + |010\rangle + |001\rangle, \\
\phi_2 &:= |011\rangle + |101\rangle + |110\rangle, \\
\phi_3 &:= |111\rangle.
\end{aligned}$$

In general, states $|\psi\rangle$ in the symmetric sector can be written as

$$|\psi\rangle = \sum_{j=0}^{n} c_j \phi_j, \tag{2}$$

for complex coefficients c_j. These states can also be obtained by applying the symmetrizer operator $\sum_{P \in S_n} P$ to a (nonsymmetric) state.

REMARK 1. The states in (1) are also called (symmetric) *Dicke states*. The separability of *mixed* states which are statistical mixtures of Dicke states was studied in [11] and criteria for separability and entanglement were given there. These are mixed states of the type $\sum_{j=0}^{n} \chi_j \tilde{\phi}_j \tilde{\phi}_j^{\dagger}$, where $\tilde{\phi}_j = \frac{\phi_j}{\|\phi_j\|}$ are normalized Dicke states, and $0 \leq \chi_j \leq 1$, with $\sum_{j=0}^{n} \chi_j = 1$. They are also called *diagonal symmetric states*. These are more general than the states we consider here because not necessarily pure. However, they are less general in that they are 'diagonal'.

[1] The above mentioned criterion of separability based on the density matrix simplifies in the case of symmetric states since because of symmetry the reduced density matrix has to be computed only for one of the qubits, being all the reduced density matrices equal.

For example the symmetric pure state $|\psi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, when written in the density matrix form $|\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|)$, contains off diagonal terms $|00\rangle\langle 11|$ and $|11\rangle\langle 00|$.

Separability of symmetric states can also be inferred by an application of the entanglement classification results for symmetric states in [2]. An alternative test can be obtained by specializing to the qubit case the results of [9] which are based on the concept of entanglement witnesses. The goal of this paper is to present a direct and simple test. This is done in the next section.

2. A test of separability for symmetric states

Our result is the following simple test of separability for symmetric states given directly in terms of the coefficients c_j in the expansion (2).

THEOREM 1. *A symmetric state $|\psi\rangle$ in* (2) *is separable if and only if for every two couples of indexes,* (l_1, l_2), (m_1, m_2), *such that*

$$l_1 + l_2 = m_1 + m_2,$$

it holds

$$c_{l_1} c_{l_2} = c_{m_1} c_{m_2}. \tag{3}$$

REMARK 2. Notice that, consistently with quantum mechanics postulates, condition (3) is independent of a common factor for the state. The differences of the quantities equated in (3) may be used to express measures of entanglement. This happens for the case $n = 3$ treated in [1].

The result was proven in [1] for the case $n = 3$. We generalize it here to an arbitrary number n of subsystems. To check the conditions (3), one can, for every $0 \leq r \leq 2n$, list all the pairs (l_1, l_2) with $l_1 \leq l_2$ such that $l_1 + l_2 = r$, and verify that the quantity $c_{l_1} c_{l_2}$ is the same for every pair. For example for $n = 4$ we obtain the pairs in the following table

r	
0	(0,0)
1	(0,1)
2	(0,2) (1,1)
3	(0,3) (1,2)
4	(0,4) (1,3) (2,2)
5	(1,4) (2,3)
6	(2,4) (3,3)
7	(3,4)
8	(4,4)

The equalities to check according to the Theorem 1 are

$$c_0c_2 = c_1^2, \quad c_0c_3 = c_1c_2, \quad c_0c_4 = c_1c_3 = c_2^2, \quad c_1c_4 = c_2c_3, \quad c_2c_4 = c_3^2.$$

3. Proof of Theorem 1: Necessity

We will prove that, for a symmetric product state $|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)^{\otimes n} = \sum_j c_j\phi_j$, we have the following[2]

$$c_j = \beta^j\alpha^{n-j}, \qquad j = 0, 1, \dots, n. \tag{4}$$

From this, it follows that

$$c_jc_{r-j} = \beta^j\alpha^{n-j}\beta^{r-j}\alpha^{n-(r-j)} = \beta^r\alpha^{2n-r},$$

independent of j, from which (3) follows.

To prove (4) we use induction on n. For $n = 1$, (4) is true because $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$, that is, $c_0 = \alpha$ and $c_1 = \beta$. In order to prove the inductive step, denote by ϕ_j^{n-1} the vector ϕ_j defined in (1) when n is replaced by $n-1$. Using the inductive assumption, we have

$$\begin{aligned}
|\psi\rangle &= \left(\sum_{j=0}^{n-1}\beta^j\alpha^{n-1-j}\phi_j^{n-1}\right) \otimes (\alpha|0\rangle + \beta|1\rangle) \\
&= \sum_{j=0}^{n-1}\beta^j\alpha^{n-j}\phi_j^{n-1} \otimes |0\rangle + \sum_{j=0}^{n-1}\beta^{j+1}\alpha^{n-1-j}\phi_j^{n-1} \otimes |1\rangle \\
&= \alpha^n\phi_0^{n-1} \otimes |0\rangle + \sum_{j=1}^{n-1}\beta^j\alpha^{n-j}\phi_j^{n-1} \otimes |0\rangle + \sum_{j=1}^{n}\beta^j\alpha^{n-1-(j-1)}\phi_{j-1}^{n-1} \otimes |1\rangle \\
&= \alpha^n\phi_0^{n-1} \otimes |0\rangle + \sum_{j=1}^{n-1}\beta^j\alpha^{n-j}\left(\phi_j^{n-1} \otimes |0\rangle + \phi_{j-1}^{n-1} \otimes |1\rangle\right) + \beta^n\phi_{n-1}^{n-1} \otimes |1\rangle.
\end{aligned}$$

Formula (4) follows from $\phi_0^{n-1}\otimes|0\rangle = \phi_0$, $\phi_j^{n-1}\otimes|0\rangle+\phi_{j-1}^{n-1}\otimes|1\rangle = \phi_j$, for $j = 1, \dots, n-1$ and $\phi_{n-1}^{n-1} \otimes |1\rangle = \phi_n$. □

4. Proof of Theorem 1: Sufficiency

First consider the special case when $c_0 = 0$. In this case, from

$$c_0c_2 = c_1^2,$$

we get $c_1 = 0$. Proceeding inductively from $c_1c_3 = c_2^2$, we get $c_2 = 0$, from $c_2c_4 = c_3^2$ we get $c_3 = 0$, and so on up to $c_{n-2}c_n = c_{n-1}^2$ which gives $c_{n-1} = 0$. Thus, the only coefficient different from zero is c_n and the state is $\phi_n = |1\,1\,\cdots 1\rangle$ which is

[2]Here and in the following we agree that $0^0 = 1$.

a separable state. If $c_0 \neq 0$, we can assume without loss of generality that $c_0 = 1$ in the (un-normalized) state (cf. Remark 2). If $c_1 = 0$, from $c_0 c_j = c_j = c_1 c_{j-1}$, we get $c_j = 0$ for $j = 1, \ldots, n$ and the state is $\phi_0 = |0\,0 \cdots 0\rangle$, which is also separable. Assume therefore that $c_0 = 1$ and $c_1 \neq 0$. Inductively, on j from $c_{j-1} c_{j+1} = c_j^2$, we get $c_j = c_1^j$, for $j = 1, \ldots, n$. This corresponds to the (un-normalized) state

$$|\psi\rangle := (|0\rangle + c_1|1\rangle)^{\otimes n},$$

which is separable. □

Remark 3. From the proof of the above theorem, it follows that every symmetric separable state is either equal to ϕ_n or ϕ_0 or it can be written, up to a normalization factor, as

$$|\psi\rangle = \sum_{j=0}^{n} z^j \phi_j, \tag{5}$$

which is obtained by setting $\alpha^n = 1$ and $z := c_1 := \beta\alpha^{n-1}$ in (4). This fact was already proved in [9] (cf. Proposition 2 and formula (15) in that paper) with a different proof and in a more general context of *D-symmetric states* for multipartite systems of dimension $d \geq 2$ (qudits). A basis of such space reduces to the basis of Dicke states $\{\phi_j\}$ in the case of qubits. The sufficiency part of the above theorem shows that the conditions (3) imply such a situation. Sufficient conditions of separability where also given in [9] based on entanglement witnesses and the PPT criterion.

5. Discussion

Using the test of Theorem 1, we can immediately assess, for example, that a state for $n = 2$,

$$|\psi\rangle = \frac{e^{2i\gamma}}{3}\phi_0 + \frac{\sqrt{2}}{3}e^{i\gamma}\phi_1 + \frac{2}{3}\phi_2,$$

is separable since $c_0 c_2 = \frac{2e^{2i\gamma}}{9} = c_1^2$. On the other hand, the state

$$|\psi\rangle = \frac{e^{i\gamma}}{\sqrt{2}}\phi_0 + \frac{1}{\sqrt{2}}\phi_1$$

is not separable since $c_0 c_2 = 0 \neq c_1^2 = \frac{1}{2}$.

In [1], for the case $n = 3$, the entanglement measures were expressed in terms of the differences of the quantities mentioned in Theorem 1, that is, the quantities (for $n = 3$) $X_2 := c_0 c_2 - c_1^2$, $X_3 := c_0 c_3 - c_1 c_2$, and $X_4 := c_1 c_3 - c_2^2$, which are all zero in the case of separable states. It is possible that similar results can be obtained for entanglement measures in the case of general n. This will be a topic for future research.

Acknowledgement

This research was supported by the NSF under Grant ECCS 1710558. The author would like to thank Emmanuel Abebe and Jesse Slater who found an error in a previous version of the paper.

REFERENCES

[1] F. Albertini and D. D'Alessandro: Symmetric States and Dynamics of Three Quantum Bits, preprint September 2021, to appear in *Quantum Information and Computation*.
[2] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata and E. Solano: Operational families of entanglement classes for symmetric N-qubit states, *Phys. Rev. Lett.* **103**, 070503 (2009).
[3] T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano and G. S. Agarwal: Operational determination of multiqubit entanglement classes via tuning of local operations, *Phys. Rev. Lett.* **102**, 053601 (2009).
[4] I. Bengtsson and K. Zyczkowski: *Geometry of Quantum States: an Introduction to Quantum Entanglement*, Cambridge University Press, Cambridge, New York 2006.
[5] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki: Quantum entanglement, *Rev. Mod. Phys.* **81**, 865-942 (2009).
[6] R. Hübener, M. Kleinmann, T-C. Wei, C. González-Guillén and O. Gühne: Geometric measure of entanglement for symmetric states, *Phys. Rev. A* **80**, 032324 (2009).
[7] M. A. Nielsen and I. L. Chuang: *Quantum Computation and Quantum Information*, Cambridge University Press,, Cambridge, U.K., New York 2000.
[8] P. Ribeiro and M. Mosseri: Entanglement in the symmetric sector of three qubits, *Phys. Rev. Lett.* **106**, 180502 (2011).
[9] A. Rutkowski, M. Banacki and M. Marciniak: Necessary and sufficient conditions of separability for D-symmetric diagonal states, *Phys. Rev. A* **99**, 022309 (2019).
[10] J. J. Sakurai: *Modern Quantum Mechanics*, Addison-Wesley Pub. Co., Reading MA, 1994.
[11] N. Yu: Separability of a mixture of Dicke states, *Phys. Rev. A* **94**, 060101(R) (2016).
[12] P. Woit: *Quantum Theory, Groups and Representations; An Introduction*, Springer International Publishing, 2017.