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INTERNALLY HANKEL k-POSITIVE SYSTEMS*

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Abstract. There has been an increased interest in the variation diminishing properties of controlled linear time-invariant (LTI) systems and time-varying linear systems without inputs. In controlled LTI systems, these properties have recently been studied from the external perspective of k-positive Hankel operators. Such systems have Hankel operators that diminish the number of sign changes (the variation) from past input to future output if the input variation is at most $k-1$. For $k=1$, this coincides with the classical class of externally positive systems. For linear systems without inputs, the focus has been on the internal perspective of k-positive state-transition matrices, which diminish the variation of the initial system state. In the LTI case and for $k=1$, this corresponds to the classical class of (unforced) positive systems. This paper bridges the gap between the internal and external perspectives of k-positivity by analyzing *internally* Hankel k-positive systems, which we define as state-space LTI systems where controllability and observability operators as well as the state-transition matrix are k-positive. We show that the existing notions of external Hankel and internal k-positivity are subsumed under internal Hankel k-positivity, and we derive tractable conditions for verifying this property in the form of internal positivity of the first k compound systems. As such, this class provides new means to verify external Hankel k-positivity, and lays the foundation for future investigations of variation diminishing controlled linear systems. As an application, we use our framework to derive new bounds for the number of over- and undershoots in the step responses of LTI systems. Since our characterization defines a new positive realization problem, we also discuss geometric conditions for the existence of minimal internally Hankel k-positive realizations.

Key words. positive systems, total positivity, k-positivity, variation diminishing, step response analysis

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1. Introduction.

Externally positive linear time-invariant (LTI) systems

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t), \\ y(t) &= cx(t) \end{aligned} \tag{1.1}$$

mapping nonnegative inputs $u(t)$ to nonnegative outputs $y(t)$ have been recognized as an important system class at least since the exposition by Luenberger [26], but many of their favorable properties have only recently been exploited [33, 38, 14, 36]. Particular emphasis has been given to the subclass of internally positive systems, that is, externally positive systems such that $x(t)$ remains in the nonnegative orthant for nonnegative $u(t)$. As such systems are characterized by nonnegative system matrices A, b, and c, they can be studied with finite-dimensional nonnegative matrix analysis [7], an advantage that motivated the search for conditions under which an externally positive system admits an internally positive realization [30, 2, 6].

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At the same time, externally positive systems are central to the study of *variation-diminishing* convolution operators

$$y(t) = \sum_{\tau=-\infty}^{\infty} g(t-\tau)u(\tau), \tag{1.2}$$

with nonnegative kernels g, that bound the *variation* (number of sign changes) of $y(t)$ by the variation of $u(t)$. The theory of variation-diminishing transformations has a rich history rooted in the seminal works of Schoenberg [34], Gantmacher and Krein [17], and Karlin [24]. A linear mapping $u \mapsto Gu$ is called *k-variation diminishing* (VD_k) if it maps an input u with at most k sign changes to an output Gu whose number of sign changes do not exceed those of u; if the order in which sign changes occur is preserved whenever u and Gu share the same number of sign variations, the VD_k property is said to be *order-preserving* (OVD_k).

The framework of *total positivity* [24] provides a characterization of variation diminishment: a linear mapping is OVD_{k-1} if and only if its matrix representation is k-positive, that is, all the minors of order up to k in that matrix are nonnegative [20, 24]; *total* positivity refers to the case when this is true for all k. Under this framework, externally positive LTI systems are associated with OVD_0 Hankel and Toeplitz operators, while internally positive systems are associated with OVD_0 controllability operators, observability operators, and the linear mappings $x \to Ax$.

Despite the link between positive systems and variation-diminishing operators, it was not until very recently that OVD_{k-1} and k-positivity have been studied as properties of LTI systems when $k > 1$. New results, with applications in nonlinear systems analysis and model order reduction, have so far focused on two distinct cases: the *external case* [19, 20, 18], dealing with controlled LTI systems; and the *unforced case* [27, 1, 43], dealing with state-space systems without inputs. The former concerns the k-positivity of Toeplitz and Hankel operators, while the latter, when translated to the LTI setting of the present paper, concerns k-positivity of the state-space matrix A, with $b = 0$. In contrast to the present work, [1, 43] also consider the case where the minors of order k (but not necessarily those of order smaller than k) of A are sign-definite (i.e., A is *sign consistent of order k* [24]). This line of research has lead to several nonlinear extensions [40, 39, 43, 42] leading, for instance, to a generalization of cooperative systems (that is, unforced systems whose linearizations are uniformly 1-positive [3, 22, 35]).

A major result of [20] characterizes *(externally) Hankel k-positive systems*, i.e., systems with OVD_{k-1} Hankel operators, in terms of the external positivity of all so-called *j-compound systems*, $1 \leq j \leq k$, whose impulse responses are given by consecutive j-minors of the Hankel operator's matrix representation.

In this paper, we develop a realization theory of Hankel k-positivity based on the notion of *internally Hankel k-positive systems*, which we define as state-space systems where the controllability and observability operators as well as A are OVD_{k-1}. Not only does this theory enable the study of variation-diminishing systems through finite-dimensional analysis, but it also establishes an important first bridge between the aforementioned unforced and external notions. Our main result is a finite-dimensional, tractable condition for the verification of the OVD_{k-1} property of the controllability and observability operators. To prove this result, we rely on a new extension of classical k-positivity verification using consecutive minors. Consequently, internal Hankel k-positivity can be completely characterized in terms of the existence of a realization that renders all j-compound systems internally positive, $1 \leq j \leq k$.

We then use these insights to discuss geometric conditions for the existence of minimal internally Hankel k-positive realizations, as previously done for the special case $k = 1$ in [30]. In particular, it is easy to verify then that all relaxation systems [41] ($k = \infty$) have a minimal internally Hankel totally positive realization.

As a practical application, we show how our results can be used to obtain upper bounds on the number of over- and undershoots in the step response of an LTI system. This is a classical control problem that lies at the heart of the rise-time-settling-time trade-off [5], and for which several lower bounds [8, 37, 10, 9], but few upper bounds [10, 9], have been found. Although our approach produces upper bounds which are not necessarily tight (a question we leave for future work), it improves on existing results by directly generalizing the approach in [10, 9]. Nonlinear extensions of this problem are of interest both in control [25] and online learning in the form of (static) regret [31]; we thus envision our work as the basis for possible interdisciplinary applications. Other possible contributions resulting from nonlinear extension are discussed in [20].

This paper is organized as follows. In the preliminaries, we recap total positivity theory and externally Hankel k-positive systems. Then, we introduce the concept of internal Hankel k-positivity and present our main results on its characterization. Subsequently, extensions to continuous-time and applications to the determination of impulse response zero-crossings are discussed. We conclude with illustrative examples and a summary of open problems.

2. Preliminaries.

This work lies at the interface between positive control systems and total positivity theory. Alongside some standard notation, this section briefly introduces key concepts and results from these fields, including recent results on externally k-positive LTI systems, which are crucial to the motivation of our main results.

2.1. Notation.

We write $\mathbb{Z}$ for the set of integers and $\mathbb{R}$ for the set of reals, with $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ standing for the respective subsets of nonnegative elements; the corresponding notation with strict inequality is also used for positive elements. The set of real sequences with indices in $\mathbb{Z}$ is denoted by $\mathbb{R}^{\mathbb{Z}}$. For matrices $X = (x_{ij}) \in \mathbb{R}^{n \times m}$, we say that X is *nonnegative*, $X \geq 0$, or $X \in \mathbb{R}^{n \times m}_{\geq 0}$ if all elements $x_{ij} \in \mathbb{R}_{\geq 0}$; again, we use the corresponding notation in case of *positivity*. The notation also applies to sequences $x = (x_i) \in \mathbb{R}^{\mathbb{Z}}$. Submatrices of $X \in \mathbb{R}^{n \times m}$ are denoted by $X[I, J] := (x_{ij})_{i \in I, j \in J}$, where $I \subseteq \{1, \dots, n\}$ and $J \subseteq \{1, \dots, m\}$. We also use the notation $X[i : i + r, j : j + s] := X[\{i, \dots, i + r\}, \{j, \dots, j + s\}]$ for $i \leq n - r$ and $j \leq m - s$. If I and J have cardinality r, then $\det(X[I, J])$ is called an *r-minor*; furthermore, $\det(X[i : i + r - 1, j : j + r - 1])$ is called a *consecutive r-minor*. For $X \in \mathbb{R}^{n \times n}$, $\sigma(X) = \{\lambda_1(X), \dots, \lambda_n(X)\}$ denotes the *spectrum* of X, where the eigenvalues are ordered by descending absolute value, i.e., $\lambda_1(X)$ is the eigenvalue with the largest magnitude, counting multiplicity. In case the magnitude of two eigenvalues coincides, we subsort them by decreasing real part. If there exists a permutation matrix $P = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$ so that $P_2^T X P_1 = 0$, then X is called *reducible* and otherwise *irreducible*. Further, X is said to be *positive semidefinite*, $X \succeq 0$, if $X = X^{\mathsf{T}}$ and $\sigma(X) \subset \mathbb{R}_{\geq 0}$. We use I_n to denote the identity matrix in $\mathbb{R}^{n \times n}$ and write $\mathrm{im}(X)$ for the *image/range* of a matrix $X \in \mathbb{R}^{n \times m}$. For $\mathcal{S} \subset \mathbb{R}^n$, we denote its *closure*, *convex hull*, and *convex conic hull* by $\mathrm{cl}(\mathcal{S})$, $\mathrm{conv}(\mathcal{S})$, and $\mathrm{cone}(\mathcal{S})$, respectively. $\mathcal{S}$ is a *polyhedral cone* if there exists $k \in \mathbb{Z}_{>0}$ and $P \in \mathbb{R}^{n \times k}$ such that $\mathcal{S} = \{Px : x \in \mathbb{R}^k_{\geq 0}\} =: \mathrm{cone}(P)$. For $A \in \mathbb{R}^{n \times n}$, $\mathcal{S}$ is said to be *A-invariant*, $A\mathcal{S} \subseteq \mathcal{S}$, if $Ax \in \mathcal{S}$ for all $x \in \mathcal{S}$. For a subset $\mathcal{S} \subset \mathbb{Z}$, we write $g \geq 0$ or $g \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$ if $g : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a *nonnegative function* *(sequence)* and

$$\mathbb{1}_{\mathcal{S}}(t) := \begin{cases} 1, & t \in \mathcal{S}, \\ 0, & t \notin \mathcal{S} \end{cases}$$

for the (1-0) indicator function. In particular, we denote the *Heaviside function* by $s(t) := \mathbb{1}_{\mathbb{R}_{\geq 0}}(t)$ and the *unit pulse function* by $\delta(t) := \mathbb{1}_{\{0\}}(t)$. The set of all *absolutely summable sequences* is denoted by ℓ_1 and the set of *bounded sequences* by ℓ_∞.

2.2. Linear discrete-time systems. We consider *finite-dimensional, discrete-time LTI systems* with single input u and single output y. The output $g(t) = y(t)$ corresponding to $u(t) = \delta(t)$ is called the *impulse response*. The *transfer function* of the system is given by

$$G(z) = \sum_{t=0}^{\infty} g(t) z^{-t} = \frac{r \prod_{i=1}^{m}(z - z_i)}{\prod_{j=1}^{n}(z - p_i)}, \tag{2.1}$$

where $r \in \mathbb{R}$, and p_i and z_i are referred to as *poles* and *zeros*, both of which are sorted in the same way as the eigenvalues of a matrix. Without loss of generality, we assume that $g(0) = 0$ ($m < n$). The tuple (A, b, c) is referred to as a *state-space realization* of $G(z)$ if (1.1) holds, with $A \in \mathbb{R}^{n \times n}$, and $b, c^\mathsf{T} \in \mathbb{R}^n$ It then holds that

$$g(t) = cA^{t-1}b\, s(t-1).$$

We assume that the set of poles and the set of zeros of a transfer function are disjoint, and define the *order of a system* as the number of poles of $G(z)$. A realization (A, b, c) is called *minimal* if the eigenvalues of A are precisely the poles of $G(z)$. Throughout this work, we assume that A is asymptotically stable (that is, $\lambda_1(A) < 1$) and $u \in \ell_\infty$, which implies $g \in \ell_1$ and $y \in \ell_\infty$. For $t \geq 0$, the *Hankel operator* (see, e.g., [4, sections 4.4 and 5.4] for an introduction)

$$(\mathcal{H}_g u)(t) := \sum_{\tau=-\infty}^{-1} g(t-\tau)u(\tau) = \sum_{\tau=1}^{\infty} g(t+\tau)u(-\tau) \tag{2.2}$$

describes the evolution of y after u has been turned off at $t = 0$, i.e., $u(t) = u(t)(1 - s(t))$. It obeys the factorization

$$\mathcal{H}_g u = \mathcal{O}(A, c)(\mathcal{C}(A, b)u) \tag{2.3}$$

with the *controllability and observability operators* given by

$$x(0) = \mathcal{C}(A, b)u := \sum_{\tau=-\infty}^{-1} A^{-\tau-1} b u(\tau),\ u \in \ell_\infty, \tag{2.4a}$$

$$(\mathcal{O}(A, c)x_0)(t) := cA^t x_0,\ x_0 \in \mathbb{R}^n,\ t \in \mathbb{Z}_{\geq 0}. \tag{2.4b}$$

Asymptotic stability of A implies that both (2.2) and (2.4a) are well defined. Finally, for $t, j \in \mathbb{Z}_{>0}$, we will make use of the *Hankel matrix*

(2.5a)

$$H_g(t, j) := \begin{pmatrix} g(t) & g(t+1) & \dots & g(t+j-1) \\ g(t+1) & g(t+2) & \dots & g(t+j) \\ \vdots & \vdots & \ddots & \vdots \\ g(t+j-1) & g(t+j) & \dots & g(t+2(j-1)) \end{pmatrix} = \mathcal{O}^j(A, c)A^{t-1}\mathcal{C}^j(A, b),$$

where

$$\mathcal{C}^j(A,b) := \begin{pmatrix} b & Ab & \dots & A^{j-1}b \end{pmatrix}, \tag{2.5b}$$

$$\mathcal{O}^j(A,c) := \mathcal{C}^j(A^\mathsf{T}, c^\mathsf{T})^\mathsf{T}. \tag{2.5c}$$

2.3. Total positivity and the variation diminishing property. A central idea in this work is that positivity is an instance of the variation diminishing property. The *variation* of a sequence or vector u is defined as the number of sign-changes in u, i.e.,

$$\mathrm{S}(u) := \sum_{i \geq 1} \mathbb{1}_{\mathbb{R}_{<0}}(\tilde{u}_i \tilde{u}_{i+1}), \quad \mathrm{S}(0) := 0,$$

where $\tilde{u}$ is the vector resulting from deleting all zeros in u.

DEFINITION 2.1. *A linear map $u \mapsto Xu$ is said to be* order-preserving k-variation diminishing *(OVD$_k$), $k \in \mathbb{Z}_{\geq 0}$, if for all u with $S(u) \leq k$ it holds that*

i. *$S(Xu) \leq S(u)$.*
ii. *The sign of the first nonzero elements in u and Xu coincide whenever $S(u) = S(Xu)$.*

If the second item is dropped, then $u \mapsto Xu$ is called k-variation diminishing *(VD$_k$). For brevity, we simply say X is (O)VD$_k$.*

The OVD$_k$ property extends the cone-invariance of nonnegative matrices, namely $X \in \mathbb{R}^{n \times m}_{\geq 0}$ is OVD$_0$, because $X\mathbb{R}^m_{\geq 0} \subseteq \mathbb{R}^n_{\geq 0}$. For generic k, *total positivity theory* provides algebraic conditions for the OVD$_k$ property by means of compound matrices. To define these, let the ith elements of the r-tuples in

$$\mathcal{I}_{n,r} := \{v = \{v_1, \dots, v_r\} \subset \mathbb{N} : 1 \leq v_1 < v_2 < \dots < v_r \leq n\}$$

be defined by *lexicographic ordering*. Then, the (i,j)th entry of the *rth multiplicative compound matrix* $X_{[r]} \in \mathbb{R}^{\binom{n}{r} \times \binom{m}{r}}$ of $X \in \mathbb{R}^{n \times m}$ is defined by $\det(X[I,J])$, where I is the ith and J is the jth element in $\mathcal{I}_{n,r}$ and $\mathcal{I}_{m,r}$, respectively. For example, if $X \in \mathbb{R}^{3 \times 3}$, then $X_{[2]}$ reads

$$\begin{pmatrix} \det(X[\{1,2\},\{1,2\}]) & \det(X[\{1,2\},\{1,3\}]) & \det(X[\{1,2\},\{2,3\}]) \\ \det(X[\{1,3\},\{1,2\}]) & \det(X[\{1,3\},\{1,3\}]) & \det(X[\{1,3\},\{2,3\}]) \\ \det(X[\{2,3\},\{1,2\}]) & \det(X[\{2,3\},\{1,3\}]) & \det(X[\{2,3\},\{2,3\}]) \end{pmatrix}.$$

Notice a nonnegative matrix verifies $X_{[1]} = X \geq 0$, which is equivalent to X being OVD$_0$. This can be generalized through the compound matrix as follows (see [20, Proposition 4]).

DEFINITION 2.2. *Let $X \in \mathbb{R}^{n \times m}$ and $k \leq \min\{m,n\}$. X is called* k-positive *if $X_{[j]} \geq 0$ for $1 \leq j \leq k$, and* strictly k-positive *if $X_{[j]} > 0$ for $1 \leq j \leq k$. In case $k = \min\{m,n\}$, X is called* (strictly) totally positive.

PROPOSITION 2.3. *Let $X \in \mathbb{R}^{n \times m}$ with $n \geq m$. Then, X is k-positive with $1 \leq k \leq m$ if and only if X is OVD$_{k-1}$.*

The following properties of the multiplicative compound matrix will be elementary to our discussion (see, e.g., [15, section 6] and [23, subsection 0.8.1]).

LEMMA 2.4. *Let $X \in \mathbb{R}^{n \times p}$ and $Y \in \mathbb{R}^{p \times m}$.*

(i) $(XY)_{[r]} = X_{[r]} Y_{[r]}$ *(Cauchy–Binet formula).*

(ii) $\sigma(X_{[r]}) = \{\prod_{i \in I} \lambda_i(X) : I \in \mathcal{I}_{n,r}\}$.
(iii) $X^{\mathsf{T}}{}_{[r]} = (X_{[r]})^{\mathsf{T}}$.

In conjunction with the Perron–Frobenius theorem [32, 16], this yields a spectral characterization of k-positive matrices as follows (see, e.g., [15, Chapter 6]).

Corollary 2.5. *Let $X \in \mathbb{R}^{n \times n}$ be k-positive such that $X_{[j]}$ is irreducible for $1 \leq j \leq k$. Then,*

i. $\lambda_1(X) > \cdots > \lambda_k(X) > 0$.
ii. $\lambda_1(X_{[j]}) = \prod_{i=1}^{j} \lambda_i(X) > 0$.
iii. $\begin{pmatrix} \xi_1 & \dots & \xi_j \end{pmatrix}_{[j]} \in \mathbb{R}^{\binom{n}{j}}_{>0}$, $1 \leq j \leq k$, *where ξ_i is the eigenvector associated with $\lambda_i(X)$ for $1 \leq i \leq k$.*

The next result shows that it often suffices to check consecutive minors to verify k-positivity vis-a-vis a combinatorial number of minors (see, e.g., [12, Theorem 2.3 and its proof]).

Proposition 2.6. *Let $X \in \mathbb{R}^{n \times m}$, $k \leq \min\{n, m\}$ be such that*

i. *all consecutive r-minors of X are positive, $1 \leq r \leq k-1$,*
ii. *all consecutive k-minors of X are nonnegative (positive).*

Then, X is (strictly) k-positive.

Finally, to be able to apply Proposition 2.6 to matrices lacking strictly positive intermediate j-minors, we will make use of the following important ancillary result, also used extensively in [17, 24].

Proposition 2.7. *Let $F(\sigma) \in \mathbb{R}^{n \times n}$ be given by $F(\sigma)_{ij} = e^{-\sigma(i-j)^2}$, with $\sigma > 0$, and let $X \in \mathbb{R}^{n \times m}$ with $m \leq n$. Then for $r \leq m$, the following hold:*

i. *$F(\sigma)$ is strictly totally positive.*
ii. *$F(\sigma) \to I$ as $\sigma \to \infty$, and $F(\sigma)X \to X$ as $\sigma \to \infty$.*
iii. *If $X_{[r]} \geq 0$, and if $\operatorname{rank} X = m$, then $(F(\sigma)X)_{[r]} > 0$ for all $\sigma > 0$.*
iv. *If $(F(\sigma)X)_{[r]} \geq 0$ for all $\sigma > 0$, then $X_{[r]} \geq 0$.*

Proof. Proposition 2.7 (i)–(iii) are proven in [24, p. 220], while Proposition 2.7 (iv) is a consequence of Proposition 2.7 (ii) and the fact that the minors of a matrix are polynomial in the matrix's entries (by the well-known Laplace expansion) and thus also continuous in the entries. ∎

Corollary 2.8. *Let $X \in \mathbb{R}^{n \times m}$, $k \leq \min\{n, m\}$ be such that*

i. $X[1:n, i:i+r-1]_{[k]} \geq 0$ *for* $1 \leq i \leq m+1-r$ *and* $1 \leq r \leq k$,
ii. *all consecutive $k-1$ columns of X have full rank.*

Then, X is k-positive.

Proof. Let $\sigma > 0$ be arbitrary and $F(\sigma)$ as in Proposition 2.7. By Corollary 2.8 (i) and (ii), it follows from Proposition 2.7 (iii) that all consecutive r-minors of $F(\sigma)X$ are positive for $1 \leq r \leq k-1$. In addition, by Corollary 2.8 (i) and Proposition 2.7 (i), all consecutive k-minors of $F(\sigma)X$ are nonnegative. Thus, $F(\sigma)X$ fulfills the assumptions of Proposition 2.6, and the result follows from Proposition 2.7 (iv). ∎

Remark 2.9. Note that, in general, neither the positivity assumption in Proposition 2.6 (i) nor the rank assumption in Corollary 2.8 (ii) can be dropped. For example,

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

fulfills Corollary 2.8 (i), but $\det(X[\{1,2\},\{1,3\}]) = -1$. A suitable extension of Corollary 2.8 that avoids this problem is given in Theorem 3.6.

2.4. Hankel k-positivity and compound systems. The OVD_k property of LTI systems (1.1) has been studied in [20], where a distinction is made between LTI systems with OVD_k Toeplitz and Hankel operators. The latter are particularly relevant to this work.

DEFINITION 2.10. *A system $G(z)$ is called* Hankel k-positive *if $\mathcal{H}_g$ is OVD_{k-1} ($k \geq 1$). If $k = \infty$, then $G(z)$ is said to be* Hankel totally positive.

In other words, $G(z)$ is OVD_{k-1} from past inputs to future outputs. Note that if $G(z)$ is Hankel k-positive, then it is also Hankel j-positive, $1 \leq j \leq k$. Since an OVD_{k-1} $\mathcal{H}_g$ maps nonnegative inputs to nonnegative outputs, it can be verified that Hankel 1-positivity coincides with the familiar property of external positivity.

DEFINITION 2.11. *$G(z)$ is* externally positive *if $y \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}$ for all $u \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}$ (and $x(0) = 0$).*

A central observation of [20, Lemma 2] is the following characterization involving k-positive matrices.

LEMMA 2.12. *A system $G(z)$ is Hankel k-positive if and only if for all $j \in \mathbb{Z}_{\geq k}$, $H_g(1, j)$ is k-positive.*

Using Corollary 2.8 and Proposition 2.7, it is easy to show that k-positivity of Hankel matrices only require checking the nonnegativity of consecutive minors [12]. From (2.5a), each of these consecutive minors is given by

$$g_{[j]}(t) := \det(H_g(t, j)),$$

which is interpreted as the impulse response of an LTI system $G_{[j]}(z)$, called the *jth compound system*. The compound systems feature in the following characterization [20, Theorem 1].

PROPOSITION 2.13. *Given $G(z)$ and $1 \leq k \leq n$, the following are equivalent:*

i. *$G(z)$ is Hankel k-positive.*
ii. *$G_{[j]}(z)$ is externally positive for $1 \leq j \leq k$.*
iii. *$H_g(1, k-1) \succ 0$, $H_g(2, k-1) \succeq 0$ and $G_{[k]}(z)$ is externally positive.*
iv. *$G_{[j]}$ is Hankel $k - j + 1$-positive for $1 \leq j \leq k$.*

In particular, the equivalence between Hankel OVD_0 and external positivity becomes evident as both properties require $g_{[1]} = g \geq 0$ [14].

A key fact for our new investigations is that if (A, b, c) is a realization of $G(z)$, then $G_{[j]}(z)$ can be realized as

$$(A_{[j]}, \mathcal{C}^j(A, b)_{[j]}, \mathcal{O}^j(A, c)_{[j]}), \tag{2.6}$$

since

$$\det(H_g(t, j)) = H_g(t, j)_{[j]} = \mathcal{O}^j(A, c)_{[j]}(A_{[j]})^{t-1}\mathcal{C}^j(A, b)_{[j]}$$

by (2.5a) and Lemma 2.4. Note that by (2.5a), $g_{[j]} = 0$ if $j > n$, which is why $k = n$ coincides with the case $k = \infty$. The following pole constraints of Hankel k-positive systems will also be important for our new developments [20, Proposition 6].

Proposition 2.14. *Let* $G(z) = \sum_{a=1}^{l} \sum_{b=1}^{m_a} \frac{r_{ba}}{(z-p_a)^b}$ *be Hankel* k*-positive. Then,* $m_1 = \cdots = m_{k-1} = 1$ *and* $p_{k-1} > 0$ *if* $k \leq \sum_{a=1}^{l} m_a$*. In particular,* $G(z)$ *is Hankel totally positive if and only if all poles are nonnegative and simple.*

3. Internally Hankel k-positive systems.

In this section, we introduce and study a subclass of Hankel k-positive systems which admit state-space realizations such that the OVD_{k-1} property also holds internally.

Definition 3.1. (A, b, c) *is called* internally Hankel k-positive *if* A*,* $\mathcal{C}(A, b)$*, and* $\mathcal{O}(A, c)$ *are* OVD_{k-1} $(1 \leq k \leq n)$*. If* $k = n$*, we say that* (A, b, c) *is* internally Hankel totally positive.

Internally Hankel k-positive systems are, therefore, OVD_{k-1} from past input u to $x(0)$, and from $x(0)$ to all future $x(t)$ and future output y. In particular, by (2.3), all internally Hankel k-positive systems are also (externally) Hankel k-positive, and setting $u \equiv 0$ recovers the k-positive property of unforced systems as partially studied in [27, 1, 43]. Thus, Definition 3.1 bridges the external and the autonomous notions of variation diminishing LTI systems. In the remainder of this section, we aim to answer the following main questions:

I. How does internal Hankel k-positivity manifest as tractable algebraic properties of (A, b, c)?

II. When does a system have a minimal internally Hankel k-positive realization?

Our answers will generalize the well-known case of $k = 1$ [30, 2, 6, 14, 26], which we will see coincides with the familiar class of internally positive systems [14].

Definition 3.2. (A, b, c) *is said to be* internally positive *if for all* $u \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}$ *and all* $x(0) \geq 0$*, it follows that* $y \in \mathbb{R}_{\geq 0}^{\mathbb{Z}_{\geq 0}}$ *and* $x(t) \geq 0$ *for all* $t \geq 0$*.*

In section 4, our findings are extended to continuous-time systems, and we use our result to establish a framework that upper bounds the variation of the impulse response in arbitrary LTI systems.

3.1. Characterization of internally Hankel k-positive systems.

We start by recalling the following well-known characterization of internal positivity in terms of system matrix properties [26].

Proposition 3.3. (A, b, c) *is internally positive if and only if* $A, b, c \geq 0$*.*

Therefore, internal positivity indeed implies that (A, b, c) is internally Hankel 1-positive (through Proposition 2.3). The converse can be seen from the following equivalences, which give a first characterization of internal Hankel k-positivity.

Lemma 3.4. *For* (A, b, c)*, the following are equivalent:*

i. $\mathcal{C}(A, b)$ *and* $\mathcal{O}(A, c)$ *are* OVD_{k-1}*, respectively.*

ii. *For all* $t \geq k$*,* $\mathcal{C}^t(A, b)$*, and* $\mathcal{O}^t(A, c)$ *are* k*-positive, respectively.*

In particular, (A, b, c) *is internally Hankel* k*-positive if and only if* A*,* $\mathcal{C}^t(A, b)$ *and* $\mathcal{O}^t(A, b)$ *are* k*-positive for all* $t \geq k$*.*

Proof. By Proposition 2.3, it suffices to show that $\mathcal{C}(A, b)$ is OVD_{k-1} if and only if $\mathcal{C}^t(A, b)$ is OVD_{k-1} for all $t \geq k$. For $\mathcal{O}(A, c)$, the proof is analogous via the duality (2.5c).

$\Rightarrow$: Follows by considering inputs u with $u(\tau) = 0$ for $\tau < -t$.

$\Leftarrow$: Let u be an input with $\mathrm{S}(u) \leq k - 1$. Since

$$\mathrm{S}(\mathcal{C}^t(A, b) \begin{pmatrix} u(-1) & \dots & u(-t) \end{pmatrix}^\mathsf{T}) \leq k - 1$$

for all $t > 0$. In the limit $t \to \infty$ we obtain $\mathrm{S}(\mathcal{C}(A,b)u) \leq k-1$. ∎

Next, we want to find a finite-dimensional and, thus, certifiable characterization of internal Hankel k-positivity. To this end, we derive our *first main result*: a sufficient condition for k-positivity of the controllablility and observability operators.

THEOREM 3.5. *Let (A,b,c) be a realization of $G(z)$ such that A is k-positive. The following hold:*

i. *If $\mathcal{C}^j(A,b)_{[j]} \geq 0$ for $1 \leq j \leq k$, then $\mathcal{C}^t(A,b)$ is k-positive for all $t \geq k$.*

ii. *If $\mathcal{O}^j(A,c)_{[j]} \geq 0$ for $1 \leq j \leq k$, then $\mathcal{O}^t(A,c)$ is k-positive for all $t \geq k$.*

To prove this result, we need the following extension of Corollary 2.8.

THEOREM 3.6. *Let $X \in \mathbb{R}^{n\times m}_{\geq 0}$, $m \geq n$, $2 \leq k \leq n$, and $Y \in \mathbb{R}^{n \times p}$ be such that*

i. *$X[1:n, i:i+k-1]_{[k]} \geq 0$ for $1 \leq i \leq m+1-k$,*

ii. *all consecutive $k-1$ columns of X have full rank.*

1. *$X_Y := \begin{pmatrix} X & Y \end{pmatrix}$ is $k-1$-positive.*

iii. *$Y[1:n,i] \in \operatorname{im}(X[1:n, m-k+2:m])$, $1 \leq i \leq p$.*

iv. *If $Y[1:n,j] = 0$, then $Y[1:n,i] = 0$, $j \leq i \leq p$.*

Then, X_Y is k-positive.

Proof. Since X is k-positive by Theorem 3.6 (i) and (ii) and Corollary 2.8, and Y is k-positive by Theorem 3.6 (iii) and (iv), to show that X_Y is k-positive we only need to prove the nonnegativity of all k-minors which involve at least one column of X and one column of Y. We shall prove this fact by induction on the number of columns of Y involved in the computation of those minors. It turns out this is equivalent to an induction on p by the following argument.

Our assumptions, and the claim of the theorem, are invariant to deletions of columns in Y. Thus, checking the nonnegativity of k-minors that involve $N \leq k-1$ columns of Y is equivalent to proving the claim for all $Y \in \mathbb{R}^{n\times N}$ that fulfill Theorem 3.6 (iii)–(v). Further, if we have shown our claim for $p = N < k-1$, then proving it for $p = N+1$ only requires checking all minors that involve $N+1$ columns of Y. Thus, we perform induction on p, with the induction assumptions and claim being those of the theorem.

The base case $p = 1$. By Theorem 3.6 (iv), $Y[1:n,1] = X[1:n, m-k+2:m]\alpha$ for some $\alpha \in \mathbb{R}^{k-1}$. If $Y[1:n,1] = 0$, then the claim follows trivially. Otherwise, by Theorem 3.6 (ii), there exists a nonsingular $X[\mathcal{I}, m-k+2:m]$ such that $X[\mathcal{I}, m-k+2:m]\alpha = Y[\mathcal{I},1]$. Then, applying *Cramer's Rule* together with the $k-1$-positivity of X_Y gives

$$\alpha_{m-i}(-1)^i \geq 0,\ 0 \leq i \leq k-2. \tag{3.1}$$ ∎

Next, let $\mathcal{P} := \{i : \alpha_i \neq 0\}$ and X_Y^i denote the matrix that is identical to X_Y, except that x_i is deleted. Then, if $i \in \mathcal{P}$, all the consecutive $k-1$-columns of X_Y^i have full rank and by (3.1), it is readily seen that

$$(X_Y^i)_{[k]} = |\alpha_i| X[1:n, m-k+2:m]_{[k]} \geq 0.$$

Hence, X_Y^i is k-positive by Corollary 2.8 for all $i \in \mathcal{P}$, which is why the only k-minors that are not verified to be nonnegative are those that contain y_1 and all x_i, $i \in \mathcal{P}$. Since these minors are zero by construction (of a low-rank matrix), the base case is proven.

Induction step. Let us assume that the claim holds true for $p = N < k-1$. We want to show that X_Y is k-positive also when $p = N+1$. To this end, we have already noticed that only the nonnegativity of minors that involve all columns of $Y \in \mathbb{R}^{n\times N+1}$ needs to be verified. As in the base case, $Y = 0$ is trivial and otherwise, consider the matrices X_Y^i, $i \in \mathcal{P}$, which by the induction assumption are k-positive. Then, the only possible k-minors of X_Y that remain to be checked are those that are made up by all x_i, $i \in \mathcal{P}$ and all columns in Y. Since these minors are zero by construction, this concludes the proof.

Proof to Theorem 3.5. We will prove the first item by induction over the minor size. The second item follows by duality. We begin by noticing that all consecutive all consecutive j columns are of the form $A^i\mathcal{C}^j(A,b)$, $i \geq 0$, which by assumption fulfill $A^i\mathcal{C}^j(A,b)_{[j]} \geq 0$. Further, note that the nonnegativity of the j-minors with $j = 1$ (base case) and $j > n_c := \operatorname{rank}(\mathcal{C}^t(A,b))$ is trivial.

It therefore suffices to check j-minors with $1 < j \leq \min\{k, n_c\}$ under the assumption that $\mathcal{C}^t(A,b)$ is $j-1$-positive for all $t \geq j-1$. If all consecutive $j-1$ columns of $\mathcal{C}^t(A,b)$ have full rank, then $\mathcal{C}^t(A,b)$ is j-positive by Corollary 2.8. Otherwise, if $\operatorname{rank}(A^i\mathcal{C}^{j-1}(A,b)) < j-1$, then $\operatorname{rank}(A^l\mathcal{C}^{j-1}(A,b)) < j-1$ for all $l \geq i$, and the following hold:

i. The first $\min\{t, n_c\}$ columns of $\mathcal{C}^t(A,b)$ are linearly independent.
ii. If i^* is the smallest integer for which $\operatorname{rank}(A^{i^*}\mathcal{C}^{j-1}(A,b)) < j-1$, then $A^lb \in \operatorname{span}(A^{i^*}b, \dots, A^{i^*+j-3}$, $l \geq i^*+j-1)$.
iii. If $A^ib = 0$, then $A^lb = 0$ for all $l \geq i$.

For sufficiently large $t \geq j$, $X = \mathcal{C}^{i^*+j-2}(A,b)$ and $Y = A^{i^*+j-2}\mathcal{C}^{t-i^*-j+2}(A,b)$ then define a splitting $\mathcal{C}^t(A,b) = \begin{pmatrix} X & Y \end{pmatrix}$ with X and Y fulfilling Theorem 3.6. This concludes the proof. ∎

Combining Lemma 3.4 and Theorem 3.5 gives the following characterization of internal Hankel k-positivity.

THEOREM 3.7. *(A,b,c) is internally Hankel k-positive if and only if the realizations of the first k compound systems of (A,b,c) in* (2.6) *are (simultaneously) internally positive.*

3.2. Internally Hankel *k*-positive realizations. To approach the question of the existence of (minimal) Hankel k-positive realizations, we turn to an invariant cone approach, which has proven to be useful in dealing with the case $k = 1$ [30, 6]. The following is a classical result.

PROPOSITION 3.8. *For $G(z)$, the following are equivalent:*
1. *$G(z)$ is externally positive with minimal realization (A,b,c).*
2. *There exists an A-invariant proper convex cone $\mathcal{K}$ such that $b \in \mathcal{K}$ and $c^\mathsf{T} \in \mathcal{K}^*$.*

In particular, $G(z)$ has an internally positive realization if and only if $\mathcal{K}$ can be chosen to be polyhedral.

Several algorithms for finding such an invariant polyhedral cone can be found, e.g., in [13, 14]. Internal positivity is, therefore, a finite-dimensional means to verify external positivity. However, since not every externally positive system admits an internally positive realization [30, 14], we cannot expect that all externally positive compound systems have internally positive realizations, and, as a consequence, internal Hankel k-positivity does not follow from its external counterpart. For Hankel *total* positivity, however, the two notions are equivalent.

PROPOSITION 3.9. *$G(z)$ is Hankel totally positive if and only if there exists a minimal realization that is internally Hankel totally positive.*

Proof. By Proposition 2.14, it holds that $G(z) = \sum_{i=1}^{n} \frac{r_i}{z-p_i}$ with $p_i \geq 0$ and $r_i > 0$. This admits a realization $A = \operatorname{diag}(p_n, \dots, p_1)$ and $b = c^\mathsf{T}$ with $b_i = \sqrt{r_{n-i+1}}$, $1 \leq i \leq n$. Thus, A is totally positive, and by applying [20, Lemma 22] to the submatrices of $\mathcal{C}^j(A,b)$, also $\mathcal{C}^j(A,b) = \mathcal{O}^j(A,c)^\mathsf{T}$ is totally positive for all $j \geq 1$. Thus, the result follows by Theorem 3.7. ∎

To bridge the gap between external and internal Hankel k-positivity, we address the existence of minimal internal realizations.

THEOREM 3.10. *$G(z)$ with order n and minimal realization (A,b,c) has a minimal internally Hankel k-positive realization, $k \leq n$, if and only if there exists a $P \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank}(P) = n$ *such that for all $1 \leq j \leq k$*

$$AP = PN \text{ for some } k\text{-positive } N, \tag{3.2a}$$

$$\mathcal{C}^j(A,b)_{[j]} \in \operatorname{cone}(P_{[j]}), \tag{3.2b}$$

$$\mathcal{O}^j(A,c)_{[j]}^\mathsf{T} \in \operatorname{cone}(P_{[j]})^*. \tag{3.2c}$$

Proof. ⇒: Let (A_+, b_+, c_+) be a minimal internally Hankel internally k-positive realization. By the similarity of minimal realizations there exists an invertible $P \in \mathbb{R}^{n\times n}$ such that for $1 \leq j \leq k$

$$AP = PN \text{ with } k\text{-positive } N = A_+,$$
$$\mathcal{C}^j(A,b)_{[j]} = P_{[j]}\mathcal{C}^j(A_+,b_+)_{[j]},$$
$$\mathcal{O}^j(A,c)_{[j]}P_{[j]} = \mathcal{O}^j(A_+,c_+)_{[j]},$$

which by Lemma 3.4 shows the claim.

⇐: If (3.2a)–(3.2c) hold, then there exists a minimal internally positive realization (N,g,h) with nonnegative

$$\mathcal{C}^j(N,g)_{[j]} = P_{[j]}^{-1}\mathcal{C}^j(A,b)_{[j]},$$
$$\mathcal{O}^j(N,h)_{[j]} = \mathcal{O}^j(A,c)_{[j]}P_{[j]}$$

for $1 \leq j \leq k$ and k-positive N. Thus, by Theorem 3.7, (N,g,h) is internally Hankel k-positive. ∎

Remark 3.11. From Proposition 3.8, we know that in case of $k = 1$, Theorem 3.10 remains true even if we drop minimality, i.e., $P \in \mathbb{R}^{K\times K}$ with $K \geq n$. The reason for this lies in the fact that there always exists a controllable, internally positive (A_+, b_+, c_+) [14, 30]. To be able to conclude the same for $k > 1$, we would need to show that (3.2a) and (3.2b) are sufficient for the existence of b_+ with $b = Pb_+$ and $\mathcal{C}^j(A_+,b_+)_{[j]} \geq 0$ for $1 \leq j \leq k$. Together with Theorem 4.6, it is possible to show then that Theorem 3.10 extends to nonminimal internally Hankel k-positive realizations, i.e., $P \in \mathbb{R}^{n\times K}$ with $K > n$.

Finally, under an irreducibility condition, all autonomous k-positive systems give rise to an internally Hankel k-positive system.

PROPOSITION 3.12. *Let $A \in \mathbb{R}^{n\times n}$ be k-positive with irreducible $A_{[j]}$, $1 \leq j \leq k$. Then there exists a $b \in \mathbb{R}^n$ such that $\mathcal{C}^j(A,b)_{[j]} > 0$ for all $1 \leq j \leq k$ and (A,b) is controllable.*

Proof. By Corollary 2.5, $\lambda_1(A) > \cdots > \lambda_k(A) > 0$. Let $\xi_1, \dots, \xi_k$ denote the associated eigenvectors. Our goal is to show that there exists $\alpha \in \mathbb{R}^k$ with $\alpha_1 \geq \cdots \geq$

$\alpha_k > 0$ such that $b = \sum_{j=1}^{k} \alpha_j \xi_j$ fulfills the first part of the claim. Then, by continuity of the determinant there also exists such a b with (A, b) controllable.

We begin by writing

$$\mathcal{C}^j(A,b) = \begin{pmatrix} \alpha_1\xi_1 & \dots & \alpha_k\xi_k \end{pmatrix} V^j,$$

where V^j is the Vandermonde matrix

$$V^j = \begin{pmatrix} 1 & \lambda_1(A) & \dots & \lambda_1(A)^{j-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_k(A) & \dots & \lambda_k(A)^{j-1} \end{pmatrix},$$

so that Lemma 2.4 implies

$$\mathcal{C}^j(A,b)_{[j]} = \begin{pmatrix} \alpha_1\xi_1 & \dots & \alpha_k\xi_k \end{pmatrix}_{[j]} {V^j}_{[j]}.$$

Since ${V^j}_{[j]}$ is a positive vector [11, Example 0.1.4], we can absorb its contribution into the choice of α and assume without loss of generality that

$$\mathcal{C}^j(A,b)_{[j]} = \begin{pmatrix} \xi_1 & \dots & \xi_k \end{pmatrix}_{[j]} \operatorname{diag}(\alpha_1, \dots, \alpha_k)_{[j]} e, \tag{3.3}$$

where e is the vector of all ones. Thus, $\mathcal{C}^j(A,b)_{[j]}$ is a linear combination of the columns in $\begin{pmatrix} \xi_1 & \dots & \xi_k \end{pmatrix}_{[j]}$, where each column is multiplied by the diagonal entry in $\operatorname{diag}(\alpha_1, \dots, \alpha_k)_{[j]}$. In particular, the first column of $\begin{pmatrix} \xi_1 & \dots & \xi_k \end{pmatrix}_{[j]}$ is positive by Corollary 2.5 and multiplied by the largest factor $\prod_{i=1}^{j} \alpha_i$. Therefore, by choosing inductively sufficiently large $\alpha_1 \geq \dots \geq \alpha_k > 0$, the vector $\mathcal{C}^j(A,b)_{[j]}$ is dominated by the contribution from $\prod_{i=1}^{j} \alpha_i \begin{pmatrix} \xi_1 & \dots & \xi_j \end{pmatrix}_{[j]}$ (as part of the linear combination (3.3)), proving their positivity for $1 \leq j \leq k$. ∎

An example why the irreducibility in Proposition 3.12 cannot, in general, be dropped is given at the end of section 5.

4. Extensions.

In this section, we first discuss extensions of our results in discrete-time (DT) to continuous-time (CT) systems, followed by applications to step-response analysis.

4.1. Continuous-time systems.

The tuple (A, b, c) is a CT state-space realization if

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \\ y(t) &= cx(t), \end{aligned} \tag{4.1}$$

with $A \in \mathbb{R}^{n \times n}$, $b, c^\mathsf{T} \in \mathbb{R}^n$. Its impulse response is

$$g(t) = ce^{At}bs(t) \tag{4.2}$$

and the controllability and observability operators are given by

$$\mathcal{C}_{\text{CT}}(A,b)u := \int_{-\infty}^{0_-} e^{-A\tau} bu(\tau) d\tau, \tag{4.3a}$$

$$(\mathcal{O}_{\text{CT}}(A,c)x_0)(t) := ce^{At}x_0, \ x_0 \in \mathbb{R}^n, \ t \geq 0. \tag{4.3b}$$

As for DT systems, we assume g to be absolutely integrable and u to be bounded. By defining the variation of a continuous-time signal $u : \mathbb{R} \to \mathbb{R}$ as

$$\mathrm{S}_{\mathrm{CT}}(u) := \sup_{\substack{n \in \mathbb{Z}_{>0} \\ t_1 < \cdots < t_n}} \mathrm{S}([u(t_1), \ldots, u(t_n)]),$$

we can define CT internal Hankel k-positivity as follows.

Definition 4.1. *A CT system (A, b, c) is called CT internally Hankel k-positive if e^{At}, $\mathcal{C}_{CT}(A, b)$, and $\mathcal{O}_{CT}(A, c)$ are OVD_{k-1} for all $t \geq 0$.*

As in DT, we seek to characterize these systems through finite-dimensional k-positive constraints. We will do so by discretization of (4.3), which allows us to apply our DT results. To this end, consider for $h, j > 0$, the *(Riemann sum) sampled controllability operator*

$$\mathcal{C}^{j,h}_{\mathrm{CT}}(A, b)u := h \sum_{i=-j}^{-1} e^{-Aih} bu(ih) = he^{Ah} \mathcal{C}^j(e^{Ah}, b) \begin{pmatrix} u(-h) \\ u(-2h) \\ \vdots \\ u(-jh) \end{pmatrix} \tag{4.4a}$$

and the *sampled observability operator*

$$\begin{pmatrix} y(h) \\ y(2h) \\ \vdots \\ y(jh) \end{pmatrix} = \mathcal{O}^j(e^{Ah}, c)e^{Ah}x_0 =: \mathcal{O}^{j,h}_{\mathrm{CT}}(A, c)x_0. \tag{4.4b}$$

Note that for each u and x_0 with finite variation there exist sufficiently large j and small h such that $\mathrm{S}_{\mathrm{CT}}(\mathcal{C}_{\mathrm{CT}}(A, b)u) = \mathrm{S}(\mathcal{C}^{j,h}_{\mathrm{CT}}(A, b)u)$ and $\mathrm{S}_{\mathrm{CT}}(\mathcal{O}_{\mathrm{CT}}(A, c)x_0) = \mathrm{S}(\mathcal{O}^{j,h}_{\mathrm{CT}}(A, c)x_0)$. Thus, Proposition 2.3 allows connecting the OVD_{k-1} property of the CT operators (4.3) to k-positivity of the matrices $\mathcal{C}^{j,h}_{\mathrm{CT}}(A, b)$ and $\mathcal{O}^{j,h}_{\mathrm{CT}}(A, c)$, where we consider all $j \geq k$. Since, by Lemma 2.4, k-positivity of these matrices follows from k-positivity of e^{Ah}, $\mathcal{C}^j(e^{Ah}, b)$, and $\mathcal{O}^j(e^{Ah}, c)$, we arrive at the following CT analogue of Lemma 3.4.

Lemma 4.2. *A CT system (A, b, c) is CT internally Hankel k-positive if and only if there exists a $\varepsilon > 0$ such that the DT system (e^{Ah}, b, c) is DT internally Hankel k-positive for all $h \in (0, \varepsilon)$.*

Using Theorem 3.7, CT internal Hankel k-positivity can be verified by checking that the realizations

$$((e^{Ah})_{[j]}, \mathcal{C}^j(e^{Ah}, b)_{[j]}, \mathcal{O}^j(e^{Ah}, c)_{[j]})$$

are internally positive for $0 \leq j \leq k$. However, it is undesirable to do this for all sufficiently small h. Next, we will discuss how to eliminate this variable from the above characterization. A classical result in that direction states that $e^{Ah} \geq 0$ for all $h \in (0, \varepsilon)$, $\varepsilon > 0$ (and, in fact, all $h \geq 0$) if and only if A is Metzler (i.e., A has nonnegative off-diagonal entries) (see, e.g., [14, Theorem 2]). In general, the (multiplicative) compound matrix of e^{At} can be expressed in terms of the *additive compound matrix* [28, section 1]

$$A^{[j]} := \log(\exp(A)_{[j]}) = \frac{d}{dh} {e^{Ah}}_{[j]} \Big|_{h=0}$$

which satisfies

$$(e^{Ah})_{[j]} = e^{A^{[j]}h}. \tag{4.5}$$

In other words, e^{Ah} is k-positive for all $h \geq 0$ if and only if $A^{[j]}$ is Metzler for $1 \leq j \leq k$. A special feature of CT systems is that the latter holds if and only if $A^{[j]}$ is Metzler for $1 \leq j \leq \min\{k, 2\}$ (see [39, Theorem 4 and Corollary 1] and [29]).

The next result will also allow us to remove h from the conditions involving $\mathcal{C}^j(e^{Ah}, b)$ and $\mathcal{O}^j(e^{Ah}, c)$.

THEOREM 4.3. *Let (A, b, c) be a CT system such that $A^{[j]}$ is Metzler for $1 \leq j \leq k$. Then, the following holds:*

i. *$\mathcal{C}^j(A, b)_{[j]} \geq 0$ for $1 \leq j \leq k$ if and only if there exists a sufficiently small $\varepsilon > 0$ such that $\mathcal{C}^j(e^{Ah}, b)_{[j]} \geq 0$ for all $1 \leq j \leq k$ and all $h \in (0, \varepsilon)$.*
ii. *$\mathcal{O}^j(A, c)_{[j]} \geq 0$ for $1 \leq j \leq k$ if and only if there exists a sufficiently small $\varepsilon > 0$ such that $\mathcal{O}^j(e^{Ah}, c)_{[j]} \geq 0$ for all $1 \leq j \leq k$ and all $h \in (0, \varepsilon)$.*

Proof. We only show the first item, as the second follows analogously. Let us begin by showing that we can assume $\operatorname{rank}(\mathcal{C}^j(A, b)) = \operatorname{rank}(\mathcal{C}^j(e^{Ah}, b))$. To see this, note that if $\operatorname{rank}(\mathcal{C}^j(A, b)) < j$, then all $A^i b \in \operatorname{im}(\mathcal{C}^{j-1}(A, b))$ for all $i \geq j-1$. In particular, $e^{Ah}b = \sum_{i=0}^{\infty} \frac{(Ah)^i}{i!} b \in \operatorname{im}(\mathcal{C}^{j-1}(A, b))$ for all $h > 0$ and thus, $\operatorname{rank}(\mathcal{C}^j(e^{Ah}, b)) < j$. Conversely, if $\operatorname{rank}(\mathcal{C}^j(e^{Ah}, b)) < j$, then by suitable column additions we also have

$$\operatorname{rank}(\mathcal{C}^j(I - e^{Ah}, b)\operatorname{diag}(1, h, \dots, h^{j-1})^{-1}) = \operatorname{rank}(\mathcal{C}^j(h^{-1}(I - e^{Ah}), b)) < j,$$

which, due to the lower semicontinuity of the rank [21], proves in the limit $h \to 0$ that $\operatorname{rank}(\mathcal{C}^j(A, b)) < j$. Hence, we can assume that $\operatorname{rank}(\mathcal{C}^j(A, b)) = \operatorname{rank}(\mathcal{C}^j(e^{Ah}, b)) = j$, as otherwise $\mathcal{C}^j(A, b)_{[j]} = 0$ and the claim holds trivially.

Next, let $\mathcal{C}^j(A, b)_{[j]} \geq 0$ for $1 \leq j \leq k$, which by Proposition 2.7 (ii) and (iii) is then equivalent to $(F(\sigma)\mathcal{C}^j(A, b))_{[j]} > 0$ for all $\sigma > 0$. Further, suitable column additions within $\mathcal{C}^j(A, b)$ yield the equivalence to $(F(\sigma)\mathcal{C}^j(I + hA, b))_{[j]} > 0$ for all $\sigma,\ h > 0$. Since $(I + hA)b$ can be approximated arbitrarily well by $e^{Ah}b$ for sufficiently small $h > 0$, the continuity of the determinant implies the equivalence to $(F(\sigma)\mathcal{C}^j(e^{Ah}, b))_{[j]} > 0$ for all $\sigma > 0$ and all sufficiently small $h > 0$. Hence, our claim follows for sufficiently small $h > 0$ by invoking Proposition 2.7 (iv). ∎

In conjunction with Theorem 4.3 and Lemma 4.2, it follows that Theorem 3.7 remains true in CT.

THEOREM 4.4. *Let (A, b, c) be a CT system. Then, (A, b, c) is CT internally Hankel k-positive if and only if*

$$(A^{[j]}, \mathcal{C}^j(A, b)_{[j]}, \mathcal{O}^j(A, c)_{[j]}) \tag{4.6}$$

is CT internally positive for all $1 \leq j \leq k$

Note that our defined compound system realizations are indeed the CT compound systems, whose external positivity can be used to verify CT (external) Hankel k-positivity. Finally, an analogue to Theorem 3.10 can be obtained by substituting (3.2a) with the condition stated in the following lemma, extending the corresponding result in [30].

LEMMA 4.5. *Let $A \in \mathbb{R}^{n \times n}$ and $P \in \mathbb{R}^{n \times K}$. Then, $e^{At}P = Pe^{Nt}$ with e^{Nt} k-positive for all $t \geq 0$ if and only if there exits a $\lambda > 0$ such that $(A + \lambda I_n)P = P(N + \lambda I_K)$ with $(N + \lambda I_K)^{[j]} \geq 0$ for all $1 \leq j \leq k$.*

Proof. We begin by remarking the following properties of the additive compound matrix [28]: let $X, Y \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{n\times K}$:

1. $(X+Y)^{[j]} = X^{[j]} + Y^{[j]}$,
2. $(e^{Xt}Z)_{[j]} = e^{X^{[j]}t}Z_{[j]}$.

$\Leftarrow$: Let $(A+\lambda I_n)P = P(N+\lambda I_K)$ for some $\lambda > 0$ such that $(N+\lambda I_K)^{[j]} \geq 0$ for all $1 \leq j \leq k$. Then, for all $\frac{i}{t} \geq \lambda$ with $t \geq 0$ and $i \in \mathbb{N}$, it holds that $(A+\frac{i}{t}I_n)P = P(N+\frac{i}{t}I_K)$, and consequently,

$$e^{At}P = \lim_{i\to\infty}\left(I_n + \frac{At}{i}\right)^i P = P\lim_{i\to\infty}\left(I_K + \frac{Nt}{i}\right)^i = Pe^{Nt}.$$

Moreover, by the first property above $N^{[j]}$ is Metzler, $1 \leq j \leq k$, which by (4.5) implies that e^{Nt} is k-positive for $t \geq 0$.

$\Rightarrow$: Let $e^{At}P = Pe^{Nt}$ with k-positive e^{Nt} for all $t \geq 0$. Then, by definition of the additive compound matrix and the properties above, it holds that

$$\begin{aligned}(A+\lambda I_n)^{[j]}P_{[j]} &= (A^{[j]} + \lambda I_n^{[j]})P_{[j]}\\ &= P_{[j]}(N^{[j]} + \lambda I_K^{[j]}) = P_{[j]}(N+\lambda I_K)^{[j]}\end{aligned}$$

for all $\lambda \geq 0$, $1 \leq j \leq k$, where $N^{[j]}$ is Metzler by (4.5). Thus, by choosing λ sufficiently large, we conclude that $(N+\lambda I_K)^{[j]}$ is nonnegative for all $1 \leq j \leq k$. ∎

4.2. Impulse and step response analysis. Next, we apply our results to the analysis of over- and undershooting in a step response, a classical problem in control (see, e.g., [5]). For LTI systems, the total number of over- and undershoots equals the number of sign-changes in the impulse response. While several lower bounds for these sign changes have been derived [8, 37, 10, 9], fewer results seem to exist on upper bounds [9, 10].

In our new framework, we observe that the impulse response of (A, b, c) fulfils $g(t) = (\mathcal{O}(A, c)b)(t)$. Therefore, if $\mathcal{O}(A, c)$ is OVD_{k-1}, then the impulse response of (A, b, c) changes its sign at most $\mathrm{S}(b)$ times for all $\mathrm{S}(b) \leq k-1$, and has the same sign-changing order as b in case of an equal number of sign-changes. Similarly to Theorem 3.10, we conclude the following result.

THEOREM 4.6. *Let (A, b, c) be a minimal realization of $G(z)$. Then, $G(z)$ admits a realization (A_+, b_+, c_+) such that A_+ and $\mathcal{O}(A_+, c_+)$ are OVD_{k-1} if and only if there exist a k-positive $N \in \mathbb{R}^{K\times K}$ and a $P \in \mathbb{R}^{n\times K}$ with $K \geq n$ such that $AP = PN$ and $\mathcal{O}^j(A, c)_{[j]}^{\mathsf{T}} \in \mathrm{cone}(P_{[j]})^*$ for $1 \leq j \leq k$, where $k \leq n$.*

Proof. $\Leftarrow$: Let N and P be as above. Defining $A_+ := N$, we have that A_+ is k-positive and thus OVD_{k-1} by Proposition 2.3. Furthermore, defining $c_+ := cP \geq 0$, it follows from the assumptions that $AP = PA_+$ and $\mathcal{O}^j(A, c)_{[j]}^{\mathsf{T}} \in \mathrm{cone}(P_{[j]})^*$ $1 \leq j \leq k$, that $\mathcal{O}^j(A_+, c_+)_{[j]} = \mathcal{O}^j(A, c)_{[j]}P_{[j]} \geq 0$, $1 \leq j \leq k$. Hence, $\mathcal{O}(A_+, c_+)$ is OVD_{k-1} by Theorem 3.5 and Lemma 3.4.

$\Rightarrow$: Let A_+ and $\mathcal{O}(A_+, c_+)$ be OVD_{k-1}. Analogously to the construction in the necessity proof of [30, Theorem 5], there exists a $P \in \mathbb{R}^{n\times K}$ such that $AP = PA_+$ and $c_+ = cP$. Therefore, also $\mathcal{O}^j(A, c)_{[j]}P_{[j]} = \mathcal{O}^j(A_+, c_+)_{[j]}$, which concludes the proof. ∎

Since the realization (A_+, b_+, c_+) may not be unique, it remains an open question how to minimize the sign changes in b_+ in order to make the upper bound the least conservative. We leave an answer to this question for future work. It should be noted that the approach in [9] essentially corresponds to the case where a realization with a totally positive observability operator exists, because it assumes positive distinct real poles and real zeros, apart from multiple poles at zero.

5. Examples.

5.1. Internal Hankel *k*-positivity. Consider a system given by the realization

$$A_+ = \begin{pmatrix} 0.25 & 0.25 & 0.20 \\ 0.25 & 0.30 & 0.30 \\ 0.10 & 0.35 & 0.40 \end{pmatrix}, \quad b_+ = c_+^\mathsf{T} = \begin{pmatrix} 1 \\ 0.1 \\ 0 \end{pmatrix}.$$

For this realization, we have

$$\mathcal{C}^3(A_+, b+) = 10^{-2} \begin{pmatrix} 100 & 27.5 & 16.575 \\ 10 & 28 & 19.325 \\ 0 & 13.5 & 17.95 \end{pmatrix},$$

$$\mathcal{C}^3(A_+, b+)_{[2]} = 10^{-3} \begin{pmatrix} 252.5 & 176.675 & 6.73375 \\ 135 & 179.5 & 26.98625 \\ 13.5 & 17.95 & 24.17125 \end{pmatrix},$$

and $\mathcal{C}^3(A_+, b+)_{[3]} = 21.472625 \cdot 10^{-3}$. Furthermore, we have

$$A_{+[2]} = 10^{-2} \begin{pmatrix} 1.25 & 2.5 & 1.5 \\ 6.25 & 8 & 3 \\ 5.75 & 7 & 1.5 \end{pmatrix}$$

and $A_{+[3]} = \det A = -2.25 \cdot 10^{-3}$ (all numbers above are exact). Several facts can be stated regarding this realization. First, $\operatorname{rank} A = \operatorname{rank} \mathcal{C}^3(A_+, b_+) = 3$, and thus the system is controllable. Furthermore, A_+ is 2-positive, but not 3-positive, while $\mathcal{C}^3(A_+, b_+)$ is 3-positive. It immediately follows from Theorem 3.5 that the controllability operator $\mathcal{C}(A_+, b_+)$ is 2-positive, which can readily be verified numerically. Second, it can be verified (we omit the details) that $\mathcal{O}^3(A_+, c_+)$ is full-rank and 3-positive; we conclude from Theorems 3.7 and 3.5 that the (minimal) realization (A_+, b_+, c_+) is internally Hankel 2-positive, but not 3-positive (since $A_{+[3]} = \det A < 0$). The canonical controllable realization of $G(z) = c_+(zI_3 - A_+)^{-1} b_+$ reads

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.00225 & -0.1075 & 0.95 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0.0058 \\ -0.6565 \\ 1.01 \end{pmatrix}^\mathsf{T} \tag{5.1}$$

which is not internally Hankel k-positive for any $k \geq 1$. For the two realizations above, the P matrix from Theorem 3.10 is simply the canonical controllability state-transformation matrix, given by

$$P = \mathcal{C}^3(A_+, b_+) \mathcal{C}^3(A, b)^{-1}.$$

To illustrate the variation-diminishing property, we show in Figures 1 and 2 the time evolution of $y_+(t) = (\mathcal{O}(A_+, b_+)x_0)(t)$ and $y(t) = (\mathcal{O}(A, b)x_0)(t)$ for the initial condition $x_0 = (-40.5, 0.9, 0.015)^\mathsf{T}$. It can be seen that given $\mathrm{S}(x_0) = 1$, the internally Hankel 2-positive realization yields $\mathrm{S}(y_+) = 0$, and the sign variation in x_0 is diminished; the controllability canonical realization yields $\mathrm{S}(y) = 3$, and the variation in x_0 is increased.

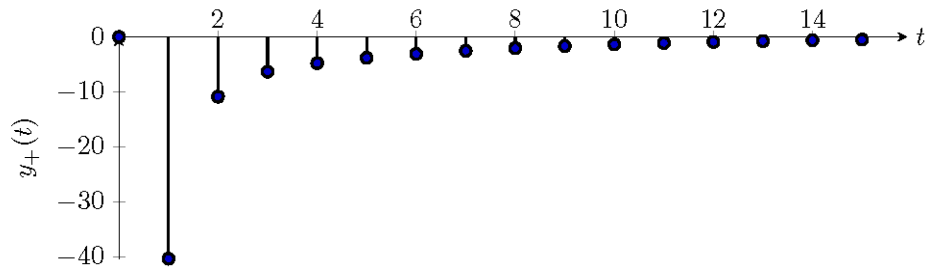

FIG. 1. *The output of the internally Hankel* 2*-positive realization,* $y_+(t) = (\mathcal{O}(A_+, b_+)x_0)(t)$*, has a smaller variation than* $x_0 = (-40.5, 0.9, 0.015)^\mathsf{T}$.

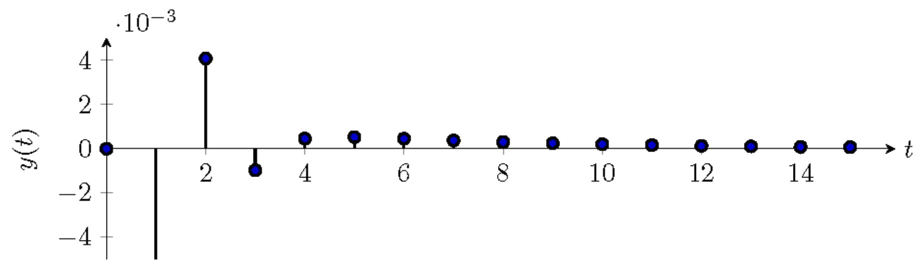

FIG. 2. *The output of the canonical controllable form realization* (5.1)*,* $y(t) = (\mathcal{O}(A, b)x_0)(t)$*, has a larger variation than* $x_0 = (-40.5, 0.9, 0.015)^\mathsf{T}$.

5.2. Impulse response analysis. Consider the following system, previously shown as an example in [9]:

$$G(z) = \frac{(z-0.22)(z-0.6)}{z^3(z-0.7)}.$$

The transfer function $G(z)$ has a realization given by

$$A = \begin{pmatrix} 0.7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ -0.82 \\ 0.132 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\mathsf{T}. \tag{5.2}$$

It can be verified that this realization has totally positive A and $\mathcal{O}^4(A, c)$. By Theorem 3.5 and Lemma 3.4, $\mathcal{O}(A, c)$ is then totally positive and the number of sign changes in the impulse response of $G(z)$ (and, hence, the number of extrema in its step response) is upper bounded by $\mathrm{S}(b) = 2$; the same upper bound was previously obtained by [9]. Figure 3 shows that this bound is tight.

However, in contrast to [9], our framework does not assume real poles or zeros. In particular, the modified transfer function

$$G_m(z) = \frac{(z-0.5+i)(z-0.5-i)}{z^3(z-0.7)},$$

can be realized with the same A and c as in (5.2) and with $b = \begin{pmatrix} 0 & 1 & -1 & 1.25 \end{pmatrix}^\mathsf{T}$, which again provides a tight upper bound on the variation of the impulse response.

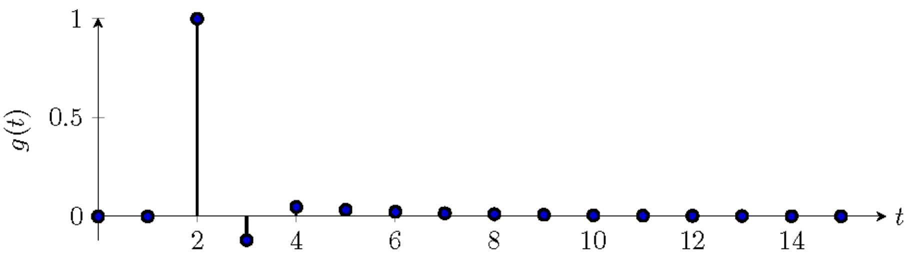

FIG. 3. *Impulse response of the system in* (5.2) *has two zero crossings, which coincides with our derived upper bound.*

Finally, note that by Proposition 2.14, there cannot be any b such that $\mathcal{C}(A, b)$ is 2-positive, because otherwise (A, b, c) would be Hankel 2-positive. This illustrates the importance of the irreducibility condition in Proposition 3.12.

6. Conclusion. Under the assumption of k-positive autonomous dynamics, this work has derived tractable conditions for which the controllability and observability operators are k-positive. These results have been used in two ways.

First, we introduced and studied the notion of internally Hankel k-positive systems, i.e., systems which are variation diminishing from past inputs with at most $k-1$ variations to future states to future outputs. It has been shown that these properties are tractable through internal positivity of the associated compound systems. In particular, internal Hankel k-positivity provides a means of studying external Hankel k-positivity with finite-dimensional tools. As a result, this systems class combines and extends two important system classes: (i) the celebrated class of internally positive systems ($k = 1$) [14] and (ii) the class of relaxation systems [41] ($k = \infty$); the latter has also been shown to admit minimal internally Hankel totally positive realizations. Moreover, our results lay the groundwork for future work linking unforced variation diminishing systems, as considered in [27, 1, 43], with the theory of externally variation diminishing systems [20]. Finally, as a generalization of the case $k = 1$ found in [30], a characterization of when an externally Hankel k-positive system possesses a minimal internally Hankel k-positive realization has been discussed. In future work, the characterizations for nonminimal realizations and realization algorithms shall be addressed. Noticeably, we have not introduced an internal notion for externally Toeplitz k-positive systems: this is a consequence of the nonseparability of the Toeplitz operator. Thus, contrary to the standard definition of internal positivity, this suggests that the Hankel operator is a more natural object with which to associate internal positivity.

Second, we have developed a new framework for upper-bounding the number of sign-changes in the impulse response of an LTI system. In particular, while the results of [9] are recovered in the case $k = \infty$, our framework allows considering generic k. In future work, we plan to address the conservatism of our analysis, its numerical tractability, and the theoretical implication of the location of zeros. Further, we believe that a nonlinear extension of our framework, by means of the well-known controlled cooperative/monotone systems class (see, e.g., [3]), will be of timely importance. For instance, the cumulative difference between a step response and the output, called the (static) regret, is a common tractable measure in online learning [31] and adaptive control problems [25]. However, its meaningfulness depends on a small variability such as a small variance or a bounded variation.

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