# Superconvergent HDG methods for Maxwell's equations via the $M$-decomposition

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**Abstract**

The concept of the $M$-decomposition was introduced by Cockburn et al. in Math. Comp. vol. 86 (2017), pp. 1609-1641 to provide criteria to guarantee optimal convergence rates for the Hybridizable Discontinuous Galerkin (HDG) method for coercive elliptic problems. In that paper they systematically constructed superconvergent hybridizable discontinuous Galerkin (HDG) methods to approximate the solutions of elliptic PDEs on unstructured meshes. In this paper, we use the $M$-decomposition to construct HDG methods for the Maxwell's equations on unstructured meshes in two dimension. In particular, we show the any choice of spaces having an $M$-decomposition, together with sufficiently rich auxiliary spaces, has an optimal error estimate and superconvergence even though the problem is not in general coercive. Motivated by the elliptic case, we obtain a superconvergent rate for the curl and flux of the solution, and this is confirmed by our numerical experiments.

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## 1. Introduction

A large number of computational techniques have been developed for solving Maxwell's equations in both the frequency and time domains. In the frequency domain, and in the presence of inhomogeneous penetrable media, the finite element method is often used. It has an additional advantage compared to finite differences in that it can handle complex geometries.

Methods using $\boldsymbol{H}(\mathrm{curl};\Omega)$-conforming edge elements have been widely studied, see for example [1–6]. The implementation of the conforming method, particularly higher order elements, is complicated. Hence, non-conforming methods provide an interesting alternative for this kind of problem that may also be attractive for nonlinear problems. In particular, Discontinuous Galerkin (DG) methods have been used to approximate the solution of the Maxwell's equations for a long time. The first DG method for solving Maxwell's equations with high frequency was analyzed in [7]. A local discontinuous Galerkin (LDG) scheme was proposed for the time-harmonic Maxwell's equations with low frequency was studied in [8] (see also [9] for this problem using mixed DG methods). These methods tend to have many more degrees of freedom than conforming methods so it is interesting to consider hybridizable methods.

This paper is concerned with developing a class of methods for Maxwell's equations in 2D. Obviously Maxwell's equations are usually studied in three dimensions, but if the domain and data functions are translation invariant in one direction, the full problem can be decoupled into a pair of problems posed in two dimensions. To see how this is possible, consider the usual time harmonic Maxwell system for the electric field $\boldsymbol{E}$ (a complex valued vector function):

$$\mathrm{curl}\,\mu_r^{-1}\mathrm{curl}\,\boldsymbol{E} - \kappa^2\epsilon_r\boldsymbol{E} = \boldsymbol{F}.$$

Here $\mu_r$ is the relative magnetic permeability, $\kappa > 0$ is the wave number, $\epsilon_r$ is the relative electric permittivity which may be complex valued. In addition $\boldsymbol{F} = ik\epsilon_0\boldsymbol{j}$, where $\boldsymbol{j}$ is the given current density and $\epsilon_0$ is the permittivity of vacuum.

If $\epsilon_r$, $\mu_r$ and $\boldsymbol{F}$ are independent of $x_3$, and if we seek a solution $\boldsymbol{E}$ that is also independent of $x_3$ we obtain a simpler partial differential equation for $\boldsymbol{u} = (E_1, E_2)^T$ given in (2a) below. To define the problem for $\boldsymbol{u}$ we need some notation. Because we are now working in two dimensions the curl operator can be defined in two ways depending on whether its argument is a scalar or a vector. We therefore introduce the following standard definitions where $\boldsymbol{v}$ is a smooth vector function and $p$ is a smooth scalar function:

$$\nabla \times \boldsymbol{v} = (-\partial_y, \partial_x)^T \cdot \boldsymbol{v}, \qquad \nabla \times p = (\partial_y, -\partial_x)^T p, \tag{1}$$

Similarly there are two definitions for the cross product again depending on the use of scalar or vector functions. If $\boldsymbol{n} \in \mathbb{R}^2$ is a unit vector (in practice the normal vector to a domain in $\mathbb{R}^2$), we define

$$\boldsymbol{n} \times \boldsymbol{v} = (-n_2, n_1)^T \cdot \boldsymbol{v}, \qquad \boldsymbol{n} \times p = (n_2, -n_1)^T p,$$

We can now state the problem we shall study. Let $\Omega$ be a bounded simply-connected Lipschitz polygon in $\mathbb{R}^2$ with connected boundary $\partial\Omega$. Then we seek to approximate the solution $u$ of the following interior problem:

$$\nabla \times (\mu_r^{-1} \nabla \times \boldsymbol{u}) - \kappa^2 \epsilon_r \boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2a}$$

$$\boldsymbol{n} \times \boldsymbol{u} = g \qquad \text{on } \partial\Omega, \tag{2b}$$

where the right hand side is $\boldsymbol{f} = (F_1, F_2)^T$. To ensure the uniqueness of the solution to this problem (and hence existence via the Fredholm alternative), we assume that $\mu_r$ is real valued and positive. In addition, either $\Im(\epsilon_r) > 0$, or $\Im(\epsilon_r) = 0$ and $\kappa^2$ is not a Maxwell eigenvalue, where $\Im(\epsilon_r)$ denotes the imaginary part of $\epsilon_r$.

Note that an alternative approach is to solve directly for the scalar variable $q := \mu_r^{-1} \nabla \times \boldsymbol{u}$. Straightforward manipulation shows that $q$ satisfies the Helmholtz equation

$$\nabla \cdot \left( \frac{1}{\epsilon_r} \nabla q \right) + \kappa^2 \mu_r q = \nabla \cdot \left( \frac{1}{\epsilon_r} \boldsymbol{f}^{\perp} \right).$$

where $\boldsymbol{f}^{\perp} = (f_2, -f_1)^T$. For studies of HDG applied to the Helmholtz equation see for example [10, 11] although variable coefficients are not discussed there. Our approach using (2) results in a different formulation of the HDG discretized problem and is motivated by the numerical study in [12]. Note in addition that using the vector form of the problem has also been advocated for example in [13, 14] and those papers further motivate the current study.

In this paper we shall study hybridizable discontinuous Galerkin (HDG) methods applied to Maxwell's equations (2). HDG methods for elliptic problems were first proposed in 2009 in [15] and an analysis using special projections was developed in [16]. HDG methods have several distinct advantages including: allowing static condensation and hence less global degrees of freedom, flexibility in meshing (inherited from DG methods), ease of design and implementation, and local conservation of physical quantities. As a result, HDG methods have been proposed for a large number of problems, see, e.g., [17–23].

An important property of HDG methods is the superconvergence of some quantities on unstructured meshes (after element by element post-processing). One way to guarantee the existence of an HDG projection and superconvergence is to ensure that the particular discretization spaces used in the HDG method satisfy an $M$-decomposition [24]. This method of analysis has been extended to other applications, see for example [25–29].

The HDG method has been applied to Maxwell's equations in [12] but without an error analysis. Later on, an error analysis was provided in [30, 31] for zero frequency and in [32, 33] for impedance boundary conditions and high wave number. These papers did not use the $M$-decomposition and only considered simplicial elements.

The aim of this paper is to use the concept of the $M$-decomposition to analyze the time-harmonic Maxwell's equations with Dirichlet boundary condition in 2D. The main novelty of our paper is that we show that provided the HDG spaces satisfy the conditions for an $M$-decomposition, and certain auxiliary spaces contain piecewise constant polynomials, an optimal error estimate will hold as well as a super-convergence of the curl of the field (as was observed in [12]). Note that in our context superconvergence of the curl of the field is important because this implies that both the electric and magnetic fields can be approximated at the same rate. We then use the $M$-decomposition to exhibit finite element spaces with optimal convergence on triangles, parallelograms and squares. Our convergence theory is supported by numerical examples in each case.

The outline of the paper is as follows. In Section 2, we set some notation and give the HDG formulation of (2). In Section 3, we follow the seminal work [24] to introduce the concept of the $M$-decomposition for Maxwell's equation. The error analysis is given in Section 4, we obtain optimal convergence rate for the electric field $\boldsymbol{u}$ and superconvergence rate for $\nabla\times\boldsymbol{u}$. The construction of example spaces and numerical experiments are provided to confirm our theoretical results in Section 5. We end with a conclusion.

## 2. The HDG method

We start by defining some notation. For any sufficiently smooth bounded domain $\Lambda \subset \mathbb{R}^2$, let $H^m(\Lambda)$ denote the usual $m^{th}$-order Sobolev space of scalar functions on $\Lambda$, and $\|\cdot\|_{m,\Lambda}$, $|\cdot|_{m,\Lambda}$ denote the corresponding norm and semi-norm. We use $(\cdot,\cdot)_\Lambda$ to denote the complex inner product on $L^2(\Lambda)$. Similarly, for the boundary $\partial\Lambda$ of $\Lambda$, we use $\langle\cdot,\cdot\rangle_{\partial\Lambda}$ to denote the $L^2$ inner product. Note that bold face fonts will be used for vector analogues of the Sobolev spaces along with vector-valued functions.

We define the negative norm $\|\cdot\|_{H^{-s}(\Omega)}$ by

$$\|u\|_{H^{-s}(\Omega)} = \sup_{v\in H^s(\Omega)} \frac{|(u,v)_\Omega|}{\|v\|_{s,\Omega}}.$$

The negative norm $\|\cdot\|_{H^{-s}(\partial\Omega)}$ can be defined by the same way.

Recalling the definition of the curl operators in 2D in (1), for $\Lambda \subset \mathbb{R}^2$ we next define

$$\begin{aligned}
\boldsymbol{H}(\mathrm{curl};\Lambda) &:= \{\boldsymbol{u} \in \boldsymbol{L}^2(\Lambda) : \nabla \times \boldsymbol{u} \in L^2(\Lambda)\},\\
\boldsymbol{H}_0(\mathrm{curl};\Lambda) &:= \{\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl};\Lambda) : \boldsymbol{n} \times \boldsymbol{u} = 0 \text{ on } \partial\Lambda\}\\
H(\mathrm{curl};\Lambda) &:= \{u \in L^2(\Lambda) : \nabla \times u \in \boldsymbol{L}^2(\Lambda)\},\\
H_0(\mathrm{curl};\Lambda) &:= \{u \in H(\mathrm{curl};\Lambda) : \boldsymbol{n} \times u = \boldsymbol{0} \text{ on } \partial\Lambda\},\\
\boldsymbol{H}(\mathrm{div}_{\epsilon_r};\Lambda) &:= \{\boldsymbol{u} \in \boldsymbol{L}^2(\Lambda) : \nabla \cdot (\epsilon_r \boldsymbol{u}) \in L^2(\Lambda)\},\\
\boldsymbol{H}(\mathrm{div}^0_{\epsilon_r};\Lambda) &:= \{\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}_{\epsilon_r};\Lambda) : \nabla \cdot (\epsilon_r \boldsymbol{u}) = 0\}.
\end{aligned}$$

where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Lambda$ and $\epsilon_r$ is a smooth enough function.

Let $\mathcal{T}_h := \{K\}$ denote a conforming mesh of $\Omega$, where $K$ is a Lipschitz polygonal element with finitely many edges. For each $K \in \mathcal{T}_h$, we let $h_K$ be the infimum of the diameters of circles containing $K$ and denote the mesh size $h := \max_{K\in\mathcal{T}_h} h_K$. We shall need more assumptions on the mesh to perform our analysis. These assumptions replace the usual "shape regularity" assumption but we delay a discussion of this point until Section 4.1. Let $\partial\mathcal{T}_h$ denote the set of edges $F \subset \partial K$ of the elements $K \in \mathcal{T}_h$ (i.e. edges of distinct triangles are counted separately) and let $\mathcal{E}_h$ denote the set of edges in the mesh $\mathcal{T}_h$. We denote by $h_F$ the length of the edge $F$. We abuse notation by using $\nabla\times$, $\nabla\cdot$ and $\nabla$ for broken curl, div and gradient operators with respect to mesh partition $\mathcal{T}_h$, respectively. To simplify the notation, we also define a function $\mathbf{h}$ on $\mathcal{T}_h$, $\partial\mathcal{T}_h$ and $\mathcal{E}_h$ which dependenting on circumstances:

$$\mathbf{h}|_K = h_K, \quad \forall K \in \mathcal{T}_h, \qquad \mathbf{h}|_{\partial K} = h_K, \quad \forall K \in \mathcal{T}_h, \qquad \mathbf{h}|_F = h_F, \quad \forall F \in \mathcal{E}_h.$$

For $u, v \in L^2(\mathcal{T}_h)$ and $\rho, \theta \in L^2(\partial\mathcal{T}_h)$, we define the following inner product and norm

$$\begin{aligned}
(u,v)_{\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h} (u,v)_K, \quad \|v\|^2_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \|v\|^2_K,\\
\langle \rho,\theta\rangle_{\partial\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h} \langle \rho,\theta\rangle_{\partial K}, \quad \|\theta\|^2_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \|\theta\|^2_{\partial K}.
\end{aligned}$$

Given a choice of three finite dimensional polynomial spaces $V(K) \subset H^1(K)$, $\boldsymbol{W}(K) \subset \boldsymbol{H}(\text{curl}; K)$ and $\boldsymbol{M}(F) \subset \boldsymbol{L}^2(F)$, where $K$ is an arbitrary element in the mesh and $F$ is an arbitrary edge, we define the global spaces by

$$
\begin{aligned}
V_h &:= \{v \in L^2(\mathcal{T}_h) : w|_K \in V(K), K \in \mathcal{T}_h\},\\
\boldsymbol{W}_h &:= \{\boldsymbol{w} \in \boldsymbol{L}^2(\mathcal{T}_h) : \boldsymbol{w}|_K \in \boldsymbol{W}(K), K \in \mathcal{T}_h\},\\
\boldsymbol{M}_h &:= \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h) : \boldsymbol{\mu}|_F \in \boldsymbol{M}(F), F \in \mathcal{E}_h\}.
\end{aligned}
$$

For later use, for any non-negative integer $k$, let $\mathcal{P}_k(K)$ denote the standard space of polynomials in two variables have total degree less than or equal to $k$.

Next, to give the HDG formulation of (2), we need to rewrite it into a mixed form. Let $q = \mu_r^{-1}\nabla \times \boldsymbol{u}$ in (2) to get the following mixed form

$$
\begin{aligned}
\mu_r q - \nabla \times \boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(3a)}\\
\nabla \times q - \kappa^2 \epsilon_r \boldsymbol{u} &= \boldsymbol{f} && \text{in } \Omega, && \text{(3b)}\\
\boldsymbol{n} \times \boldsymbol{u} &= g && \text{on } \partial\Omega. && \text{(3c)}
\end{aligned}
$$

As usual for HDG, the upcoming method and analysis are based on the above mixed form.

For the convenience, we next give the following integration by parts formula for each curl operator in two-dimensions. The proof is followed by a standard density argument and hence we omit it here.

**Lemma 2.1.** Let $K$ be an element in the mesh $\mathcal{T}_h$, and let $\boldsymbol{u} \in \boldsymbol{H}(\text{curl}; K)$ and $r \in H(\text{curl}; K)$. Then we have

$$
\begin{aligned}
(\nabla \times \boldsymbol{u}, r)_K &= \langle \boldsymbol{n} \times \boldsymbol{u}, r\rangle_{\partial K} + (\boldsymbol{u}, \nabla \times r)_K, && \text{(4a)}\\
(\nabla \times r, \boldsymbol{u})_K &= \langle \boldsymbol{n} \times r, \boldsymbol{u}\rangle_{\partial K} + (r, \nabla \times \boldsymbol{u})_K, && \text{(4b)}
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit outward normal to $K$.

We can now derive the HDG method for (3) by multiplying each equation by the appropriate discrete test function, integrating element by element and use integration by parts (see (4)) element by element in the usual way (c.f. [15]).

Summing the results over all elements, the HDG methods seeks an approximation to $(q, \boldsymbol{u}, \boldsymbol{u}|_{\mathcal{E}_h})$, by $(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$, such that

$$(\mu_r q_h, r_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times r_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times \widehat{\boldsymbol{u}}_h, r_h \rangle_{\partial \mathcal{T}_h} = 0, \tag{5a}$$

$$(q_h, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \boldsymbol{n} \times \widehat{q}_h, \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} - (\kappa^2 \epsilon_r \boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{T}_h}, \tag{5b}$$

$$\langle \boldsymbol{n} \times \widehat{q}_h, \widehat{\boldsymbol{v}}_h \rangle_{\mathcal{F}_h / \partial \Omega} = 0, \tag{5c}$$

$$\langle \boldsymbol{n} \times \widehat{\boldsymbol{u}}_h, \boldsymbol{n} \times \widehat{\boldsymbol{v}}_h \rangle_{\partial \Omega} = \langle g, \boldsymbol{n} \times \widehat{\boldsymbol{v}}_h \rangle_{\partial \Omega} \tag{5d}$$

for all $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$, and the choice of $\boldsymbol{n} \times \widehat{q}_h$ follows the usual HDG pattern,

$$\boldsymbol{n} \times \widehat{q}_h = \boldsymbol{n} \times q_h + \tau \boldsymbol{n} \times (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \times \boldsymbol{n}, \tag{5e}$$

where $\tau$ is a penalization parameter taken to be positive and piecewise constant on the edges of the mesh (more details will be given later).

It is well-known that the 2D Maxwell's equations can be rewritten as the Helmholtz equation. For a given $\boldsymbol{w} := (w_1, w_2)$, let $\boldsymbol{w}^{\perp} := (-w_2, w_1)$, then we have that

$$\begin{aligned}
&\nabla \times \boldsymbol{u} := (-\partial_y, \partial_x) \cdot (u_1, u_2) = (\partial_x, \partial_y) \cdot (u_2, -u_1) = \nabla \cdot (-\boldsymbol{u}^{\perp}), \\
&\boldsymbol{n} \times \boldsymbol{u} := (-n_2, n_1)^T \cdot (u_1, u_2) = (n_1, n_2) \cdot (u_2, -u_1) = \boldsymbol{n} \cdot (-\boldsymbol{u}^{\perp}), \\
&\nabla \times q := (-\partial_y, \partial_x) q = \nabla^{\perp}(-q).
\end{aligned}$$

Hence, we can rewrite the equation (3) as

$$\kappa^2 \epsilon_r \boldsymbol{u}^{\perp} + \nabla(-q) = \boldsymbol{f}^{\perp} \quad \text{in } \Omega, \tag{6a}$$

$$\nabla \cdot \boldsymbol{u}^{\perp} - \mu_r(-q) = 0 \quad \text{in } \Omega, \tag{6b}$$

$$\boldsymbol{n} \cdot (\boldsymbol{u}^{\perp}) = -g \quad \text{on } \partial\Omega. \tag{6c}$$

We can see that the Helmholtz equation is equivalent with the Maxwell equation in 2D. The main goal of this paper is to follow paper [12], which focuses on the 2D Maxwell equation written as in our paper. Furthermore, it is pointed out in [13, 14], that for the 2D Maxwell equations, it maybe better not to rewrite the problem as a Helmholtz equation.

## 3. $\boldsymbol{M}$-decompositions

In this section, we follow the seminal paper [24] to give the concept of the $M$-decomposition for Maxwell's equation in two dimensions. To do this, we need an appropriate combined trace operator $\mathrm{tr} : V(K) \times \boldsymbol{W}(K) \mapsto L^2(\partial K)$ defined as follows:

$$\mathrm{tr}(v, \boldsymbol{w}) := (\boldsymbol{n} \times v + \boldsymbol{n} \times \boldsymbol{w} \times \boldsymbol{n})|_{\partial K}. \tag{7}$$

**Definition 3.1.** We say that $V(K) \times \boldsymbol{W}(K)$ admits an $M$-decomposition when the following conditions are met:

$$\boldsymbol{n} \times V(K) \subset \boldsymbol{M}(\partial K), \qquad \boldsymbol{n} \times \boldsymbol{W}(K) \times \boldsymbol{n} \subset \boldsymbol{M}(\partial K), \tag{8a}$$

and there exists a subspace $\widetilde{V}(K) \times \widetilde{\boldsymbol{W}}(K)$ of $V(K) \times \boldsymbol{W}(K)$ satisfying

$$\nabla \times V(K) \subset \widetilde{\boldsymbol{W}}(K), \qquad \nabla \times \boldsymbol{W}(K) \subset \widetilde{V}(K), \tag{8b}$$

$$\mathrm{tr} : \left(\widetilde{V}^{\perp}(K) \times \widetilde{\boldsymbol{W}}^{\perp}(K)\right) \to \boldsymbol{M}(\partial K) \text{ is an isomorphism,} \tag{8c}$$

$$\text{for any } \boldsymbol{\mu} \in \boldsymbol{M}(\partial K), \text{ if } \boldsymbol{n} \times \boldsymbol{\mu} = 0, \text{ it holds } \boldsymbol{\mu} = \boldsymbol{0}. \tag{8d}$$

Here $\widetilde{V}^{\perp}(K)$ and $\widetilde{\boldsymbol{W}}^{\perp}(K)$ are the $L^2(K)$-orthogonal complements of $\widetilde{V}(K)$ in $V(K)$, and $\widetilde{\boldsymbol{W}}(K)$ in $\boldsymbol{W}(K)$, respectively.

Since the 2D Maxwell equation can be rewritten as a Helmholtz equation, we can use the spaces which satisfy the M-decomposition for second-order elliptic operators.

Let $\boldsymbol{V}_e(K) \times W_e(K)$ admits an $M_e(K)$ decomposition for second-order elliptic operators in 2D, then by [24, Definition 2.1] we must have

(a) $\mathrm{tr}(\boldsymbol{V}_e(K) \times W_e(K)) \subset M_e(K)$ and there exists a subspace $\boldsymbol{V}_e\tilde{(}K) \times \widetilde{W}_e(K)$ of $\boldsymbol{V}_e(K) \times W_e(K)$ satisfying

(b) $\nabla W_e(K) \times \nabla \cdot \boldsymbol{V}_e(K) \subset \tilde{\boldsymbol{V}}_e(K) \times \widetilde{W}_e(K)$

(c) $\mathrm{tr} : \tilde{\boldsymbol{V}}_e(K)^{\perp} \times \widetilde{W}_e(K)^{\perp} \to M_e(K)$ is an isomorphism.

We define

$$\begin{aligned} V(K) &:= W_e(K), \\ \boldsymbol{W}(K) &:= \{\boldsymbol{v}^\perp : \boldsymbol{v} \in \boldsymbol{V}_e(K)\}, \\ \boldsymbol{M}(\partial K) &:= \boldsymbol{n}^\perp M_e(\partial K). \end{aligned}$$

It is easy to check that if $\boldsymbol{V}_e(K) \times W_e(K)$ admits an $M_e(K)$ decomposition for second-order elliptic operators, then $V(K) \times \boldsymbol{W}(K)$ admits an $\boldsymbol{M}(K)$-decomposition for 2D Maxwell operators.

## 4. Error Analysis

In this section, we present our main result, an error analysis for the HDG approximation to Maxwell's equations given by (5). To simplify the derivation we shall assume that $\mu_r$ and $\epsilon_r$ are smooth functions and $\mathrm{Re}(\mu_r) > 0$. First we discuss the extra conditions on the spaces $V(K)$ and $W(K)$ needed for this analysis. These conditions arise because at this point each element $K \in \mathcal{T}_h$ is a general polygon, yet we need certain properties for functions in these spaces (that hold for standard elements including triangles, parallelograms and squares that are considered later in this paper). For triangles these conditions follow if the mesh is assumed to be regular, and the spaces $V(K)$ and $\boldsymbol{W}(K)$ are sufficiently rich. After this discussion, we consider an adjoint problem needed for the analysis and finally present the error analysis.

Throughout this section, we use $C$ to denote a positive constant independent of mesh size, which may take on different values at each occurrence.

### *4.1. Additional assumptions on the approximation spaces*

Throughout this section we assume that the following conditions on the local spaces $V(K)$ and $\boldsymbol{W}(K)$ hold:

1. Most importantly, we assume that the space $V(K) \times \boldsymbol{W}(K)$ admits an $M$-decomposition.

2. The spaces $V(K)$ and $\boldsymbol{W}(K)$ must satisfy

$$\mathcal{P}_0(K) \in \widetilde{V}(K) \text{ and } [\mathcal{P}_0(K)]^2 \in \widetilde{\boldsymbol{W}}(K) \tag{9}$$

for all elements $K$. In addition, we assume that if $\Pi_0$ (respectively $\boldsymbol{\Pi}_0$) denotes the $L^2(K)$ (respectively $\boldsymbol{L}^2(K)$) orthogonal projection onto $V(K)$ (respectively $\boldsymbol{W}(K)$) then the following estimates hold:

$$\|\boldsymbol{w} - \boldsymbol{\Pi}_0\boldsymbol{w}\|_K \leq Ch_K^s\|\boldsymbol{w}\|_{\boldsymbol{H}^s(K)} \text{ and } \|p - \Pi_0 p\|_K \leq Ch_K^s\|p\|_{H^s(K)}$$

for any sufficiently smooth $\boldsymbol{w}$ or $p$ and $0 \leq s \leq 1$.

3. Let $\mathcal{T}_h^\star$ be a refined mesh of $\mathcal{T}_h$ consisting of simplices obtained by subdividing each element $K \in \mathcal{T}_h$ using triangles. We assume that the number of triangles used in each element is bounded independent of $h$ (i.e. there is a fixed maximum number of triangles covering each $K$ independent of $h$). Next we define $\boldsymbol{W}_h^\star = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{u}|_K \in [\mathcal{P}_\ell(K)]^2, \forall K \in \mathcal{T}_h^\star\}$ and $\ell \geq 1$ is some integer such that $\boldsymbol{W}_h \subset \boldsymbol{W}_h^\star$. We assume that $\mathcal{T}_h^\star$ is shape-regular. This assumption implies that standard scaling estimates can be used for $V(K)$ and $\boldsymbol{W}(K)$, because scaling can be used triangle by triangle on the $\mathcal{T}_h^\star$. In addition, standard finite element spaces constructed on this mesh have the usual approximation properties.

   Note that this notion of shape regularity for the general mesh is the analogue of that used to define shape regularity for a quadrilateral mesh in [34].

### 4.2. The dual problems

To obtain the superconvergent rates of $\nabla \times \boldsymbol{u}$ and $q$, we need to consider the following two dual problems. First, we find $(\psi, \boldsymbol{\phi}) \in H(\text{curl};\Omega) \times \boldsymbol{H}_0(\text{curl};\Omega)$ such that

$$\mu_r\psi - \nabla \times \boldsymbol{\phi} = \theta \qquad \text{in } \Omega, \tag{10a}$$

$$\nabla \times \psi - \kappa^2\overline{\epsilon_r}\boldsymbol{\phi} = \mathbf{0} \qquad \text{in } \Omega, \tag{10b}$$

$$\boldsymbol{n} \times \boldsymbol{\phi} = 0 \qquad \text{on } \Gamma. \tag{10c}$$

Inserting (10b) into (10a) and (10c) to get the following Hemlholtz equation:

$$\mu_r \psi - \nabla \times (\kappa^{-2} \overline{\epsilon_r}^{-1} \nabla \times \psi) = \theta \qquad \text{in } \Omega,$$
$$\boldsymbol{n} \times (\kappa^{-2} \overline{\epsilon_r}^{-1} \nabla \times \psi) = 0 \qquad \text{on } \Gamma.$$

By the standard regularity result for the equivalent Helmholtz equation, for some $s > 1/2$ we have

$$\|\psi\|_{H^{1+s}(\Omega)} \leq C \|\theta\|_{L^2(\Omega)}.$$

Next, we consider the following dual problem: find $(\Psi, \boldsymbol{\Phi}) \in H(\text{curl}; \Omega) \times \boldsymbol{H}_0(\text{curl}; \Omega)$ such that

$$\mu_r \Psi - \nabla \times \boldsymbol{\Phi} = 0 \qquad \text{in } \Omega, \tag{11a}$$
$$\nabla \times \Psi - \kappa^2 \overline{\epsilon_r} \boldsymbol{\Phi} = \overline{\epsilon_r} \boldsymbol{\Theta} \qquad \text{in } \Omega, \tag{11b}$$
$$\boldsymbol{n} \times \boldsymbol{\Phi} = 0 \qquad \text{on } \Gamma, \tag{11c}$$

where $\boldsymbol{\Theta} \in \boldsymbol{H}(\text{div}^0_{\overline{\epsilon_r}}; \Omega)$, and $\overline{\epsilon_r}$ is the complex conjugate of $\epsilon_r$. Under our assumptions on $\mu_r$, $\epsilon_r$ and $\kappa$, this problem has a unique solution. Notice that $\boldsymbol{\Theta} \in \boldsymbol{H}(\text{div}^0_{\overline{\epsilon_r}}; \Omega)$ implies that $\boldsymbol{\Phi} \in \boldsymbol{H}(\text{div}^0_{\overline{\epsilon_r}}; \Omega)$. The regularity of the solution of (11) is given in Theorem 4.5.

We recall the following result, where $L^2_0(\Omega)$ denotes the space of functions in $L^2(\Omega)$ with average value zero.

**Lemma 4.1** (c.f [34, Corollary 2.4])**.** Let $\Omega$ be a bounded connected Lipschitz domain in $\mathbb{R}^2$, then for any $f \in L^2_0(\Omega)$, there exists a $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ such that

$$\nabla \cdot \boldsymbol{v} = f, \qquad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

With the above result, we are ready to prove the following lemma:

**Lemma 4.2.** Let $\Omega$ be a bounded connected Lipschitz domain in $\mathbb{R}^2$, then for any $f \in L^2(\Omega)$, there exists a $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that

$$\nabla \cdot \boldsymbol{v} = f, \qquad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

*Proof.* Let $\bar{f} = |\Omega|^{-1}(f,1)_\Omega$ be the mean value of $f$, then $f - \bar{f} \in L_0^2(\Omega)$. By Lemma 4.1, there exists a $\boldsymbol{w} = (w_1, w_2)^T \in \boldsymbol{H}_0^1(\Omega)$, such that $\nabla \cdot \boldsymbol{w} = f - \bar{f}$, and $\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f - \bar{f}\|_{L^2(\Omega)}$.

Let $(x_0, y_0)$ be a point in the domain $\Omega$. Define $\boldsymbol{v} = (v_1, v_2)^T$ with $v_1 = w_1 + (x - x_0)\bar{f}$, $v_2 = w_2$, then $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and

$$
\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \nabla \cdot \boldsymbol{w} + \nabla \cdot ((x - x_0)\bar{f}, 0)^T = f, \\
\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} &\leq \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} + \|(x - x_0)\bar{f}\|_{\boldsymbol{H}^1(\Omega)} \\
&\leq C\|f - \bar{f}\|_{L^2(\Omega)} + \|(x - x_0)\bar{f}\|_{L^2(\Omega)} + \|\bar{f}\|_{L^2(\Omega)} \\
&\leq C(\|f\|_{L^2(\Omega)} + \|\bar{f}\|_{L^2(\Omega)}) \\
&\leq C\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

□

The previous result can be used to prove the existence of a vector potential as follows

**Lemma 4.3.** Let $f \in L^2(\Omega)$, then there exists a function $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$, such that

$$
\nabla \times \boldsymbol{w} = f, \qquad \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
$$

*Proof.* By Lemma 4.2, there exists a $\boldsymbol{v} = (v_1, v_2)^T \in \boldsymbol{H}^1(\Omega)$, such that $\nabla \cdot \boldsymbol{v} = f$ and $\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. We take $\boldsymbol{w} = (w_1, w_2)^T$ with $w_1 = -v_2$ and $w_2 = w_1$, hence

$$
\begin{aligned}
\nabla \times \boldsymbol{w} = -\partial_y w_1 + \partial_x w_2 = \partial_y v_2 + \partial_x v_1 = \nabla \cdot \boldsymbol{v} = f, \\
\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

□

The proof of the next theorem follows that of [35, Proposition 3.7] and [4, Theorem 3.50], which deal with the 3D case.

**Theorem 4.4.** Let $\Omega$ be a simply connected Lipschitz domain in $\mathbb{R}^2$, then the space $\boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}\ _{\epsilon_r};\Omega)$ is imbedded in the space $\boldsymbol{H}^s(\Omega)$ with some $s \in (\frac{1}{2}, 1]$, and the following estimate holds

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\Omega)} \leq C\left(\|\nabla \times \boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla \cdot (\ \epsilon_r \boldsymbol{u})\|_{L^2(\Omega)}\right), \tag{12}$$

for all $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}\ _{\epsilon_r};\Omega)$.

*Proof.* Let $\mathcal{O}$ be a smooth open set with a connected boundary (a circle for instance), which contains $\bar{\Omega}$. Let $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}\ _{\epsilon_r};\Omega)$, we extend $\boldsymbol{u}$ to $\mathcal{O}$ by zero, so $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{curl};\mathcal{O})$, therefore $\nabla \times \boldsymbol{u} \in L^2(\mathcal{O})$. From Lemma 4.3, there exists a function $\boldsymbol{w} \in \boldsymbol{H}^1(\mathcal{O})/\mathbb{R}$ such that $\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{u}$ in $\mathcal{O}$ and $\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathcal{O})} \leq C\|\nabla \times \boldsymbol{u}\|_{L^2(\mathcal{O})}$. Since $\nabla \times (\boldsymbol{u} - \boldsymbol{w}) = 0$ in $\mathcal{O}$, then there is a function $\chi \in H^1(\mathcal{O})/\mathbb{R}$ such that $\boldsymbol{u} - \boldsymbol{w} = \nabla\chi$ in $\mathcal{O}$. Since $\boldsymbol{u} = \boldsymbol{0}$ in $\mathcal{O}/\bar{\Omega}$, we have $-\boldsymbol{w} = \nabla\chi$ in $\mathcal{O} \setminus \bar{\Omega}$, therefore, $\chi \in H^2(\mathcal{O} \setminus \bar{\Omega})$. Then $\chi \in H^1(\Omega)/\mathbb{R}$ satisfies

$$\nabla \cdot (\epsilon_r \nabla\chi) = \nabla \cdot (\epsilon_r \boldsymbol{u}) - \nabla \cdot (\epsilon_r \boldsymbol{w}) \quad \text{in } \Omega,$$

where $\nabla\chi|_{\partial\Omega}$ takes the exterior value of $\nabla\chi = -\boldsymbol{w}$. So $\chi|_{\partial\Omega} \in H^{\frac{1}{2}+s}(\partial\Omega)$ with some $s \in (\frac{1}{2}, 1]$. By [36, Corollary 18.15], we have $\chi \in H^{1+s}(\Omega)$ and

$$\|\chi\|_{H^{1+s}(\Omega)} \leq C\left(\|\ \nabla \cdot (\epsilon_r \boldsymbol{u}) - \nabla \cdot (\epsilon_r \boldsymbol{w})\|_{H^{-1+s}(\Omega)} + \|\nabla\chi\|_{H^{-\frac{1}{2}+s}(\partial\Omega)}\right).$$

Therefore, using the fact that $\nabla\chi = -\boldsymbol{w}$ on $\mathcal{O} \setminus \bar{\Omega}$ it holds

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\Omega)} &= \|\boldsymbol{w} + \nabla\chi\|_{\boldsymbol{H}^s(\Omega)} \\
&\leq \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} + C\left(\|\ \nabla \cdot (\epsilon_r \boldsymbol{u}) - \nabla \cdot (\epsilon_r \boldsymbol{w})\|_{H^{-1+s}(\Omega)} + \|\nabla\chi\|_{H^{-\frac{1}{2}+s}(\partial\Omega)}\right) \\
&\leq C\left(\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} + \|\ \nabla \cdot (\epsilon_r \boldsymbol{u})\|_{L^2(\Omega)}\right) \\
&\leq C\left(\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathcal{O})} + \|\ \nabla \cdot (\epsilon_r \boldsymbol{u})\|_{L^2(\Omega)}\right) \\
&\leq C\left(\|\nabla \times \boldsymbol{u}\|_{L^2(\Omega)} + \|\ \nabla \cdot (\epsilon_r \boldsymbol{u})\|_{L^2(\Omega)}\right).
\end{aligned}$$

Thus we finish our proof. □

Now we can state a complete regularity result for the adjoint problem:

**Theorem 4.5.** Under the assumptions on the domain and coefficients of (11), we have the following regularity for the solution of problem (11)

$$\|\Psi\|_{H^1(\Omega)} + \|\boldsymbol{\Phi}\|_{\boldsymbol{H}^s(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}, \tag{13}$$

for some $s \in (\frac{1}{2}, 1]$ depending on $\Omega$.

*Proof.* To simplify the notation, we define

$$a^+(\boldsymbol{u}, \boldsymbol{v}) = (\mu_r^{-1}\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v})_\Omega + (\boldsymbol{u}, \boldsymbol{v})_\Omega.$$

Let $\widetilde{\boldsymbol{\Phi}} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0{}_{\epsilon_r};\Omega)$ be the solution of

$$a^+(\widetilde{\boldsymbol{\Phi}}, \boldsymbol{v}) = (\boldsymbol{\Theta}, \boldsymbol{v})_\Omega, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0{}_{\epsilon_r};\Omega). \tag{14}$$

Let $\mathcal{K} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0{}_{\epsilon_r};\Omega)$ be such that for any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ the function $\mathcal{K}\boldsymbol{w}$ satisfies

$$a^+(\mathcal{K}\boldsymbol{w}, \boldsymbol{v}) = -(\kappa^2\overline{\epsilon_r}+1)(\boldsymbol{w}, \boldsymbol{v})_\Omega, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0{}_{\epsilon_r};\Omega). \tag{15}$$

Obviously, $\widetilde{\boldsymbol{\Phi}}$ and $\mathcal{K}$ are well-defined and

$$a^+((\mathcal{I} + \mathcal{K})\boldsymbol{\Phi}, \boldsymbol{v}) = a^+(\widetilde{\boldsymbol{\Phi}}, \boldsymbol{v}),$$

where $\mathcal{I}$ is the identity operator. This gives

$$(\mathcal{I} + \mathcal{K})\boldsymbol{\Phi} = \widetilde{\boldsymbol{\Phi}}. \tag{16}$$

Form (14) and (15), we get

$$\|\widetilde{\boldsymbol{\Phi}}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \times \widetilde{\boldsymbol{\Phi}}\|_{L^2(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)},$$
$$\|\mathcal{K}\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \times \mathcal{K}\boldsymbol{\Phi}\|_{L^2(\Omega)} \le C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)}.$$

From Theorem 4.4, we know that $\boldsymbol{H}_0(\mathrm{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div}^0{}_{\epsilon_r};\Omega)$ is compactly imbedded in the space $\boldsymbol{L}^2(\Omega)$. Using (15) we see that $\mathcal{K}$ is a self-adjoint and compact operator on $\boldsymbol{L}^2(\Omega)$. Hence, since our assumptions on $\epsilon_r$ and $\kappa^2$ guarantee at most one solution, by the Fredholm Alternative, (16) has a unique solution.

Therefore,

$$\begin{aligned}\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)} &= \|(\mathcal{I}+\mathcal{K})^{-1}\widetilde{\boldsymbol{\Phi}}\|_{\boldsymbol{L}^2(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)},\\ \|\nabla\times\boldsymbol{\Phi}\|_{L^2(\Omega)} &\le \|\nabla\times\mathcal{K}\boldsymbol{\Phi}\|_{L^2(\Omega)} + \|\nabla\times\widetilde{\boldsymbol{\Phi}}\|_{L^2(\Omega)}\\ &\le C\left(\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)}\right) \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

By the Equation (11), we have

$$\|\nabla\times(\mu_r^{-1}\nabla\times\boldsymbol{\Phi})\|_{\boldsymbol{L}^2(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}.$$

Since $\mu$ is smooth, then we have $\nabla\times\boldsymbol{\Phi}\in H^1(\Omega)$ and

$$\|\nabla\times\boldsymbol{\Phi}\|_{H^1(\Omega)} \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}.$$

Since Theorem 4.4 ensures $\boldsymbol{H}_0(\mathrm{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div}^0;\Omega)$ is imbeded in the space $\boldsymbol{H}^s(\Omega)$ with $(\frac{1}{2},1]$, then we have

$$\|\boldsymbol{\Phi}\|_{\boldsymbol{H}^s(\Omega)} \le C\left(\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\times\boldsymbol{\Phi}\|_{L^2(\Omega)}\right) \le C\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}. \tag{17}$$

This finishes our proof. □

Let $P_V, P_{\widetilde{V}}, \boldsymbol{P_W}$ and $\boldsymbol{P}_{\widetilde{\boldsymbol{W}}}$ denote the $L^2$- projections on the spaces $V_h, \widetilde{V}_h, \boldsymbol{W}_h$ and $\widetilde{\boldsymbol{W}}_h$, respectively.

Now we state the main result of the paper. The proof is found in Section 4.4.

**Theorem 4.6.** Suppose the spaces $(V_h, \boldsymbol{W}_h, \boldsymbol{M}_h)$ have an $M$-decomposition and the assumptions in Section 4.1 are satisfied. Let $(q,\boldsymbol{u})\in H(\mathrm{curl};\Omega)\times\boldsymbol{H}(\mathrm{curl};\Omega)$ and $(q_h,\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\in V_h\times\boldsymbol{W}_h\times\boldsymbol{M}_h$ be the solution of (3) and (5), respectively. Then there exists an $h_0>0$ such that for all $h\le h_0$, we have the error estimate

$$\begin{aligned}\|q-q_h\|_{\mathcal{T}_h} &\le C\left(\|q-P_Vq\|_{\mathcal{T}_h} + \|\boldsymbol{u}-\boldsymbol{P_W}\boldsymbol{u}\|_{\mathcal{T}_h}\right),\\ \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathcal{T}_h} &\le C\left(\|q-P_Vq\|_{\mathcal{T}_h} + \|\boldsymbol{u}-\boldsymbol{P_W}\boldsymbol{u}\|_{\mathcal{T}_h}\right).\end{aligned}$$

Furthermore, the post processed solution $\boldsymbol{u}_h^\star\in\boldsymbol{W}^\star(\mathcal{T}_h)$ defined later in (66) satisfies the estimate

$$\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\|_{\mathcal{T}_h} \le C \left(\|q - P_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}} \boldsymbol{u}\|_{\mathcal{T}_h} \right.$$
$$\left. + \inf_{\boldsymbol{w}_h^\star \in \boldsymbol{W}^\star(\mathcal{T}_h)} \|\nabla \times (\boldsymbol{u} - \boldsymbol{w}_h^\star)\|_{\mathcal{T}_h} \right),$$

and the post processed solution $q_h^\star \in V^\star(\mathcal{T}_h)$ defined later in (69) satisfies the estimate

$$\|q - q_h^\star\|_{\mathcal{T}_h} \le Ch^s \left(\|\Pi_V q - q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u}\|_{\mathcal{T}_h} \right.$$
$$\left. + \inf_{v_h^\star \in V^\star(\mathcal{T}_h)} \|\nabla \times (q - v_h^\star)\|_{\mathcal{T}_h} \right) + Ch\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathcal{T}_h}.$$

*4.3. The HDG Projection*

An appropriate HDG projection plays a key role in the derivation of optimal error estimates and superconvergence (see for example [16, 37–43]). In the case of Maxwell's equations, we define the following HDG projection: find $(\Pi_V q, \boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u}) \in V(K) \times \boldsymbol{W}(K)$ such that

$$(\Pi_V q, v_h)_K = (q, v_h)_K, \tag{18a}$$

$$(\boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u}, \boldsymbol{w}_h)_K = (\boldsymbol{u}, \boldsymbol{w}_h)_K, \tag{18b}$$

$$\langle \Pi_V q - \tau \boldsymbol{n} \times \boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u}, \boldsymbol{n} \times \boldsymbol{\mu}_h \rangle_F = \langle q - \tau \boldsymbol{n} \times \boldsymbol{u}, \boldsymbol{n} \times \boldsymbol{\mu}_h \rangle_F \tag{18c}$$

for all $v_h \in \widetilde{V}(K)$, $\boldsymbol{w}_h \in \widetilde{\boldsymbol{W}}(K)$, $\boldsymbol{\mu}_h \in \boldsymbol{M}(F)$ and for all edges $F \subset \partial K$. The following theorem proves that the above definition uniquely specifies the projections and provides optimal error estimates for this projection.

**Theorem 4.7.** System (18) defines a unique projection $(\Pi_V q, \boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u})$. Moreover, we have the following error estimate:

$$\|\boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u} - \boldsymbol{u}\|_K \le C(\|\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}} \boldsymbol{u}\|_K + h_K \|\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}} \nabla \times q\|_K$$
$$+ h_K^{1/2} \|\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}} \boldsymbol{u})\|_{\partial K}), \tag{19a}$$

$$\|\Pi_V q - q\|_K \le C(h_K^{1/2} \|q - P_V q\|_{\partial K} + h_K^{1/2} \|\boldsymbol{n} \times (\boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u} - \boldsymbol{u})\|_{\partial K}$$
$$+ \|q - P_V q\|_K). \tag{19b}$$

We only give a proof for (19a) in the following three lemmas, since (19b) is very similar.

**Lemma 4.8** (Existence and Uniqueness)**.** System (18) defines a unique projection $(\Pi_V q, \boldsymbol{\Pi_W u})$.

*Proof.* By Definition 3.1 we have

$$\dim V(K) + \dim \boldsymbol{W}(K) = \dim \widetilde{V}(K) + \dim \widetilde{\boldsymbol{W}}(K) + \dim \boldsymbol{M}(\partial K).$$

This means that system (18) is square, hence we only need to prove uniqueness. We set the right hand sides of (18) to zero, i.e., $q = 0$ and $\boldsymbol{u} = \boldsymbol{0}$. By (18a) and (18b), we have

$$\Pi_V q \in \widetilde{V}^{\perp}(K) \quad \text{and} \quad \boldsymbol{\Pi_W u} \in \widetilde{\boldsymbol{W}}^{\perp}(K). \tag{20}$$

Since $\boldsymbol{n} \times \boldsymbol{W}(K) \times \boldsymbol{n} \subset \boldsymbol{M}(\partial K)$, then we can take $\boldsymbol{\mu}_h = \boldsymbol{n} \times \boldsymbol{\Pi_W u} \times \boldsymbol{n}$ in (18c) to get

$$\begin{aligned}
\langle \tau \boldsymbol{n} \times \boldsymbol{\Pi_W u}, &\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\Pi_W u} \times \boldsymbol{n}) \rangle_{\partial K} \\
&= \langle \Pi_V q, \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\Pi_W u} \times \boldsymbol{n}) \rangle_{\partial K} \\
&= \langle \Pi_V q, \boldsymbol{n} \times \boldsymbol{\Pi_W u} \rangle_{\partial K} \\
&= (\Pi_V q, \nabla \times \boldsymbol{\Pi_W u})_K - (\nabla \times \Pi_V q, \boldsymbol{\Pi_W u})_K \\
&= 0.
\end{aligned}$$

Since $\tau$ is piecewise constant and positive, then

$$\boldsymbol{n} \times \boldsymbol{\Pi_W u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial K. \tag{21}$$

Moreover, $\boldsymbol{n} \times \boldsymbol{\Pi_W u} = 0$ on $\partial K$. Since $\boldsymbol{n} \times V(K) \subset \boldsymbol{M}(\partial K)$, then we can take $\boldsymbol{\mu}_h = \boldsymbol{n} \times \Pi_V q$ in (18c) to get

$$\boldsymbol{n} \times \Pi_V q = 0 \quad \text{on} \quad \partial K. \tag{22}$$

We combine (20), (21), (22) and (8c) to conclude that $\boldsymbol{\Pi_W u} = \boldsymbol{0}$ and $\Pi_V q = 0$. This proves the system (18) defines a unique projection $(\Pi_V q, \boldsymbol{\Pi_W u})$. $\square$

To estimate $\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}-\boldsymbol{u}$, we decouple the projection $\mathbf{\Pi}_{\boldsymbol{W}}$ from $\Pi_V$ in (18) as follows.

**Lemma 4.9.** The projection $\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}$ satisfies

$$(\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u},\boldsymbol{v}_h)_K = (\boldsymbol{u},\boldsymbol{v}_h)_K,, \tag{23a}$$

$$\langle\tau\boldsymbol{n}\times\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{w}_h\rangle_{\partial K} = (\nabla\times q,\boldsymbol{w}_h)_K + \langle\tau\boldsymbol{n}\times\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{w}_h\rangle_{\partial K} \tag{23b}$$

for all $\boldsymbol{v}_h\in\widetilde{\boldsymbol{W}}(K)$, $\boldsymbol{w}_h\in\widetilde{\boldsymbol{W}}^{\perp}(K)$.

*Proof.* Noticing that (18c) can be rewritten as

$$\langle\tau\boldsymbol{n}\times\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{\mu}_h\rangle_{\partial K} = \langle\Pi_V q - q,\boldsymbol{n}\times\boldsymbol{\mu}_h\rangle_{\partial K} + \langle\tau\boldsymbol{n}\times\boldsymbol{u},\boldsymbol{n}\times\boldsymbol{\mu}_h\rangle_{\partial K}. \tag{24}$$

Since $\boldsymbol{n}\times\boldsymbol{W}(K)\times\boldsymbol{n}\subset\boldsymbol{M}(\partial K)$, then we take $\boldsymbol{\mu}_h=\boldsymbol{n}\times\boldsymbol{w}_h\times\boldsymbol{n}$ in (24) to get

$$\langle\tau\boldsymbol{n}\times\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{w}_h\rangle_{\partial K} = \langle\boldsymbol{n}\times(q-\Pi_V q),\boldsymbol{w}_h\rangle_{\partial K} + \langle\tau\boldsymbol{n}\times\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{w}_h\rangle_{\partial K}. \tag{25}$$

Then, for all $\boldsymbol{w}_h\in\widetilde{\boldsymbol{W}}^{\perp}(K)$, by (8b) and (18a), we have

$$(\nabla\times\Pi_V q,\boldsymbol{w}_h)_K = 0, \tag{26a}$$

$$(q-\Pi_V q,\nabla\times\boldsymbol{w}_h)_K = 0. \tag{26b}$$

Next, we use the integration by parts identity (4b) to get

$$\begin{aligned}
&\langle\boldsymbol{n}\times(q-\Pi_V q),\boldsymbol{w}_h\rangle_{\partial K} && (27)\\
&\quad= (\nabla\times(q-\Pi_V q),\boldsymbol{w}_h)_K - (q-\Pi_V q,\nabla\times\boldsymbol{w}_h)_K \\
&\quad= (\nabla\times(q-\Pi_V q),\boldsymbol{w}_h)_K, && \text{by (26b)}\\
&\quad= (\nabla\times q,\boldsymbol{w}_h)_K. && \text{by (26a)} \quad (28)
\end{aligned}$$

Therefore, (18b), (25) and (27) gives the system (23). ☐

Now we can give the proof of (19a).

*Proof of* (19a). By the definition of $\boldsymbol{P}_{\boldsymbol{W}}$ and $\boldsymbol{P}_{\widetilde{\boldsymbol{W}}}$, we can rewrite equation (23) as follows:

$$(\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u}, \boldsymbol{v}_h)_K = 0, \tag{29a}$$

$$\begin{aligned}\langle \tau\boldsymbol{n} \times (\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u}), \boldsymbol{n} \times \boldsymbol{w}_h\rangle_{\partial K} &= (\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}}\nabla \times q, \boldsymbol{w}_h)_K \\ &\quad + \langle \tau\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u}), \boldsymbol{n} \times \boldsymbol{w}_h\rangle_{\partial K}, \end{aligned} \tag{29b}$$

for all $(\boldsymbol{v}_h, \boldsymbol{w}_h) \in (\widetilde{\boldsymbol{W}}(K) \times \widetilde{\boldsymbol{W}}^{\perp}(K)$. By the same arguments as in the proof of Lemma 4.8, we can prove that $\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u} \in \boldsymbol{W}(K)$ is uniquely determined by the right hand side of (29). Using a standard scaling estimate (this can be used because of the assumption on $\mathcal{T}_h^*$ in Section 4.1) we have

$$\begin{aligned}\|\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u}\|_K \leq{}& Ch_K\|\tau^{-1}(\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}}\nabla \times q)\|_K \\ &+ Ch_K^{1/2}\|\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{u})\|_{\partial K}.\end{aligned}$$

Thus, the triangle inequality gives the desired result. □

Next, we extend the error estimates (19) to fractional order Sobolev spaces. To do this we use a local inverse inequality. For any function $\boldsymbol{w}_h \in \boldsymbol{W}(K)$ or $p_h \in V(K)$ the following inverse estimate holds:

$$\|\boldsymbol{w}_h\|_{H^s(K)} \leq Ch_K^{-s}\|\boldsymbol{w}_h\|_K, \text{ and } \|\boldsymbol{p}_h\|_{H^s(K)} \leq Ch_K^{-s}\|\boldsymbol{p}_h\|_K$$

with $0 \leq s \leq 1$. The constant $C$ is independent of the function, element and mesh size. Note that this assumption follows from our assumption on the auxiliary mesh $\mathcal{T}_h^*$ when $s = 1$ and trivially holds when $s = 0$. Hence by interpolation it holds for general $0 \leq s \leq 1$.

**Lemma 4.10.** For any $s \in [0,1]$, we have

$$\begin{aligned}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} \leq C h_K^{-s}(&\|\boldsymbol{u} - \boldsymbol{P_W u}\|_K + h_K\|\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}}\nabla \times q\|_K \\ &+ h_K^{1/2}\|\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P_W u})\|_{\partial K}) + \|\boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)},\end{aligned} \tag{30a}$$

$$\begin{aligned}\|\Pi_V q - q\|_{H^s(K)} \leq C(&h_K^{1/2-s}\|q - P_V q\|_{\partial K} + h_K^{-s}\|q - P_V q\|_K \\ &+ \|q - P_V q\|_{H^s(K)} + \|\boldsymbol{u} - \boldsymbol{P_W u}\|_{\boldsymbol{H}^s(K)} \\ &+ h_K^{-s}\|\boldsymbol{u} - \boldsymbol{P_W u}\|_K + h_K^{1-s}\|\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}}\nabla \times q\|_K \\ &+ h_K^{1/2-s}\|\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P_W u})\|_{\partial K}).\end{aligned} \tag{30b}$$

*Proof.* Using the fact that $\boldsymbol{P_W}$ is the $\boldsymbol{L}^2$ orthogonal projection on $\boldsymbol{W}(K)$ and applying the local inverse inequality discussed before the statement of the lemma, we get

$$\begin{aligned}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} &= \|\boldsymbol{\Pi_W u} - \boldsymbol{P_W u} + \boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} \\ &\leq \|\boldsymbol{\Pi_W u} - \boldsymbol{P_W u}\|_{\boldsymbol{H}^s(K)} + \|\boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} \\ &\leq C h_K^{-s}\|\boldsymbol{\Pi_W u} - \boldsymbol{P_W u}\|_K + \|\boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} \\ &\leq C h_K^{-s}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_K + C h_K^{-s}\|\boldsymbol{P_W u} - \boldsymbol{u}\|_K \\ &\quad + \|\boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)}.\end{aligned}$$

Combining the estimate (19a) and the above inequality we have

$$\begin{aligned}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)} \leq C h_K^{-s}(&\|\boldsymbol{u} - \boldsymbol{P_W u}\|_K + h_K\|\nabla \times q - \boldsymbol{P}_{\widetilde{\boldsymbol{W}}}\nabla \times q\|_K \\ &+ h_K^{1/2}\|\boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{P_W u})\|_{\partial K}) + \|\boldsymbol{P_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)}.\end{aligned}$$

This proves (30a).

Next, we prove (30b). By the same arguments we have

$$\|\Pi_V q - q\|_{H^s(K)} \leq C h_K^{-s}\|\Pi_V q - q\|_K + C h_K^{-s}\|P_V q - q\|_K + \|P_V q - q\|_{H^s(K)}.$$

By Lemma 7.2 in [44] to get

$$\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_{\partial K} \leq C\left(h_K^{-1/2}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_K + h_K^{s-1/2}\|\boldsymbol{\Pi_W u} - \boldsymbol{u}\|_{\boldsymbol{H}^s(K)}\right). \tag{31}$$

Using estimates (19b), (19a), (31) and (30a), we can obtain (30b). □

Since $\mathcal{P}_0(K) \in \widetilde{V}(K)$ and $[\mathcal{P}_0(K)]^2 \in \widetilde{\boldsymbol{W}}(K)$ with appropriate projection error bounds (see Section 4.1), by Theorem 4.6 and Lemma 4.10, we have the following corollary.

**Corollary 4.11.** Let $(\Psi, \boldsymbol{\Phi}) \in H(\text{curl};\Omega) \times [\boldsymbol{H}_0(\text{curl};\Omega) \cap \boldsymbol{H}(\text{div}^0;\Omega)]$ be the solution of (11) and assume that the regularity result (13) holds, then for $s \in (1/2, 1]$, we have

$$\|\boldsymbol{\Pi_W \Phi} - \boldsymbol{\Phi}\|_{\mathcal{T}_h} + \|\Pi_V \Psi - \Psi\|_{\mathcal{T}_h} \le Ch^s \|\boldsymbol{\Theta}\|_{\mathcal{T}_h}, \tag{32a}$$

$$\|\boldsymbol{\Pi_W \Phi} - \boldsymbol{\Phi}\|_{\boldsymbol{H}^s(\mathcal{T}_h)} + \|\Pi_V \Psi - \Psi\|_{H^s(\mathcal{T}_h)} \le C \|\boldsymbol{\Theta}\|_{\mathcal{T}_h}. \tag{32b}$$

We can now prove our main result: Theorem 4.6.

### *4.4. Proof of Theorem 4.6*

First, we define the following HDG operator $\mathscr{B} : [V_h \times \boldsymbol{W}_h \times M_h]^2 \to \mathbb{C}$

$$\begin{aligned}
&\mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \\
&\qquad = (\mu_r q_h, r_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times r_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times \widehat{\boldsymbol{u}}_h, r_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad\quad + (\nabla \times q_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle q_h, \boldsymbol{n} \times \widehat{\boldsymbol{v}}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad\quad + \langle \tau \boldsymbol{n} \times (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h}.
\end{aligned} \tag{33}$$

By the definition of $\mathscr{B}$ in (33), we can rewrite the HDG formulation of the system (5) in a compact form, as follows:

**Lemma 4.12.** The HDG method seeks $(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h^g$ such that

$$\mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - (\kappa^2 \epsilon_r \boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{T}_h}, \tag{34}$$

for all $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h^0$, in which $\boldsymbol{M}_h^g$ and $\boldsymbol{M}_h^0$ are defined as

$$\begin{aligned}
\boldsymbol{M}_h^g &= \{\boldsymbol{\mu} \in \boldsymbol{M}_h : \boldsymbol{n} \times \boldsymbol{\mu}|_{\partial\Omega} = (\boldsymbol{P_M}(\boldsymbol{n} \times g)) \times \boldsymbol{n}\}, \\
\boldsymbol{M}_h^0 &= \{\boldsymbol{\mu} \in \boldsymbol{M}_h : \boldsymbol{n} \times \boldsymbol{\mu}|_{\partial\Omega} = 0\},
\end{aligned}$$

where $\boldsymbol{P_M}$ denotes the $L^2$-projection from $\boldsymbol{L}^2(F)$ onto space $\boldsymbol{M}(F)$. Thus if $\boldsymbol{u} \in \boldsymbol{L}^2(F)$ then $\boldsymbol{P_M u} \in \boldsymbol{M}(F)$ satisfies

$$\langle \boldsymbol{P_M u}, \boldsymbol{v}_h \rangle_F = \langle \boldsymbol{u}, \boldsymbol{v}_h \rangle_F \qquad\qquad \forall \boldsymbol{v}_h \in \boldsymbol{M}(F). \tag{35}$$

Next, we give some properties of the operator $\mathscr{B}$ below, the proof of the following lemma is very simple and we omit it here.

**Lemma 4.13.** For any $(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in [V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h]^2$, we have

$$\mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; r_h, -\boldsymbol{v}_h, -\widehat{\boldsymbol{v}}_h) = \overline{\mathscr{B}(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h; q_h, -\boldsymbol{u}_h, -\widehat{\boldsymbol{u}}_h)}.$$

**Lemma 4.14.** If for all $r_h \in V_h$, $(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$ satisfies

$$\mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h; r_h, \mathbf{0}, \mathbf{0}) = (G, r_h)_{\mathcal{T}_h},$$

where $G \in L^2(\Omega)$, then we have

$$\|\nabla \times \boldsymbol{u}_h\|_{\mathcal{T}_h} \le C \left( \|q_h\|_{\mathcal{T}_h} + \|\mathbf{h}^{-1/2} \boldsymbol{n} \times (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\|_{\partial \mathcal{T}_h} + \|G\|_{\mathcal{T}_h} \right). \tag{36}$$

*Proof.* By the definition of $\mathscr{B}$ in (33), we have

$$(\mu_r q_h, r_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times r_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times \widehat{\boldsymbol{u}}_h, r_h \rangle_{\partial \mathcal{T}_h} = (G, \boldsymbol{r}_h)_{\mathcal{T}_h}. \tag{37}$$

We take $r_h = \nabla \times \boldsymbol{u}_h$ in (37) and integrate by parts to get

$$\begin{aligned} (\mu_r q_h, \nabla \times \boldsymbol{u}_h)_{\mathcal{T}_h} - (\nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{u}_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \nabla \times \boldsymbol{u}_h \rangle_{\partial \mathcal{T}_h} \\ = (G, \nabla \times \boldsymbol{u}_h)_{\mathcal{T}_h}. \end{aligned}$$

After apply the Cauchy-Schwartz inequality and the local inverse inequality can get our desired result. □

Now, we give the proof of Theorem 4.6, splitting it into three steps.

#### *4.4.1. Step 1: Error equations and energy arguments*

**Lemma 4.15.** Let $(q, \boldsymbol{u}) \in H(\mathrm{curl}; \Omega) \times \boldsymbol{H}(\mathrm{curl}; \Omega)$ be the weak solution of (3), then for all $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h^0$, we have

$$\mathscr{B}(\Pi_V q, \boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{u}, \boldsymbol{P}_{\boldsymbol{M}} \boldsymbol{u}; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (\mu_r (\Pi_V q - q), r_h)_{\mathcal{T}_h} + (\nabla \times q, \boldsymbol{v}_h)_{\mathcal{T}_h}. \tag{38}$$

*Proof.* By the definition of $\mathscr{B}$ in (33) and use (8a) we have

$$
\begin{aligned}
&\mathscr{B}(\Pi_V q, \boldsymbol{\Pi_W u}, \boldsymbol{P_M u}; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \\
&\quad = (\mu_r \Pi_V q, r_h)_{\mathcal{T}_h} - (\boldsymbol{\Pi_W u}, \nabla \times r_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times \boldsymbol{P_M u}, r_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + (\nabla \times \Pi_V q, \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \Pi_V q, \boldsymbol{n} \times \widehat{\boldsymbol{v}}_h \rangle_{\partial \mathcal{T}_h} \\
&\qquad + \langle \tau \boldsymbol{n} \times (\boldsymbol{\Pi_W u} - \boldsymbol{P_M u}), \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h} \\
&\quad = (\mu_r \Pi_V q, r_h)_{\mathcal{T}_h} - (\boldsymbol{u}, \nabla \times r_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \times \boldsymbol{u}, r_h \rangle_{\partial \mathcal{T}_h} && \text{by (18b), (35)} \\
&\qquad + (\Pi_V q, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \Pi_V q, \boldsymbol{n} \times (\widehat{\boldsymbol{v}}_h - \boldsymbol{v}_h) \rangle_{\partial \mathcal{T}_h} && \text{by (4a)} \\
&\qquad + \langle \tau \boldsymbol{n} \times (\boldsymbol{\Pi_W u} - \boldsymbol{u}), \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h} \\
&\quad = (\mu_r \Pi_V q - \nabla \times \boldsymbol{u}, r_h)_{\mathcal{T}_h} && \text{by (4a)} \\
&\qquad + (q, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \Pi_V q, \boldsymbol{n} \times (\widehat{\boldsymbol{v}}_h - \boldsymbol{v}_h) \rangle_{\partial \mathcal{T}_h} && \text{by (18a)} \\
&\qquad + \langle \tau \boldsymbol{n} \times (\boldsymbol{\Pi_W u} - \boldsymbol{u}), \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h}.
\end{aligned}
$$

Since $\widehat{\boldsymbol{v}}_h$ is single valued on interior edges and equal to zero on boundary faces, then $\langle q, \boldsymbol{n} \times \widehat{\boldsymbol{v}}_h \rangle_{\partial \mathcal{T}_h} = 0$. Moreover, by (4a) we have

$$
\begin{aligned}
&(q, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \Pi_V q, \boldsymbol{n} \times (\widehat{\boldsymbol{v}}_h - \boldsymbol{v}_h) \rangle_{\partial \mathcal{T}_h} \\
&\quad = (\nabla \times \boldsymbol{q}, \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle q - \Pi_V q, \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h} \\
&\quad = (\nabla \times \boldsymbol{q}, \boldsymbol{v}_h)_{\mathcal{T}_h} - \langle \tau \boldsymbol{n} \times (\boldsymbol{\Pi_W u} - \boldsymbol{u}), \boldsymbol{n} \times (\boldsymbol{v}_h - \widehat{\boldsymbol{v}}_h) \rangle_{\partial \mathcal{T}_h} \quad \text{by (18c).}
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\mathscr{B}(\Pi_V q, \boldsymbol{\Pi_W u}, \boldsymbol{P_M u}; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \\
&\quad = (\mu_r \Pi_V q - \nabla \times \boldsymbol{u}, r_h)_{\mathcal{T}_h} + (\nabla \times q, \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\quad = (\mu_r (\Pi_V q - q), r_h)_{\mathcal{T}_h} + (\nabla \times q, \boldsymbol{v}_h)_{\mathcal{T}_h}, && \text{by (3)}
\end{aligned}
$$

and completes the proof. □

To simplify notation, we define

$$
\varepsilon_h^q = \Pi_V q - q_h, \quad \varepsilon_h^{\boldsymbol{u}} = \boldsymbol{\Pi_W u} - \boldsymbol{u}_h, \quad \varepsilon_h^{\widehat{\boldsymbol{u}}} = \boldsymbol{P_M u} - \widehat{\boldsymbol{u}}_h. \tag{39}
$$

We subtract (34) from (38) to get the following error equations.

**Lemma 4.16.** Using the notation (39), for any $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in V_h, \times \boldsymbol{W}_h \times \boldsymbol{M}_h^0$, we have

$$\begin{aligned}\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - (\kappa^2\epsilon_r\varepsilon_h^{\boldsymbol{u}}, \boldsymbol{v}_h)_{\mathcal{T}_h} \\ = (\mu_r(\Pi_V q - q), r_h)_{\mathcal{T}_h} + (\kappa^2\epsilon_r(\boldsymbol{u} - \boldsymbol{\Pi_W u}), \boldsymbol{v}_h)_{\mathcal{T}_h}.\end{aligned} \tag{40}$$

*Proof.* By the definition of $\mathscr{B}$ in (33) and Lemma 4.15, we get

$$\begin{aligned}
&\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - (\kappa^2\epsilon_r\varepsilon_h^{\boldsymbol{u}}, \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\quad = \mathscr{B}(\Pi_V q, \boldsymbol{\Pi_W}, \boldsymbol{P_M u}, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - \mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \\
&\qquad - (\kappa^2\epsilon_r\boldsymbol{\Pi_W u}, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\kappa^2\epsilon_r\boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\quad = \mathscr{B}(\Pi_V q, \boldsymbol{\Pi_W}, \boldsymbol{P_M u}, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - (\kappa^2\epsilon_r\boldsymbol{\Pi_W u}, \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\qquad - \left[\mathscr{B}(q_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) - (\kappa^2\epsilon_r\boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathcal{T}_h}\right] \\
&\quad = (\mu_r(\Pi_V q - q), r_h)_{\mathcal{T}_h} + (\nabla \times q, \boldsymbol{v}_h)_{\mathcal{T}_h} - (\kappa^2\epsilon_r\boldsymbol{\Pi_W u}, \boldsymbol{v}_h)_{\mathcal{T}_h} - (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\quad = (\mu_r(\Pi_V q - q), r_h)_{\mathcal{T}_h} + (\kappa^2\epsilon_r(\boldsymbol{u} - \boldsymbol{\Pi_W u}), \boldsymbol{v}_h)_{\mathcal{T}_h},
\end{aligned}$$

where we used (3b) in the last inequality.

□

**Lemma 4.17.** Using definition (39), we have the error estimate

$$\begin{aligned}\|\sqrt{\mu_r}\varepsilon_h^q\|_{\mathcal{T}_h} + \|\sqrt{\tau}\boldsymbol{n} \times (\varepsilon_h^{\boldsymbol{u}} - \varepsilon_h^{\widehat{\boldsymbol{u}}})\|_{\partial\mathcal{T}_h} \\ \le C\left(\|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}\right).\end{aligned} \tag{41}$$

*Proof.* First, we take $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (\varepsilon_h^q, \boldsymbol{0}, \boldsymbol{0})$ in (40) to get

$$\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; \varepsilon_h^q, \boldsymbol{0}, \boldsymbol{0}) = (\mu_r(\Pi_V q - q), \varepsilon_h^q)_{\mathcal{T}_h}. \tag{42}$$

Next, we take $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (0, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}})$ in (40) to get

$$\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; 0, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}) - (\kappa^2\epsilon_r\varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h} = (\kappa^2\epsilon_r(\boldsymbol{u} - \boldsymbol{\Pi_W u}), \varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}. \tag{43}$$

By the equations (42) and (43), we get

$$\begin{aligned}\overline{\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; \varepsilon_h^q, \boldsymbol{0}, \boldsymbol{0})} + \mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; 0, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}) - (\kappa^2\epsilon_r\varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h} \\ = \overline{(\mu_r(\Pi_V q - q), \varepsilon_h^q)_{\mathcal{T}_h}} + (\kappa^2\epsilon_r(\boldsymbol{u} - \boldsymbol{\Pi_W u}), \varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}.\end{aligned} \tag{44}$$

On the other hand, by the definition of $\mathscr{B}$ in (33) to get

$$\begin{aligned}&\overline{\mathscr{B}(\varepsilon_h^q,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}};\varepsilon_h^q,\boldsymbol{0},\boldsymbol{0})}+\mathscr{B}(\varepsilon_h^q,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}};0,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}})\\&\quad=\|\sqrt{\mu_r}\varepsilon_h^q\|_{\mathcal{T}_h}^2+\|\sqrt{\tau}\boldsymbol{n}\times(\varepsilon_h^{\boldsymbol{u}}-\varepsilon_h^{\widehat{\boldsymbol{u}}})\|_{\partial\mathcal{T}_h}^2.\end{aligned}\tag{45}$$

Hence, by the equation (44) and (45), we have

$$\begin{aligned}&\|\sqrt{\mu_r}\varepsilon_h^q\|_{\mathcal{T}_h}^2+\|\sqrt{\tau}\boldsymbol{n}\times(\varepsilon_h^{\boldsymbol{u}}-\varepsilon_h^{\widehat{\boldsymbol{u}}})\|_{\partial\mathcal{T}_h}^2\\&\quad=\overline{(\mu_r(\Pi_V q-q),\varepsilon_h^q)_{\mathcal{T}_h}}+(\kappa^2\epsilon_r(\boldsymbol{u}-\boldsymbol{\Pi_W}\boldsymbol{u}),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\&\quad\le C\|\Pi_V q-q\|_{\mathcal{T}_h}\|\sqrt{\mu_r}\varepsilon_h^q\|_{\mathcal{T}_h}+C\|\boldsymbol{u}-\boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.\end{aligned}$$

Use of Young's inequality gives our desired result. □

*4.4.2. Step 2: Duality argument*

Similarly to Lemma 4.15, we have:

**Lemma 4.18.** Let $(\Psi,\boldsymbol{\Phi})\in H(\mathrm{curl};\Omega)\times\boldsymbol{H}_0(\mathrm{curl};\Omega)$ be the weak solution of (11), then for all $(r_h,\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\in V_h\times\boldsymbol{W}_h\times\boldsymbol{M}_h^0$, we have

$$\mathscr{B}(\Pi_V\Psi,\boldsymbol{\Pi_W}\boldsymbol{\Phi},\boldsymbol{P_M}\boldsymbol{\Phi};r_h,\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)=(\mu_r(\Pi_V\Psi-\Psi),r_h)_{\mathcal{T}_h}+(\nabla\times\Psi,\boldsymbol{v}_h)_{\mathcal{T}_h}.$$

The next lemma gives a partial error estimate:

**Lemma 4.19.** Assume $\boldsymbol{\Theta}\in\boldsymbol{H}(\mathrm{div}^0_{\overline{\epsilon_r}};\Omega)$, and that the regularity estimate (13) holds, then we have

$$|(\varepsilon_h^{\boldsymbol{u}},\ \overline{\epsilon_r}\boldsymbol{\Theta})_{\mathcal{T}_h}|\le Ch^s\left(\|q-\Pi_V q\|_{\mathcal{T}_h}+\|\boldsymbol{u}-\boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}\right)\|\boldsymbol{\Theta}\|_{\mathcal{T}_h}+Ch^s\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}\|\boldsymbol{\Theta}\|_{\mathcal{T}_h}.$$

*Proof.* First, we take $(r_h,\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)=(-\Pi_V\Psi,\boldsymbol{\Pi_W}\boldsymbol{\Phi},\boldsymbol{P_M}\boldsymbol{\Phi})$ in (40) we get

$$\begin{aligned}&\mathscr{B}(\varepsilon_h^q,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}};-\Pi_V\Psi,\boldsymbol{\Pi_W}\boldsymbol{\Phi},\boldsymbol{P_M}\boldsymbol{\Phi})-(\kappa^2\epsilon_r\varepsilon_h^{\boldsymbol{u}},\boldsymbol{\Pi_W}\boldsymbol{\Phi})_{\mathcal{T}_h}\\&\quad=-(\mu_r(\Pi_V q-q),\Pi_V\Psi)_{\mathcal{T}_h}+(\kappa^2\epsilon_r(\boldsymbol{u}-\boldsymbol{\Pi_W}\boldsymbol{u}),\boldsymbol{\Pi_W}\boldsymbol{\Phi})_{\mathcal{T}_h}.\end{aligned}\tag{46}$$

By Lemma 4.13 and Lemma 4.18 and using (11b) we have

$$\begin{aligned}&\mathscr{B}(\varepsilon_h^q,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}};-\Pi_V\Psi,\boldsymbol{\Pi_W}\boldsymbol{\Phi},\boldsymbol{P_M}\boldsymbol{\Phi})\\&\quad=\overline{\mathscr{B}(\Pi_V\Psi,\boldsymbol{\Pi_W}\boldsymbol{\Phi},\boldsymbol{P_M}\boldsymbol{\Phi};-\varepsilon_h^q,\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\widehat{\boldsymbol{u}}})}\\&\quad=-(\varepsilon_h^q,\mu_r(\Pi_V\Psi-\Psi))_{\mathcal{T}_h}+(\varepsilon_h^{\boldsymbol{u}},\nabla\times\Psi)_{\mathcal{T}_h}\\&\quad=-(\varepsilon_h^q,\mu_r(\Pi_V\Psi-\Psi))_{\mathcal{T}_h}+(\varepsilon_h^{\boldsymbol{u}},\ \overline{\epsilon_r}\boldsymbol{\Theta}+\kappa^2\overline{\epsilon_r}\boldsymbol{\Phi})_{\mathcal{T}_h}.\end{aligned}\tag{47}$$

Comparing (46) with (47) to get

$$
\begin{aligned}
(\varepsilon_h^{\boldsymbol{u}},\ \overline{\epsilon_r}\boldsymbol{\Theta})_{\mathcal{T}_h} &= (\mu_r \varepsilon_h^q, \Pi_V \Psi - \Psi)_{\mathcal{T}_h} - (\mu_r(\Pi_V q - q), \Pi_V \Psi)_{\mathcal{T}_h} \\
&\quad + (\kappa^2 \epsilon_r(\boldsymbol{u} - \boldsymbol{\Pi_W u}), \boldsymbol{\Pi_W \Phi})_{\mathcal{T}_h} - (\kappa^2 \epsilon_r \varepsilon_h^{\boldsymbol{u}}, \boldsymbol{\Phi} - \boldsymbol{\Pi_W \Phi})_{\mathcal{T}_h} \\
&= T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

Next, we estimate $\{T_i\}_{i=1}^4$ one by one. For the terms $T_1$ and $T_4$, by (32a) and estimate for $\varepsilon_h^q$ in Lemma 4.17, we have

$$
\begin{aligned}
|T_1| &\le C h^s \|\boldsymbol{\Theta}\|_{\mathcal{T}_h} (\|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}), \\
|T_4| &\le C h^s \|\boldsymbol{\Theta}\|_{\mathcal{T}_h} \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.
\end{aligned}
$$

For the remain two terms $T_2$ and $T_3$, since $\mathcal{P}_0(K) \in \widetilde{V}(K)$ and $[\mathcal{P}_0(K)]^2 \in \widetilde{\boldsymbol{W}}(K)$ with appropriate estimates for the projection (see Section 4.1), then by (32b) we have

$$
\begin{aligned}
T_2 &= |(\Pi_V q - q, \mu_r \Pi_V \Psi - \Pi_0(\mu_r \Pi_V \Psi))_{\mathcal{T}_h}| \le C h^s \|q - \Pi_V q\|_{\mathcal{T}_h} \|\Pi_V \Psi\|_{H^s(\Omega)} \\
&\le C h^s \|q - \Pi_V q\|_{\mathcal{T}_h} \|\boldsymbol{\Theta}\|_{\mathcal{T}_h}, \\
|T_3| &= |(\boldsymbol{u} - \boldsymbol{\Pi_W u}, \kappa^2 \overline{\epsilon_r} \boldsymbol{\Pi_W \Phi} - \boldsymbol{\Pi}_0 \left[\kappa^2 \overline{\epsilon_r} \boldsymbol{\Pi_W \Phi}\right])_{\mathcal{T}_h}| \\
&\le C h^s \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} (\|\boldsymbol{\Pi_W \Phi}\|_{\boldsymbol{H}^s(\Omega)}) \\
&\le C h^s \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} \|\boldsymbol{\Theta}\|_{\mathcal{T}_h}.
\end{aligned}
$$

By the above estimates of $\{T_i\}_{i=1}^4$ we get

$$
\begin{aligned}
|(\varepsilon_h^{\boldsymbol{u}},\ \overline{\epsilon_r}\boldsymbol{\Theta})_{\mathcal{T}_h}| &\le C h^s \left(\|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h}\right) \|\boldsymbol{\Theta}\|_{\mathcal{T}_h} \\
&\quad + C h^s \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h} \|\boldsymbol{\Theta}\|_{\mathcal{T}_h}.
\end{aligned} \tag{48}
$$

This completes the proof. □

We cannot set $\boldsymbol{\Theta} = \varepsilon_h^{\boldsymbol{u}}$ to get an estimate of $\varepsilon_h^{\boldsymbol{u}}$ since $\varepsilon_h^{\boldsymbol{u}} \notin \boldsymbol{H}(\mathrm{div}^0_{\overline{\epsilon_r}}; \Omega)$, hence we need to modify the analysis.

Recall the shape-regular submesh $\mathcal{T}_h^\star$ defined in Section 4.1. We define $\boldsymbol{W}_h^\star = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{u}|_K \in [\mathcal{P}_\ell(K)]^2, \forall K \in \mathcal{T}_h^\star\}$ and $\ell \ge 1$ is some integer such that $\boldsymbol{W}_h \subset \boldsymbol{W}_h^\star$.

Next, we recall the $\boldsymbol{H}(\mathrm{curl};\Omega)$ conforming element in 2D. For any $\boldsymbol{v}$ smooth enough, with $K$ being a simplex, find $\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_\ell(K)$ such that

$$\langle \boldsymbol{n} \times \boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, p_\ell\rangle_E = \langle \boldsymbol{n} \times \boldsymbol{v}, p_\ell\rangle_E, \qquad \forall p_\ell \in \mathcal{P}_\ell(E), \tag{49a}$$

$$(\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, \nabla \times p_{\ell-1})_K = (\boldsymbol{v}, \nabla \times p_{\ell-1})_K, \qquad \forall p_{\ell-1} \in \mathcal{P}_{\ell-1}(K), \tag{49b}$$

and, when $\ell \geq 2$

$$(\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, \nabla(b_K p_{\ell-2}))_K = (\boldsymbol{v}, \nabla(b_K p_{\ell-2}))_K, \qquad \forall p_{\ell-2} \in \mathcal{P}_{\ell-1}(K) \tag{49c}$$

for all edges $F$ of $K$, where $b_K$ is the bubble function of $K$ of order three.

Following a standard procedure in [45, Lemma 3.2, Theorem 3.1], we have the following theorem:

**Theorem 4.20.** Equation (49) defines a unique $\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_\ell(K)$, and the following estimate holds:

$$\|\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v} - \boldsymbol{v}\|_{0,K} \leq C h_K^m \|\boldsymbol{v}\|_{m,K}, \tag{50}$$

with $\boldsymbol{v} \in \boldsymbol{H}^m(\Omega)$, and $m \in (\frac{1}{2}, \ell+1]$. We define $\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}} = \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}|_K$ for all $K \in \mathcal{T}_h^*$, then $\boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega)$. In addition, $\boldsymbol{n} \times \boldsymbol{v}|_{\partial\Omega} = 0$ implies that $\boldsymbol{n} \times \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{v}|_{\partial\Omega} = 0$.

Furthermore, the previously defined interpolation operator commutes with curl.

**Lemma 4.21.** Suppose $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega)$ is smooth enough that $\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}$ is well defined. Let $\Pi_{K,\ell-1}$ be the $L^2$ projection onto space $\mathcal{P}_{\ell-1}(K)$, then we have the commutativity property

$$\nabla \times \boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v} = \Pi_{K,\ell-1}\nabla \times \boldsymbol{v}. \tag{51}$$

*Proof.* For any $p_{\ell-1} \in \mathcal{P}_{\ell-1}(K)$, we have

$$\begin{aligned}(\nabla \times \boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, p_{\ell-1})_K &= (\boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, \nabla \times p_{\ell-1})_K + \langle \boldsymbol{n} \times \boldsymbol{\Pi}_{K,\ell}^{\mathrm{curl}}\boldsymbol{v}, p_{\ell-1}\rangle_{\partial K} \\ &= (\boldsymbol{v}, \nabla \times p_{\ell-1})_K + \langle \boldsymbol{n} \times \boldsymbol{v}, p_{\ell-1}\rangle_{\partial K} \\ &= (\nabla \times \boldsymbol{v}, p_{\ell-1})_K.\end{aligned}$$

□

Following the same techniques in [46, Proposition 4.5] of 3D case, we have the following result for 2D.

**Lemma 4.22** (c.f [46, Proposition 4.5])**.** For any $\boldsymbol{v}_h \in \boldsymbol{W}_h$, there exists $\boldsymbol{\Pi}_h^{\mathrm{curl,c}}\boldsymbol{v}_h \in \boldsymbol{W}_h^\star \cap \boldsymbol{H}_0(\mathrm{curl};\Omega)$ such that

$$\|\boldsymbol{v}_h - \boldsymbol{\Pi}_h^{\mathrm{curl,c}}\boldsymbol{v}_h\|_{\mathcal{T}_h} \le C\|\mathbf{h}^{1/2}[\![\boldsymbol{n}\times\boldsymbol{v}_h]\!]\|_{\mathcal{E}_h}, \tag{52a}$$

$$\|\nabla\times(\boldsymbol{v}_h - \boldsymbol{\Pi}_h^{\mathrm{curl,c}}\boldsymbol{v}_h)\|_{\mathcal{T}_h} \le C\|\mathbf{h}^{-1/2}[\![\boldsymbol{n}\times\boldsymbol{v}_h]\!]\|_{\mathcal{E}_h}, \tag{52b}$$

where $\boldsymbol{W}_h^\star = \{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega) : \boldsymbol{u}|_K \in [\mathcal{P}_\ell(K)]^2, \forall K\in\mathcal{T}_h^\star\}$ and $\ell$ is some integer such that $\boldsymbol{W}_h \subset \boldsymbol{W}_h^\star$.

**Definition 4.23.** Suppose the solution of (3) is smooth enough. Let $Q_h^\star = H_0^1(\Omega)\cap\mathcal{P}_{\ell+1}(\mathcal{T}_h^\star)$ be a finite element space with respect to the mesh $\mathcal{T}_h^\star$ (therefore, $\nabla Q_h^\star \subset \boldsymbol{H}_0(\mathrm{curl};\Omega)\cap\boldsymbol{W}_h^*$) with $\sigma_h \in H_0^1(\Omega)\cap Q_h^\star$ satisfy

$$(\,\overline{\epsilon_r}\nabla\sigma_h, \nabla q_h)_{\mathcal{T}_h} = (\,\overline{\epsilon_r}\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{u}_h - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}), \nabla q_h)_{\mathcal{T}_h} \tag{53}$$

for all $q_h \in H_0^1(\Omega)\cap Q_h^\star$. Then we define

$$\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) = \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} + \nabla\sigma_h. \tag{54}$$

It is easy to check the following lemma using Definition 4.23 and Lemma 4.22, hence we omit the proof.

**Lemma 4.24.** Suppose the solution of (3) is smooth enough, then we have

$$\nabla\times\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) = \nabla\times\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}, \qquad [\![\boldsymbol{n}\times\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)]\!] = [\![\boldsymbol{n}\times\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}]\!], \tag{55a}$$

$$\nabla\times\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{u}_h - \boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)) = \nabla\times\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{u}_h - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}), \tag{55b}$$

$$(\,\overline{\epsilon_r}\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{u}_h - \boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)), \nabla q_h)_{\mathcal{T}_h} = 0, \quad \forall q_h \in H_0^1(\Omega)\cap Q_h^*. \tag{55c}$$

In addition, we have the following estimates:

**Lemma 4.25.** We have the following estimates:

$$\begin{aligned}
&\|\nabla\times(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h))\|_{\mathcal{T}_h} \\
&\le C\left(\|\mathbf{h}^{-1/2}(q - \Pi_V q)\|_{\mathcal{T}_h} + \|\mathbf{h}^{-1/2}(\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u})\|_{\mathcal{T}_h} + \|\mathbf{h}^{-1/2}\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}\right),
\end{aligned} \tag{56a}$$

$$\begin{aligned}
&\|(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h) - \boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)\|_{\mathcal{T}_h} \\
&\le C\left(\|\mathbf{h}^{1/2}(q - \Pi_V q)\|_{\mathcal{T}_h} + \|\mathbf{h}^{1/2}(\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u})\|_{\mathcal{T}_h} + \|\mathbf{h}^{1/2}\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}\right).
\end{aligned} \tag{56b}$$

*Proof.* We use definition Definition 4.23 and Lemma 4.22. By the definition of $\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}$ in (54) and the approximation property of $\mathbf{\Pi}_h^{\mathrm{curl,c}}$ in Lemma 4.22 to get

$$
\begin{aligned}
&\|\nabla \times (\mathbf{\Pi}_h^{\mathrm{curl,c}}(\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h))\|_{\mathcal{T}_h} \\
&\quad \le \|\nabla \times (\mathbf{\Pi}_h^{\mathrm{curl,c}}(\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)) - \nabla \times (\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)\|_{\mathcal{T}_h} \\
&\qquad + \|\nabla \times (\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)\|_{\mathcal{T}_h} \\
&\quad = \|\nabla \times (\mathbf{\Pi}_h^{\mathrm{curl,c}}(\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}_h)) - \nabla \times (\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{T}_h} && \text{by (55b)} \\
&\qquad + \|\nabla \times (\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{T}_h} && \text{by (55a)} \\
&\quad \le C\|\mathbf{h}^{-1/2}[\![\boldsymbol{n} \times (\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}_h)]\!]\|_{\mathcal{E}_h} + \|\nabla \times (\mathbf{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{T}_h} && \text{by (52b)} \\
&\quad = C\|\mathbf{h}^{-1/2}[\![\boldsymbol{n} \times \varepsilon_h^{\boldsymbol{u}}]\!]\|_{\mathcal{E}_h} + \|\nabla \times \varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.
\end{aligned}
$$

Since $\varepsilon_h^{\widehat{\boldsymbol{u}}}$ is single valued on the interior faces and zero on the boundary, then we have

$$
\begin{aligned}
&\|\nabla \times (\mathbf{\Pi}_h^{\mathrm{curl,c}}(\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h))\|_{\mathcal{T}_h} \\
&\quad \le C\|\mathbf{h}^{-1/2}[\![\boldsymbol{n} \times (\varepsilon_h^{\boldsymbol{u}} - \varepsilon_h^{\widehat{\boldsymbol{u}}})]\!]\|_{\mathcal{E}_h} + \|\nabla \times \varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.
\end{aligned}
$$

Hence, (36) and (41) give the proof of (56a).

Next, by the approximation of $\mathbf{\Pi}_h^{\mathrm{curl,c}}$ in Lemma 4.22 and (55a) to get

$$
\begin{aligned}
&\|(\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h) - \mathbf{\Pi}_h^{\mathrm{curl,c}}(\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)\|_{\mathcal{T}_h} \\
&\quad \le C\|\mathbf{h}^{1/2}[\![\boldsymbol{n} \times (\mathbf{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h) - \boldsymbol{u}_h)]\!]\|_{\mathcal{E}_h} \\
&\quad = C\|\mathbf{h}^{1/2}[\![\boldsymbol{n} \times (\varepsilon_h^{\boldsymbol{u}} - \varepsilon_h^{\widehat{\boldsymbol{u}}})]\!]\|_{\mathcal{E}_h}.
\end{aligned}
$$

Finally, (41) gives the proof of (56b). □

Next, we prove the following lemma which is similar in [3, Lemma 4.5].

**Lemma 4.26.** Let $\mathbf{\Theta} \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{H}(\mathrm{div}^0_{\overline{\epsilon_r}};\Omega)$ satisfy

$$\nabla \times \mathbf{\Theta} = \nabla \times \boldsymbol{w}_h \quad \text{in } \Omega, \tag{57}$$

where $\boldsymbol{w}_h \in \boldsymbol{H}_0(\mathrm{curl};\Omega) \cap \boldsymbol{W}_h^\star$ and $(\overline{\epsilon_r}\boldsymbol{w}_h, \nabla q_h)_\Omega = 0$ for all $q_h \in Q_h^\star$. Then we have

$$\|\boldsymbol{w}_h - \mathbf{\Theta}\|_{L^2(\Omega)} \le Ch^s\|\nabla \times \mathbf{\Theta}\|_{L^2(\Omega)}, \tag{58}$$

where $s \in (\frac{1}{2}, 1]$ is defined in Theorem 4.4. The following stability result also holds:

$$\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)}. \tag{59}$$

*Proof.* We define $\Pi_{\ell-1}|_K := \Pi_{K,\ell-1}$, then the following holds

$$\begin{aligned}
\nabla \times (\boldsymbol{w}_h - \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta}) &= \nabla \times \boldsymbol{w}_h - \Pi_{\ell-1}\nabla \times \boldsymbol{\Theta} \\
&= \nabla \times \boldsymbol{w}_h - \Pi_{\ell-1}\nabla \times \boldsymbol{w}_h \\
&= 0.
\end{aligned}$$

Thus there is a $q_h \in Q_h^\star = \mathcal{P}_{\ell+1}(\mathcal{T}_h^\star) \cap H_0^1(\Omega)$ such that $\boldsymbol{w}_h - \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta} = \nabla q_h$. By a direct calculation, one can obtain

$$\begin{aligned}
\|\boldsymbol{w}_h - \boldsymbol{\Theta}\|_{L^2(\Omega)}^2 \leq\ & C\mathrm{Re}(\epsilon_r(\boldsymbol{w}_h - \boldsymbol{\Theta}), \boldsymbol{w}_h - \boldsymbol{\Theta})_\Omega \\
=\ & C\mathrm{Re}((\epsilon_r(\boldsymbol{w}_h - \boldsymbol{\Theta}), \boldsymbol{w}_h - \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta} + \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta} - \boldsymbol{\Theta})_\Omega \\
=\ & C\mathrm{Re}((\epsilon_r(\boldsymbol{w}_h - \boldsymbol{\Theta}), \nabla q_h + \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta} - \boldsymbol{\Theta})_\Omega \\
=\ & C\mathrm{Re}((\epsilon_r(\boldsymbol{w}_h - \boldsymbol{\Theta}), \boldsymbol{\Pi}_{h,\ell}^{\mathrm{curl}}\boldsymbol{\Theta} - \boldsymbol{\Theta})_\Omega,
\end{aligned}$$

where we have used that $\boldsymbol{w}_h$ is discrete divergence free, and $\boldsymbol{\Theta}$ is divergence free. Now using Theorems 4.4 and 4.20 we get

$$\begin{aligned}
\|\boldsymbol{w}_h - \boldsymbol{\Theta}\|_{L^2(\Omega)}^2 &\leq Ch^s\|\boldsymbol{w}_h - \boldsymbol{\Theta}\|_{L^2(\Omega)}\|\boldsymbol{\Theta}\|_{\boldsymbol{H}^s(\Omega)} \\
&\leq Ch^s\|\boldsymbol{w}_h - \boldsymbol{\Theta}\|_{L^2(\Omega)}\|\nabla \times \boldsymbol{\Theta}\|_{L^2(\Omega)},
\end{aligned}$$

where $s \in (\frac{1}{2}, 1]$ is specified in Theorem 4.4.

By the Helmoltz decomposition in two dimensions, there is a $\phi \in H_0^1(\Omega)$ and $\psi \in H^1(\Omega)$ such that

$$\boldsymbol{\Theta} = \epsilon_r\nabla\phi + \nabla \times \psi, \qquad \|\phi\|_{H^1(\Omega)} \leq \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}, \qquad \|\psi\|_{H^1(\Omega)} \leq \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}.$$

Then we use the integration by parts and (57) to get

$$\begin{aligned}
\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}^2 &= (\boldsymbol{\Theta}, \boldsymbol{\Theta})_\Omega = (\epsilon_r\nabla\phi + \nabla \times \psi, \boldsymbol{\Theta})_\Omega = -(\phi, \nabla \cdot (\overline{\epsilon_r}\boldsymbol{\Theta}))_\Omega + (\psi, \nabla \times \boldsymbol{\Theta})_\Omega \\
&= (\psi, \nabla \times \boldsymbol{w}_h)_\Omega = (\nabla \times \psi, \boldsymbol{w}_h)_\Omega \leq \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Thus we obtain our result. □

**Lemma 4.27.** Let $(q,\boldsymbol{u}) \in H(\mathrm{curl};\Omega)\times \boldsymbol{H}(\mathrm{curl};\Omega)$ and $(q_h,\boldsymbol{u}_h)\in V_h\times \boldsymbol{W}_h$ be the solution of (3) and (5), respectively. Then there exists an $h_0>0$ such that for all $h\le h_0$, we have the error estimate

$$
\begin{aligned}
\|q-q_h\|_{\mathcal{T}_h} &\le C\left(\|q-\Pi_V q\|_{\mathcal{T}_h}+\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\|_{\mathcal{T}_h}\right),\\
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathcal{T}_h} &\le C\left(h^{s-1/2}\|q-\Pi_V q\|_{\mathcal{T}_h}+C\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\|_{\mathcal{T}_h}\right).
\end{aligned}
$$

*Proof.* First, let $\boldsymbol{\Theta}\in \boldsymbol{H}_0(\mathrm{curl};\Omega)\cap \boldsymbol{H}(\mathrm{div}^0_{\overline{\epsilon_r}};\Omega)$ be the solution of

$$
\nabla\times\boldsymbol{\Theta} = \nabla\times(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)) \quad \text{in } \Omega.
$$

By Lemma 4.26 and (56a), one has

$$
\begin{aligned}
&\|\boldsymbol{\Theta}-(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h))\|_{\mathcal{T}_h}\\
&\quad\le C\|\mathbf{h}^s\nabla\times(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h))\|_{\mathcal{T}_h}\\
&\quad\le Ch^{s-1/2}\left(\|q-\Pi_V q\|_{\mathcal{T}_h}+\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\|_{\mathcal{T}_h}+\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}\right).
\end{aligned} \tag{60}
$$

Therefore, by the triangle inequality, (59) and Definition 4.23 we have

$$
\begin{aligned}
\|\boldsymbol{\Theta}\|_{\mathcal{T}_h} &\le \|(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h))\|_{\mathcal{T}_h}\\
&\le 2\|\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathcal{T}_h}\le C\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.
\end{aligned} \tag{61}
$$

Next, we rewrite $\|\varepsilon_h^{\boldsymbol{u}}\|^2_{\mathcal{T}_h}$ as follows:

$$
\begin{aligned}
\|\varepsilon_h^{\boldsymbol{u}}\|^2_{\mathcal{T}_h} &\le C\mathrm{Re}(\,\overline{\epsilon_r}\varepsilon_h^{\boldsymbol{u}},\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&= C\mathrm{Re}[(\,\overline{\epsilon_r}(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)-\boldsymbol{\Theta}),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}+(\,\overline{\epsilon_r}\boldsymbol{\Theta},\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&\quad+(\,\overline{\epsilon_r}((\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)-\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&\quad+(\,\overline{\epsilon_r}(\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}]\\
&= C\mathrm{Re}[(\,\overline{\epsilon_r}(\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)-\boldsymbol{\Theta}),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}+(\,\overline{\epsilon_r}\boldsymbol{\Theta},\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&\quad+(\,\overline{\epsilon_r}((\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)-\boldsymbol{\Pi}_h^{\mathrm{curl,c}}(\boldsymbol{\Pi}_{\boldsymbol{W}}^{\mathrm{m}}(\boldsymbol{u},\boldsymbol{u}_h)-\boldsymbol{u}_h)),\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&\quad-(\,\overline{\epsilon_r}\nabla\sigma_h,\varepsilon_h^{\boldsymbol{u}})_{\mathcal{T}_h}]\\
&= S_1+S_2+S_3+S_4.
\end{aligned}
$$

The first three terms $S_1$, $S_2$ and $S_3$ have been estimated in (60), Lemma 4.19, (61), and (56b), respectively. We next estimate the last term $S_4$ by taking $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (0, \nabla\sigma_h, \nabla\sigma_h - (\boldsymbol{n}\cdot\nabla\sigma_h)\boldsymbol{n})$ in (34) to get

$$(\kappa^2\epsilon_r \boldsymbol{u}_h, \nabla\sigma_h)_{\mathcal{T}_h} = -(\boldsymbol{f}, \nabla\sigma_h)_{\mathcal{T}_h}. \tag{62}$$

Moreover, we have $-(\boldsymbol{f}, \nabla\sigma_h)_{\mathcal{T}_h} = (\kappa^2\epsilon_r \boldsymbol{u}, \nabla\sigma_h)_{\mathcal{T}_h}$, therefore

$$(\,\overline{\epsilon_r}\nabla\sigma_h, \boldsymbol{u}_h - \boldsymbol{u})_{\mathcal{T}_h} = 0.$$

This implies

$$\begin{aligned}
|S_4| &= |(\,\overline{\epsilon_r}\nabla\sigma_h, \boldsymbol{u}_h - \boldsymbol{u})_{\mathcal{T}_h} + (\,\overline{\epsilon_r}\nabla\sigma_h, \boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u})_{\mathcal{T}_h}| \\
&= |(\,\overline{\epsilon_r}\nabla\sigma_h, \boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u})_{\mathcal{T}_h}| \\
&\le C\|\boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}.
\end{aligned}$$

By the above estimations of $\{S_i\}_{i=1}^4$, there exists an $h_0 > 0$ such that for all $h \le h_0$, we have

$$\|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h} \le C\left(h^{s-1/2}\|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}\right). \tag{63}$$

By the above estimate and Lemma 4.17 we get

$$\|\varepsilon_h^q\|_{\mathcal{T}_h} \le C\left(\|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}\right).$$

Combining the above estimates with the triangle inequality gives the desired result. □

Similarly to Lemma 4.15, we have

**Lemma 4.28.** Let $(\psi, \boldsymbol{\phi}) \in H(\mathrm{curl};\Omega) \times \boldsymbol{H}_0(\mathrm{curl};\Omega)$ be the solution of (10), then for all $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h^0$, we have

$$\mathscr{B}(\Pi_V\psi, \boldsymbol{\Pi_W}\boldsymbol{\phi}, \boldsymbol{P_M}\boldsymbol{\phi}; r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (\mu_r(\Pi_V\psi - \psi) - \theta, r_h)_{\mathcal{T}_h} + (\nabla\times\psi, \boldsymbol{v}_h)_{\mathcal{T}_h}.$$

**Lemma 4.29.** Let $(q, \boldsymbol{u}) \in H(\mathrm{curl};\Omega) \times \boldsymbol{H}(\mathrm{curl};\Omega)$ and $(q_h, \boldsymbol{u}_h) \in V_h \times \boldsymbol{W}_h$ be the solution of (3) and (5), respectively. Then there exists an $h_0 > 0$ such that for all $h \le h_0$, we have the error estimate

$$\|q_h - \Pi_V q\|_{\mathcal{T}_h} \le Ch^s(\|\Pi_V q - q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W}\boldsymbol{u}\|_{\mathcal{T}_h}).$$

*Proof.* First, we take $(r_h, \boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) = (-\Pi_V \psi, \boldsymbol{\Pi_W \phi}, \boldsymbol{P_M \phi})$ in (40) we get

$$\begin{aligned} &\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; -\Pi_V \psi, \boldsymbol{\Pi_W \phi}, \boldsymbol{P_M \phi}) - (\kappa^2 \epsilon_r \varepsilon_h^{\boldsymbol{u}}, \boldsymbol{\Pi_W \phi})_{\mathcal{T}_h} \\ &\quad = -(\mu_r(\Pi_V q - q), \Pi_V \psi)_{\mathcal{T}_h} + (\kappa^2 \epsilon_r (\boldsymbol{u} - \boldsymbol{\Pi_W u}), \boldsymbol{\Pi_W \phi})_{\mathcal{T}_h}. \end{aligned} \tag{64}$$

On the other hand, using Lemma 4.13 we have

$$\begin{aligned} &\mathscr{B}(\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}}; -\Pi_V \psi, \boldsymbol{\Pi_W \phi}, \boldsymbol{P_M \phi}) \\ &\quad = \overline{\mathscr{B}(\Pi_V \psi, \boldsymbol{\Pi_W \phi}, \boldsymbol{P_M \phi}; -\varepsilon_h^q, \varepsilon_h^{\boldsymbol{u}}, \varepsilon_h^{\widehat{\boldsymbol{u}}})} \\ &\quad = -(\varepsilon_h^q, \mu_r(\Pi_V \psi - \psi) - \theta)_{\mathcal{T}_h} + (\varepsilon_h^{\boldsymbol{u}}, \nabla \times \psi)_{\mathcal{T}_h} \\ &\quad = -(\varepsilon_h^q, \mu_r(\Pi_V \psi - \psi))_{\mathcal{T}_h} + (\varepsilon_h^{\boldsymbol{u}}, \kappa^2 \overline{\epsilon_r} \boldsymbol{\phi})_{\mathcal{T}_h} + (\varepsilon_h^q, \theta)_{\mathcal{T}_h}. \end{aligned}$$

We take $\theta = \varepsilon_h^q$ in the above equation and use (64) to get

$$\begin{aligned} \|\varepsilon_h^q\|_{\mathcal{T}_h}^2 &= (\varepsilon_h^q, \mu_r(\Pi_V \psi - \psi))_{\mathcal{T}_h} - (\varepsilon_h^{\boldsymbol{u}}, \kappa^2 \overline{\epsilon_r} (\boldsymbol{\phi} - \boldsymbol{\Pi_W \phi}))_{\mathcal{T}_h} \\ &\quad - (\mu_r(\Pi_V q - q), \Pi_V \psi)_{\mathcal{T}_h} + (\kappa^2 \epsilon_r (\boldsymbol{u} - \boldsymbol{\Pi_W u}), \boldsymbol{\Pi_W \phi})_{\mathcal{T}_h} \\ &\le C h^s \|\varepsilon_h^q\|_{\mathcal{T}_h} (\|\psi\|_{H^s(\Omega)} + \|\phi\|_{\boldsymbol{H}^s(\Omega)}) + C h^s \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h} (\|\psi\|_{H^s(\Omega)} + \|\phi\|_{\boldsymbol{H}^s(\Omega)}) \\ &\quad + C h^s (\|\Pi_V q - q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h}). \end{aligned}$$

Then, there exists an $h_0 > 0$ such that for all $h \le h_0$, we have

$$\|\varepsilon_h^q\|_{\mathcal{T}_h} \le C h^s \left( \|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{u}}\|_{\mathcal{T}_h} \right).$$

By the above estimate and (63) we get

$$\|\varepsilon_h^q\|_{\mathcal{T}_h} \le C \left( \|q - \Pi_V q\|_{\mathcal{T}_h} + \|\boldsymbol{u} - \boldsymbol{\Pi_W u}\|_{\mathcal{T}_h} \right).$$

□

#### *4.4.3. Step 3: Post-processing for the vector variable*

Let $\boldsymbol{W}^\star(K)$ be a finite element space, we first define the following space:

$$Q^\star(K) = \{v : \nabla v \in \boldsymbol{W}^\star(K)\}. \tag{65}$$

The post-processing method reads: we seek $\boldsymbol{u}_h^\star \in \boldsymbol{W}^\star(K)$ such that

$$(\nabla \times \boldsymbol{u}_h^\star, \nabla \times \boldsymbol{w})_K = (q_h, \nabla \times \boldsymbol{w})_K, \qquad \text{for all } \boldsymbol{w} \in \boldsymbol{W}^\star(K), \tag{66a}$$

$$(\boldsymbol{u}_h^\star, \nabla v)_K = (\boldsymbol{u}_h, \nabla v)_K, \qquad \text{for all } v \in Q^\star(K). \tag{66b}$$

Now, we state the main result in this section.

**Lemma 4.30.** Let $(q, \boldsymbol{u})$ be the solution of (3). Then the system (66) is well-defined and there exists an $h_0 > 0$ such that for all $h \le h_0$, we have the error estimate

$$\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\|_{\mathcal{T}_h} \le C \left( \|q_h - q\|_{\mathcal{T}_h} + \inf_{\boldsymbol{w}_h \in \boldsymbol{W}^\star(\mathcal{T}_h)} \|\nabla \times (\boldsymbol{u} - \boldsymbol{w}_h)\|_{\mathcal{T}_h} \right).$$

*Proof.* Since the constraints for (66a) and (66b) are $\dim \boldsymbol{W}^\star(K) - \dim V^\star(K) + 1$ and $\dim Q^\star(K) - 1$, then (66) is a square system. Therefore, we only need to prove uniqueness for (66). Let $q_h = 0$ and $\boldsymbol{u}_h = 0$ in (66) and we take $\boldsymbol{w} = \boldsymbol{u}_h^\star$ in (66a) to get $\nabla \times \boldsymbol{u}_h^\star = 0$. Next, by the definition of $Q^\star(K)$ in (65), there is a $v \in Q^\star(K)$ such that $\nabla v = \boldsymbol{u}_h^\star$. By (66b), we get $\boldsymbol{u}_h^\star = \boldsymbol{0}$. This proves the uniqueness.

For any $\boldsymbol{w}_h \in \boldsymbol{W}^\star(K)$, we rewrite the system (66) into

$$(\nabla \times (\boldsymbol{u}_h^\star - \boldsymbol{w}_h), \nabla \times \boldsymbol{w})_K = (q_h - \nabla \times \boldsymbol{w}_h, \nabla \times \boldsymbol{w})_K \quad \forall \boldsymbol{w} \in \boldsymbol{W}^\star(K), \tag{67a}$$

$$(\boldsymbol{u}_h^\star - \boldsymbol{w}_h, \nabla v)_K = (\boldsymbol{u}_h - \boldsymbol{w}_h, \nabla v)_K \quad \forall v \in Q^\star(K). \tag{67b}$$

By (67a) te get

$$\begin{aligned} \|\nabla \times (\boldsymbol{u}_h^\star - \boldsymbol{w}_h)\|_{\mathcal{T}_h} &\le C \|q_h - \nabla \times \boldsymbol{w}_h\|_{\mathcal{T}_h} \\ &\le C \left( \|q_h - q\|_{\mathcal{T}_h} + \|\nabla \times (\boldsymbol{u} - \boldsymbol{w}_h)\|_{\mathcal{T}_h} \right). \end{aligned}$$

Using the above estimate and the triangle inequality gives the desired result. □

In practice, problem (66) is complicated to implement. The next lemma provides a simple way to do this, that is equivalent to (66).

**Lemma 4.31.** The post-processing problem (66) is equivalent to the following system: find $(\boldsymbol{u}_h^\star, \eta_h, \gamma_h) \in \boldsymbol{W}^\star(K) \times Q^\star(K) \times \mathcal{P}_0(K)$, such that

$$(\nabla \times \boldsymbol{u}_h^\star, \nabla \times \boldsymbol{w})_K + (\nabla \eta_h, \boldsymbol{w})_K = (q_h, \nabla \times \boldsymbol{w})_K \quad \forall \boldsymbol{w} \in \boldsymbol{W}^\star(K), \tag{68a}$$

$$(\boldsymbol{u}_h^\star, \nabla v)_K + (\gamma_h, v)_K = (\boldsymbol{u}_h, \nabla v)_K \quad \forall v \in Q^\star(K), \tag{68b}$$

$$(\eta_h, s)_K = 0 \quad \forall s \in \mathcal{P}_0(K). \tag{68c}$$

*Proof.* To prove this, we only need to prove (68) is well-defined and $\eta_h = \gamma_h = 0$. It is obvious to see that the system (68) is a square system, hence we only need to prove the uniqueness. We take $\boldsymbol{w} = \nabla \eta_h$, $v = \gamma_h$ and $s = 1$ in (68) to get $\nabla \eta_h = 0, \gamma_h = 0$ and $(\eta_h, 1) = 0$. Hence $\eta_h = \gamma_h = 0$. $\square$

### *4.4.4. Step 4: Post-processing for the scalar variable*

Let $V^\star(K)$ be a finite element space defined on $K$ which contains $\mathcal{P}_0(K)$. The post-processing method reads: we seek $(q_h^\star, \gamma_h) \in V^\star(K) \times \mathcal{P}_0(K)$ such that

$$(\nabla \times q_h^\star, \nabla \times w)_K + (\gamma_h, w)_K = (\boldsymbol{f} + \kappa^2 \epsilon_r \boldsymbol{u}_h, \nabla \times w_h)_K \quad \forall w \in V^\star(K), \tag{69a}$$

$$(q_h^\star, v)_K = (q_h, v)_K, \quad \forall v \in \mathcal{P}_0(K). \tag{69b}$$

**Lemma 4.32.** Let $(q, \boldsymbol{u})$ be the solution of (3). Then the system (69) is well-defined and there exists an $h_0 > 0$ such that for all $h \le h_0$, we have the error estimate

$$\begin{aligned} \|q_h^\star - q\|_{\mathcal{T}_h} &\le Ch\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathcal{T}_h} + Ch_K^s\|\Pi_V q - q\|_{\mathcal{T}_h} \\ &\quad + Ch^s(\|\boldsymbol{u} - \boldsymbol{\Pi_W} \boldsymbol{u}\|_{\mathcal{T}_h} + \|\nabla \times (r_h - q)\|_{\mathcal{T}_h}), \end{aligned}$$

where $r_h|_K \in V^\star(K)$ be the $L^2$-projection of $q$ on the element $K$.

*Proof.* First, by the triangle inequality we get

$$\|q_h^\star - r_h\|_{\mathcal{T}_h} \le \|q_h^\star - q_h + \Pi_V q - r_h\|_{\mathcal{T}_h} + \|q_h - \Pi_V q\|_{\mathcal{T}_h}.$$

Since $(q_h^\star - q_h, 1)_{\mathcal{T}_h} = 0$, then

$$(q_h^\star - q_h + \Pi_V q - r_h, 1)_{\mathcal{T}_h} = 0.$$

This implies

$$\begin{aligned} &\|q_h^\star - q_h + \Pi_V q - r_h\|_{\mathcal{T}_h} \\ &\quad = \|q_h^\star - q_h + \Pi_V q - r_h - \Pi_0(q_h^\star - q_h + \Pi_V q - r_h)\|_{\mathcal{T}_h} \\ &\quad \le Ch\|\nabla \times (q_h^\star - q_h + \Pi_V q - r_h)\|_{\mathcal{T}_h}, \end{aligned} \tag{70}$$

where $\Pi_0$ is the $L^2$ projection on $\mathcal{P}_0(K)$. Next, we apply the equation (69a) to get

$$\begin{aligned} &(\nabla \times (q_h^\star - q_h + \Pi_V q - r_h), \nabla \times w_h)_K + (\gamma_h, w_h)_{\mathcal{T}_h} \\ &\quad = (\boldsymbol{f} + \kappa^2 \epsilon_r \boldsymbol{u}_h + \nabla \times (-q_h + \Pi_V q - r_h), \nabla \times w_h)_{\mathcal{T}_h}. \end{aligned}$$

We take $w_h = q_h^\star - q_h + \Pi_V q - r_h$ in the above equation and use (69b) to get

$$
\begin{aligned}
\|\nabla \times w_h\|_{L^2(K)}^2 &= (\boldsymbol{f} + \kappa^2\epsilon_r \boldsymbol{u}_h + \nabla \times (-q_h + \Pi_V q - r_h), \nabla \times w_h)_{\mathcal{T}_h} \\
&= (\nabla \times q - \kappa^2 \epsilon_r \boldsymbol{u} + \kappa^2\epsilon_r \boldsymbol{u}_h + \nabla \times (-q_h + \Pi_V q - r_h), \nabla \times w_h)_{\mathcal{T}_h} \\
&\le C(\|\nabla \times (q - r_h)\|_{\mathcal{T}_h} + \|\nabla \times (q_h - \Pi_V q)\|_{\mathcal{T}_h} \\
&\qquad + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathcal{T}_h})\|\nabla \times w_h\|_{\mathcal{T}_h},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\|\nabla \times (q_h^\star - q_h + \Pi_V q - r_h)\|_{\mathcal{T}_h} \\
&\quad \le C(\|\nabla \times (q - r_h)\|_{\mathcal{T}_h} + \|\nabla \times (q_h - \Pi_V q)\|_{\mathcal{T}_h} + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{\mathcal{T}_h}).
\end{aligned} \tag{71}
$$

Finally, combining (70), (71), Lemma 4.29 and the triangle inequality can get our result.

□

## 5. Numerical experiments

In this section, we shall present some concrete examples of spaces $V(K)$, $\boldsymbol{W}(K)$, $\boldsymbol{M}(\partial K)$ that satisfy the definition of the $M$-decomposition; see Definition 3.1 and hence that are predicted to have optimal convergence rate and even superconvergence. The construction of the spaces can be found in [29]. In all numerical experiments, we take $\mu_r = 1$, the stabilization parameter $\tau = 1$, and $\kappa^2\epsilon_r = 10.5$. We test two kinds of problems: in the first case the exact solution is smooth and is given by

$$
u_1 = \sin(2\pi x)\sin(2\pi y), \quad u_2 = \sin(\pi x)\sin(\pi y),
$$
$$
q = \pi\cos(\pi x)\sin(\pi y) - 2\pi\sin(2\pi x)\cos(2\pi y),
$$

and in the second case the solution is less regular:

$$
u_1 = r^{\frac{2}{3}}\sin\left(\frac{2}{3}\theta\right), \quad u_2 = 0, \quad q = -\frac{2}{3}r^{-\frac{4}{3}}\left(\sin\left(\frac{2}{3}\theta\right)y + \cos\left(\frac{2}{3}\theta\right)\right).
$$

In this second case the solution is less regular and $\boldsymbol{u} = [u_1, u_2] \in \boldsymbol{H}^{5/3-\epsilon}(\Omega)$ with $\epsilon$ arbitrarily small. Boundary data is chosen so that the above functions satisfy (2)

The post-processing spaces in all experiments are taken as

$$Q^{\star}(K) = \mathcal{P}_{k+2}(K), \qquad \boldsymbol{W}^{\star}(K) = \boldsymbol{\mathcal{P}}_{k+1}(K), \qquad V^{\star}(K) = \mathcal{P}_{k+1}(K).$$

*5.1. Triangle Mesh*

We assume that the mesh $\mathcal{T}_h$ consists of shape regular triangles and choose $\mathcal{T}_h^* = \mathcal{T}_h$ (see Section 4.1). We might hope that standard $\mathcal{P}_k$ polynomial spaces could work. For any integer $k \geq 1$, let

$$V(K) = \mathcal{P}_k(K), \qquad \boldsymbol{W}(K) = \boldsymbol{\mathcal{P}}_k(K),$$

$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F = \boldsymbol{n} \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K\}.$$

In Tables 1 and 2, we show numerical results on the unit square with a uniform triangular mesh for smooth solution, i.e., case 1. We obtain an optimal convergence rate for the solution $\boldsymbol{u}$ and superconvergence rate for $\nabla \times \boldsymbol{u}$.

Table 1: Uniform triangular mesh and degree $k$ elements on the unit square $\Omega = (0,1) \times (0,1)$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\Vert \boldsymbol{u} - \boldsymbol{u}_h \Vert_{\mathcal{T}_h}$ | | $\Vert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h) \Vert_{\mathcal{T}_h}$ | | $\Vert q - q_h \Vert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 2.14e-1 | | 8.07e+0 | | 1.71e-1 | |
| | $2^4$ | 4.43e-2 | 2.27 | 3.53e+0 | 1.19 | 3.58e-2 | 2.26 |
| | $2^5$ | 1.03e-2 | 2.10 | 1.67e+0 | 1.08 | 8.46e-3 | 2.08 |
| | $2^6$ | 2.51e-3 | 2.04 | 8.10e-1 | 1.04 | 2.09e-3 | 2.02 |
| | $2^7$ | 6.18e-4 | 2.02 | 4.00e-1 | 1.02 | 5.20e-4 | 2.00 |
| 2 | $2^3$ | 2.20e-2 | | 1.47e+0 | | 1.46e-2 | |
| | $2^4$ | 2.51e-3 | 3.13 | 3.40e-1 | 2.12 | 1.81e-3 | 3.02 |
| | $2^5$ | 3.00e-4 | 3.06 | 8.15e-2 | 2.06 | 2.26e-4 | 3.00 |
| | $2^6$ | 3.67e-5 | 3.03 | 2.00e-2 | 2.03 | 2.82e-5 | 3.00 |
| | $2^7$ | 4.54e-6 | 3.02 | 4.94e-3 | 2.02 | 3.52e-6 | 3.00 |

In Tables 3 and 4, we show numerical results on the unit square with a uniform triangular mesh for the non-smooth solution, i.e., case 2. The variable $q_h$ and the post-processed approximation $\nabla \times \boldsymbol{u}_h^{\star}$ converge at the optimal with

Table 2: Uniform triangular mesh and degree $k$ elements on the unit square $\Omega = (0,1) \times (0,1)$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\|\boldsymbol{u} - \boldsymbol{u}_h^\star\|_{\mathcal{T}_h}$ | | $\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\|_{\mathcal{T}_h}$ | | $\|q - q_h^\star\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.80e-1 | | 1.71e-1 | | 1.17e-1 | |
| | $2^4$ | 3.64e-2 | 2.31 | 3.58e-2 | 2.26 | 1.44e-2 | 3.02 |
| | $2^5$ | 8.44e-3 | 2.11 | 8.46e-3 | 2.08 | 1.76e-3 | 3.03 |
| | $2^6$ | 2.04e-3 | 2.05 | 2.09e-3 | 2.02 | 2.17e-4 | 3.02 |
| | $2^7$ | 5.03e-4 | 2.02 | 5.20e-4 | 2.00 | 2.69e-5 | 3.01 |
| 2 | $2^3$ | 1.80e-2 | | 1.46e-2 | | 4.38e-3 | |
| | $2^4$ | 2.04e-3 | 3.14 | 1.81e-3 | 3.02 | 2.46e-4 | 4.15 |
| | $2^5$ | 2.44e-4 | 3.07 | 2.26e-4 | 3.00 | 1.47e-5 | 4.06 |
| | $2^6$ | 2.98e-5 | 3.03 | 2.82e-5 | 3.00 | 9.04e-7 | 4.03 |
| | $2^7$ | 3.68e-6 | 3.02 | 3.52e-6 | 3.00 | 5.60e-8 | 4.01 |

respect to the regularity of the exact solution. However, the variable $\boldsymbol{u}_h$ is not optimally convergent. This is also as predicted by by our numerical analysis, see Theorems 4.6 and 4.7.

### *5.2. Parallelogram Mesh*

The mesh $\mathcal{T}_h$ is assumed to consist of parallelograms. For this mesh we construct $\mathcal{T}_h^*$ by subdividing each parallelogram into two subtriangles. The triangular mesh is assumed to be shape regular so satisfying the requirements from Section 4.1.

For any integer $k \geq 0$, let

$$V(K) = \mathcal{P}_k(K), \quad \boldsymbol{W}(K) = \boldsymbol{\mathcal{P}}_k(K) + \nabla \mathrm{span}\{x^{k+1}y, xy^{k+1}\},$$
$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F = \boldsymbol{n} \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K\},$$
$$\widetilde{V}(K) = \mathcal{P}_{k-1}(K), \qquad \widetilde{\boldsymbol{W}}(K) = \nabla \times V(K) \oplus \boldsymbol{W}_0(K).$$

Then $V(K)$ and $\boldsymbol{W}(K)$ admit an $M$-decomposition with respect the spaces $\widetilde{V}(K)$ and $\widetilde{\boldsymbol{W}}(K)$.

Table 3: Uniform triangular mesh and degree $k$ elements on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_{\mathcal{T}_h}$ | | $\\|\nabla\times(\boldsymbol{u}-\boldsymbol{u}_h)\\|_{\mathcal{T}_h}$ | | $\\|q-q_h\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 4.59e-1 | 6.08 | 2.86e+0 | 6.08 | 1.32e+0 | 5.72 |
| | $2^4$ | 5.33e-1 | -0.22 | 2.57e+0 | 0.15 | 1.58e+0 | -0.26 |
| | $2^5$ | 3.29e-1 | 0.70 | 1.63e+0 | 0.66 | 1.06e+0 | 0.57 |
| | $2^6$ | 2.48e-1 | 0.41 | 1.59e+0 | 0.03 | 8.06e-1 | 0.40 |
| | $2^7$ | 1.73e-1 | 0.52 | 1.52e+0 | 0.06 | 5.61e-1 | 0.52 |
| | $2^8$ | 1.14e-1 | 0.60 | 1.76e+0 | -0.21 | 3.70e-1 | 0.60 |
| | $2^9$ | 7.43e-2 | 0.62 | 2.15e+0 | -0.29 | 2.37e-1 | 0.64 |
| 2 | $2^3$ | 3.09e+0 | -1.29 | 1.56e+1 | -0.94 | 5.80e+0 | -0.69 |
| | $2^4$ | 3.75e-1 | 3.04 | 1.77e+0 | 3.14 | 1.21e+0 | 2.26 |
| | $2^5$ | 3.16e-1 | 0.25 | 1.75e+0 | 0.01 | 1.03e+0 | 0.23 |
| | $2^6$ | 2.45e-1 | 0.37 | 1.82e+0 | -0.05 | 7.97e-1 | 0.37 |
| | $2^7$ | 1.72e-1 | 0.51 | 2.08e+0 | -0.19 | 5.58e-1 | 0.51 |
| | $2^8$ | 1.15e-1 | 0.58 | 2.51e+0 | -0.27 | 3.69e-1 | 0.60 |
| | $2^9$ | 7.73e-2 | 0.58 | 3.11e+0 | -0.31 | 2.37e-1 | 0.64 |

Now, we give another construction: For any integer $k \geq 0$, let

$$V(K) = \mathcal{P}_k(K), \quad \boldsymbol{W}(K) = \boldsymbol{\mathcal{P}}_k(K) + \nabla \mathrm{span}\{x^{k+1}y, xy^{k+1}\} + \binom{y}{-x}\widetilde{\mathcal{P}}_k(K),$$
$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F = \boldsymbol{n} \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K\},$$
$$\widetilde{\boldsymbol{W}}(K) = \nabla \times V(K) \oplus \boldsymbol{W}_0(K).$$

Then $V(K)$ and $\boldsymbol{W}(K)$ admit an $M$-decomposition with respect the spaces $\widetilde{V}(K)$ and $\widetilde{\boldsymbol{W}}(K)$.

In Tables 5 to 8, we show numerical results on a parallelogram with a uniform parallelogram mesh. We obtain the optimal convergence rate for the solution $\boldsymbol{u}$ and superconvergence rate for $\nabla \times \boldsymbol{u}$.

### 5.3. Rectangle Mesh

The mesh $\mathcal{T}_h$ is assumed to consist of squares. For this mesh we construct $\mathcal{T}_h^*$

Table 4: Uniform triangular mesh and degree $k$ elements on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\lVert \boldsymbol{u} - \boldsymbol{u}_h^\star \rVert_{\mathcal{T}_h}$ | | $\lVert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star) \rVert_{\mathcal{T}_h}$ | | $\lVert q - q_h^\star \rVert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 4.53e-1 | 6.01 | 1.32e+0 | 5.72 | 1.32e+0 | 5.76 |
| | $2^4$ | 5.32e-1 | -0.23 | 1.58e+0 | -0.26 | 1.58e+0 | -0.26 |
| | $2^5$ | 3.29e-1 | 0.69 | 1.06e+0 | 0.57 | 1.06e+0 | 0.57 |
| | $2^6$ | 2.48e-1 | 0.40 | 8.06e-1 | 0.40 | 8.06e-1 | 0.40 |
| | $2^7$ | 1.73e-1 | 0.52 | 5.61e-1 | 0.52 | 5.61e-1 | 0.52 |
| | $2^8$ | 1.14e-1 | 0.60 | 3.70e-1 | 0.60 | 3.70e-1 | 0.60 |
| | $2^9$ | 7.43e-2 | 0.62 | 2.37e-1 | 0.64 | 2.37e-1 | 0.64 |
| 2 | $2^3$ | 3.07e+0 | -1.31 | 5.80e+0 | -0.69 | 5.79e+0 | -0.69 |
| | $2^4$ | 3.75e-1 | 3.04 | 1.21e+0 | 2.26 | 1.21e+0 | 2.26 |
| | $2^5$ | 3.15e-1 | 0.25 | 1.03e+0 | 0.23 | 1.03e+0 | 0.23 |
| | $2^6$ | 2.45e-1 | 0.37 | 7.97e-1 | 0.37 | 7.97e-1 | 0.37 |
| | $2^7$ | 1.72e-1 | 0.51 | 5.58e-1 | 0.51 | 5.58e-1 | 0.51 |
| | $2^8$ | 1.15e-1 | 0.58 | 3.69e-1 | 0.60 | 3.69e-1 | 0.60 |
| | $2^9$ | 7.73e-2 | 0.58 | 2.37e-1 | 0.64 | 2.37e-1 | 0.64 |

by subdividing each square into two subtriangles. The triangular mesh is shape regular so satisfying the requirements from Section 4.1 (for a general rectangular mesh, the triangular mesh must be shape regular).

In this section, we assume that all elements $K$ are rectangles with edges parallel to the coordinate axes. We denote by $\mathcal{Q}_k$ the standard space of polynomials in two variables with maximum degree $k$ in each variable. Unlike in the parallelogram case, we consider the use of $\mathcal{Q}_k$ based elements as these are often used for square elements. Our first lemma shows that simple $\mathcal{Q}_k$ elements alone do not suffice.

Table 5: Parallelogram mesh and enriched case I on $\Omega = \{(x, y) : 0 \leq x - \sqrt{3}y \leq 1, 0 \leq y \leq 1/2\}$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\mathcal{T}_h}$ | | $\lVert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h) \rVert_{\mathcal{T}_h}$ | | $\lVert q - q_h \rVert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.30e+1 | | 1.15e+2 | | 4.35e+0 | |
| | $2^4$ | 1.57e-1 | 6.37 | 5.11e+0 | 4.49 | 6.49e-2 | 6.07 |
| | $2^5$ | 1.48e-2 | 3.41 | 2.11e+0 | 1.28 | 1.08e-2 | 2.59 |
| | $2^6$ | 2.52e-3 | 2.55 | 1.02e+0 | 1.04 | 2.37e-3 | 2.19 |
| | $2^7$ | 5.23e-4 | 2.27 | 5.07e-1 | 1.01 | 5.71e-4 | 2.05 |
| 2 | $2^3$ | 7.19e-1 | | 9.18e+0 | | 1.40e-1 | |
| | $2^4$ | 1.46e-2 | 5.62 | 8.16e-1 | 3.49 | 2.68e-3 | 5.71 |
| | $2^5$ | 1.35e-3 | 3.44 | 1.66e-1 | 2.29 | 3.23e-4 | 3.05 |
| | $2^6$ | 1.49e-4 | 3.18 | 3.85e-2 | 2.11 | 4.03e-5 | 3.00 |
| | $2^7$ | 1.76e-5 | 3.08 | 9.29e-3 | 2.05 | 5.04e-6 | 3.00 |

Table 6: Parallelogram mesh and enriched case I on $\Omega = \{(x, y) : 0 \leq x - \sqrt{3}y \leq 1, 0 \leq y \leq 1/2\}$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\lVert \boldsymbol{u} - \boldsymbol{u}_h^\star \rVert_{\mathcal{T}_h}$ | | $\lVert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star) \rVert_{\mathcal{T}_h}$ | | $\lVert q - q_h^\star \rVert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.29e+1 | | 4.35e+0 | | 4.35e+0 | |
| | $2^4$ | 1.54e-1 | 6.39 | 6.49e-2 | 6.07 | 5.58e-2 | 6.28 |
| | $2^5$ | 1.32e-2 | 3.55 | 1.08e-2 | 2.59 | 5.97e-3 | 3.22 |
| | $2^6$ | 1.93e-3 | 2.77 | 2.37e-3 | 2.19 | 7.36e-4 | 3.02 |
| | $2^7$ | 3.34e-4 | 2.53 | 5.71e-4 | 2.05 | 9.18e-5 | 3.00 |
| 2 | $2^3$ | 7.15e-1 | | 1.40e-1 | | 1.13e-1 | |
| | $2^4$ | 1.43e-2 | 5.65 | 2.68e-3 | 5.71 | 1.54e-3 | 6.19 |
| | $2^5$ | 1.31e-3 | 3.45 | 3.23e-4 | 3.05 | 6.75e-5 | 4.51 |
| | $2^6$ | 1.44e-4 | 3.18 | 4.03e-5 | 3.00 | 3.64e-6 | 4.21 |
| | $2^7$ | 1.70e-5 | 3.09 | 5.04e-6 | 3.00 | 2.13e-7 | 4.09 |

Table 7: Parallelogram mesh and enriched case II on $\Omega = \{(x, y) : 0 \le x - \sqrt{3}y \le 1, 0 \le y \le 1/2\}$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{T}_h}$ | | $\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{T}_h}$ | | $\|q - q_h\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.20e+1 | | 1.75e+2 | | 3.61e+0 | |
| | $2^4$ | 1.60e-1 | 6.23 | 5.18e+0 | 5.08 | 6.52e-2 | 5.79 |
| | $2^5$ | 1.47e-2 | 3.44 | 2.11e+0 | 1.30 | 1.08e-2 | 2.59 |
| | $2^6$ | 2.51e-3 | 2.55 | 1.02e+0 | 1.05 | 2.37e-3 | 2.19 |
| | $2^7$ | 5.22e-4 | 2.27 | 5.05e-1 | 1.01 | 5.71e-4 | 2.05 |
| 2 | $2^3$ | 7.69e+0 | | 1.35e+2 | | 1.04e+0 | |
| | $2^4$ | 1.46e-2 | 9.04 | 8.17e-1 | 7.37 | 2.66e-3 | 8.61 |
| | $2^5$ | 1.35e-3 | 3.44 | 1.66e-1 | 2.30 | 3.22e-4 | 3.04 |
| | $2^6$ | 1.49e-4 | 3.18 | 3.84e-2 | 2.11 | 4.02e-5 | 3.00 |
| | $2^7$ | 1.76e-5 | 3.08 | 9.27e-3 | 2.05 | 5.03e-6 | 3.00 |

Table 8: Parallelogram mesh and enriched case II on $\Omega = \{(x, y) : 0 \le x - \sqrt{3}y \le 1, 0 \le y \le 1/2\}$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\|\boldsymbol{u} - \boldsymbol{u}_h^\star\|_{\mathcal{T}_h}$ | | $\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\|_{\mathcal{T}_h}$ | | $\|q - q_h^\star\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.19e+1 | | 3.61e+0 | | 3.85e+0 | |
| | $2^4$ | 1.56e-1 | 6.24 | 6.52e-2 | 5.79 | 5.60e-2 | 6.10 |
| | $2^5$ | 1.31e-2 | 3.58 | 1.08e-2 | 2.59 | 5.98e-3 | 3.23 |
| | $2^6$ | 1.91e-3 | 2.78 | 2.37e-3 | 2.19 | 7.37e-4 | 3.02 |
| | $2^7$ | 3.32e-4 | 2.53 | 5.71e-4 | 2.05 | 9.19e-5 | 3.00 |
| 2 | $2^3$ | 7.61e+0 | | 1.04e+0 | | 6.74e-1 | |
| | $2^4$ | 1.43e-2 | 9.06 | 2.66e-3 | 8.61 | 1.54e-3 | 8.77 |
| | $2^5$ | 1.31e-3 | 3.45 | 3.22e-4 | 3.04 | 6.75e-5 | 4.51 |
| | $2^6$ | 1.44e-4 | 3.18 | 4.02e-5 | 3.00 | 3.64e-6 | 4.21 |
| | $2^7$ | 1.70e-5 | 3.09 | 5.03e-6 | 3.00 | 2.13e-7 | 4.09 |

For any integer $k \geq 1$, let

$$V(K) = \mathcal{Q}_k(K), \quad \boldsymbol{W}(K) = \boldsymbol{\mathcal{Q}}_k(K) + \nabla \text{span}\{x^{k+1}y, xy^{k+1}\},$$
$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F = \boldsymbol{n} \times p_k, \text{ for some } p_k \in \mathcal{P}_k(F) \text{ and for each edge } F \subset \partial K\},$$
$$\widetilde{V}(K) = \mathcal{Q}_{k-1}(K), \qquad \widetilde{\boldsymbol{W}}(K) = \nabla \times V(K) \oplus \boldsymbol{W}_0(K).$$

Then $V(K)$ and $\boldsymbol{W}(K)$ admit an $M$-decomposition with respect the spaces $\widetilde{V}(K)$ and $\widetilde{\boldsymbol{W}}(K)$.

Now, we give another construction: For any integer $k \geq 0$, let

$$V(K) = \mathcal{Q}_k(K), \quad \boldsymbol{W}(K) = \boldsymbol{\mathcal{Q}}_k(K) + \nabla \text{span}\{x^{k+1}y, xy^{k+1}\} + \text{span}\left\{\begin{pmatrix} x^k y^{k+1} \\ x^{k+1} y^k \end{pmatrix}\right\},$$
$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} \mid \boldsymbol{\mu}|_F = \boldsymbol{n} \times \mathcal{P}_k(F) \text{ for each edge } F \subset \partial K\},$$
$$\widetilde{V}(K) = \mathcal{Q}_k(K), \qquad \widetilde{\boldsymbol{W}}(K) = \nabla \times V(K) \oplus \boldsymbol{W}_0(K).$$

Then $V(K)$ and $\boldsymbol{W}(K)$ admit an $M$-decomposition with respect the spaces $\widetilde{V}(K)$ and $\widetilde{\boldsymbol{W}}(K)$.

The next element family can be found in the third exact sequence in [47, Theorem 3.1].

For any integer $k \geq 0$, let

$$V(K) = \mathcal{Q}_k(K), \quad \boldsymbol{W}(K) = \boldsymbol{\mathcal{Q}}_k(K) + \nabla \text{span}\{x^{k+1}y, xy^{k+1}\} + \text{span}\left\{\begin{pmatrix} -x^k y^{k+1} \\ x^{k+1} y^k \end{pmatrix}\right\},$$
$$\boldsymbol{M}(\partial K) = \{\boldsymbol{\mu} \mid \boldsymbol{\mu}|_F = \boldsymbol{n} \times \mathcal{P}_k(F) \text{ for each edge } F \subset \partial K\},$$
$$\widetilde{V}(K) = \mathcal{Q}_k(K), \qquad \widetilde{\boldsymbol{W}}(K) = \nabla \times V(K) \oplus \boldsymbol{W}_0(K).$$

Then $V(K)$ and $\boldsymbol{W}(K)$ admit an $M$-decomposition with respect the spaces $\widetilde{V}(K)$ and $\widetilde{\boldsymbol{W}}(K)$.

In Tables 9 and 10, we show the numerical results on unit square with rectangle mesh and we obtain optimal convergence rate for the solution $\boldsymbol{u}$ and superconvergence rate for $\nabla \times \boldsymbol{u}$ using Enrichment Construction I elements when the solution is smooth enough. In Tables 11 and 12, we show the numerical results on unit square with rectangle mesh and we obtain optimal convergence rate for both $q$ and the postprocess of $\boldsymbol{\nabla} \times \boldsymbol{u}$ using Enrichment Construction I elements when the solution is non-smooth.

Table 9: Uniform square mesh with Enrichment case I on the unit square $\Omega = (0,1) \times (0,1)$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathcal{T}_h}$ | | $\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mathcal{T}_h}$ | | $\|q - q_h\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.78e-1 | | 6.93e+0 | | 1.41e-1 | |
| | $2^4$ | 3.90e-2 | 2.19 | 3.05e+0 | 1.19 | 2.88e-2 | 2.29 |
| | $2^5$ | 9.19e-3 | 2.08 | 1.44e+0 | 1.09 | 6.83e-3 | 2.08 |
| | $2^6$ | 2.23e-3 | 2.04 | 6.98e-1 | 1.04 | 1.68e-3 | 2.02 |
| | $2^7$ | 5.51e-4 | 2.02 | 3.44e-1 | 1.02 | 4.20e-4 | 2.00 |
| 2 | $2^3$ | 1.86e-2 | | 8.44e-1 | | 8.50e-3 | |
| | $2^4$ | 1.10e-3 | 4.08 | 1.37e-1 | 2.62 | 8.76e-4 | 3.28 |
| | $2^5$ | 1.34e-4 | 3.05 | 3.31e-2 | 2.05 | 1.09e-4 | 3.00 |
| | $2^6$ | 1.65e-5 | 3.02 | 8.13e-3 | 2.02 | 1.36e-5 | 3.00 |
| | $2^7$ | 2.04e-6 | 3.01 | 2.02e-3 | 2.01 | 1.70e-6 | 3.00 |

Numerical results for Enrichment Construction II elements show that exactly the same error as for Enrichment Case I and so we do not reproduce them here. Results for Enrichment Construction III are shown in Tables 13 to 16. The results exhibit the expected convergence rates from theory.

## 6. Conclusion

In this paper we have shown that the $M$-decomposition, together with sufficiently rich auxilary spaces, is sufficient to guarantee optimal order convergence for the vector 2D problem arising from Maxwell's equations. This can be used to evaluate and construct HDG schemes on two commonly occurring elements (triangles and squares).

As pointed out by Cockburn and Fu [29], it is not possible to carry out the construction of the spaces under consideration by using only polynomials for more general elements $K$. Thus the extension of this theory to more general elements is a challenging project.

An interesting problem is to devise a similar theory for the full Maxwell's

Table 10: Uniform square mesh with Enrichment case I on the unit square $\Omega = (0,1) \times (0,1)$ for smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u} - \boldsymbol{u}_h^\star\\|_{\mathcal{T}_h}$ | | $\\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\\|_{\mathcal{T}_h}$ | | $\\|q - q_h^\star\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 4.09e-2 | | 2.17e-1 | | 1.00e-1 | |
| | $2^4$ | 5.38e-3 | 2.93 | 5.11e-2 | 2.09 | 1.16e-2 | 3.11 |
| | $2^5$ | 9.12e-4 | 2.56 | 1.26e-2 | 2.02 | 1.39e-3 | 3.07 |
| | $2^6$ | 1.96e-4 | 2.22 | 3.14e-3 | 2.00 | 1.70e-4 | 3.03 |
| | $2^7$ | 4.70e-5 | 2.06 | 7.85e-4 | 2.00 | 2.10e-5 | 3.02 |
| 2 | $2^3$ | 1.37e-2 | | 2.44e-2 | | 3.41e-3 | |
| | $2^4$ | 4.76e-4 | 4.84 | 3.09e-3 | 2.98 | 2.07e-4 | 4.04 |
| | $2^5$ | 5.64e-5 | 3.08 | 3.88e-4 | 2.99 | 1.30e-5 | 3.99 |
| | $2^6$ | 6.90e-6 | 3.03 | 4.85e-5 | 3.00 | 8.16e-7 | 4.00 |
| | $2^7$ | 8.53e-7 | 3.02 | 6.07e-6 | 3.00 | 5.10e-8 | 4.00 |

Table 11: Uniform square mesh with Enrichment case I on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u} - \boldsymbol{u}_h\\|_{\mathcal{T}_h}$ | | $\\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\\|_{\mathcal{T}_h}$ | | $\\|q - q_h\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 8.98e-1 | 1.77 | 1.91e+0 | 3.91 | 7.61e-1 | 2.25 |
| | $2^4$ | 4.53e-1 | 0.99 | 1.91e+0 | -0.00 | 1.38e+0 | -0.86 |
| | $2^5$ | 3.20e-1 | 0.50 | 1.36e+0 | 0.49 | 1.04e+0 | 0.41 |
| | $2^6$ | 2.45e-1 | 0.38 | 1.18e+0 | 0.21 | 7.98e-1 | 0.38 |
| | $2^7$ | 1.75e-1 | 0.49 | 1.78e+1 | -3.92 | 5.60e-1 | 0.51 |
| | $2^8$ | 1.14e-1 | 0.62 | 1.29e+0 | 3.79 | 3.69e-1 | 0.60 |
| 2 | $2^3$ | 1.27e+0 | 0.13 | 2.93e+1 | -2.26 | 1.81e+0 | 1.27 |
| | $2^4$ | 5.91e-1 | 1.10 | 1.55e+1 | 0.92 | 1.28e+0 | 0.50 |
| | $2^5$ | 3.19e-1 | 0.89 | 1.84e+0 | 3.07 | 1.04e+0 | 0.31 |
| | $2^6$ | 2.69e-1 | 0.24 | 1.40e+1 | -2.93 | 7.97e-1 | 0.38 |
| | $2^7$ | 2.33e-1 | 0.21 | 4.06e+1 | -1.53 | 5.58e-1 | 0.51 |
| | $2^8$ | 1.16e-1 | 1.01 | 2.00e+0 | 4.35 | 3.69e-1 | 0.60 |

Table 12: Uniform square mesh with Enrichment case I on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\lVert \boldsymbol{u} - \boldsymbol{u}_h^\star \rVert_{\mathcal{T}_h}$ | | $\lVert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star) \rVert_{\mathcal{T}_h}$ | | $\lVert q - q_h^\star \rVert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 8.95e-1 | 0.22 | 7.57e-1 | 2.26 | 7.56e-1 | 2.45 |
| | $2^4$ | 4.52e-1 | 0.99 | 1.38e+0 | -0.87 | 1.38e+0 | -0.87 |
| | $2^5$ | 3.20e-1 | 0.50 | 1.04e+0 | 0.41 | 1.04e+0 | 0.41 |
| | $2^6$ | 2.45e-1 | 0.38 | 7.98e-1 | 0.38 | 7.98e-1 | 0.38 |
| | $2^7$ | 1.72e-1 | 0.51 | 5.60e-1 | 0.51 | 5.59e-1 | 0.52 |
| | $2^8$ | 1.14e-1 | 0.59 | 3.69e-1 | 0.60 | 3.69e-1 | 0.60 |
| 2 | $2^3$ | 8.56e-1 | 0.67 | 1.77e+0 | 1.30 | 1.78e+0 | 1.30 |
| | $2^4$ | 5.37e-1 | 0.67 | 1.27e+0 | 0.48 | 1.27e+0 | 0.48 |
| | $2^5$ | 3.19e-1 | 0.75 | 1.04e+0 | 0.30 | 1.04e+0 | 0.30 |
| | $2^6$ | 2.64e-1 | 0.27 | 7.97e-1 | 0.38 | 7.97e-1 | 0.38 |
| | $2^7$ | 2.19e-1 | 0.27 | 5.57e-1 | 0.52 | 5.57e-1 | 0.52 |
| | $2^8$ | 1.16e-1 | 0.92 | 3.69e-1 | 0.60 | 3.69e-1 | 0.60 |

Table 13: Results for a uniform square mesh with Enrichment case III on the unit square $\Omega = (0,1) \times (0,1)$

| $k$ | $\frac{\sqrt{2}}{h}$ | $\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\mathcal{T}_h}$ | | $\lVert \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h) \rVert_{\mathcal{T}_h}$ | | $\lVert q - q_h \rVert_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.77e-1 | | 6.93e+0 | | 1.41e-1 | |
| | $2^4$ | 3.90e-2 | 2.19 | 3.05e+0 | 1.19 | 2.88e-2 | 2.29 |
| | $2^5$ | 9.19e-3 | 2.08 | 1.44e+0 | 1.09 | 6.83e-3 | 2.08 |
| | $2^6$ | 2.23e-3 | 2.04 | 6.98e-1 | 1.04 | 1.68e-3 | 2.02 |
| | $2^7$ | 5.51e-4 | 2.02 | 3.44e-1 | 1.02 | 4.20e-4 | 2.00 |
| 2 | $2^3$ | 9.68e-3 | | 5.93e-1 | | 7.14e-3 | |
| | $2^4$ | 1.10e-3 | 3.13 | 1.37e-1 | 2.11 | 8.76e-4 | 3.03 |
| | $2^5$ | 1.34e-4 | 3.05 | 3.31e-2 | 2.05 | 1.09e-4 | 3.00 |
| | $2^6$ | 1.65e-5 | 3.02 | 8.13e-3 | 2.02 | 1.36e-5 | 3.00 |
| | $2^7$ | 2.04e-6 | 3.01 | 2.02e-3 | 2.01 | 1.70e-6 | 3.00 |

Table 14: Results for a uniform square mesh with Enrichment case III on the unit square $\Omega = (0,1) \times (0,1)$

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u} - \boldsymbol{u}_h^\star\\|_{\mathcal{T}_h}$ | | $\\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\\|_{\mathcal{T}_h}$ | | $\\|q - q_h^\star\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 4.06e-2 | | 2.17e-1 | | 1.00e-1 | |
| | $2^4$ | 5.37e-3 | 2.92 | 5.11e-2 | 2.09 | 1.16e-2 | 3.11 |
| | $2^5$ | 9.12e-4 | 2.56 | 1.26e-2 | 2.02 | 1.39e-3 | 3.07 |
| | $2^6$ | 1.96e-4 | 2.22 | 3.14e-3 | 2.00 | 1.70e-4 | 3.03 |
| | $2^7$ | 4.70e-5 | 2.06 | 7.85e-4 | 2.00 | 2.10e-5 | 3.02 |
| 2 | $2^3$ | 4.54e-3 | | 2.44e-2 | | 3.24e-3 | |
| | $2^4$ | 4.64e-4 | 3.29 | 3.09e-3 | 2.98 | 2.07e-4 | 3.96 |
| | $2^5$ | 5.60e-5 | 3.05 | 3.88e-4 | 2.99 | 1.30e-5 | 3.99 |
| | $2^6$ | 6.88e-6 | 3.03 | 4.85e-5 | 3.00 | 8.16e-7 | 4.00 |
| | $2^7$ | 8.52e-7 | 3.01 | 6.07e-6 | 3.00 | 5.10e-8 | 4.00 |

Table 15: Uniform square mesh with Enrichment case III on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u} - \boldsymbol{u}_h\\|_{\mathcal{T}_h}$ | | $\\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\\|_{\mathcal{T}_h}$ | | $\\|q - q_h\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.07e+0 | 1.24 | 3.93e+0 | 2.53 | 1.89e+0 | 0.59 |
| | $2^4$ | 3.89e-1 | 1.46 | 1.73e+0 | 1.18 | 1.25e+0 | 0.61 |
| | $2^5$ | 3.14e-1 | 0.31 | 1.34e+0 | 0.36 | 1.02e+0 | 0.29 |
| | $2^6$ | 2.43e-1 | 0.37 | 1.17e+0 | 0.20 | 7.90e-1 | 0.37 |
| | $2^7$ | 1.71e-1 | 0.51 | 1.15e+0 | 0.03 | 5.55e-1 | 0.51 |
| | $2^8$ | 1.27e-1 | 0.42 | 2.65e+0 | -1.21 | 4.09e-1 | 0.44 |
| 2 | $2^3$ | 2.30e-1 | 2.56 | 1.67e+0 | 1.85 | 5.41e-1 | 2.99 |
| | $2^4$ | 4.67e+0 | -4.34 | 1.62e+2 | -6.61 | 1.99e+0 | -1.88 |
| | $2^5$ | 3.17e-1 | 3.88 | 1.58e+0 | 6.68 | 1.03e+0 | 0.95 |
| | $2^6$ | 5.81e-1 | -0.87 | 1.86e+1 | -3.55 | 7.91e-1 | 0.38 |
| | $2^7$ | 1.72e-1 | 1.75 | 1.64e+0 | 3.50 | 5.57e-1 | 0.51 |
| | $2^8$ | 1.16e-1 | 0.57 | 1.96e+0 | -0.26 | 3.68e-1 | 0.60 |

Table 16: Uniform square mesh with Enrichment case III on the unit square $\Omega = (0,1) \times (0,1)$ for non-smooth test

| $k$ | $\frac{\sqrt{2}}{h}$ | $\\|\boldsymbol{u} - \boldsymbol{u}_h^\star\\|_{\mathcal{T}_h}$ | | $\\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h^\star)\\|_{\mathcal{T}_h}$ | | $\\|q - q_h^\star\\|_{\mathcal{T}_h}$ | |
|---|---|---|---|---|---|---|---|
| | | Error | Rate | Error | Rate | Error | Rate |
| 1 | $2^3$ | 1.02e+0 | 0.11 | 1.88e+0 | 0.59 | 1.89e+0 | 0.80 |
| | $2^4$ | 3.87e-1 | 1.40 | 1.25e+0 | 0.59 | 1.25e+0 | 0.60 |
| | $2^5$ | 3.14e-1 | 0.30 | 1.02e+0 | 0.29 | 1.02e+0 | 0.29 |
| | $2^6$ | 2.42e-1 | 0.37 | 7.90e-1 | 0.37 | 7.90e-1 | 0.37 |
| | $2^7$ | 1.71e-1 | 0.51 | 5.55e-1 | 0.51 | 5.55e-1 | 0.51 |
| | $2^8$ | 1.27e-1 | 0.43 | 4.09e-1 | 0.44 | 4.09e-1 | 0.44 |
| 2 | $2^3$ | 2.21e-1 | 2.60 | 5.41e-1 | 2.99 | 5.42e-1 | 2.99 |
| | $2^4$ | 3.85e+0 | -4.12 | 1.40e+0 | -1.37 | 1.39e+0 | -1.36 |
| | $2^5$ | 3.17e-1 | 3.61 | 1.03e+0 | 0.44 | 1.03e+0 | 0.44 |
| | $2^6$ | 5.66e-1 | -0.84 | 7.91e-1 | 0.38 | 7.91e-1 | 0.38 |
| | $2^7$ | 1.72e-1 | 1.72 | 5.57e-1 | 0.51 | 5.57e-1 | 0.51 |
| | $2^8$ | 1.16e-1 | 0.57 | 3.68e-1 | 0.60 | 3.68e-1 | 0.60 |

equations in three dimensions. Not only is this more complicated, but it is also essentially different compared to 2D. This will be explored in our future work.


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## References


[1] J.-C. Nédélec, A new family of mixed finite elements in $\mathbf{R}^3$, Numer. Math. 50 (1) (1986) 57–81. doi:10.1007/BF01389668.

[2] J.-C. Nédélec, Mixed finite elements in $\mathbf{R}^3$, Numer. Math. 35 (3) (1980) 315–341. doi:10.1007/BF01396415.

[3] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer. 11 (2002) 237–339. doi:10.1017/S0962492902000041.

[4] P. Monk, Finite element methods for Maxwell's equations, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003. doi:10.1093/acprof:oso/9780198508885.001.0001.

[5] P. Monk, A finite element method for approximating the time-harmonic Maxwell equations, Numer. Math. 63 (2) (1992) 243–261. doi:10.1007/BF01385860.

[6] L. Zhong, S. Shu, G. Wittum, J. Xu, Optimal error estimates for Nedelec edge elements for time-harmonic Maxwell's equations, J. Comput. Math. 27 (5) (2009) 563–572. doi:10.4208/jcm.2009.27.5.011.

[7] I. Perugia, D. Schötzau, P. Monk, Stabilized interior penalty methods for the time-harmonic Maxwell equations, Comput. Methods Appl. Mech. Engrg. 191 (41-42) (2002) 4675–4697. doi:10.1016/S0045-7825(02)00399-7.

[8] I. Perugia, D. Schötzau, The *hp*-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations, Math. Comp. 72 (243) (2003) 1179–1214. doi:10.1090/S0025-5718-02-01471-0.

[9] P. Houston, I. Perugia, D. Schötzau, Mixed discontinuous Galerkin approximation of the Maxwell operator, SIAM J. Numer. Anal. 42 (1) (2004) 434–459. doi:10.1137/S003614290241790X.

[10] R. Griesmaier, P. Monk, Error analysis for a hybridizable discontinuous Galerkin method for the Helmholtz equation, J. Sci. Comput. 49 (3) (2011) 291–310. doi:10.1007/s10915-011-9460-z.

[11] J. Cui, W. Zhang, An analysis of HDG methods for the Helmholtz equation, IMA J. Numer. Anal. 34 (1) (2014) 279–295. doi:10.1093/imanum/drt005.

[12] N. C. Nguyen, J. Peraire, B. Cockburn, Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations, J. Comput. Phys. 230 (19) (2011) 7151–7175. doi:10.1016/j.jcp.2011.05.018.

[13] S. C. Brenner, J. Cui, Z. Nan, L.-y. Sung, Hodge decomposition for divergence-free vector fields and two-dimensional Maxwell's equations, Math. Comput. 81 (2012) 643–659. doi:10.2307/23267967.

[14] S. Brenner, J. Gedicke, L.-Y. Sung, Hodge decomposition for two-dimensional time-harmonic Maxwell's equations: impedance boundary condition, Math. Meth. Appl. Sci. 40 (2) (2015) 370–390.

[15] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2) (2009) 1319–1365. doi:10.1137/070706616.

[16] B. Cockburn, J. Gopalakrishnan, F.-J. Sayas, A projection-based error analysis of HDG methods, Math. Comp. 79 (271) (2010) 1351–1367. doi:10.1090/S0025-5718-10-02334-3.

[17] G. Chen, W. Hu, J. Shen, J. R. Singler, Y. Zhang, X. Zheng, An HDG method for distributed control of convection diffusion PDEs, J. Comput. Appl. Math. 343 (2018) 643–661. doi:10.1016/j.cam.2018.05.028.

[18] W. Hu, J. Shen, J. R. Singler, Y. Zhang, X. Zheng, A Superconvergent HDG Method for Distributed Control of Convection Diffusion PDEs, J. Sci. Comput. 76 (3) (2018) 1436–1457. doi:10.1007/s10915-018-0668-z.

[19] W. Gong, W. Hu, M. Mateos, J. Singler, X. Zhang, Y. Zhang, A New HDG Method for Dirichlet Boundary Control of Convection Diffusion PDEs II: Low Regularity, SIAM J. Numer. Anal. 56 (4) (2018) 2262–2287. doi:10.1137/17M1152103.

[20] A. Cesmelioglu, B. Cockburn, W. Qiu, Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations, Math. Comp. 86 (306) (2017) 1643–1670. doi:10.1090/mcom/3195.

[21] S. Muralikrishnan, M.-B. Tran, T. Bui-Thanh, iHDG: an iterative HDG framework for partial differential equations, SIAM J. Sci. Comput. 39 (5) (2017) S782–S808. doi:10.1137/16M1074187.

[22] T. Bui-Thanh, Construction and analysis of HDG methods for linearized shallow water equations, SIAM J. Sci. Comput. 38 (6) (2016) A3696–A3719. doi:10.1137/16M1057243.

[23] J. P. Sheldon, S. T. Miller, J. S. Pitt, A hybridizable discontinuous Galerkin method for modeling fluid-structure interaction, J. Comput. Phys. 326 (2016) 91–114. doi:10.1016/j.jcp.2016.08.037.

[24] B. Cockburn, G. Fu, F. J. Sayas, Superconvergence by $M$-decompositions. Part I: General theory for HDG methods for diffusion, Math. Comp. 86 (306) (2017) 1609–1641. doi:10.1090/mcom/3140.

[25] B. Cockburn, G. Fu, W. Qiu, Discrete $H^1$-inequalities for spaces admitting M-decompositions, SIAM J. Numer. Anal. 56 (6) (2018) 3407–3429. doi:10.1137/17M1144830.

[26] B. Cockburn, G. Fu, Devising superconvergent HDG methods with symmetric approximate stresses for linear elasticity by $M$-decompositions, IMA J. Numer. Anal. 38 (2) (2018) 566–604. doi:10.1093/imanum/drx025.

[27] B. Cockburn, G. Fu, W. Qiu, A note on the devising of superconvergent HDG methods for Stokes flow by $M$-decompositions, IMA J. Numer. Anal. 37 (2) (2017) 730–749. doi:10.1093/imanum/drw029.

[28] B. Cockburn, G. Fu, Superconvergence by $M$-decompositions. Part III: Construction of three-dimensional finite elements, ESAIM Math. Model. Numer. Anal. 51 (1) (2017) 365–398. doi:10.1051/m2an/2016023.

[29] B. Cockburn, G. Fu, Superconvergence by $M$-decompositions. Part II: Construction of two-dimensional finite elements, ESAIM Math. Model. Numer. Anal. 51 (1) (2017) 165–186. doi:10.1051/m2an/2016016.

[30] H. Chen, W. Qiu, K. Shi, M. Solano, A superconvergent HDG method for the Maxwell equations, J. Sci. Comput. 70 (3) (2017) 1010–1029. doi:10.1007/s10915-016-0272-z.

[31] H. Chen, W. Qiu, K. Shi, A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations, Comput. Methods Appl. Mech. Engrg. 333 (2018) 287–310. doi:10.1016/j.cma.2018.01.030.

[32] X. Feng, P. Lu, X. Xu, A hybridizable discontinuous Galerkin method for the time-harmonic Maxwell equations with high wave number, Comput. Methods Appl. Math. 16 (3) (2016) 429–445. doi:10.1515/cmam-2016-0021.

[33] P. Lu, H. Chen, W. Qiu, An absolutely stable *hp*-HDG method for the time-harmonic Maxwell equations with high wave number, Math. Comp. 86 (306) (2017) 1553–1577. doi:10.1090/mcom/3150.

[34] V. Girault, P.-A. Raviart, Finite element methods for Navier-Stokes equations, Vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986, theory and algorithms. doi:10.1007/978-3-642-61623-5.

[35] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (9) (1998) 823–864. doi:10.1002/(SICI)1099-1476(199806)21:9<823::AID-MMA976>3.0.CO;2-B.

[36] M. Dauge, Elliptic boundary value problems on corner domains, Vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1988, smoothness and asymptotics of solutions. doi:10.1007/BFb0086682.

[37] B. Cockburn, B. Dong, J. Guzmán, A hybridizable and superconvergent discontinuous Galerkin method for biharmonic problems, J. Sci. Comput. 40 (1-3) (2009) 141–187. doi:10.1007/s10915-009-9279-z.

[38] B. Cockburn, J. Gopalakrishnan, N. C. Nguyen, J. Peraire, F.-J. Sayas, Analysis of HDG methods for Stokes flow, Math. Comp. 80 (274) (2011) 723–760. doi:10.1090/S0025-5718-2010-02410-X.

[39] B. Chabaud, B. Cockburn, Uniform-in-time superconvergence of HDG methods for the heat equation, Math. Comp. 81 (277) (2012) 107–129. doi:10.1090/S0025-5718-2011-02525-1.

[40] F. Celiker, B. Cockburn, K. Shi, A projection-based error analysis of HDG methods for Timoshenko beams, Math. Comp. 81 (277) (2012) 131–151. doi:10.1090/S0025-5718-2011-02522-6.

[41] B. Cockburn, W. Qiu, K. Shi, Conditions for superconvergence of HDG methods for second-order elliptic problems, Math. Comp. 81 (279) (2012) 1327–1353. doi:10.1090/S0025-5718-2011-02550-0.

[42] Y. Chen, B. Cockburn, Analysis of variable-degree HDG methods for convection-diffusion equations. Part I: general nonconforming meshes, IMA J. Numer. Anal. 32 (4) (2012) 1267–1293. doi:10.1093/imanum/drr058.

[43] A. Cesmelioglu, B. Cockburn, N. C. Nguyen, J. Peraire, Analysis of HDG methods for Oseen equations, J. Sci. Comput. 55 (2) (2013) 392–431. doi:10.1007/s10915-012-9639-y.

[44] A. Ern, J.-L. Guermond, Finite element quasi-interpolation and best approximation, ESAIM Math. Model. Numer. Anal. 51 (4) (2017) 1367–1385.

[45] D. Boffi, F. Brezzi, L. F. Demkowicz, R. G. Durán, R. S. Falk, M. Fortin, Mixed finite elements, compatibility conditions, and applications, Vol. 1939 of Lecture Notes in Mathematics, Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence, 2008, lectures given at the C.I.M.E. Summer School held in Cetraro, June 26–July 1, 2006, Edited by Boffi and Lucia Gastaldi. doi:10.1007/978-3-540-78319-0.

[46] P. Houston, I. Perugia, A. Schneebeli, D. Schötzau, Interior penalty method for the indefinite time-harmonic Maxwell equations, Numer. Math. 100 (3) (2005) 485–518. doi:10.1007/s00211-005-0604-7.

[47] B. Cockburn, G. Fu, A systematic construction of finite element commuting exact sequences, SIAM J. Numer. Anal. 55 (4) (2017) 1650–1688. doi:10.1137/16M1073352.