SIAM J. NUMER. ANAL.
Vol. 60, No. 1, pp. 450–474

ANALYSIS AND APPROXIMATIONS OF DIRICHLET BOUNDARY CONTROL OF STOKES FLOWS IN THE ENERGY SPACE*

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Abstract. We study Dirichlet boundary control of Stokes flows in 2D polygonal domains. We consider cost functionals with two different boundary control regularization terms: the $\boldsymbol{L}^2(\Gamma)$-norm and an energy space seminorm. We prove well-posedness, provide first order optimality conditions, derive regularity results, and develop finite element discretizations for both problems, and we also prove finite element error estimates for the latter problem. The motivation to study the energy space problem follows from our analysis: we prove that the choice of the control space $\boldsymbol{L}^2(\Gamma)$ can lead to an optimal control with discontinuities at the corners, even when the domain is convex. This phenomenon is also observed in numerical experiments. This behavior does not occur in Dirichlet boundary control problems for the Poisson equation on convex polygonal domains, and it may not be desirable in real applications. For the energy space problem, we show that the solution of the control problem is more regular than the solution of the problem with the $\boldsymbol{L}^2(\Gamma)$-regularization. The improved regularity enables us to prove a priori error estimates for the control in the energy norm. We present several numerical experiments for both control problems on convex and nonconvex domains.

Key words. Dirichlet boundary control, Stokes flows, energy space, regularity, finite element method, error estimates

AMS subject classifications. 49J20, 65N30

DOI. 10.1137/21M1406799

1. Introduction.

PDE-constrained optimal control is an active research area and has been popular for the last several decades. Interest in analysis and computation for problems in this area has been generated by a wide variety of applications and the fast development of computational resources. There are already several monographs and chapters devoted to various aspects of the field, including theoretical analysis, computational methods, and application areas; see, e.g., [33, 44, 6].

Boundary control problems for PDEs are a very important part of this field since for many applications control may only be applied at the boundary of the physical domain. Dirichlet boundary control problems are especially important in application areas, but the problems can be difficult to analyze mathematically—especially when

*Received by the editors March 22, 2021; accepted for publication (in revised form) October 11, 2021; published electronically February 15, 2022.
https://doi.org/10.1137/21M1406799
Funding: The work of the first author was partially supported by the Strategic Priority Research Program of Chinese Academy of Sciences grant XDB 41000000, the National Key Basic Research Program grant 2018YFB0704304, and the National Natural Science Foundation of China grants 11671391, 12071468. The work of the second author was supported by the Spanish Ministerio de Economía y Competitividad project MTM2017-83185-P. The work of the fourth author was partially supported by the National Science Foundation (NSF) grant DMS-1818867.
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the physical domain has a nonsmooth boundary. One of the key points in the study of Dirichlet boundary control problems is the choice of the control penalty in the cost functional. A natural goal in many applications is to minimize the "amount" of control used, which naturally leads to a boundary control penalty using the $L^2(\Gamma)$-norm. This also appears to be a reasonable choice from a numerical approximation point of view. However, in the analysis of such a problem the governing state equation is typically understood in a very weak sense since the Dirichlet boundary condition is only in $L^2(\Gamma)$.

Despite this difficulty, many researchers have considered problems using the $L^2(\Gamma)$ control penalty and developed numerical methods and numerical analysis results for problems governed by elliptic scalar equations. See [19, 8, 47, 17, 28, 1, 2] for different advances in the theory and classical finite element methods; in the recent works [10, 12, 36, 25, 11] the hybridizable discontinuous Galerkin (HDG) has been succesfully applied to problems governed by elliptic scalar equations posed on convex polygonal domains. We also refer to [23, 26] for error estimates for parabolic Dirichlet boundary control problems, to [46] for state-constrained problems, and to [7] for a Robin penalization approach.

On the other hand, $H^{1/2}(\Gamma)$ appears to be a natural choice to study the state equation in the standard variational formulation. There are also some numerical analysis results in this direction. See [49, 48, 53]. In [13, 39] a different formulation of this method is proposed where the control penalty now involves the harmonic extension of the control into the domain; a posteriori error estimates and the convergence of the adaptive finite element method are studied in [27]. There are also other ways to deal with the inhomogeneous Dirichlet boundary condition. In [41, 42, 43] elliptic Dirichlet boundary control problems are studied in the energy space setting using wavelet schemes for the spatial discretization and using a Lagrange multiplier for the inhomogeneous Dirichlet boundary condition.

Dirichlet boundary control problems are of great interest for applications in fluid dynamics; see, for example, [20, 21, 31, 30, 34, 35, 16, 37, 51]. Although many numerical algorithms and simulation results can be found in the literature, there are very few well-posedness, regularity, and numerical analysis results for Dirichlet boundary control problems for fluid flows in polygonal domains.

In this work, we study Dirichlet Stokes flow control problems in 2D polygonal domains using both L^2 and $H^{1/2}$ for the control spaces. We give precise well-posedness and regularity results for both problems, and we show that the L^2-regularized optimal control can be discontinuous at the corners of a convex domain. We prove higher regularity for the energy space control problem. We also develop finite element methods for both problems, present an efficient way to compute the gradient of the objective functional, and prove a priori error estimates for the energy space problem.

We believe that the present work is the first to give regularity results and also convergence rates for standard finite element methods for a Dirichlet boundary flow control problems on polygonal domains. The only other work that we are aware of that proves convergence rates for a numerical method for a Dirichlet boundary flow problem on a polygonal domain is our recent work [24], which considers a *tangential* Stokes boundary control problem. We improve on this work in a number of ways. First, we do not restrict to the case of a tangential control. Second, we do not require the polygonal domain to be convex. Third, we use standard finite element methods for Stokes flows in this work, instead of the HDG method considered in [24]. HDG methods are very flexible and powerful and have many advantages (see [24] for more information); however, standard finite element methods for Stokes flows are more widely used and are readily available in many existing software packages.

Below, we give precise formulations of the Dirichlet Stokes control problems we consider and give a brief overview of related work.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with polygonal boundary Γ. We let $H^m(\Omega)$ denote the standard Sobolev space with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$, and we use bold font to denote vector valued spaces. Set $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^2$ and $\boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{v} = 0 \text{ on } \Gamma\}$. We denote the L^2-inner products on $\boldsymbol{L}^2(\Omega)$, $L^2(\Omega)$, $\boldsymbol{L}^2(\Gamma)$, and $L^2(\Gamma)$ by

$$(\boldsymbol{y}, \boldsymbol{z}) = \sum_{j=1}^{2} \int_\Omega y_j z_j, \qquad (p, q) = \int_\Omega pq, \qquad (\boldsymbol{y}, \boldsymbol{z})_\Gamma = \sum_{j=1}^{2} \int_\Gamma y_j z_j, \qquad (u, v)_\Gamma = \int_\Gamma uv.$$

We use $\langle \cdot, \cdot \rangle$ to denote the duality product between $H^{-s}(\Omega)$ and $H^s(\Omega)$. We let $H^s(\Gamma)$ denote the space of traces of $H^{s+1/2}(\Omega)$ for $0 < s < 3/2$, and we note that $H^s(\Gamma)$ for $1/2 < s < 3/2$ is given by $H^s(\Gamma) = \{u \in \Pi_{i=1}^m H^s(\Gamma_i) : u \in C(\Gamma)\}$; see [29, Theorem 1.5.2.8]. (This definition does not make sense for $s = 3/2$.) Here, Γ_i are the sides of the polygon. For $0 < s < 3/2$, we use $\langle \cdot, \cdot \rangle_\Gamma$ to denote the duality product between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$.

For the Stokes problem, we use the standard spaces

$$\boldsymbol{H}(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \quad \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\}, \quad L_0^2(\Omega) = \left\{p \in L^2(\Omega), \quad (p, 1) = 0\right\},$$

as well as the velocity spaces (see [52, section 2.1])

$$\boldsymbol{V}^s(\Omega) = \{\boldsymbol{y} \in \boldsymbol{H}^s(\Omega) : \ \nabla \cdot \boldsymbol{y} = 0, \ \ \langle \boldsymbol{y} \cdot \boldsymbol{n}, 1 \rangle_\Gamma = 0\}, \quad s \geqslant 0,$$

which are Banach spaces with the $\boldsymbol{H}^s(\Omega)$-norm. For $0 \leqslant s < 3/2$, define

$$\boldsymbol{V}^s(\Gamma) = \{\boldsymbol{u} \in \boldsymbol{H}^s(\Gamma) : \ (\boldsymbol{u} \cdot \boldsymbol{n}, 1)_\Gamma = 0\},$$

and let $\boldsymbol{V}^{-s}(\Gamma)$ denote the dual space. A proper definition of $\boldsymbol{V}^s(\Gamma)$ for all $0 \leqslant s < 3/2$ was needed to obtain the regularity results in [24].

For the control problem, consider a target state $\boldsymbol{y}_d \in \boldsymbol{H}$, a velocity penalty space $\boldsymbol{H} \hookrightarrow \boldsymbol{L}^2(\Omega)$, and a control penalty space $\boldsymbol{U} \hookrightarrow \boldsymbol{V}^0(\Gamma)$. Let $\alpha > 0$ denote a Tikhonov regularization parameter, and consider the optimal control problem

$$\min_{\boldsymbol{u} \in \boldsymbol{U}} J(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y}_{\boldsymbol{u}} - \boldsymbol{y}_d\|_{\boldsymbol{H}}^2 + \frac{\alpha}{2}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2, \tag{1.1}$$

where $\boldsymbol{y}_{\boldsymbol{u}} \in \boldsymbol{V}^0(\Omega)$ is the unique solution (either in the transposition sense—see Definition 2.3 below—or standard variational solution) of the Stokes system

$$-\Delta \boldsymbol{y} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \quad \nabla \cdot \boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = \boldsymbol{u} \quad \text{on } \Gamma. \tag{1.2}$$

We note that similar Dirichlet control problems with various choices of the spaces $\boldsymbol{H}$ and $\boldsymbol{U}$ have been considered in the literature for both the Stokes and Navier–Stokes equations. The choices $\boldsymbol{H} = \boldsymbol{L}^4(\Omega)$ and $\boldsymbol{U} = \boldsymbol{V}^1(\Gamma)$ were used in the early work [31]. In [16], the spaces $\boldsymbol{H} = \boldsymbol{V}^1(\Omega)$ and $\boldsymbol{U} = \boldsymbol{L}^2(\Gamma)$ are used for the objective functional; however, the optimal control problem looks for admissible optimal controls in $\boldsymbol{U}_{\mathrm{ad}} = \boldsymbol{V}^{1/2}(\Gamma)$, which is the natural space for the controls to obtain a variational solution of the state equation (1.2). In [37], the authors consider a smooth domain and choose $\boldsymbol{H} = \boldsymbol{V}^0(\Omega)$ and $\boldsymbol{U} = \boldsymbol{V}^0(\Gamma)$. We show in polygonal domains that this approach leads to optimal controls that are *discontinuous* at the corners; see section 3 for the well-posedness and regularity results. However, a better regularity result for these spaces is obtained if we consider tangential control, i.e., we impose the condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ pointwise instead of $(\boldsymbol{u} \cdot \boldsymbol{n}, 1)_\Gamma = 0$; see [24] for more details.

Here we focus on the energy space method for the problem in polygonal domains. In section 4 we formulate the Dirichlet boundary control problem of Stokes equation with velocity space $\boldsymbol{H} = \boldsymbol{V}^0(\Omega)$ and control space $\boldsymbol{U} = \boldsymbol{V}^{1/2}(\Gamma)$, and we derive the first order optimality condition by using the Steklov–Poincaré operator. Higher regularity of the solutions is shown compared to the $\boldsymbol{L}^2(\Gamma)$ setting. In section 5 we give finite element approximations and error estimates for the energy space method. Numerical experiments are carried out in section 6 for both choices $\boldsymbol{U} = \boldsymbol{V}^0(\Gamma)$ and $\boldsymbol{U} = \boldsymbol{V}^{1/2}(\Gamma)$ in both convex and nonconvex polygonal domains.

As can be seen in the numerical experiments, our estimates are not sharp. Sharp error estimates for Lagrange linear elements have been obtained for the case of the Poisson equation in [2] for L^2-regularization and in [53] for $H^{1/2}$-regularization using a detailed study of the regularity of the adjoint state: a splitting into a regular plus a singular part in the first reference and regularity in the weighted Sobolev space $W^{2,\infty}_{\boldsymbol{\alpha}}(\Omega)$ in the second one. We have noticed that, for problems governed by the Stokes equation, orders of convergence higher than the ones that could be obtained using those techniques are possible using higher order elements; see Table 2. We do not attempt those techniques in this paper and defer its study to a future work.

Remark 1.1. For $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, if we let $\boldsymbol{y}^f \in \boldsymbol{V}^1(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$ be the unique solution of (1.2) for $\boldsymbol{u} = 0$ and redefine $\boldsymbol{y}_d := \boldsymbol{y}_d - \boldsymbol{y}^f$, we can formulate an equivalent problem to (1.1) with $\boldsymbol{f} = 0$, in the sense that the optimal control will be the same for both problems and the optimal states will differ by $\boldsymbol{y}^f$. Thus, in the rest of the work, we assume $\boldsymbol{f} = 0$.

Remark 1.2. The introduction of control constraints does not lead to any differences in the regularity of the solutions or the rates of convergence. Control-constrained problems can be treated by means of variational inequalities instead of equalities, and there are plenty of examples about this in the literature. We focus on the unconstrained problem in order to avoid additional technicalities.

2. Regularity results. We first summarize the result we presented in [24] about the concept of solution for Dirichlet data in $\boldsymbol{V}^0(\Gamma)$ and its precise regularity.

To introduce the definition of solution of the state equation, we first need some results about the following *compressible* Stokes equation. For data $(\boldsymbol{g}, h) \in \boldsymbol{H}^{-1}(\Omega) \times \boldsymbol{L}^2_0(\Omega)$, we say that $(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h}) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega)$ is a solution of

$$-\Delta \boldsymbol{z} + \nabla q = \boldsymbol{g} \text{ in } \Omega, \quad \nabla \cdot \boldsymbol{z} = h \text{ in } \Omega, \quad \boldsymbol{z} = 0 \text{ on } \Gamma \tag{2.1}$$

if

$$\begin{aligned} (\nabla \boldsymbol{z}_{\boldsymbol{g},h}, \nabla \boldsymbol{\zeta}) - (q_{\boldsymbol{g},h}, \nabla \cdot \boldsymbol{\zeta}) &= \langle \boldsymbol{g}, \boldsymbol{\zeta} \rangle \quad && \forall \boldsymbol{\zeta} \in \boldsymbol{H}^1_0(\Omega), \\ (\chi, \nabla \cdot \boldsymbol{z}_{\boldsymbol{g},h}) &= (h, \chi) \quad && \forall \chi \in L^2_0(\Omega). \end{aligned}$$

Following [15], we define ξ, the singular exponent for the Stokes operator, as the smallest nonzero real part of all roots of the equation

$$(\lambda - 1)^{-1} \lambda^{-2} \left(\sin^2(\lambda\omega) - \lambda^2 \sin^2 \omega \right) = 0, \tag{2.2}$$

where ω denotes the greatest interior angle of Γ. It is known (see page 80 of [15]) that $\xi > \pi/\omega$ when $\omega \in (0,\pi)$ and $\xi < \pi/\omega$ when $\omega \in (\pi, 2\pi)$. Let

$$s^\star = \min\{\xi - 1/2, 1/2\}. \tag{2.3}$$

THEOREM 2.1 ([15, Theorem 5.5(a)]). *Let s satisfy $-1/2<s<s^\star$. If $\boldsymbol{g} \in \boldsymbol{H}^{s-1/2}(\Omega)$ and $h \in H^{s+1/2}(\Omega) \cap L^2_0(\Omega)$, then* (2.1) *has a unique solution $(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h}) \in [\boldsymbol{H}^{3/2+s}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)] \times [H^{1/2+s}(\Omega) \cap L^2_0(\Omega)]$. Moreover, we have*

$$\|\boldsymbol{z}_{\boldsymbol{g},h}\|_{\boldsymbol{H}^{3/2+s}(\Omega)} + \|q_{\boldsymbol{g},h}\|_{H^{1/2+s}(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{g}\|_{\boldsymbol{H}^{s-1/2}(\Omega)} + \|h\|_{H^{s+1/2}(\Omega)/\mathbb{R}}\big). \tag{2.4}$$

It is important to note that Theorem 2.1 only holds for $s < 1/2$. This means that even in convex domains one cannot expect in general to have $\boldsymbol{H}^2(\Omega)$-regularity of $\boldsymbol{z}$. Later, we also require a regularity result for the case $h \equiv 0$ in a convex domain. Let $(\boldsymbol{z}(\boldsymbol{g}), q(\boldsymbol{g}))$ denote the solution of (2.1) for $h = 0$, i.e., $\boldsymbol{z}(\boldsymbol{g}) = \boldsymbol{z}_{\boldsymbol{g},0}$ and $q(\boldsymbol{g}) = q_{\boldsymbol{g},0}$.

THEOREM 2.2 ([15, Theorem 5.5(b)(c)]). *Suppose $\boldsymbol{g} \in \boldsymbol{H}^{t-1}(\Omega)$ for some $0 \leqslant t < \xi$ and $h = 0$. Then* (2.1) *has a unique solution $\boldsymbol{z}(\boldsymbol{g}) \in \boldsymbol{V}^{t+1}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$, $q(\boldsymbol{g}) \in H^t(\Omega) \cap L^2_0(\Omega)$, and there exists a constant $C > 0$ independent of $\boldsymbol{g}$ such that*

$$\|\boldsymbol{z}(\boldsymbol{g})\|_{\boldsymbol{H}^{1+t}(\Omega)} + \|q(\boldsymbol{g})\|_{H^t(\Omega)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{H}^{t-1}(\Omega)}.$$

Below, we define transposition solutions of the state equation (1.2) with $\boldsymbol{f} = \boldsymbol{0}$ (see Remark 1.1) in the case $\boldsymbol{u} \in \boldsymbol{V}^{-s}(\Gamma)$ for $0 < s < s^\star$. Elements of this space do not necessarily satisfy any condition analogous to $(\boldsymbol{u}\cdot\boldsymbol{n}, 1)_\Gamma = 0$. In order to account for the constants, we follow [52, equation (2.2)], and for $(\boldsymbol{z}, q) \in \boldsymbol{H}^{3/2+s}(\Omega) \times H^{1/2+s}(\Omega)$ with $s > 0$ we define the constant

$$\lambda(\boldsymbol{z}, q) = \frac{1}{|\Gamma|}(\partial_{\boldsymbol{n}}\boldsymbol{z}\cdot\boldsymbol{n} - q, 1)_\Gamma. \tag{2.5}$$

This constant satisfies

$$\|\partial_{\boldsymbol{n}}\boldsymbol{z} - q\boldsymbol{n}\|_{L^2(\Gamma)/\mathbb{R}} = \|\partial_{\boldsymbol{n}}\boldsymbol{z} - q\boldsymbol{n} - \lambda(\boldsymbol{z}, q)\boldsymbol{n}\|_{L^2(\Gamma)},$$

and we have

$$\partial_{\boldsymbol{n}}\boldsymbol{z} - (q + \lambda(\boldsymbol{z}, q))\boldsymbol{n} \in \boldsymbol{V}^0(\Gamma).$$

This fact, the continuity of the normal trace in $\boldsymbol{H}^{3/2+s}(\Omega)$, the continuity of the trace in $\boldsymbol{H}^s(\Omega)$, and (2.4) give that for $0 < s < 1/2$ we have

$$\|\partial_{\boldsymbol{n}}\boldsymbol{z}_{\boldsymbol{g},h} - (q_{\boldsymbol{g},h} + \lambda(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h}))\boldsymbol{n}\|_{H^s(\Gamma)} \leqslant C\big(\|\boldsymbol{g}\|_{\boldsymbol{H}^{s-1/2}(\Omega)} + \|h\|_{H^{s+1/2}(\Omega)/\mathbb{R}}\big). \tag{2.6}$$

This allows us to give the following well-defined notion of transposition solution for the state equation (again, with $\boldsymbol{f} = \boldsymbol{0}$).

DEFINITION 2.3. *Suppose $0 \leqslant s < s^\star$ and $\boldsymbol{u} \in \boldsymbol{V}^{-s}(\Gamma)$. We say that $\boldsymbol{y}_{\boldsymbol{u}} \in \boldsymbol{V}^0(\Omega)$, $p_{\boldsymbol{u}} \in \big(H^1(\Omega) \cap L^2_0(\Omega)\big)'$ is a solution in the transposition sense of*

$$-\Delta\boldsymbol{y} + \nabla p = \boldsymbol{0} \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = \boldsymbol{u} \text{ on } \Gamma \tag{2.7}$$

if

$$(\boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{g}) - \langle p_{\boldsymbol{u}}, h\rangle = \langle \boldsymbol{u}, -\partial_{\boldsymbol{n}}\boldsymbol{z}_{\boldsymbol{g},h} + (q_{\boldsymbol{g},h} + \lambda(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h}))\boldsymbol{n}\rangle_\Gamma \tag{2.8}$$

for all $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and $h \in H^1(\Omega) \cap L^2_0(\Omega)$, where $(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h}) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega)$ is the unique solution of (2.1) *and $\lambda(\boldsymbol{z}_{\boldsymbol{g},h}, q_{\boldsymbol{g},h})$ is the constant given in* (2.5).

Furthermore, this definition can be rewritten in different forms when $\boldsymbol{u}$ is more regular. First, if $\boldsymbol{u} \in \boldsymbol{V}^0(\Gamma)$, then $(\boldsymbol{u}, \lambda \boldsymbol{n})_\Gamma = 0$ for every constant $\lambda \in \mathbb{R}$, and therefore (2.8) can be written as

$$(\boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{g}) - \langle p_{\boldsymbol{u}}, h\rangle = (\boldsymbol{u}, -\partial_{\boldsymbol{n}} \boldsymbol{z}_{\boldsymbol{g},h} + q_{\boldsymbol{g},h}\boldsymbol{n})_\Gamma. \tag{2.9}$$

Second, if $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$, then the transposition solution is the variational solution of the following problem: Find $(\boldsymbol{y}_{\boldsymbol{u}}, p_{\boldsymbol{u}}) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfying

$$\begin{aligned} (\nabla \boldsymbol{y}_{\boldsymbol{u}}, \nabla \boldsymbol{\zeta}) - (p_{\boldsymbol{u}}, \nabla \cdot \boldsymbol{\zeta}) &= 0 \quad \forall \boldsymbol{\zeta} \in \boldsymbol{H}^1_0(\Omega), \\ (\chi, \nabla \cdot \boldsymbol{y}_{\boldsymbol{u}}) &= 0 \quad \forall \chi \in L^2(\Omega)/\mathbb{R}, \\ \boldsymbol{y}_{\boldsymbol{u}} &= \boldsymbol{u} \quad \text{ on } \Gamma. \end{aligned} \tag{2.10}$$

Next, we give a regularity result for the state equation (2.7) on polygonal domains from [24, Theorem 2.2].

THEOREM 2.4. *If $\boldsymbol{u} \in \boldsymbol{V}^s(\Gamma)$ for $-s^\star < s < s^\star + 1$, then the solution of (2.7) satisfies*

$$\boldsymbol{y}_{\boldsymbol{u}} \in \boldsymbol{V}^{s+1/2}(\Omega) \quad \textit{and} \quad p_{\boldsymbol{u}} \in \begin{cases} H^{s-1/2}(\Omega)/\mathbb{R} & \textit{if } s \geqslant 1/2, \\ \left(H^{1/2-s}(\Omega)/\mathbb{R}\right)' & \textit{if } s \leqslant 1/2. \end{cases}$$

Also, the control-to-state mapping $\boldsymbol{u} \mapsto \boldsymbol{y}_{\boldsymbol{u}}$ is continuous from $\boldsymbol{V}^s(\Gamma)$ to $\boldsymbol{V}^{s+1/2}(\Omega)$.

We also recall here the concept of stress force on the boundary as used in [32]. Let $(\boldsymbol{\psi}, \phi)$ be the solution of the *incompressible* Stokes system with source $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and Dirichlet data $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$, i.e., $\boldsymbol{\psi} = \boldsymbol{z}(\boldsymbol{g}) + \boldsymbol{y}_{\boldsymbol{u}}$ and $\phi = q(\boldsymbol{g}) + p_{\boldsymbol{u}}$, where $(\boldsymbol{z}(\boldsymbol{g}), q(\boldsymbol{g}))$ is the solution of (2.1) with $h = 0$, and $(\boldsymbol{y}_{\boldsymbol{u}}, p_{\boldsymbol{u}})$ is the solution of (2.10).

For $\boldsymbol{g}$ and $\boldsymbol{u}$ as above, we define the stress force on the boundary $\boldsymbol{t}(\boldsymbol{g}, \boldsymbol{u})$ related to $(\boldsymbol{\psi}, \phi)$ to be the unique solution in $\boldsymbol{H}^{-1/2}(\Gamma)$ of the variational problem:

$$\langle \boldsymbol{t}(\boldsymbol{g}, \boldsymbol{u}), \boldsymbol{\zeta}\rangle_\Gamma = (\nabla \boldsymbol{\psi}, \nabla \boldsymbol{\zeta}) - (\phi, \nabla \cdot \boldsymbol{\zeta}) - (\boldsymbol{g}, \boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \boldsymbol{H}^1(\Omega). \tag{2.11}$$

Notice that for $\boldsymbol{u} \in \boldsymbol{V}^{r+1/2}(\Gamma)$ with $r > 0$, integration by parts shows that

$$\boldsymbol{t}(\boldsymbol{g}, \boldsymbol{u}) = \partial_{\boldsymbol{n}} \boldsymbol{\psi} - \phi \boldsymbol{n}. \tag{2.12}$$

For $0 \leqslant s < s^\star + 1$, we define the solution operator $\boldsymbol{E} : \boldsymbol{V}^s(\Gamma) \to \boldsymbol{L}^2(\Omega)$ by

$$\boldsymbol{E}\boldsymbol{u} = \boldsymbol{y}_{\boldsymbol{u}}. \tag{2.13}$$

Directly from (2.9) with $h = 0$ and (2.12), the adjoint $\boldsymbol{E}^\star : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}^{-s}(\Gamma)$ is defined by

$$\boldsymbol{E}^\star \boldsymbol{g} = -\partial_{\boldsymbol{n}} \boldsymbol{z}(\boldsymbol{g}) + q(\boldsymbol{g})\boldsymbol{n} = -\boldsymbol{t}(\boldsymbol{g}, \boldsymbol{0}). \tag{2.14}$$

By Theorem 2.4 we know that $\boldsymbol{E} : \boldsymbol{V}^s(\Gamma) \to \boldsymbol{L}^2(\Omega)$ is bounded, and hence $\boldsymbol{E}^\star : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}^{-s}(\Gamma)$ is also bounded. Therefore, $\boldsymbol{E}^\star \boldsymbol{E} : \boldsymbol{V}^s(\Gamma) \to \boldsymbol{V}^{-s}(\Gamma)$ is bounded. Furthermore, setting $s = 1/2$, [32, Theorem 4] gives that for all $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$ we have

$$\|\boldsymbol{E}^\star \boldsymbol{E} \boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant C \|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{2.15}$$

3. Stokes Dirichlet boundary control in $\boldsymbol{V}^0(\Gamma)$. In this section, we investigate the case $\boldsymbol{U} = \boldsymbol{V}^0(\Gamma)$. For $\boldsymbol{y}_d \in \boldsymbol{L}^2(\Omega)$ and $\alpha > 0$, our control problem reads

$$\min_{\boldsymbol{u}\in \boldsymbol{V}^0(\Gamma)} J_0(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y_u} - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Gamma)}, \tag{3.1}$$

where $\boldsymbol{y_u} \in \boldsymbol{V}^0(\Omega)$ is the solution of the state equation (2.9). By (2.13) we have

$$J_0(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{E}^\star\boldsymbol{E}\boldsymbol{u}, \boldsymbol{u})_\Gamma - (\boldsymbol{E}^\star\boldsymbol{y}_d, \boldsymbol{u})_\Gamma + \frac{c_\Omega}{2} + \frac{\alpha}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Gamma)} =: \boldsymbol{F}(\boldsymbol{u}) + \frac{\alpha}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Gamma)}, \tag{3.2}$$

where $c_\Omega = \|\boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)}$ and $\boldsymbol{F}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{E}^\star\boldsymbol{E}\boldsymbol{u}, \boldsymbol{u})_\Gamma - (\boldsymbol{E}^\star\boldsymbol{y}_d, \boldsymbol{u})_\Gamma + \dfrac{c_\Omega}{2}$ is the tracking term. Notice that here we have used the solution operator defined in (2.13) for $s = 0$: $\boldsymbol{E} : \boldsymbol{V}^0(\Gamma) \to \boldsymbol{L}^2(\Omega)$. It is straightforward to prove that

$$\boldsymbol{F}'(\boldsymbol{u})\boldsymbol{v} = (\boldsymbol{E}^\star\boldsymbol{E}\boldsymbol{u}, \boldsymbol{v})_\Gamma - (\boldsymbol{E}^\star\boldsymbol{y}_d, \boldsymbol{v})_\Gamma \quad \forall \boldsymbol{u} \in \boldsymbol{V}^0(\Gamma) \text{ and } \boldsymbol{v} \in \boldsymbol{V}^0(\Gamma). \tag{3.3}$$

Although we are mainly interested in this work in regularization in the energy space $\boldsymbol{V}^{1/2}(\Gamma)$, the solution properties of the problem with $\boldsymbol{V}^0(\Gamma)$-regularization are also of interest in order to more clearly see the advantages and disadvantages of energy space control problem. It is also interesting to see the differences between the Dirichlet boundary control of the Poisson equation (cf. [1]) and of the Stokes system.

Using the strict convexity of the functional and the continuity of the control-to-state mapping, which follows from Theorem 2.4, it is standard to prove the existence of a unique solution $\boldsymbol{u}_0 \in \boldsymbol{V}^0(\Gamma)$ of problem (3.1). We also prove regularity results below, and we show that the optimal control can be discontinuous at the corners of a convex polygonal domain.

THEOREM 3.1. *Let ξ be the singular exponent for the Stokes operator, and let $s^\star$ be the exponent defined in* (2.3). *Suppose $\boldsymbol{y}_d \in \boldsymbol{H}^m(\Omega)$ for some $0 \leqslant m < s^\star$, and let $\boldsymbol{u}_0 \in \boldsymbol{V}^0(\Gamma)$ be the solution of problem* (3.1). *Then $\boldsymbol{u}_0 \in \boldsymbol{V}^s(\Gamma)$ for all $0 \leqslant s < s^\star$, and there exist $\boldsymbol{y}_0 \in \boldsymbol{V}^{s+1/2}(\Omega)$, $p_0 \in (H^{1/2-s}(\Omega) \cap L^2_0(\Omega))'$, $\boldsymbol{z}_0 \in \boldsymbol{V}^{1+t}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$, and $q_0 \in H^t(\Omega) \cap L^2_0(\Omega)$, for all $t \leqslant 1 + m$ such that $t < \xi$, that satisfy the state equation*

$$-\Delta \boldsymbol{y}_0 + \nabla p_0 = \boldsymbol{0} \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{y}_0 = 0 \text{ in } \Omega, \quad \boldsymbol{y}_0 = \boldsymbol{u}_0 \text{ on } \Gamma; \tag{3.4}$$

the adjoint state equation

$$-\Delta \boldsymbol{z}_0 + \nabla q_0 = \boldsymbol{y}_0 - \boldsymbol{y}_d \text{ in } \Omega, \quad \nabla\cdot\boldsymbol{z}_0 = 0 \text{ in } \Omega, \quad \boldsymbol{z}_0 = 0 \text{ on } \Gamma; \tag{3.5}$$

and the optimality condition

$$(\alpha\boldsymbol{u}_0 - (\partial_{\boldsymbol{n}}\boldsymbol{z}_0 - q_0\boldsymbol{n}), \boldsymbol{v})_\Gamma = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0(\Gamma). \tag{3.6}$$

Moreover, there exists $\lambda_0 \in \mathbb{R}$ such that

$$\boldsymbol{u}_0 = \frac{1}{\alpha}(\partial_{\boldsymbol{n}}\boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n})$$

and

$$\boldsymbol{u}_0 \in \prod_{i=1}^n \boldsymbol{H}^{t-1/2}(\Gamma_i) \quad \forall\, t \leqslant m+1 \text{ such that } t < \xi.$$

Finally, if $m > 0$ and Ω is convex, then $\boldsymbol{u}_0$ is continuous at a corner x_j if and only if $q_0(x_j) + \lambda_0 = 0$.

Here, the state equation must be understood in the transposition sense (2.9), while the adjoint state equation must be understood in the variational sense.

Proof. By the definition of $J_0(\boldsymbol{u})$ in (3.2) and (3.3), the derivative of the objective functional $J_0(\boldsymbol{u})$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^0(\Gamma)$ can be written as

$$\begin{aligned} J_0'(\boldsymbol{u})\boldsymbol{v} &= (\alpha \boldsymbol{u} + \boldsymbol{E}^\star \boldsymbol{E}\boldsymbol{u}, \boldsymbol{v})_\Gamma - (\boldsymbol{E}^\star \boldsymbol{y}_d, \boldsymbol{v})_\Gamma = (\alpha \boldsymbol{u} + \boldsymbol{E}^\star(\boldsymbol{E}\boldsymbol{u} - \boldsymbol{y}_d), \boldsymbol{v})_\Gamma \\ &= (\alpha \boldsymbol{u} - (\partial_{\boldsymbol{n}} \boldsymbol{z}(\boldsymbol{y}_{\boldsymbol{u}} - \boldsymbol{y}_d) - q(\boldsymbol{y}_{\boldsymbol{u}} - \boldsymbol{y}_d)\boldsymbol{n}), \boldsymbol{v})_\Gamma, \end{aligned}$$

where we used (2.13) and (2.14) in the last equality. The optimality conditions follow in a standard way. For $\boldsymbol{v} \in \boldsymbol{V}^0(\Gamma)$ we have that $(\lambda \boldsymbol{n}, \boldsymbol{v})_\Gamma = 0$ for any $\lambda \in \mathbb{R}$. Taking λ_0 to equal the constant $\lambda(\boldsymbol{z}(\boldsymbol{y}_0 - \boldsymbol{y}_d), q(\boldsymbol{y}_0 - \boldsymbol{y}_d))$, which is defined in (2.5), and using (3.6), we also have that

$$(\alpha \boldsymbol{u}_0 - (\partial_{\boldsymbol{n}} \boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n}), \boldsymbol{v})_\Gamma = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0(\Gamma).$$

This implies that $\alpha \boldsymbol{u}_0$ is the $\boldsymbol{L}^2(\Gamma)$-projection of $\partial_{\boldsymbol{n}} \boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n}$ onto $\boldsymbol{V}^0(\Gamma)$. Since $\partial_{\boldsymbol{n}} \boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n} \in \boldsymbol{V}^0(\Gamma)$, we have

$$\boldsymbol{u}_0 = \frac{1}{\alpha}(\partial_{\boldsymbol{n}} \boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n}).$$

The regularity follows from a bootstrapping argument: From Theorem 2.4 we have that $\boldsymbol{y}_0 \in \boldsymbol{V}^{1/2}(\Omega)$. Using this and taking into account that $\boldsymbol{y}_d \in \boldsymbol{H}^m(\Omega)$, we have from Theorem 2.2 that $\boldsymbol{z}_0 \in \boldsymbol{V}^{1+t}(\Omega)$, $q_0 \in H^t(\Omega) \cap L^2_0(\Omega)$ for all $t \leqslant 1 + m$ such that $t < \xi$.

From trace theory, and since $1/2 < t$, it is clear that

$$\partial_{\boldsymbol{n}} \boldsymbol{z}_0 - (q_0 + \lambda_0)\boldsymbol{n} \in \prod_{i=1}^n \boldsymbol{H}^{t-1/2}(\Gamma_i) \quad \forall\, t \leqslant m + 1 \text{ such that } t < \xi.$$

For $t < 1$, and taking $s = t - 1/2$, we have that $s < s^\star$ and that $\prod_{i=1}^n \boldsymbol{H}^s(\Gamma_i) = \boldsymbol{H}^s(\Gamma)$. Therefore, (3.6) gives that $u_0 \in \boldsymbol{H}^s(\Gamma)$ for all $s < s^\star$. The regularity of the optimal state follows from Theorem 2.4.

If $m > 0$ and Ω is convex, then the gradient of the dual pressure q_0 is a function in $H^{t-1}(\Omega)$ with $t - 1 > 0$. So we have that each component z^i, $i = 1, 2$, of $\boldsymbol{z}_0$ satisfies $\Delta z^i \in H^{t-1}(\Omega)$ and $z^i = 0$ on Γ. Therefore, we have that $\partial_{\boldsymbol{n}} z^i(x_j) = 0$, $i = 1, 2$, for every convex corner x_j (cf. [7, Appendix A]); also, from [7, Lemma A2] and the Sobolev embedding theorem we have that the normal derivative of $\boldsymbol{z}_0$ is a continuous function. For the pressure, the situation is slightly different. From trace theory we have that $q_0 \in H^{t-1/2}(\Gamma)$, and by Sobolev embeddings we know q_0 is a continuous function. Nevertheless, the vector $\boldsymbol{n}$ is discontinuous at the corners, and hence the $(q_0 + \lambda_0)\boldsymbol{n}$ can only be continuous at x_j if $q_0(x_j) = -\lambda_0$. ∎

Remark 3.2. This regularity of the optimal control in a convex domain is essentially different from the regularity achieved by the optimal control of problems related to the Poisson equation. The solution of a problem governed by the Poisson equation must be a continuous function, which is also zero at the corners. In our case, the optimal control may show discontinuities. See Figure 2 for an example with a continuous control and Figure 3 for a problem example with *discontinuous* control.

Remark 3.3. Notice that the pressure is determined up to a constant. We choose the pressure such that $(q_0, 1) = 0$, but any other representative is of course possible. The value of λ_0 would change accordingly, so that $q_0 + \lambda_0$ does not vary.

4. Stokes Dirichlet boundary control in the energy space. Next, we consider Stokes Dirichlet boundary control with a different regularization term:

$$\min_{\boldsymbol{u}\in \boldsymbol{V}^{1/2}(\Gamma)} J_{1/2}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y}_{\boldsymbol{u}} - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}|\boldsymbol{u}|^2_{\boldsymbol{H}^{1/2}(\Gamma)}, \tag{4.1}$$

where again we assume $\boldsymbol{y}_d \in \boldsymbol{L}^2(\Omega)$ and $\alpha > 0$.

There are different kinds of definitions for the $\boldsymbol{H}^{1/2}(\Gamma)$-norm; e.g., one may use the Sobolev–Slobodeckii norm or the Fourier transform. The key point to the study of the optimization problem (4.1) is to find an appropriate representation for the $\boldsymbol{H}^{1/2}(\Gamma)$-norm that enables us to derive the first order optimality condition. Here we follow the idea of [49] and introduce a Stokes version of the Steklov–Poincaré operator (cf. [3, 18]) associated with (2.9).

It follows from Theorem 2.4 that for any given control $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$, there exists a unique state $(\boldsymbol{y}_{\boldsymbol{u}}, p_{\boldsymbol{u}}) \in \boldsymbol{V}^1(\Omega) \times L^2_0(\Omega)$ that satisfies

$$\|\boldsymbol{y}_{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} + \|p_{\boldsymbol{u}}\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{4.2}$$

Given $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$, we define $\boldsymbol{D}\boldsymbol{u} \in \boldsymbol{H}^{-1/2}(\Gamma)$ by

$$\langle \boldsymbol{D}\boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma = (\nabla \boldsymbol{y}_{\boldsymbol{u}}, \nabla \boldsymbol{R}\boldsymbol{v}) - (p_{\boldsymbol{u}}, \nabla\cdot \boldsymbol{R}\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma), \tag{4.3}$$

where $\boldsymbol{R}$ is any continuous extension operator from $\boldsymbol{H}^{1/2}(\Gamma)$ to $\boldsymbol{H}^1(\Omega)$.

LEMMA 4.1. *The definition of $\boldsymbol{D}$ is independent of the chosen extension $\boldsymbol{R}$ and*

$$\boldsymbol{D}\boldsymbol{u} = \boldsymbol{t}(\boldsymbol{0}, \boldsymbol{u}), \tag{4.4a}$$

$$\|\boldsymbol{D}\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \quad \forall \boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma). \tag{4.4b}$$

Proof. First of all, writing the PDE in divergence form as

$$-\nabla\cdot\big((\nabla + \nabla^T)\boldsymbol{y}_{\boldsymbol{u}} - p_{\boldsymbol{u}}\mathcal{I}\big) = 0$$

gives $(\nabla + \nabla^T)\boldsymbol{y}_{\boldsymbol{u}} - p_{\boldsymbol{u}}\mathcal{I} \in \boldsymbol{H}(\mathrm{div};\Omega)$, and so this function has a well-defined normal trace in $\boldsymbol{H}^{-1/2}(\Gamma)$. It is remarkable too that it is possible to define a variational normal derivative $\partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}} \in \boldsymbol{H}^{-1/2}(\Gamma)$ (cf. [7, Lemma A6]), and hence $p_{\boldsymbol{u}}\boldsymbol{n}$ is also a well-defined element in $\boldsymbol{H}^{-1/2}(\Gamma)$.

Next, for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma)$, integrating by parts in the definition of $\boldsymbol{D}\boldsymbol{u}$ gives

$$\begin{aligned}\langle \boldsymbol{D}\boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma &= \int_\Omega \big(\nabla \boldsymbol{y}_{\boldsymbol{u}}\nabla \boldsymbol{R}\boldsymbol{v} - p_{\boldsymbol{u}}\nabla\cdot \boldsymbol{R}\boldsymbol{v}\big)\\ &= \int_\Omega \big(-\Delta \boldsymbol{y}_{\boldsymbol{u}} + \nabla p_{\boldsymbol{u}}\big)\boldsymbol{R}\boldsymbol{v} + \langle \partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}} - p_{\boldsymbol{u}}\boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma = \langle \partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}} - p_{\boldsymbol{u}}\boldsymbol{n}, \boldsymbol{v}\rangle_\Gamma,\end{aligned} \tag{4.5}$$

where we used $-\Delta \boldsymbol{y}_{\boldsymbol{u}} + \nabla p_{\boldsymbol{u}} = 0$. This proves that the definition of $\boldsymbol{D}$ is independent of the chosen extension $\boldsymbol{R}$, and (4.4a) holds by (2.12) and (4.5).

Finally, we prove (4.4b). Using the definition of $\boldsymbol{D}$ in (4.3), the bound in (4.2), and the continuity of $\boldsymbol{R} : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{H}^1(\Omega)$ gives

$$\begin{aligned}\|\boldsymbol{D}\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} &= \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \boldsymbol{H}^{1/2}(\Gamma)} \frac{\langle \boldsymbol{D}\boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma}{\|\boldsymbol{v}\|_{\boldsymbol{H}^{1/2}(\Gamma)}}\\ &\leqslant C \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \boldsymbol{H}^{1/2}(\Gamma)} \frac{(\|\boldsymbol{y}_{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} + \|p_{\boldsymbol{u}}\|_{L^2(\Omega)})|\boldsymbol{R}\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)}}{\|\boldsymbol{v}\|_{\boldsymbol{H}^{1/2}(\Gamma)}}\\ &\leqslant C(\|\boldsymbol{y}_{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} + \|p_{\boldsymbol{u}}\|_{L^2(\Omega)})\\ &\leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)}.\end{aligned}$$

∎

LEMMA 4.2. *The mapping* $\langle \boldsymbol{D}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma^{1/2}$ *is a seminorm in* $\boldsymbol{V}^{1/2}(\Gamma)$ *equivalent to the* $\boldsymbol{H}^{1/2}(\Gamma)$ *seminorm.*

Proof. Let $\boldsymbol{Q}$ be the projection of $\boldsymbol{H}^{1/2}(\Gamma)$ onto $\boldsymbol{V}^{1/2}(\Gamma)$, and set $\boldsymbol{R}\boldsymbol{v} = \boldsymbol{y}_{\boldsymbol{Q}\boldsymbol{v}}$. Notice that $\nabla\cdot\boldsymbol{R}\boldsymbol{v} = 0$, and if $\boldsymbol{v}\in\boldsymbol{V}^{1/2}(\Gamma)$, then $\boldsymbol{R}\boldsymbol{v} = \boldsymbol{y}_{\boldsymbol{v}}$. By (4.3) we have

$$\langle \boldsymbol{D}\boldsymbol{u},\boldsymbol{v}\rangle_\Gamma = (\nabla\boldsymbol{y}_{\boldsymbol{u}},\nabla\boldsymbol{y}_{\boldsymbol{v}}) \quad \forall\boldsymbol{v}\in\boldsymbol{V}^{1/2}(\Gamma), \tag{4.6}$$

and thus we have that $\langle \boldsymbol{D}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma^{1/2}$ is a seminorm in $\boldsymbol{V}^{1/2}(\Gamma)$ equivalent to the $\boldsymbol{H}^{1/2}(\Gamma)$ seminorm. ∎

Proceeding similarly to the derivation of (3.2), the precise formulation of our control problem is given by

(4.7)
$$\min_{\boldsymbol{u}\in\boldsymbol{V}^{1/2}(\Gamma)} J_{1/2}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y}_{\boldsymbol{u}}-\boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}\langle \boldsymbol{D}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma = \frac{1}{2}\langle \boldsymbol{T}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma - \langle \boldsymbol{w},\boldsymbol{u}\rangle_\Gamma + \frac{c_\Omega}{2},$$

where $c_\Omega = \|\boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)}$ and

$$\boldsymbol{T} = \alpha\boldsymbol{D} + \boldsymbol{E}^\star\boldsymbol{E}, \quad \boldsymbol{w} = \boldsymbol{E}^\star\boldsymbol{y}_d \in \boldsymbol{V}^{-1/2}(\Gamma). \tag{4.8}$$

Notice that here we have used the solution operator defined in (2.13) for $s = 1/2$: $\boldsymbol{E}:\boldsymbol{V}^{1/2}(\Gamma)\to\boldsymbol{L}^2(\Omega)$. The functional being convex and coercive implies that problem (4.7) has a unique solution $\bar{\boldsymbol{u}}\in\boldsymbol{V}^{1/2}(\Gamma)$.

We also note that, by (4.6), an alternative way to write the functional for $\boldsymbol{u}\in\boldsymbol{V}^{1/2}(\Gamma)$ is

$$J_{1/2}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y}_{\boldsymbol{u}}-\boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}\|\nabla\boldsymbol{y}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

LEMMA 4.3. *There exist constants* $C_1, C_2 > 0$ *such that for every* $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}^{1/2}(\Gamma)$,

$$\langle \boldsymbol{T}\boldsymbol{u},\boldsymbol{v}\rangle_\Gamma \le C_1\|\boldsymbol{u}\|_{\boldsymbol{V}^{1/2}(\Gamma)}\|\boldsymbol{v}\|_{\boldsymbol{V}^{1/2}(\Gamma)}$$

and

$$\langle \boldsymbol{T}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma \ge C_2\|\boldsymbol{u}\|^2_{\boldsymbol{V}^{1/2}(\Gamma)}.$$

Proof. The first property follows immediately from the definition of $\boldsymbol{T}$. Notice that $\boldsymbol{D}$ maps $\boldsymbol{V}^{1/2}(\Gamma)$ into $\boldsymbol{H}^{-1/2}(\Gamma)$ which is continuously embedded in $\boldsymbol{V}^{-1/2}(\Gamma)$ by duality.

Next, by (4.8), (2.13), and (4.6) we have

$$\begin{aligned}\langle \boldsymbol{T}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma &= \|\boldsymbol{E}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha\langle \boldsymbol{D}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma = \|\boldsymbol{y}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha\|\nabla\boldsymbol{y}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\geqslant \min(1,\alpha)\|\boldsymbol{y}_{\boldsymbol{u}}\|^2_{\boldsymbol{H}^1(\Omega)} \geqslant C_2\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{1/2}(\Gamma)} = C_2\|\boldsymbol{u}\|^2_{\boldsymbol{V}^{1/2}(\Gamma)},\end{aligned}$$

where we used the trace theorem in the last inequality. ∎

Next, we give more insights into the structure of the solution to problem (4.7). The functional $J_{1/2}$ in problem (4.7) is Fréchet differentiable with respect to $\boldsymbol{u}$. Furthermore, for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}^{1/2}(\Gamma)$, by (4.7) and (4.8) we have

$$\begin{aligned}J'_{1/2}(\boldsymbol{u})\boldsymbol{v} &= \langle \boldsymbol{T}\boldsymbol{u}-\boldsymbol{w},\boldsymbol{v}\rangle_\Gamma = \langle \alpha\boldsymbol{D}\boldsymbol{u} + \boldsymbol{E}^\star(\boldsymbol{E}\boldsymbol{u}-\boldsymbol{y}_d),\boldsymbol{v}\rangle_\Gamma\\ &= \langle \alpha(\partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}} - p_{\boldsymbol{u}}\boldsymbol{n}) - (\partial_{\boldsymbol{n}}\boldsymbol{z}(\boldsymbol{y}_{\boldsymbol{u}}-\boldsymbol{y}_d) - q(\boldsymbol{y}_{\boldsymbol{u}}-\boldsymbol{y}_d)\boldsymbol{n}),\boldsymbol{v}\rangle_\Gamma,\end{aligned}$$

where we used (4.5), (2.13), and (2.14) in the last equality.

Now we are in the position to derive the regularity of the solution to the minimization problem (4.7).

THEOREM 4.4. *Let ξ be the singular exponent for the Stokes operator defined in section 2. Assume $\boldsymbol{y}_d \in \boldsymbol{H}^m(\Omega)$ for some $0 \leqslant m < \min\{2, 1+\xi\}$, and let $\bar{\boldsymbol{u}} \in \boldsymbol{V}^{1/2}(\Gamma)$ be the optimal solution of problem (4.7). Then $\bar{\boldsymbol{u}} \in \boldsymbol{V}^{1/2+r}(\Gamma)$ for all $r < \min\{1, \xi\}$ and there exist $\bar{\boldsymbol{y}} \in \boldsymbol{V}^{1+r}(\Omega)$, $\bar{p} \in H^r(\Omega) \cap L^2_0(\Omega)$, $\bar{\boldsymbol{z}} \in \boldsymbol{V}^{1+t}(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$, and $\bar{q} \in H^t(\Omega) \cap L^2_0(\Omega)$ for all $t \leqslant 1+m$ such that $t < \xi$ that satisfy the state equation*

$$-\Delta \bar{\boldsymbol{y}} + \nabla \bar{p} = \boldsymbol{0} \text{ in } \Omega, \quad \nabla \cdot \bar{\boldsymbol{y}} = 0 \text{ in } \Omega, \quad \bar{\boldsymbol{y}} = \bar{\boldsymbol{u}} \text{ on } \Gamma;$$

the adjoint state equation

$$-\Delta \bar{\boldsymbol{z}} + \nabla \bar{q} = \bar{\boldsymbol{y}} - \boldsymbol{y}_d \text{ in } \Omega, \quad \nabla \cdot \bar{\boldsymbol{z}} = 0 \text{ in } \Omega, \quad \bar{\boldsymbol{z}} = 0 \quad \text{on } \Gamma;$$

and the optimality condition

$$\langle \alpha(\partial_{\boldsymbol{n}} \bar{\boldsymbol{y}} - \bar{p}\boldsymbol{n}) - (\partial_{\boldsymbol{n}} \bar{\boldsymbol{z}} - \bar{q}\boldsymbol{n}), \boldsymbol{v} \rangle_\Gamma = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{1/2}(\Gamma).$$

Moreover, there exists $\bar{\lambda} \in \mathbb{R}$ such that

$$\alpha(\partial_{\boldsymbol{n}} \bar{\boldsymbol{y}} - \bar{p}\boldsymbol{n}) = \partial_{\boldsymbol{n}} \bar{\boldsymbol{z}} - (\bar{q} + \bar{\lambda})\boldsymbol{n}. \tag{4.9}$$

Here, both the state equation and the adjoint state equation must be understood in the variational sense.

Proof. The minimization problem, being a convex problem, is equivalent to the following Euler–Lagrange equation:

$$J'_{1/2}(\boldsymbol{u})\boldsymbol{v} = \langle \boldsymbol{T}\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{v} \rangle_\Gamma = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{1/2}(\Gamma). \tag{4.10}$$

The existence of a unique solution follows immediately from the Lax–Milgram theorem and Lemma 4.3. First order optimality conditions follow in a standard way. Taking $\bar{\lambda} = \lambda(\bar{z}, \bar{q})$, we deduce relation (4.9) as we did for the $\boldsymbol{L}^2(\Gamma)$-regularized problem.

Since $\bar{\boldsymbol{u}} \in \boldsymbol{V}^{1/2}(\Gamma)$, by Theorem 2.4 we have that $\bar{\boldsymbol{y}} \in \boldsymbol{V}^1(\Omega)$. From Theorems 2.1 and 2.2, we obtain $\bar{\boldsymbol{z}} \in \boldsymbol{V}^{1+t}(\Omega)$ and $\bar{q} \in H^t(\Omega) \cap L^2_0(\Omega)$ for all $t \leqslant \min\{2, 1+m\}$ with $t < \xi$. Using the trace theorem (see [29, Theorem 1.5.2.1]) we arrive at

$$\boldsymbol{e} := \partial_{\boldsymbol{n}} \bar{\boldsymbol{z}} - (\bar{q} + \bar{\lambda})\boldsymbol{n} \in \prod_{i=1}^{n} \boldsymbol{H}^{t-1/2}(\Gamma_i) \subset \prod_{i=1}^{n} \boldsymbol{H}^{r-1/2}(\Gamma_i) \quad \forall r < \min\{1, \xi\}.$$

From the trace theorem again on polygons (see [29, Theorem 1.5.2.1] and also [22, Remark 1.1, Chapter 1]), we know that there exists some $\boldsymbol{Y} \in \boldsymbol{H}^{1+r}(\Omega)$ such that $\partial_{\boldsymbol{n}} \boldsymbol{Y} = \boldsymbol{e}/\alpha$ on Γ. So we have that $\boldsymbol{F} = \Delta \boldsymbol{Y} \in \boldsymbol{H}^{r-1}(\Omega)$ and $H = -\nabla \cdot \boldsymbol{Y} \in H^r(\Omega)$. Using the state equation and the optimality condition (4.9), we deduce that the pair $(\bar{\boldsymbol{y}} - \boldsymbol{Y}, \bar{p})$ satisfies

$$-\Delta(\bar{\boldsymbol{y}} - \boldsymbol{Y}) + \nabla \bar{p} = \boldsymbol{F} \text{ in } \Omega, \quad \nabla \cdot (\bar{\boldsymbol{y}} - \boldsymbol{Y}) = H \text{ in } \Omega, \quad \partial_{\boldsymbol{n}}(\bar{\boldsymbol{y}} - \boldsymbol{Y}) - \bar{p}\boldsymbol{n} = 0 \text{ on } \Gamma.$$

This problem has a variational solution, which is unique up to a constant. Noticing that the singular exponents for the Stokes problem with Neumann boundary conditions are the same as those for Dirichlet boundary conditions (see, e.g., [50, pages 191–192]), we deduce from Theorem 2.1 that $\bar{\boldsymbol{y}} \in \boldsymbol{H}^{1+r}(\Omega)$. From the standard trace theorem, we have that $\bar{\boldsymbol{u}} \in \boldsymbol{H}^{r+1/2}(\Gamma)$. ∎

Remark 4.5. In this case, the optimal control is a continuous function even for problems posed on nonconvex domains; see the second subfigure of Figure 4 in Example 6.3 below.

In order to use the Aubin–Nitsche technique to obtain error estimates in $\boldsymbol{L}^2(\Gamma)$ for the control variable, we are also going to study, for any given $\boldsymbol{\eta} \in \boldsymbol{L}^2(\Gamma)$, the regularity of the unique solution $\boldsymbol{u}_{\boldsymbol{\eta}} \in \boldsymbol{V}^{1/2}(\Gamma)$ of the problem

$$\langle \boldsymbol{T}\boldsymbol{u}_{\boldsymbol{\eta}}, \boldsymbol{v}\rangle_\Gamma = (\boldsymbol{\eta}, \boldsymbol{v})_\Gamma \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{1/2}(\Gamma).$$

A straightforward computation, using the definitions of $\boldsymbol{T}$, $\boldsymbol{D}$, and $\boldsymbol{E}$, gives

$$\langle \boldsymbol{T}\boldsymbol{u}_{\boldsymbol{\eta}} - \boldsymbol{\eta}, \boldsymbol{v}\rangle_\Gamma = \langle \alpha(\partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}} - p_{\boldsymbol{u}_{\boldsymbol{\eta}}}\boldsymbol{n}) - (\partial_{\boldsymbol{n}}\boldsymbol{z}(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}) - q(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}})\boldsymbol{n}) - \boldsymbol{\eta}, \boldsymbol{v}\rangle_\Gamma$$

for all $\boldsymbol{v} \in \boldsymbol{V}^{1/2}(\Gamma)$. So we have that there exists some $\lambda \in \mathbb{R}$ such that $(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}, p_{\boldsymbol{u}_{\boldsymbol{\eta}}})$ solves the following Neumann problem:

$$\begin{gathered} -\Delta \boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}} + \nabla p_{\boldsymbol{u}_{\boldsymbol{\eta}}} = 0 \text{ in } \Omega, \quad \nabla \cdot \boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}} = 0 \quad \text{in } \Omega, \\ \alpha(\partial_{\boldsymbol{n}}\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}} - p_{\boldsymbol{u}_{\boldsymbol{\eta}}}\boldsymbol{n}) = \partial_{\boldsymbol{n}}\boldsymbol{z}(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}) - (q(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}) + \lambda)\boldsymbol{n} + \boldsymbol{\eta} \text{ on } \Gamma. \end{gathered}$$

Now we can follow the reasoning of Theorem 4.4. In this case

$$\boldsymbol{e} := \partial_{\boldsymbol{n}}\boldsymbol{z}(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}) - (q(\boldsymbol{y}_{\boldsymbol{u}_{\boldsymbol{\eta}}}) + \lambda)\boldsymbol{n} + \boldsymbol{\eta} \in \boldsymbol{L}^2(\Gamma),$$

so we are in the same situation as before, but with $t = 1/2$, which leads to $\boldsymbol{u}_{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\Gamma)$. Notice that we do not need convexity to obtain this result.

5. Finite element method for the Stokes Dirichlet energy space control problem.

In this section, we consider finite element approximations to the optimal control problem (4.7). We also briefly mention finite element approximations to the problem (3.1) in Remarks 5.5, 5.6, and 5.12.

5.1. Discretization of the problem.

First, we assume that the finite dimensional spaces $\boldsymbol{Y}_h \subset \boldsymbol{H}^1(\Omega)$ and $W_h \subset L^2(\Omega)$ satisfy the inf-sup condition: For each $p_h \in W_h$ there exists a $\boldsymbol{y}_h \in \boldsymbol{Y}_h$ such that

$$\int_\Omega p_h \nabla \cdot \boldsymbol{y}_h dx = \|p_h\|^2_{L^2(\Omega)} \text{ and } \|\boldsymbol{y}_h\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\|p_h\|_{L^2(\Omega)}.$$

It is well known that the $\mathcal{P}_1+$ bubble -$\mathcal{P}_1$ "Mini" element or the $\mathcal{P}_{k+1} - \mathcal{P}_k$, $k \geqslant 1$, "Taylor–Hood" element satisfies the inf-sup condition.

Let $\boldsymbol{Y}_h^0 := \boldsymbol{Y}_h \cap \boldsymbol{H}_0^1(\Omega)$, $W_h^0 = W_h \cap L_0^2(\Omega)$, and $\boldsymbol{Y}_h(\Gamma) \subset \boldsymbol{H}^{1/2}(\Gamma)$ be the trace of $\boldsymbol{Y}_h$. Let the discrete control space be given by

$$\boldsymbol{U}_h := \{\boldsymbol{u}_h \in \boldsymbol{Y}_h(\Gamma) : \ (\boldsymbol{u}_h \cdot \boldsymbol{n}, 1)_\Gamma = 0\}. \tag{5.1}$$

Next, we define the discrete optimization problem:

$$\min_{\boldsymbol{u}_h \in \boldsymbol{U}_h} J_h(\boldsymbol{u}_h) = \frac{1}{2}\|\boldsymbol{E}_h\boldsymbol{u}_h - \boldsymbol{y}_{d,h}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}(\boldsymbol{D}_h\boldsymbol{u}_h, \boldsymbol{u}_h)_\Gamma, \tag{5.2}$$

where $\boldsymbol{y}_{d,h} \in \boldsymbol{Y}_h$ is a suitable approximation of $\boldsymbol{y}_d$ in the sense that $\|\boldsymbol{y}_{d,h} - \boldsymbol{y}_d\|_{\boldsymbol{L}^2(\Omega)} \leqslant Ch^r$, and the discrete operators $\boldsymbol{D}_h$ and $\boldsymbol{E}_h$ are given below. Here, and in the rest of the paper, $r < \min\{1, \xi\}$ is the exponent obtained in Theorem 4.4.

We define the operators $\boldsymbol{E}_h : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{L}^2(\Omega)$ and $P_h : \boldsymbol{H}^{1/2}(\Gamma) \to W_h^0$ by

$$\boldsymbol{E}_h\boldsymbol{u} = \boldsymbol{y}_h, \quad P_h\boldsymbol{u} = p_h. \tag{5.3}$$

Here $(\boldsymbol{y}_h, p_h) \in \boldsymbol{Y}_h \times W_h^0$ is the finite element approximation of $(\boldsymbol{y}_{\boldsymbol{u}}, p_{\boldsymbol{u}})$, i.e., $(\boldsymbol{y}_h, p_h)$ satisfies

$$
\begin{aligned}
(\nabla \boldsymbol{y}_h, \nabla \boldsymbol{\zeta}_h) - (p_h, \nabla \cdot \boldsymbol{\zeta}_h) &= 0 \qquad \forall \boldsymbol{\zeta}_h \in \boldsymbol{Y}_h^0, \\
(\chi_h, \nabla \cdot \boldsymbol{y}_h) &= 0 \qquad \forall \chi_h \in W_h^0, \\
\boldsymbol{y}_h &= \boldsymbol{Q}_h \boldsymbol{u} \quad \text{on } \Gamma,
\end{aligned} \tag{5.4}
$$

where $\boldsymbol{Q}_h \boldsymbol{u}$ is the $\boldsymbol{L}^2$-projection of $\boldsymbol{u}$ onto $\boldsymbol{U}_h$.

Next, we give the discrete approximation of the stress force on the boundary, as introduced in [32, section 3]. For any $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, we define $(\boldsymbol{z}_h(\boldsymbol{g}), q_h(\boldsymbol{g})) \in \boldsymbol{Y}_h^0 \times W_h^0$ to be the unique solution of

$$
\begin{aligned}
(\nabla \boldsymbol{z}_h(\boldsymbol{g}), \nabla \boldsymbol{\zeta}_h) - (q_h(\boldsymbol{g}), \nabla \cdot \boldsymbol{\zeta}_h) &= (\boldsymbol{g}, \boldsymbol{\zeta}_h) \qquad \forall \boldsymbol{\zeta}_h \in \boldsymbol{Y}_h^0, \\
(\chi_h, \nabla \cdot \boldsymbol{z}_h(\boldsymbol{g})) &= 0 \qquad \forall \chi_h \in W_h^0.
\end{aligned}
$$

For $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$, let $\boldsymbol{\psi}_h = \boldsymbol{z}_h(\boldsymbol{g}) + \boldsymbol{E}_h \boldsymbol{u}$ and $\phi_h = q_h(\boldsymbol{g}) + P_h \boldsymbol{u}$. We define $\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{u}) \in \boldsymbol{Y}_h(\Gamma)$ as the approximation of the stress force on the boundary of the pair $(\boldsymbol{\psi}_h, \phi_h)$:

$$
(\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{u}), \boldsymbol{\zeta}_h)_\Gamma = (\nabla \boldsymbol{\psi}_h, \nabla \boldsymbol{\zeta}_h) - (\phi_h, \nabla \cdot \boldsymbol{\zeta}_h) - (\boldsymbol{g}, \boldsymbol{\zeta}_h) \qquad \forall \boldsymbol{\zeta}_h \in \boldsymbol{Y}_h. \tag{5.5}
$$

Notice that this is exactly the concept of discrete normal derivative; see [8] or, better suited for our purposes, [53].

Remark 5.1. It is also important to notice that, for $\boldsymbol{v}_h \in \boldsymbol{Y}_h(\Gamma)$, we have that

$$
(\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{u}), \boldsymbol{v}_h)_\Gamma = (\nabla \boldsymbol{\psi}_h, \nabla \boldsymbol{R}_h \boldsymbol{v}_h) - (\phi_h, \nabla \cdot \boldsymbol{R}_h \boldsymbol{v}_h) - (\boldsymbol{g}, \boldsymbol{R}_h \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{Y}_h(\Gamma) \tag{5.6}
$$

for any linear extension operator $\boldsymbol{R}_h : \boldsymbol{Y}_h(\Gamma) \to \boldsymbol{Y}_h$. For instance, $\boldsymbol{R}_h$ could be the discrete harmonic extension, the operator $\boldsymbol{E}_h$ as in Lemma 5.3, or the zero extension, which we use in section 5.2.

For $\boldsymbol{u} \in \boldsymbol{V}^{1/2}(\Gamma)$ we define $\boldsymbol{D}_h$ as the approximation of the stress force on the boundary of the pair $(\boldsymbol{E}_h \boldsymbol{u}, P_h \boldsymbol{u})$:

$$
\boldsymbol{D}_h \boldsymbol{u} = \boldsymbol{t}_h(\boldsymbol{0}, \boldsymbol{u}). \tag{5.7}
$$

LEMMA 5.2. *The mapping $\boldsymbol{u}_h \mapsto (\boldsymbol{D}_h \boldsymbol{u}_h, \boldsymbol{u}_h)_\Gamma^{1/2}$ is a seminorm in $\boldsymbol{U}_h$.*

Proof. Notice that for $\boldsymbol{u}_h \in \boldsymbol{U}_h \subset \boldsymbol{V}^{1/2}(\Gamma)$, using that $\boldsymbol{E}_h \boldsymbol{u}_h \in \boldsymbol{Y}_h$ and $P_h \boldsymbol{u}_h \in W_h^0$, by (5.7), (5.3), and (5.5) we have

$$
\begin{aligned}
(\boldsymbol{D}_h \boldsymbol{u}_h, \boldsymbol{u}_h)_\Gamma = (\boldsymbol{t}_h(\boldsymbol{0}, \boldsymbol{u}_h), \boldsymbol{E}_h \boldsymbol{u}_h)_\Gamma &= (\nabla \boldsymbol{E}_h \boldsymbol{u}_h, \nabla \boldsymbol{E}_h \boldsymbol{u}_h) - (P_h \boldsymbol{u}_h, \nabla \cdot \boldsymbol{E}_h \boldsymbol{u}_h) \\
&= (\nabla \boldsymbol{E}_h \boldsymbol{u}_h, \nabla \boldsymbol{E}_h \boldsymbol{u}_h),
\end{aligned}
$$

where we used $(q_h, \nabla \cdot \boldsymbol{E}_h \boldsymbol{u}_h) = 0$ for all $q_h \in W_h^0$ in the last equality. The assertion now follows trivially from the linearity of $\boldsymbol{E}_h$. ∎

LEMMA 5.3. *For every $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{v}_h \in \boldsymbol{U}_h$, we have that*

$$
(\boldsymbol{g}, \boldsymbol{E}_h \boldsymbol{v}_h) = (-\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{0}), \boldsymbol{v}_h)_\Gamma,
$$

and the adjoint of the restriction of $\boldsymbol{E}_h$ to $\boldsymbol{U}_h$ is given by

$$
\boldsymbol{E}_h^\star \boldsymbol{g} = -\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{0}). \tag{5.8}
$$

Proof. We define $\boldsymbol{G}_h\boldsymbol{g}$ as the discrete approximation of the negative stress force on the boundary of the pair $(\boldsymbol{z}_h(\boldsymbol{g}), q_h(\boldsymbol{g}))$:

$$\boldsymbol{G}_h\boldsymbol{g} = -\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{0}).$$

Consider $\boldsymbol{v}_h \in \boldsymbol{U}_h$; notice that $\boldsymbol{E}_h\boldsymbol{v}_h \in \boldsymbol{Y}_h$, and by definition it equals $\boldsymbol{v}_h$ on the boundary. Using (5.6) for $\boldsymbol{R}_h = \boldsymbol{E}_h$ and the facts $q_h(\boldsymbol{g})$, $P_h\boldsymbol{v}_h \in W_h^0$, which imply that both $(q_h(\boldsymbol{g}), \nabla \cdot \boldsymbol{E}_h\boldsymbol{v}_h) = 0$ and $(P_h\boldsymbol{v}_h, \nabla \cdot z_h(\boldsymbol{g})) = 0$, we obtain

$$\begin{aligned}(\boldsymbol{G}_h\boldsymbol{g}, \boldsymbol{v}_h)_\Gamma &= -(\nabla \boldsymbol{z}_h(\boldsymbol{g}), \nabla \boldsymbol{E}_h\boldsymbol{v}_h) + (q_h(\boldsymbol{g}), \nabla \cdot \boldsymbol{E}_h\boldsymbol{v}_h) + (\boldsymbol{g}, \boldsymbol{E}_h\boldsymbol{v}_h)\\ &= -(P_h\boldsymbol{v}_h, \nabla \cdot \boldsymbol{z}_h(\boldsymbol{g})) + (\boldsymbol{g}, \boldsymbol{E}_h\boldsymbol{v}_h)\\ &= (\boldsymbol{g}, \boldsymbol{E}_h\boldsymbol{v}_h),\end{aligned}$$

and the proof is complete. ∎

LEMMA 5.4. *Problem* (5.2) *has a unique solution* $\bar{\boldsymbol{u}}_h$.

Proof. By Lemma 5.2, it is standard to deduce that J_h is coercive in $\boldsymbol{U}_h$. Since it is also strictly convex, problem (5.2) has a unique solution $\bar{\boldsymbol{u}}_h$. ∎

Following the same notation in section 4, we define

$$\boldsymbol{T}_h = \alpha\boldsymbol{D}_h + \boldsymbol{E}_h^\star\boldsymbol{E}_h, \qquad \boldsymbol{w}_h = \boldsymbol{E}_h^\star\boldsymbol{y}_{d,h}. \tag{5.9}$$

Then the problem (5.2) can be rewritten as

$$J_h(\boldsymbol{u}_h) = \frac{1}{2}(\boldsymbol{T}_h\boldsymbol{u}_h, \boldsymbol{u}_h)_\Gamma - (\boldsymbol{w}_h, \boldsymbol{u}_h)_\Gamma + \frac{1}{2}\|\boldsymbol{y}_{d,h}\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{5.10}$$

and the unique solution $\bar{\boldsymbol{u}}_h$ of the discrete problem satisfies the first order optimality condition

$$(\boldsymbol{T}_h\bar{\boldsymbol{u}}_h, \boldsymbol{v}_h)_\Gamma = (\boldsymbol{w}_h, \boldsymbol{v}_h)_\Gamma \quad \forall \boldsymbol{v}_h \in \boldsymbol{U}_h. \tag{5.11}$$

Remark 5.5. The discretization of the problem (3.1) is done in the same way. The solution of the discrete problem satisfies

$$(\alpha\boldsymbol{u}_{0h} + \boldsymbol{E}_h^\star\boldsymbol{E}_h\boldsymbol{u}_{0h}, \boldsymbol{v}_h)_\Gamma = (\boldsymbol{w}_h, \boldsymbol{v}_h)_\Gamma \quad \forall \boldsymbol{v}_h \in \boldsymbol{U}_h.$$

Thanks to the remarkable result [4, Theorem 5.2], the approximation of the transposition solution can be done using the discrete weak formulation given to compute $\boldsymbol{E}_h$.

5.2. Efficient computation of the gradient of the objective functional. Define $N = \dim \boldsymbol{Y}_h$, $M = \dim W_h$, and $N_\Gamma = \dim \boldsymbol{Y}_h(\Gamma)$. Let $\mathcal{T} \in \mathbb{R}^{N_\Gamma \times N_\Gamma}$ be the matrix representation of $\boldsymbol{T}_h$, i.e., $\underline{v}^T\mathcal{T}\underline{u} = (\boldsymbol{T}_h\boldsymbol{u}_h, \boldsymbol{v}_h)_\Gamma$ for all $\boldsymbol{u}_h$ and $\boldsymbol{v}_h \in \boldsymbol{Y}_h(\Gamma)$.

Let $\mathcal{M}$ denote the mass matrix representing the standard inner product in $\boldsymbol{L}^2(\Omega)$, and let $\mathcal{K}$ denote the stiffness matrix representing the vector Laplace operator on the finite element space $\boldsymbol{Y}_h$. Additionally, $\mathcal{B}$ denotes the matrix representation of the divergence operator on the involved finite element spaces $\boldsymbol{Y}_h$ and W_h. We impose the condition $p_M = 0$, and instead of $\mathcal{B}$, we use the corresponding $\tilde{\mathcal{B}}$ eliminating row M. For $\mathcal{X} \in \{\mathcal{M}, \mathcal{K}, \tilde{\mathcal{B}}, \tilde{\mathcal{B}}^T\}$, denote $\mathcal{X}_{00}$, $\mathcal{X}_{\Gamma\Gamma}$, $\mathcal{X}_{\Gamma 0}$, $\mathcal{X}_{0\Gamma}$, respectively, the submatrices whose entries are indexed in interior and/or boundary nodes and, following MATLAB colon notation, $\mathcal{X}_{0:}$,$\mathcal{X}_{:0}$, $\mathcal{X}_{\Gamma :}$ and $\mathcal{X}_{:\Gamma}$ denote the submatrices whose row or column indexes are interior or boundary, respectively. Finally, we denote $\mathcal{S}_{\Gamma\Gamma}$ the mass matrix representing the standard inner product in $L^2(\Gamma)$. Notice also that we will use the conventions $\tilde{\mathcal{B}}_{0:}^T = (\tilde{\mathcal{B}}_{:0})^T$ and $\tilde{\mathcal{B}}_{\Gamma:}^T = (\tilde{\mathcal{B}}_{:\Gamma})^T$.

Let $\underline{w} \in \mathbb{R}^{N_\Gamma}$ be the vector representation of $\boldsymbol{w}_h$, and define $\underline{b} = \mathcal{S}_{\Gamma\Gamma}\underline{w}$ so that $\underline{u}^T\underline{b} = (\boldsymbol{w}_h, \boldsymbol{u}_h)_\Gamma$ for all $\boldsymbol{u}_h \in \boldsymbol{Y}_h(\Gamma)$. Since the normal vector $\boldsymbol{n}$ is piecewise constant, we know that there exists $\underline{c} \in \mathbb{R}^{N_\Gamma}$ such that $\underline{u}^T\underline{c} = (\boldsymbol{u}_h, \boldsymbol{n})_\Gamma$ for all $\boldsymbol{u}_h \in \boldsymbol{Y}_h(\Gamma)$. Then the problem (5.10) is equivalent to

$$\begin{cases} \min \ \dfrac{1}{2}\underline{u}^T\mathcal{T}\underline{u} - \underline{u}^T\underline{b}, \\ \underline{u} \in \mathbb{R}^{N_\Gamma}, \quad \underline{u}^T\underline{c} = 0. \end{cases} \tag{5.12}$$

We show how to compute $\underline{b}$ and $\mathcal{T}\underline{u}$.

Given $\underline{u}$, the vector representation of $\boldsymbol{u}_h \in \boldsymbol{Y}_h(\Gamma)$, its related state and pressure can be computed by solving, for $(\underline{y}_I, \underline{\tilde{p}})$,

$$\mathcal{K}_{00}\underline{y}_I + \tilde{\mathcal{B}}_{0:}^T\underline{\tilde{p}} = -\mathcal{K}_{\Gamma 0}^T\underline{u}, \qquad \tilde{\mathcal{B}}_{:0}\underline{y}_I = -\tilde{\mathcal{B}}_{:\Gamma}\underline{u}$$

and recovering $\underline{y} = (\underline{y}_I, \underline{u})^T$. Given $\underline{y}$, the vector representation of $\boldsymbol{y}_h \in \boldsymbol{Y}_h$, the dual state and pressure $(\boldsymbol{z}_h(\boldsymbol{y}_h), q_h(\boldsymbol{y}_h))$ can be computed by solving

$$\mathcal{K}_{00}\underline{z} + \tilde{\mathcal{B}}_{0:}^T\underline{\tilde{q}} = \mathcal{M}_{0:}\underline{y}, \qquad \tilde{\mathcal{B}}_{:0}\underline{z} = \underline{0}.$$

For an efficient computation, it is important to consider the discrete extension operator $\boldsymbol{R}_h\boldsymbol{v}_h \in \boldsymbol{Y}_h$ such that $\boldsymbol{R}_h\boldsymbol{v}_h = \boldsymbol{v}_h$ on Γ and $\boldsymbol{R}_h\boldsymbol{v}_h = \boldsymbol{0}$ in the interior nodes of Ω; see Remark 5.1.

By the definition of $\boldsymbol{w}_h$ in (5.9) and using (5.8) and (5.6) we have

$$\begin{aligned}(\boldsymbol{w}_h, \boldsymbol{v}_h)_\Gamma &= (\boldsymbol{E}_h^\star \boldsymbol{y}_{d,h}, \boldsymbol{v}_h)_\Gamma \\ &= -(\nabla \boldsymbol{z}_h(\boldsymbol{y}_{d,h}), \nabla \boldsymbol{R}_h\boldsymbol{v}_h) + (q_h(\boldsymbol{y}_{d,h}), \nabla\cdot \boldsymbol{R}_h\boldsymbol{v}_h) + (\boldsymbol{y}_{d,h}, \boldsymbol{R}_h\boldsymbol{v}_h) \\ &= (-\mathcal{K}_{\Gamma 0}\underline{z}_d - \tilde{\mathcal{B}}_{\Gamma:}^T\underline{\tilde{q}}_d + \mathcal{M}_{\Gamma:}\underline{y}_d)\cdot \underline{v},\end{aligned}$$

where $(\underline{z}_d, \underline{\tilde{q}}_d)$ is the vector representation of $(\boldsymbol{z}_h(\boldsymbol{y}_{d,h}), q_h(\boldsymbol{y}_{d,h}))$. In the same way, using (5.3) and denoting $(\underline{z}, \underline{\tilde{q}})$ the vector representation of $(\boldsymbol{z}_h(\boldsymbol{y}_h), q_h(\boldsymbol{y}_h))$,

$$(\boldsymbol{E}_h^\star \boldsymbol{E}_h\boldsymbol{u}_h, \boldsymbol{v}_h)_\Gamma = (\boldsymbol{E}_h^\star \boldsymbol{y}_h, \boldsymbol{v}_h)_\Gamma = (-\mathcal{K}_{\Gamma 0}\underline{z} - \tilde{\mathcal{B}}_{\Gamma:}^T\underline{\tilde{q}} + \mathcal{M}_{\Gamma:}\underline{y})\cdot\underline{v}.$$

Finally, to obtain the matrix representation of the perturbed Steklov–Poincaré operator $\boldsymbol{D}_h$ we use (5.7) and (5.6) to obtain

$$(\boldsymbol{D}_h\boldsymbol{u}_h, \boldsymbol{v}_h)_\Gamma = (\nabla \boldsymbol{E}_h\boldsymbol{u}_h, \nabla \boldsymbol{R}_h\boldsymbol{v}_h) - (P_h\boldsymbol{u}_h, \nabla\cdot\boldsymbol{R}_h\boldsymbol{v}_h) = (\mathcal{K}_{\Gamma:}\underline{y} + \tilde{\mathcal{B}}_{\Gamma:}^T\underline{\tilde{p}})\cdot\underline{v}.$$

The Lagrange multiplier related to the constraint, which plays a role in the case of $\boldsymbol{L}^2(\Gamma)$-regularization (see Remark 3.3), can also be recovered by means of

$$\lambda = \frac{\underline{c}^T(\mathcal{T}\underline{u} - \underline{b})}{\underline{c}^T\underline{c}}.$$

Remark 5.6. To solve the $\boldsymbol{L}^2$-regularized problem, the procedure is very similar. The only difference (cf. [45]) is the computation of $\mathcal{T}\underline{u}$, which is done in the following way:

$$\mathcal{T}\underline{u} = \alpha\mathcal{S}_{\Gamma\Gamma}\underline{u} + \mathcal{M}_{\Gamma:}\underline{y} - \mathcal{K}_{\Gamma 0}\underline{z} - \tilde{\mathcal{B}}_{\Gamma:}^T\underline{\tilde{q}}.$$

An approximation of the quantity λ_0 can be done using the Lagrange multiplier by means of $\lambda_0 = -\lambda/|\Gamma|$.

5.3. Error analysis. Now, we state the main result in this section.

THEOREM 5.7. *Let $\bar{\boldsymbol{u}} \in \boldsymbol{V}^{r+1/2}(\Gamma)$, with $r < \min\{1, \xi\}$, be the unique solution of problem (4.7), and let $\bar{\boldsymbol{u}}_h \in \boldsymbol{U}_h$ be the solution of (5.2). If the conditions in Theorem 4.4 are all fulfilled, then*

$$\|\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant Ch^r \|\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}.$$

We recall that ξ is the singular exponent for the Stokes operator defined in section 2.

To prove Theorem 5.7, we assume that the following approximation properties are satisfied (see [22, Chapter II, section 1.3]):

(H1) There exists an operator $r_h \in \mathcal{L}(\boldsymbol{H}^2(\Omega), \boldsymbol{Y}_h)$ such that

$$\|\boldsymbol{y} - r_h \boldsymbol{y}\|_{\boldsymbol{H}^1(\Omega)} \leqslant Ch\|\boldsymbol{y}\|_{\boldsymbol{H}^2(\Omega)} \quad \forall \boldsymbol{y} \in \boldsymbol{H}^2(\Omega),$$

r_h preserves the boundary conditions, and

$$\|\boldsymbol{u} - r_h \boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant Ch\|\boldsymbol{u}\|_{\Pi_{i=1}^n \boldsymbol{H}^{3/2}(\Gamma_i)} \quad \forall \boldsymbol{u} \in \text{trace } \boldsymbol{H}^2(\Omega).$$

(H2) There exists an operator $S_h \in \mathcal{L}(L^2(\Omega), W_h)$ such that

$$\|p - S_h p\|_{L^2(\Omega)} \leqslant Ch\|p\|_{H^1(\Omega)} \quad \forall p \in H^1(\Omega).$$

These assumptions are satisfied by typical finite element spaces used to solve the Stokes equation, such as the $\mathcal{P}_1+$ bubble -$\mathcal{P}_1$ "Mini" element or the $\mathcal{P}_{k+1} - \mathcal{P}_k$, $k \geqslant 1$, "Taylor–Hood" element; see [22, Chapter II, sections. 4.1 and 4.2], where we take r_h to be the corresponding Lagrange interpolation operator and S_h the $L^2(\Omega)$-projection. We note also that, for $r \leq 1$, the $\boldsymbol{L}^2(\Gamma)$-projection $\boldsymbol{Q}_h$ satisfies the following standard estimate:

$$\|\boldsymbol{Q}_h \boldsymbol{u} - \boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant Ch^r \|\boldsymbol{u}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)} \ \forall \boldsymbol{u} \in \boldsymbol{V}^{r+1/2}(\Gamma). \tag{5.13}$$

LEMMA 5.8. *There exists a constant $C > 0$ independent of h such that for any $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{V}^{1/2}(\Gamma)$ we have*

$$\|\boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{v})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant C(\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{H}^{1/2}(\Gamma)}).$$

Moreover, if $\boldsymbol{v} \in \boldsymbol{V}^{r+1/2}(\Gamma)$, with $r < \min\{1, \xi\}$, we have the error estimate

$$\|\boldsymbol{t}(\boldsymbol{g}, \boldsymbol{v}) - \boldsymbol{t}_h(\boldsymbol{g}, \boldsymbol{v})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant Ch^r(\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}).$$

Proof. The results follow directly from [32, Proposition 17] and Theorems 2.2 and 2.4. ◻

In the next lemma, we collect the approximation properties of $\boldsymbol{E}_h$, $\boldsymbol{E}_h^\star$, and $\boldsymbol{D}_h$ that will be used to obtain the final error estimate.

LEMMA 5.9. *The approximate solution operators $\boldsymbol{E}_h : \boldsymbol{V}^{1/2}(\Gamma) \to \boldsymbol{L}^2(\Omega)$, $\boldsymbol{E}_h^\star : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}^{-1/2}(\Gamma)$, $\boldsymbol{D}_h : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{H}^{-1/2}(\Gamma)$ are bounded, i.e., there exists a constant $C > 0$ independent of h such that*

$$\|\boldsymbol{E}_h \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)}, \tag{5.14a}$$

$$\|\boldsymbol{E}_h^\star \boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}, \tag{5.14b}$$

$$\|\boldsymbol{D}_h \boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{5.14c}$$

Moreover, for $\boldsymbol{u} \in \boldsymbol{V}^{r+1/2}(\Gamma)$ *and* $\boldsymbol{g} \in \boldsymbol{H}^r(\Omega)$*, with* $r < \min\{1, \xi\}$*, the following error estimates hold:*

$$\|\boldsymbol{E}\boldsymbol{u} - \boldsymbol{E}_h\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leqslant Ch^r\|\boldsymbol{u}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}, \tag{5.15a}$$

$$\|\boldsymbol{E}^\star\boldsymbol{g} - \boldsymbol{E}_h^\star\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant Ch^r\|\boldsymbol{g}\|_{\boldsymbol{H}^r(\Omega)}, \tag{5.15b}$$

$$\|\boldsymbol{D}\boldsymbol{u} - \boldsymbol{D}_h\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant Ch^r\|\boldsymbol{u}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}. \tag{5.15c}$$

Proof. The boundness of $\boldsymbol{E}_h$ and the approximation error follow directly from [32, Theorem 15] and the continuous embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$. The remaining estimates can be easily obtained by Lemma 5.8, (5.8), and (5.7). ∎

Next, we introduce the following auxiliary problem: find $\widehat{\boldsymbol{u}}_h \in \boldsymbol{U}_h$ such that

$$\langle \boldsymbol{T}\widehat{\boldsymbol{u}}_h, \boldsymbol{v}_h\rangle_\Gamma = \langle \boldsymbol{w}, \boldsymbol{v}_h\rangle_\Gamma \quad \forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \tag{5.16}$$

where $\boldsymbol{w} = \boldsymbol{E}^\star\boldsymbol{y}_d \in \boldsymbol{V}^{-1/2}(\Gamma)$.

LEMMA 5.10. *Let* $\bar{\boldsymbol{u}} \in \boldsymbol{V}^{r+1/2}(\Gamma)$*, with* $r < \min\{1, \xi\}$*, be the unique solution of problem* (4.7) *and* $\widehat{\boldsymbol{u}}_h \in \boldsymbol{U}_h$ *be the solution of* (5.16)*. If the conditions in Theorem* 4.4 *are all fulfilled, then*

$$\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant Ch^r\|\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2+r}(\Gamma)}. \tag{5.17}$$

Proof. First, by (4.10), (5.16), and $\boldsymbol{U}_h \subset \boldsymbol{V}^{1/2}(\Gamma)$, we have

$$\langle \boldsymbol{T}(\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h), \boldsymbol{v}_h\rangle_\Gamma = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{U}_h. \tag{5.18}$$

Next, by Lemma 4.3, we know that $\boldsymbol{T}$ is $\boldsymbol{V}^{1/2}(\Gamma)$-elliptic and continuous. For any $\boldsymbol{u}_h^\star \in \boldsymbol{U}_h$, the error estimate follows in a standard way:

$$\begin{aligned} c\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|^2_{\boldsymbol{H}^{1/2}(\Gamma)} &\leqslant \langle \boldsymbol{T}(\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h), \bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\rangle_\Gamma = \langle \boldsymbol{T}(\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h), \bar{\boldsymbol{u}} - \boldsymbol{u}_h^\star)\rangle_\Gamma \\ &\leqslant \|\boldsymbol{T}(\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h)\|_{\boldsymbol{V}^{-1/2}(\Gamma)}\|\bar{\boldsymbol{u}} - \boldsymbol{u}_h^\star\|_{\boldsymbol{H}^{1/2}(\Gamma)} \\ &\leqslant C\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}\|\bar{\boldsymbol{u}} - \boldsymbol{u}_h^\star\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \end{aligned}$$

Therefore, there exists $C > 0$ such that

$$\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant C \inf_{\boldsymbol{u}_h^\star \in \boldsymbol{U}_h} \|\bar{\boldsymbol{u}} - \boldsymbol{u}_h^\star\|_{\boldsymbol{H}^{1/2}(\Gamma)}.$$

The result follows by interpolation (see, e.g., [5, Theorem (14.3.3)]), taking $\boldsymbol{u}_h^\star = \boldsymbol{Q}_h\boldsymbol{u}$, and using the regularity of $\bar{\boldsymbol{u}}$ stated in Theorem 4.4 and estimate (5.13). ∎

Now we give the proof of Theorem 5.7.

Proof of Theorem 5.7. Due to Lemma 5.10, it is enough to obtain the error estimate for $\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$.

By the definition of $\boldsymbol{T}_h$ in (5.9) and Lemma 5.2, we know that $\boldsymbol{T}_h$ is coercive on $\boldsymbol{U}_h$. By the first order conditions satisfied by $\widehat{\boldsymbol{u}}_h$ and $\bar{\boldsymbol{u}}_h$ in (5.16) and (5.11) and by Cauchy–Schwarz inequality, we know that there exists a constant κ independent of h such that

$$\begin{aligned} \kappa\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|^2_{\boldsymbol{H}^{1/2}(\Gamma)} &\leqslant \langle \boldsymbol{T}_h(\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h), \bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\rangle_\Gamma \\ &= \langle \boldsymbol{w}_h - \boldsymbol{w}, \bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\rangle_\Gamma + \langle (\boldsymbol{T} - \boldsymbol{T}_h)\widehat{\boldsymbol{u}}_h, \bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\rangle_\Gamma \\ &\leqslant \|\boldsymbol{w}_h - \boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} \\ &\quad + \|(\boldsymbol{T} - \boldsymbol{T}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \end{aligned}$$

Hence, dividing by $\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$ and using the definitions of $\boldsymbol{T}$ and $\boldsymbol{T}_h$ in (4.8) and (5.9) we have

$$\begin{aligned}\kappa\|\bar{\boldsymbol{u}}_h - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} &\leqslant \|\boldsymbol{w}_h - \boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|(\boldsymbol{T} - \boldsymbol{T}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant \|\boldsymbol{w}_h - \boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|(\boldsymbol{D} - \boldsymbol{D}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\quad + \|(\boldsymbol{E}^\star\boldsymbol{E} - \boldsymbol{E}_h^\star\boldsymbol{E}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &= S_1 + S_2 + S_3.\end{aligned}$$

For the first term S_1, using the approximation properties of $\boldsymbol{y}_{d,h}$, (4.8), (5.9), (5.15b), and (5.14b), we get

$$\begin{aligned}S_1 &= \|\boldsymbol{w}_h - \boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} = \|\boldsymbol{E}_h^\star\boldsymbol{y}_{d,h} - \boldsymbol{E}^\star\boldsymbol{y}_d\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant \|(\boldsymbol{E}_h^\star - \boldsymbol{E}^\star)\boldsymbol{y}_d\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|\boldsymbol{E}_h^\star(\boldsymbol{y}_{d,h} - \boldsymbol{y}_d)\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \leqslant Ch^r.\end{aligned}$$

For the second term S_2, by the definition of $\boldsymbol{D}_h$ in (5.7) we know that $\boldsymbol{D}_h\boldsymbol{Q}_h = \boldsymbol{D}_h$, where $\boldsymbol{Q}_h$ is the L^2-projection. We have

$$\begin{aligned}S_2 &= \|(\boldsymbol{D} - \boldsymbol{D}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant \|\boldsymbol{D}(\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|\boldsymbol{D}\bar{\boldsymbol{u}} - \boldsymbol{D}_h\boldsymbol{Q}_h\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|\boldsymbol{D}_h(\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h)\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant C\|\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \|\boldsymbol{D}\bar{\boldsymbol{u}} - \boldsymbol{D}_h\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + C\|\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)},\end{aligned}$$

where we used (4.4b) and (5.14c) in the last inequality. Next, by (5.17), (5.15c), and (5.13) we have

$$\begin{aligned}S_2 &\leqslant C\|\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \|\boldsymbol{D}\bar{\boldsymbol{u}} - \boldsymbol{D}_h\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + C\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)} \\ &\quad + C\|\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \\ &\leqslant Ch^r\|\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}.\end{aligned}$$

Next, for the term S_3 we proceed similarly to S_2. Using the fact that $\boldsymbol{E}_h\boldsymbol{Q}_h = \boldsymbol{E}_h$, we have

$$\begin{aligned}S_3 &= \|(\boldsymbol{E}^\star\boldsymbol{E} - \boldsymbol{E}_h^\star\boldsymbol{E}_h)\widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant \|\boldsymbol{E}^\star\boldsymbol{E}(\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|(\boldsymbol{E}^\star - \boldsymbol{E}_h^\star)\boldsymbol{E}\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\quad + \|\boldsymbol{E}_h^\star(\boldsymbol{E}\bar{\boldsymbol{u}} - \boldsymbol{E}_h\boldsymbol{Q}_h\bar{\boldsymbol{u}})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|(\boldsymbol{E}_h^\star\boldsymbol{E}_h(\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h)\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leqslant C\|\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} + Ch^r\|\boldsymbol{E}\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{r+1}(\Omega)} + C\|\boldsymbol{E}\bar{\boldsymbol{u}} - \boldsymbol{E}_h\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad + C\|\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)},\end{aligned}$$

where we used (2.15), (5.15b), (5.14b), and (5.14a) in the last inequality. By (5.17), (5.15a), and (5.13) we have

$$\begin{aligned}S_3 &\leqslant C\|\widehat{\boldsymbol{u}}_h - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} + Ch^r\|\boldsymbol{E}\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{r+1}(\Omega)} + C\|\boldsymbol{E}\bar{\boldsymbol{u}} - \boldsymbol{E}_h\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad + C\|\boldsymbol{Q}_h\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \\ &\leqslant Ch^r\|\bar{\boldsymbol{u}}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}.\end{aligned}$$

Collecting all the estimates completes the proof. $\square$

Remark 5.11. The application of the Aubin–Nitsche technique to the intermediate problem leads easily to

$$\|\bar{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Gamma)} \leqslant Ch^{r+1/2}\|\boldsymbol{u}\|_{\boldsymbol{H}^{r+1/2}(\Gamma)}.$$

However, using this to obtain error estimates in $\boldsymbol{L}^2(\Gamma)$ for $\bar{\boldsymbol{u}}_h$ is not immediate because $\bar{\boldsymbol{u}}_h$ satisfies a problem with a perturbed operator and perturbed second member. Following [14, Remark 26.1], the error would be of the same order as

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\bar{\boldsymbol{u}}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)} + \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{H}^{-1/2}(\Gamma)}.$$

Using the improved error estimate for the discrete approximation of the stress force on the boundary for regular solutions in [32, Proposition 17], we find that the convergence order $r + 1/2$ for those terms can be achieved under the following two assumptions: first, that $\boldsymbol{y}_d \in \boldsymbol{H}^{r-1/2}(\Omega)$, which is quite reasonable, but also that $\bar{\boldsymbol{u}}_h$ is bounded in $\boldsymbol{H}^{1+r}(\Gamma)$. But this second assumption requires a higher regularity of the optimal solution; in such a case the order of convergence in $\boldsymbol{H}^{1/2}(\Gamma)$ would be increased by another $1/2$.

In numerical experiments, this is the behavior usually observed with the "Mini" finite element: order $3/2$ in $\boldsymbol{H}^{1/2}(\Gamma)$ and order 2 in $\boldsymbol{L}^2(\Gamma)$.

Remark 5.12. Although the discretizations of the $\boldsymbol{L}^2$-regularized problem and the $\boldsymbol{H}^{1/2}$-regularized problem are very similar, the error analysis performed for the second case cannot be carried out for the first because of the lack of regularity of the solution $\boldsymbol{u}_0 \in \boldsymbol{H}^s(\Gamma)$ for $0 \leqslant s < s^\star$, where $s^\star = \min\{1/2, \xi - 1/2\}$.

Using the general discretization error estimate of [2, Theorem 3.2], we see that the error is bounded by the best approximation error in the space, the error related to the discretization of the state equation, and the error related to the discrete approximation of the stress force on the boundary. While we have no results for the last two ones, the first one is determined by the Sobolev exponent s, so one cannot expect more than h^s for the error.

6. Numerical experiments. In this section we carry out some numerical experiments to compare the solutions of the two control problems (3.1) and (4.7), and we also illustrate how the convergence orders can vary due to the shape of the domain and the problem data. We present two examples in a square domain, the first one having a very regular solution, and one example in an L-shaped domain. We discretize each problem using the "Mini" finite element [40] and a family of meshes of size $h_i = 2^{-i}\sqrt{2}$ obtained by regular refinement of an initial coarse mesh of size $h_0 = \sqrt{2}$. For one problem, we also discretize using Taylor–Hood elements. Since we do not have the exact solution, we compare the obtained solutions for $i = 2, \ldots, I-2$ with the reference solution obtained for $i = I$, where $I = 9$ for the square (a mesh with $2 \times 2^{2\times 9} = 524288$ elements) and $I = 8$ for the L-shaped domain (a mesh with $6 \times 2^{2\times 8} = 393216$ elements).

Let $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$; we report the $\boldsymbol{L}^2(\Gamma)$-norm error and the $\boldsymbol{H}^{1/2}(\Gamma)$-seminorm error, both computed using the equivalent mesh-independent discrete norms obtained in [9].

Example 6.1. We consider the unit square domain $\Omega = (0,1)^2$ and set the regularization parameter $\alpha = 1.0e-3$. We choose the forcing $\boldsymbol{f} = (1,1)$, and for the target state we choose the large vortex given in [38],

$$\boldsymbol{y}_d = 200 \times [x_1^2(1-x_1)^2 x_2(1-x_2)(1-2x_2); -x_1(1-x_1)(1-2x_1)x_2^2(1-x_2)^2];$$

see the left of Figure 1. For a related example using tangential boundary control, see [24]. The data size in terms of the tracking functional can be measured as $\boldsymbol{F}(\mathbf{0}) = 0.302339$. Notice that $\nabla \cdot \boldsymbol{y}_d = 0$ and $\boldsymbol{y}_d = 0$ on Γ, but it cannot be the solution of the Stokes problem with data $\boldsymbol{f} = (1, 1)$ since $\boldsymbol{f} + \Delta \boldsymbol{y}_d$ is not a conservative field.

For the $\boldsymbol{H}^{1/2}(\Gamma)$-regularization, we obtain a value for the tracking term of $\boldsymbol{F}(\bar{\boldsymbol{u}}) = 0.112264$, while for the $\boldsymbol{L}^2(\Gamma)$-regularization we obtain a slightly smaller value $\boldsymbol{F}(\boldsymbol{u}_0) = 0.111576$. A graph of the state, the optimal control in the energy space, and the solution of the $\boldsymbol{L}^2(\Gamma)$-regularized problem can be found in Figure 2. In this case, $\boldsymbol{u}_0$ is a continuous function. Numerically, we find that $|q_0(x_j) + \lambda_0| < 3 \times 10^{-8}$ for all four corners x_j.

The value of the singular exponent for this domain is $\xi = 2.740$; see [15, Table 1]. This means that the exponent giving the order of convergence of the energy regularized problem in the $\boldsymbol{H}^{1/2}(\Gamma)$-norm is $r \approx 1$ and the exponent giving the best possible order of convergence of the $\boldsymbol{L}^2(\Gamma)$-regularized problem in the $\boldsymbol{L}^2(\Gamma)$-norm is $s \approx 0.5$. We obtain the results summarized in Table 1 for the optimal control problem with $\boldsymbol{H}^{1/2}$-regularization and with $\boldsymbol{L}^2(\Gamma)$-regularization. In this case the solution is very regular; the results are similar for both approaches and better than predicted by the general theory. This high regularity can also be noticed in the orders of convergence found for the other variables using higher order Taylor–Hood elements; see Table 2.

Example 6.2. Set $\Omega = (0, 1)^2$, $\alpha = 1$, $\boldsymbol{f} = \mathbf{0}$, and $\boldsymbol{y}_d = (x_1; x_2 - x_1)$. The data size is $\boldsymbol{F}(\mathbf{0}) = 0.25$, and the target does not belong to $\boldsymbol{V}^0(\Omega)$. A graph of the target field is sketched in the middle of Figure 1.

For the energy regularization, we find $\boldsymbol{F}(\bar{\boldsymbol{u}}) = 0.117607$; see the first two subfigures of Figure 3. For the $\boldsymbol{L}^2(\Gamma)$-regularized problem, we have that $\boldsymbol{F}(\boldsymbol{u}_0) = 0.158279$. The control is discontinuous at the corners (see the last subfigure of Figure 3) and hence is not in $\boldsymbol{H}^{1/2}(\Gamma)$. Finite element error results are summarized in Table 3. Again we have $r \approx 1$ and $s \approx 0.5$. In this case, the observed experimental order of convergence for the $\boldsymbol{L}^2(\Gamma)$ error of the $\boldsymbol{L}^2(\Gamma)$-regularized problem is quite close to s.

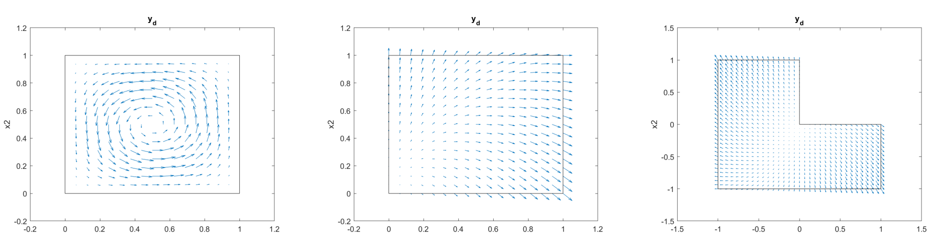

FIG. 1. *Left is the target of Example* 6.1*, middle is the target of Example* 6.2*, right is the target of Example* 6.3.

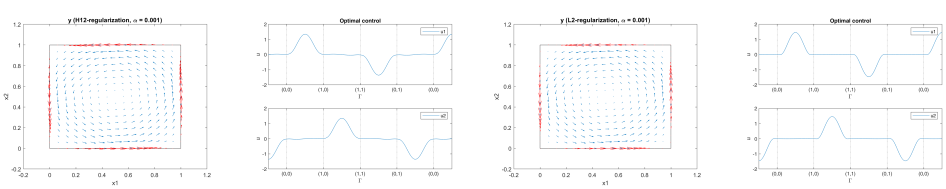

FIG. 2. *Solution of Example* 6.1*: The first two subfigures are for $\boldsymbol{H}^{1/2}(\Gamma)$-regularization; the last two subfigures are for $\boldsymbol{L}^2(\Gamma)$-regularization.*

TABLE 1
Errors and experimental order of convergence for Example 6.1.

i	$\boldsymbol{H}^{1/2}$-regularization			
	$\|\boldsymbol{e}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$	Rate	$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(\Gamma)}$	Rate
2	4.93E+0	-	8.37E-01	-
3	1.62E+0	1.61	2.56E-01	1.71
4	4.82E-01	1.75	6.80E-02	1.91
5	1.39E-01	1.79	1.75E-02	1.96
6	4.07E-02	1.78	4.37E-03	2.00
i	$\boldsymbol{L}^2$-regularization			
	$\|\boldsymbol{e}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$	Rate	$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(\Gamma)}$	Rate
2	6.17E+0	-	9.78E-01	-
3	2.01E+0	1.62	3.03E-01	1.69
4	6.37E-01	1.65	8.00E-02	1.92
5	1.87E-01	1.77	2.01E-02	1.93
6	5.54E-02	1.75	5.31E-03	1.98

TABLE 2
Errors and experimental order of convergence for the state and adjoint state for Example 6.1.

	i	$\|\boldsymbol{y}-\boldsymbol{y}_h\|_{\boldsymbol{L}^2(\Omega)}$		$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$		$\|\boldsymbol{z}-\boldsymbol{z}_h\|_{\boldsymbol{L}^2(\Omega)}$	
		Error	Rate	Error	Rate	Error	Rate
$\mathcal{P}_2-\mathcal{P}_1$	1	1.73E-03	-	2.36E-02	-	2.03E-03	-
	2	2.76E-04	2.65	8.14E-03	1.54	3.79E-04	2.42
	3	3.69E-05	2.90	2.20E-03	1.89	5.13E-05	2.89
	4	5.12E-06	2.85	6.16E-04	1.83	6.61E-06	2.95
	5	7.36E-07	2.80	1.81E-04	1.76	1.81E-07	2.98
$\mathcal{P}_3-\mathcal{P}_2$	1	4.54E-04	-	9.56E-02	-	6.28E-03	-
	2	4.17E-05	3.45	1.70E-03	2.49	5.67E-04	3.47
	3	4.39E-06	3.25	3.98E-03	2.10	4.45E-05	3.67
	4	6.50E-07	2.75	1.26E-04	1.65	3.75E-06	3.57
	5	9.61E-08	2.76	3.85E-04	1.72	3.20E-07	3.55

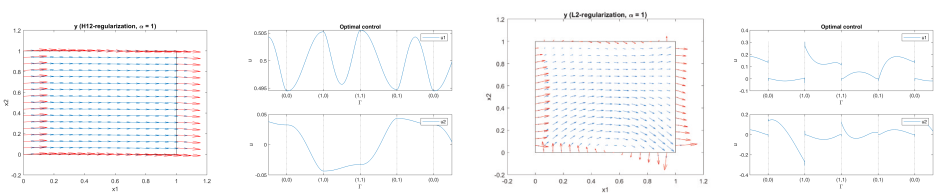

FIG. 3. *Solution of Example* 6.2*: The first two subfigures are for* $\boldsymbol{H}^{1/2}(\Gamma)$*-regularization; the last two subfigures are for* $\boldsymbol{L}^2(\Gamma)$*-regularization.*

TABLE 3
Errors and experimental order of convergence for Example 6.2.

i	$\boldsymbol{H}^{1/2}$-regularization				$\boldsymbol{L}^2$-regularization	
	$\|\boldsymbol{e}_h\|_{\boldsymbol{H}^{1/2}(\Gamma)}$	Rate	$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(\Gamma)}$	Rate	$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(\Gamma)}$	Rate
2	2.80E-02	-	3.77E-03	-	1.29E-01	-
3	9.88E-03	1.50	1.05E-03	1.85	8.90E-02	0.53
4	3.34E-03	1.57	2.81E-04	1.90	6.22E-02	0.52
5	1.10E-03	1.60	7.32E-05	1.94	4.37E-02	0.51
6	3.67E-04	1.59	1.86E-05	1.98	3.08E-02	0.51

Example 6.3. We take the same data as Example 6.2 but now consider the L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)^2$. The results on this domain are $\boldsymbol{F}(\boldsymbol{0}) = 1.75$,

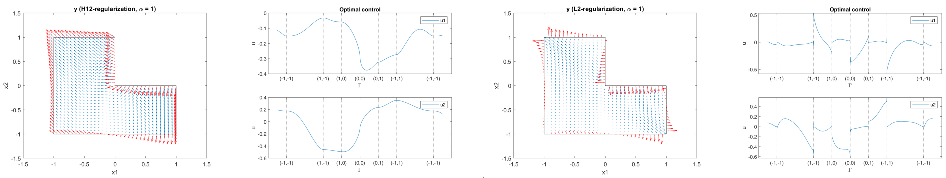

FIG. 4. *Solution of Example* 6.3*: The first two subfigures are for* $\boldsymbol{H}^{1/2}(\Gamma)$*-regularization; the last two subfigures are for* $\boldsymbol{L}^2(\Gamma)$*-regularization.*

TABLE 4
Errors and experimental order of convergence for Example 6.3.

i	$\boldsymbol{H}^{1/2}$-regularization				$\boldsymbol{L}^2$-regularization	
	$\Vert\boldsymbol{e}_h\Vert_{\boldsymbol{H}^{1/2}(\Gamma)}$	Rate	$\Vert\boldsymbol{e}_h\Vert_{\boldsymbol{L}^2(\Gamma)}$	Rate	$\Vert\boldsymbol{e}_h\Vert_{\boldsymbol{L}^2(\Gamma)}$	Rate
2	4.11E-01	-	7.40E-02	-	3.40E-01	-
3	2.49E-01	0.72	3.42E-02	1.12	2.38E-01	0.51
4	1.53E-01	0.71	1.55E-02	1.14	1.71E-01	0.48
5	9.12E-02	0.74	6.86E-03	1.18	1.24E-01	0.46
6	5.07E-02	0.85	2.83E-03	1.28	8.95E-02	0.47

$\boldsymbol{F}(\bar{\boldsymbol{u}}) = 1.107016$, $\boldsymbol{F}(\boldsymbol{u}_0) = 1.044080$. Graphs of the data and the solutions can be found in the right subfigure of Figure 1 and the first two subfigures of Figure 4. Experimental orders of convergence are in Table 4. The singular exponent for this domain is $\xi = 0.544$, so $r \approx 0.544$ and $s \approx 0.044$. The observed orders of convergence are higher.

One remarkable fact is that for the $\boldsymbol{L}^2(\Gamma)$-regularized problem the optimal control need not tend to ∞ at a nonconvex corner, as happens with Dirichlet optimal control problems governed by the Poisson equation in a nonconvex polygonal domain.

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