THE ROTATION NUMBER FOR ALMOST PERIODIC SCHRÖDINGER OPERATORS WITH δ-POTENTIALS

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Dedicated to the memory of Russell Johnson

Abstract. We consider one-dimensional Schrödinger operators with almost periodic potentials and δ-interactions supported on an almost periodic point set and with almost periodic coefficients. For operators of this kind we introduce a rotation number in the spirit of Johnson and Moser.

1. Introduction

In this paper, we consider the Schrödinger operator

$$H_{q,V,\Gamma}\psi(x) := -\psi''(x) + \left(q(x) + \sum_{i\in\mathbb{Z}} v_i\delta(x-x_i)\right)\psi(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $\delta(x-x_i)$ is the Dirac δ-function at x_i, $q(x)$ is a Bohr almost periodic function, $V = \{v_i\}_{i\in\mathbb{Z}}$ is an almost periodic sequence, and $\Gamma = \{x_i\}_{i\in\mathbb{Z}}$ is a discrete point set in $\mathbb{R}$. If the point set possesses some sort of recurrence, such as almost periodicity in a sense to be defined in detail below, our goal is to introduce the *rotation number* in the spirit of Johnson and Moser [8] for (1.1).

Let $\lambda \in \mathbb{R}$. The equation $H_{q,V,\Gamma}\psi = \lambda\psi$ can be written as the following system.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \psi' \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & q(x)-\lambda \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi' \\ \psi \end{pmatrix}, & x \in \mathbb{R}\setminus\Gamma, \\ \begin{pmatrix} \psi'(x_i+) \\ \psi(x_i+) \end{pmatrix} = \begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi'(x_i-) \\ \psi(x_i-) \end{pmatrix}, & x = x_i \in \Gamma. \end{cases} \tag{1.2}$$

The rotation number measures the average number of times the vector $(\psi', \psi)^T$ rotates around the origin per unit length. The existence of this limit and its uniformity properties are the main objective here. It is clear from (1.2) that the δ-interactions cause jumps in ψ' and hence one needs to exercise care in properly defining the limit in question. Recently, Qi and Yuan studied piecewise continuous almost periodic/automorphic solutions to (1.2) and related problems; see [15, 16].

The rotation number is a fundamental object in the study of one-dimensional Schrödinger operators with almost periodic potentials (i.e., in the case $v_i \equiv 0$); see the foundational work [8]. The key connections to the spectral analysis of these operators include a description of the spectrum as the set of points of non-constancy

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of the rotation number, as well as a canonical gap labeling by assigning the constant value the rotation number takes in a gap of the spectrum and showing that these gap labels must belong to the frequency module. The rotation number is intimately connected to another fundamental object: the integrated density of states. For example, one can see by means of oscillation theory that one is a constant multiple of the other. We refer the reader to Avron-Simon [1] for a foundational discussion of the integrated density of states associated with one-dimensional Schrödinger operators with almost periodic potentials; see also [4]. Besides the rotation number and the integrated density of states, the Lyapunov exponent is another fundamental object in the study of these operators. The celebrated Thouless formula links the integrated density of states and the Lyapunov exponent; see the survey by Damanik [2] and references therein.

In this paper we will focus on the dynamical aspects of the definition of the rotation number in our generalized setting and defer the discussion of the spectral aspects to a future publication. The primary issues we need to address are the discontinuity of the derivative of the solutions, which is resolved by the choice of a suitable homotopy, and the validity of a unique ergodicity statement and a corresponding ergodic theorem.

This paper is dedicated to the memory of Russell Johnson. Both of us have been inspired by Johnson's work, and in particular by the landmark paper [8], which serves as the motivation and the starting point of this paper. Beyond his important mathematical work, Johnson always shined bright through his kindness and humility. He was such a pleasure to be around and to interact with. He is being missed!

2. Almost Periodicity

In this section we use a unified approach to introduce almost periodic functions, almost periodic sequences, and almost periodic point sets.

Let $(Y, \|\cdot\|)$ be a complete space. We will denote by $\mathbb{K}$ either $\mathbb{Z}$ or $\mathbb{R}$, depending on the setting. We consider a $\mathbb{K}$ action on Y by shifts and denote for $y \in Y$ and $\tau \in \mathbb{K}$ the corresponding shifted element in Y by $y \cdot \tau$.

This shift action satisfies the following two conditions:

- group structure:

$$y \cdot 0 = y, \text{ and } y \cdot (\tau_1 + \tau_2) = (y \cdot \tau_1) \cdot \tau_2, \qquad \text{for all } y \in Y,\ \tau_1,\ \tau_2 \in \mathbb{K}, \tag{2.1}$$

- isometry:

$$\|y_1 \cdot \tau - y_2 \cdot \tau\| = \|y_1 - y_2\|, \qquad \text{for all } \tau \in \mathbb{K},\ y_i \in Y,\ i = 1, 2, \tag{2.2}$$

- uniform continuity: for any $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$ such that

$$\|y \cdot \tau - y\| < \varepsilon, \qquad \text{for all } y \in Y,\ \tau \in \mathbb{K} \text{ with } |\tau| < \delta. \tag{2.3}$$

Note that when $\mathbb{K} = \mathbb{Z}$, (2.3) is trivial. We say that $\Lambda \subset \mathbb{K}$ is *relatively dense* (with window size ℓ) if there exists $\ell \in \mathbb{K}^+$ such that

$$\Lambda \cap [a, a + \ell] \neq \emptyset \qquad \forall\ a \in \mathbb{K}.$$

Definition 2.1. *We say that $y \in Y$ is almost periodic if one of the following conditions holds:*

i): *for any $\varepsilon > 0$, $\mathrm{P}(y, \varepsilon) := \{\tau \in \mathbb{K} : \|y \cdot \tau - y\| < \varepsilon\}$ is relatively dense in $\mathbb{K}$;*

ii): *the hull of y defined by*

$$\mathrm{E}_y := \overline{\{y \cdot \tau : \tau \in \mathbb{K}\}}^{(Y,\|\cdot\|)},$$

is compact;

iii): *for any sequence $\{\tilde{\tau}_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$, one can extract a subsequence $\{\tau_n\} \subset \{\tilde{\tau}_n\}$ such that $\{y \cdot \tau_n\}$ is convergent in $(Y, \|\cdot\|)$.*

The equivalence of these different definitions is well known. For completeness, we give a proof with the help of the following lemmas from real analysis.

Lemma 2.2. [5] *In a metric space (X, d), the following properties are equivalent:*

- *(X, d) is complete and totally bounded;*
- *(X, d) is compact;*
- *any sequence in X has a convergent subsequence.*

Lemma 2.3. [5] *Let S be a subset of a metric space (X, d). The following properties are equivalent:*

- *S is totally bounded;*
- *the closure of S, denoted by $\overline{S}$, is totally bounded.*

Theorem 2.4. *The conditions* i), ii) *and* iii) *in* Definition 2.1 *are equivalent.*

Proof. We will show that i) and ii) are equivalent, and that ii) and iii) are equivalent.

$\boxed{\text{i)} \Longrightarrow \text{ii)}}$: By Lemma 2.2 it suffices to show that $(\mathrm{E}_y, \|\cdot\|)$ is complete and totally bounded. Since E_y is closed in the complete space $(Y, \|\cdot\|)$, $(\mathrm{E}_y, \|\cdot\|)$ is complete as well. It follows that we need to prove that $(\mathrm{E}_y, \|\cdot\|)$ is totally bounded. By Lemma 2.3 it suffices to show that $\{y \cdot \tau : \tau \in \mathbb{K}\}$ is totally bounded. By Definition 2.1: i), for any $\varepsilon > 0$, $\mathrm{P}(y, \varepsilon/2)$ is relatively dense in $\mathbb{K}$. Thus, there exists $\ell_\varepsilon \in \mathbb{K}^+$ such that for any $a \in \mathbb{K}$,

$$\mathrm{P}(y, \varepsilon/2) \cap [-a, -a + \ell_\varepsilon] \neq \emptyset.$$

Let $-a + b_{a,\varepsilon} \in \mathrm{P}(y, \varepsilon/2) \cap [-a, -a + \ell_\varepsilon]$, where $b_{a,\varepsilon} \in [0, \ell_\varepsilon] \cap \mathbb{K}$ depends on the parameters a and ε. By (2.1) and (2.2), we have

$$\|y \cdot b_{a,\varepsilon} - y \cdot a\| = \|y \cdot (-a + b_{a,\varepsilon}) - y\| < \varepsilon/2, \qquad \forall\, a \in \mathbb{K}. \tag{2.4}$$

By (2.3), for such ε, there exists $\delta_{\varepsilon/2} > 0$ such that

$$\|y \cdot \tau - y\| < \varepsilon/2, \qquad \forall\, y \in Y,\ |\tau| < \delta_{\varepsilon/2},\ \tau \in \mathbb{K}. \tag{2.5}$$

Thus the interval $[0, \ell_\varepsilon]$ can be separated into the following sub-intervals:

$$\left[0, \frac{\delta_{\varepsilon/2}}{2}\right), \cdots, \left[i\frac{\delta_{\varepsilon/2}}{2}, (i+1)\frac{\delta_{\varepsilon/2}}{2}\right), \cdots, \left[\left[\frac{2\ell_\varepsilon}{\delta_{\varepsilon/2}}\right]\frac{\delta_{\varepsilon/2}}{2}, \ell_\varepsilon\right],$$

where $[x]$ denotes the maximal integer less than x. We take one point from each sub-interval, denoted by $p_{i,\varepsilon} \in \mathbb{K}$, if the sub-interval is not empty. Then we collect these points and construct a finite set $F \subset \{y \cdot \tau : \tau \in \mathbb{K}\}$ by

$$F := \left\{y \cdot p_{i,\varepsilon} : i = 0, 1, \cdots, \left[\frac{2\ell_\varepsilon}{\delta_{\varepsilon/2}}\right]\right\}.$$

By (2.4) and (2.5), it follows that $\{y \cdot \tau : \tau \in \mathbb{K}\}$ is totally bounded.

$\boxed{\text{ii)} \Longrightarrow \text{i)}}$: Since E_y is compact, by Lemma 2.2 and Lemma 2.3, it follows that $\{y \cdot \tau : \tau \in \mathbb{K}\}$ is totally bounded. This means that for any $\varepsilon > 0$, there exists a finite subset, denoted by $I_\varepsilon := \{\tau_i \in \mathbb{K} : i = 1, 2, \cdots, n_\varepsilon\}$ such that

$$\|y - y \cdot (-a + \tau_{i_a})\| = \|y \cdot a - y \cdot \tau_{i_a}\| < \varepsilon, \qquad \forall\, a \in \mathbb{K}, \tag{2.6}$$

where $\tau_{i_a} \in I_\varepsilon$ depends on the parameter a. Let us denote $L_\varepsilon := \max_{1 \le i \le n_\varepsilon} |\tau_i|$. Then

$$-a - L_\varepsilon \le -a + \tau_{i_a} \le -a + L_\varepsilon, \qquad \forall\, a \in \mathbb{K}.$$

Combining this with (2.6), we have

$$\mathrm{P}(y, \varepsilon) \cap [-a - L_\varepsilon, -a + L_\varepsilon] \neq \emptyset.$$

Thus $\mathrm{P}(y, \varepsilon)$ is relatively dense. Here, $\ell_\varepsilon = 2L_\varepsilon$.

$\boxed{\text{ii)} \Longrightarrow \text{iii)}}$: This is obvious from Lemma 2.2.

$\boxed{\text{iii)} \Longrightarrow \text{ii)}}$: Assume that $\{y_n\}_{n \in \mathbb{N}} \subset \mathrm{E}_y$. Then there exist $\tau_n \in \mathbb{K}$ such that

$$\|y_n - y \cdot \tau_n\| < 1/n, \qquad \forall\, n \in \mathbb{N}. \tag{2.7}$$

For the sequence $\{y \cdot \tau_n\}_{n \in \mathbb{N}}$, it follows from Definition 2.1: iii) that there exists a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\{y \cdot \tau_{n_k}\}$ is convergent as $k \to +\infty$. Combining this with (2.7), we obtain that $\{y_{n_k}\} \subset \mathrm{E}_y$ is convergent as $k \to +\infty$. By Lemma 2.2, the assertion follows. □

When $(Y, \|\cdot\|)$ is $(\mathcal{C}_u(\mathbb{R}), \|\cdot\|_\infty)$, the space of all bounded and uniformly continuous functions, and the shift is

$$f \cdot \tau := f(\cdot + \tau),$$

where $f \in \mathcal{C}_u(\mathbb{R})$ and $\tau \in \mathbb{R}$, Definition 2.1 gives the characterization of *almost periodic functions*. Note that Bohr originally introduced the notion of almost periodic functions on the space of continuous functions. We know that a Bohr almost periodic function is bounded and uniformly continuous [7]. The reason that we work on the space $\mathcal{C}_u(\mathbb{R})$ is that it is convenient to take use of Definition 2.1. We denote the space of all almost periodic functions by $\mathcal{C}_{ap}(\mathbb{R})$. It is well known that $\mathcal{C}_{ap}(\mathbb{R})$ is a complete space [7].

Similarly, when $(Y, \|\cdot\|)$ is $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$, the space of all bounded bi-infinite sequences, and the shift is

$$V \cdot \tau := \{v_{i+\tau}\}_{i \in \mathbb{Z}},$$

where $V = \{v_i\}_{i \in \mathbb{Z}} \in \ell^\infty(\mathbb{R})$ and $\tau \in \mathbb{Z}$, Definition 2.1 defines the *almost periodic sequences*. Note that (2.3) is trivial because $\mathbb{K} = \mathbb{Z}$. Denote by $\ell^\infty_{ap}(\mathbb{R})$ the space of all almost periodic sequences. It is a complete space as well. The relationship between almost periodic functions and almost periodic sequences is described in the following lemma:

Lemma 2.5. [7] *If $f = f(x) \in \mathcal{C}_{ap}(\mathbb{R})$, then $\{f(i) : i \in \mathbb{Z}\} \in \ell^\infty_{ap}(\mathbb{R})$. Conversely, if $V = \{v_i\}_{i \in \mathbb{Z}} \in \ell^\infty_{ap}(\mathbb{R})$, then there exists $f \in \mathcal{C}_{ap}(\mathbb{R})$ such that $f(i) = v_i$ for all $i \in \mathbb{Z}$.*

For a generalization of Lemma 2.5, see [19] in which we can establish a similar result on almost periodic point sets instead of periodic point sets such as $\mathbb{Z}$. For almost periodic objects, we may introduce the following important quantity:

Lemma 2.6. *Let $f = f(x) \in \mathcal{C}_{ap}(\mathbb{R})$. Then the limit*

$$\mathrm{M}(f) := \lim_{x_2 - x_1 \to +\infty} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$$

exists uniformly for all $x_1,\ x_2 \in \mathbb{R}$. We call it the mean value of f.

Lemma 2.7. *Let $V = \{v_i\}_{i\in\mathbb{Z}} \in \ell^\infty_{ap}(\mathbb{R})$. Then the limit*

$$\mathrm{M}(V) := \lim_{i_2 - i_1 \to +\infty} \frac{1}{i_2 - i_1} \sum_{i=i_1}^{i_2} v_i$$

exists uniformly for all $i_1,\ i_2 \in \mathbb{Z}$. We call it the mean value of V.

Now we consider point sets. In order to describe Delone dynamical systems, Lenz and Stollmann [10] introduced the notion of almost periodic point sets in $\mathbb{R}^d$. For the general case defined on locally compact abelian groups, see [9, 11]. For the one-dimensional case and our purpose, we give the following description, which is somewhat different from what was done in the dissertation by Zhou [19]. Some part of the dissertation was published in [18, 12].

From now on, we assume that a point set $\Gamma = \{x_i\}_{i\in\mathbb{Z}}$ is discrete in the real line and satisfies the following requirement:

$$0 < \inf_{i\in\mathbb{Z}} \Delta x_i \le \sup_{i\in\mathbb{Z}} \Delta x_i < \infty, \qquad \Delta x_i := x_i - x_{i-1}.$$

Two point sets $\Gamma = \{x_i\}$ and $\hat\Gamma = \{\hat x_i\}$ are the same if there exists $k \in \mathbb{Z}$ such that $\hat x_i = x_{i+k}$ for all $i \in \mathbb{Z}$. Denote by $\mathcal{L}$ the space of all point sets in $\mathbb{R}$. It can be equipped with a metric as follows:

$$\mathrm{dist}(\Gamma^1, \Gamma^2) := \max\left\{\tilde{\mathrm{dist}}(\Gamma^1, \Gamma^2),\ \tilde{\mathrm{dist}}(\Gamma^2, \Gamma^1)\right\}, \tag{2.8}$$

where

$$\tilde{\mathrm{dist}}(\Gamma^1, \Gamma^2) := \sup_{i\in\mathbb{Z}} \min_{j\in\mathbb{Z}} |x_i^1 - x_j^2| \quad \text{for } \Gamma^k = \{x_i^k\}_{i\in\mathbb{Z}} \in \mathcal{L},\ k = 1, 2.$$

The metric $\mathrm{dist}(\cdot,\cdot)$ may be regarded as the Hausdorff metric. Note that the space $(\mathcal{L}, \mathrm{dist})$ is not complete. However, for any given $0 < m \le M < \infty$, the set

$$\mathcal{L}_{m,M} := \left\{\Gamma = \{x_i\}_{i\in\mathbb{Z}} : \Delta x_i \in [m, M]\ \forall\, i \in \mathbb{Z}\right\} \tag{2.9}$$

is a closed subset in $\mathcal{L}$ and then $(\mathcal{L}_{m,M}, \mathrm{dist})$ is a complete space [19]. We have

- $\mathcal{L} = \bigcup_{0<m\le M<\infty} \mathcal{L}_{m,M}$;
- for $\Gamma^1,\ \Gamma^2 \in \mathcal{L}_{m,M}$, $\mathrm{dist}(\Gamma^1, \Gamma^2) \le M/2$;
- when $\Gamma^k = \{x_i^k\} \in \mathcal{L}_{m,M}, k = 1, 2$ satisfy $\mathrm{dist}(\Gamma^1, \Gamma^2) < m/2$, then there exists $i_0 \in \mathbb{Z}$ such that

$$\mathrm{dist}(\Gamma^1, \Gamma^2) = \sup_{i\in\mathbb{Z}} |x_i^1 - x_{i+i_0}^2|. \tag{2.10}$$

Due to the properties above, convergence in $(\mathcal{L}_{m,M}, \mathrm{dist})$ can be characterized in the following way.

Lemma 2.8. [19] *Let $\Gamma^k = \{x_i^k\}_{i\in\mathbb{Z}} \in \mathcal{L}_{m,M},\ k \in \mathbb{Z}^+$. Then $\Gamma^k \to \Gamma^0$ in $(\mathcal{L}_{m,M}, \mathrm{dist})$ if and only if there exists a sequence $\{i_k\} \subset \mathbb{Z}$ such that*

$$\lim_{k\to\infty} \sup_{i\in\mathbb{Z}} |x_{i+i_k}^k - x_i^0| = 0.$$

Let us consider the point sets that include zero and furthermore are such that $x_0 = 0$. Denote

$$\mathcal{L}^0_{m,M} := \{\Gamma \in \mathcal{L}_{m,M} : x_0 = 0\}.$$

Obviously, this space is complete as well. Introduce

$$\mathcal{L}^0 := \{\Gamma \in \mathcal{L} : x_0 = 0\}.$$

We have

$$\mathcal{L}^0 = \bigcup_{0<m\leq M<\infty} \mathcal{L}^0_{m,M}.$$

Now we define the shift on $\mathcal{L}^0$ in a way that is a little different from, but more concise than, the way it was done in [19]. For $\Gamma \in \mathcal{L}^0$ and $k \in \mathbb{Z}$, the shift of Γ is defined by

$$\Gamma \cdot k := \{\hat{x}_i\}_{i\in\mathbb{Z}} \in \mathcal{L}^0, \qquad \hat{x}_i := x_{i+k} - x_k. \tag{2.11}$$

Since $\Gamma \in \mathcal{L}^0$, we have

$$\Gamma \cdot (k_1 + k_2) = (\Gamma \cdot k_1) \cdot k_2. \tag{2.12}$$

Then the family of shifts $\{\Gamma \cdot k\}_{k\in\mathbb{Z}}$ yields a dynamical system on $\mathcal{L}^0$, but we do not have an isometry property like (2.2). In fact, we have the following:

Lemma 2.9. *Let $k \in \mathbb{Z}$ and $\Gamma^i = \{x^i_n\}_{n\in\mathbb{Z}} \in \mathcal{L}^0$, $i = 1, 2$. Then*

$$|\operatorname{dist}(\Gamma^1 \cdot k, \Gamma^2 \cdot k) - \operatorname{dist}(\Gamma^1, \Gamma^2)| \leq |x^1_k - x^2_k|. \tag{2.13}$$

Proof. Since $\Gamma^1 \cdot k = \{x^1_{n+k} - x^1_k\}_{n\in\mathbb{Z}}$ and $\Gamma^2 \cdot k = \{x^2_{h+k} - x^2_k\}_{h\in\mathbb{Z}}$, we have

$$|(x^1_{n+k} - x^1_k) - (x^2_{h+k} - x^2_k)| \leq |x^1_{n+k} - x^2_{h+k}| + |x^1_k - x^2_k|.$$

This implies that

$$\begin{aligned} &\min_{h\in\mathbb{Z}} |(x^1_{n+k} - x^1_k) - (x^2_{h+k} - x^2_k)| \\ \leq &\min_{h\in\mathbb{Z}} |x^1_{n+k} - x^2_{h+k}| + |x^1_k - x^2_k| \\ = &\min_{h\in\mathbb{Z}} |x^1_{n+k} - x^2_h| + |x^1_k - x^2_k|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\tilde{\operatorname{dist}}(\Gamma^1 \cdot k, \Gamma^2 \cdot k) \\ = &\sup_{n\in\mathbb{Z}} \min_{h\in\mathbb{Z}} |(x^1_{n+k} - x^1_k) - (x^2_{h+k} - x^2_k)| \\ \leq &\sup_{n\in\mathbb{Z}} \min_{h\in\mathbb{Z}} |x^1_{n+k} - x^2_h| + |x^1_k - x^2_k| \\ = &\tilde{\operatorname{dist}}(\Gamma^1, \Gamma^2) + |x^1_k - x^2_k|. \end{aligned}$$

Similarly, we have

$$\tilde{\operatorname{dist}}(\Gamma^2 \cdot k, \Gamma^1 \cdot k) \leq \tilde{\operatorname{dist}}(\Gamma^2, \Gamma^1) + |x^1_k - x^2_k|.$$

By (2.8), we have

$$\operatorname{dist}(\Gamma^1 \cdot k, \Gamma^2 \cdot k) - \operatorname{dist}(\Gamma^1, \Gamma^2) \leq |x^1_k - x^2_k|. \tag{2.14}$$

On the other hand, we have

$$|x^1_{n+k} - x^2_{h+k}| \leq |(x^1_{n+k} - x^1_k) - (x^2_{h+k} - x^2_k)| + |x^1_k - x^2_k|.$$

By the argument above, we have

$$\operatorname{dist}(\Gamma^1,\Gamma^2)-\operatorname{dist}(\Gamma^1\cdot k,\Gamma^2\cdot k)\le |x_k^1-x_k^2|. \tag{2.15}$$

It follows from (2.14) and (2.15) that we obtain the desired result (2.13). □

Let us derive, as a special case, the following statement:

Lemma 2.10. *Let $\Gamma\in\mathcal{L}^0$ and $i,\ j,\ k\in\mathbb{Z}$. Then*

$$\begin{aligned}&|\operatorname{dist}(\Gamma\cdot(i+k),\Gamma\cdot(j+k))-\operatorname{dist}(\Gamma\cdot i,\Gamma\cdot j)|\\ &\le |(x_{i+k}-x_i)-(x_{j+k}-x_j)| = |(x_{i+k}-x_{j+k})-(x_i-x_j)|.\end{aligned} \tag{2.16}$$

Proof. Denote $\Gamma=\{x_n\}_{n\in\mathbb{Z}}$. Then we have $\Gamma\cdot i=\{x_{i+n}-x_i\}_{n\in\mathbb{Z}}$ and $\Gamma\cdot j=\{x_{j+h}-x_j\}_{h\in\mathbb{Z}}$. Applying Lemma 2.9 to $\Gamma\cdot i$ and $\Gamma\cdot j$, we have the desired result (2.16). □

For $\Gamma\in\mathcal{L}^0$, there exist $m,\ M\in\mathbb{R}^+$ such that $\Gamma\in\mathcal{L}^0_{m,M}$. We define almost periodic point sets as follows:

Definition 2.11. *We say that $\Gamma\in\mathcal{L}^0_{m,M}$ is almost periodic if one of the following conditions holds:*

- i): *the hull of Γ, defined by*

$$\mathrm{E}_\Gamma := \overline{\{\Gamma\cdot k : k\in\mathbb{Z}\}}^{(\mathcal{L}^0_{m,M},\operatorname{dist})},$$

  is a compact subset in $\mathcal{L}^0_{m,M}$;
- ii): *for any sequence $\{\tilde n_k\}\subset\mathbb{Z}$, one can extract a subsequence $\{n_k\}\subset\{\tilde n_k\}$ such that $\{\Gamma\cdot n_k\}$ is convergent in $(\mathcal{L}^0_{m,M},\operatorname{dist})$.*

Theorem 2.12. *The conditions* i) *and* ii) *in* Definition 2.11 *are equivalent.*

Proof. This is obvious by Lemma 2.2. □

Denote by $\mathcal{L}^0_{m,M,ap}$ the space of all almost periodic point sets in $\mathcal{L}^0_{m,M}$. Although the shift of point sets does not have the isometry property like (2.2), we still have a similar version as Definition 2.1: i) for almost periodic point sets because of the following lemma.

Lemma 2.13. *Let $\Gamma^i=\{x_n^i\}_{n\in\mathbb{Z}}\in\mathcal{L}^0_{m,M},\ i=1,2$, and $\operatorname{dist}(\Gamma^1,\Gamma^2)<m/2$. Then for all $k\in\mathbb{Z}$, we have*

$$\operatorname{dist}(\Gamma^1\cdot k,\Gamma^2\cdot k)\le 2\operatorname{dist}(\Gamma^1,\Gamma^2).$$

Proof. By (2.10), we have

$$\operatorname{dist}(\Gamma^1,\Gamma^2)=\sup_{n\in\mathbb{Z}}|x_n^1-x_n^2|\ge|x_k^1-x_k^2|\qquad\text{for all } k\in\mathbb{Z}.$$

Combining this with (2.13), we obtain the desired result. □

Lemma 2.14. *$\Gamma\in\mathcal{L}^0_{m,M,ap}$ if and only if for any $\varepsilon>0$,*

$$\mathrm{P}(\Gamma,\varepsilon):=\{\tau\in\mathbb{Z}:\operatorname{dist}(\Gamma\cdot\tau,\Gamma)<\varepsilon\}$$

is relatively dense in $\mathbb{Z}$.

Proof. We use the argument of Part i) $\Longrightarrow$ ii) and Part ii) $\Longrightarrow$ i) in the proof of Theorem 2.4, and address the difference stemming from the absence of the isometry property for point sets. Without loss of generality we assume that $\varepsilon < m$ in this proof.

First we prove the implication $\Longleftarrow$ in the statement of the lemma. It suffices to show that $\{\Gamma \cdot \tau : \tau \in \mathbb{Z}\}$ is totally bounded. Since for any $\varepsilon > 0$, $\mathrm{P}(\Gamma, \varepsilon/2)$ is relatively dense in $\mathbb{Z}$, we obtain that there exists $\ell_\varepsilon \in \mathbb{Z}^+$ such that for any $a \in \mathbb{Z}$,

$$\mathrm{P}(\Gamma, \varepsilon/2) \cap [-a, -a + \ell_\varepsilon] \neq \emptyset.$$

Let $-a + b_{a,\varepsilon} \in \mathrm{P}(\Gamma, \varepsilon/2) \cap [-a, -a + \ell_\varepsilon]$, where $b_{a,\varepsilon} \in [0, \ell_\varepsilon] \cap \mathbb{Z}$ depends on the parameters a and ε. By Lemma 2.13, we have

$$\operatorname{dist}(\Gamma \cdot a, \Gamma \cdot b_{a,\varepsilon}) \leq 2 \operatorname{dist}(\Gamma \cdot (-a + b_{a,\varepsilon}), \Gamma) < \varepsilon, \qquad \forall\, a \in \mathbb{Z}. \tag{2.17}$$

We construct a finite set $F \subset \{\Gamma \cdot \tau : \tau \in \mathbb{Z}\}$ by

$$F := \{\Gamma \cdot i : i = 0, 1, \cdots, \ell_\varepsilon\}.$$

By (2.17), it follows that $\{\Gamma \cdot \tau : \tau \in \mathbb{Z}\}$ is totally bounded.

Now we prove the implication $\Longrightarrow$ in the statement of the lemma. We know that $\{\Gamma \cdot \tau : \tau \in \mathbb{Z}\}$ is totally bounded. This means that for any $\varepsilon > 0$, there exists a finite subset, denoted by $I_\varepsilon := \{\tau_i \in \mathbb{Z} : i = 1, 2, \cdots, n_\varepsilon\}$, such that

$$\operatorname{dist}(\Gamma \cdot a, \Gamma \cdot \tau_{i_a}) < \varepsilon/2, \qquad \forall\, a \in \mathbb{Z},$$

where $\tau_{i_a} \in I_\varepsilon$ depends on the parameter a. By Lemma 2.13, we have

$$\operatorname{dist}(\Gamma, \Gamma \cdot (-a + \tau_{i_a})) \leq 2 \operatorname{dist}(\Gamma \cdot a, \Gamma \cdot \tau_{i_a}) < \varepsilon. \tag{2.18}$$

Let us denote $L_\varepsilon := \max_{1 \leq i \leq n_\varepsilon} |\tau_i|$. Then

$$-a - L_\varepsilon \leq -a + \tau_{i_a} \leq -a + L_\varepsilon, \qquad \forall\, a \in \mathbb{Z}.$$

Combining this with (2.18), we find

$$\mathrm{P}(\Gamma, \varepsilon) \cap [-a - L_\varepsilon, -a + L_\varepsilon] \neq \emptyset.$$

Thus $\mathrm{P}(\Gamma, \varepsilon)$ is relatively dense. Here, $\ell_\varepsilon = 2L_\varepsilon$. $\square$

An example of an almost periodic point set is given by

$$\Gamma_a := \{i + a \sin i\}_{i \in \mathbb{Z}},$$

where $|a| < 1$. In fact, by Lemma 2.5, we know that $\{\sin i\}_{i \in \mathbb{Z}} \in \ell^\infty_{ap}(\mathbb{R})$. It follows from Example 3.7 in [18] that Γ_a is almost periodic in the sense of the definition given in [18]. Since directly compared with the two statements, the definition in [18] is stronger than Definition 2.11 here, we obtain the desired assertion. In fact, Definition 2.11 and the corresponding definition in [18] are equivalent.

By Lemma 2.8, we know that $\mathcal{L}^0_{m,M,ap}$ is a complete metric space. In fact, i_k in Lemma 2.8 vanishes here when two point sets are very close, because the point sets we consider include zero. The following lemmas are necessary in the proof of our main result.

Lemma 2.15. *Let $\Gamma \in \mathcal{L}^0_{m,M,ap}$. Then $\Delta\Gamma := \{\Delta x_i\}_{i \in \mathbb{Z}} \in \ell^\infty_{ap}(\mathbb{R})$.*

Proof. By Definition 2.1: iii) it suffices to show that for any sequence $\{\tilde{n}_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}$, one can extract a subsequence $\{n_k\} \subset \{\tilde{n}_k\}$ such that

$$\{\Delta\Gamma \cdot n_k = \{\Delta x_{i+n_k}\}_{i\in\mathbb{Z}}\}_{k\in\mathbb{N}}$$

is convergent in $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$. Indeed, we have

$$x_{\tilde{n}_k} = x_{\tilde{n}_k} - x_0 = \sum_{j=1}^{\tilde{n}_k} \Delta x_j,$$

and

$$\Gamma \cdot \tilde{n}_k = \{x_{i+\tilde{n}_k} - x_{\tilde{n}_k}\}_{i\in\mathbb{Z}} \in \mathcal{L}^0_{m,M,ap}.$$

It follows from $\Gamma \in \mathcal{L}^0_{m,M,ap}$ that there exists a subsequence, denoted by $\{n_k\} \subset \{\tilde{n}_k\}$, such that $\{\Gamma \cdot n_k\}$ is convergent in $(\mathcal{L}^0_{m,M}, \mathrm{dist})$. Note that

$$\Gamma \cdot n_k = \{x_{i+n_k} - x_{n_k}\}_{i\in\mathbb{Z}}.$$

By Lemma 2.8, we know that $\{x_{i+n_k} - x_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence uniformly for all $i \in \mathbb{Z}$. This implies that

$$\{\Delta x_{i+n_k} = x_{i+n_k} - x_{n_k} - (x_{i-1+n_k} - x_{n_k})\}_{k\in\mathbb{N}} \subset \mathbb{R}$$

is a Cauchy sequence uniformly for all $i \in \mathbb{Z}$. Thus we obtain that $\{\Delta\Gamma \cdot n_k\}$ is convergent. □

Similar to the mean value of almost periodic objects, we may introduce the following quantity for almost periodic point sets.

Lemma 2.16. *Let $\Gamma \in \mathcal{L}^0_{m,M,ap}$. Then the limit*

$$\lim_{x\to+\infty} \frac{1}{x} \#(\tilde{\Gamma} \cap [0, x)) =: [\Gamma] = \frac{1}{\mathrm{M}(\Delta\Gamma)} \in \left[\frac{1}{M}, \frac{1}{m}\right]$$

exists uniformly for all $\tilde{\Gamma} \in \mathrm{E}_\Gamma$, where $\#(\cdot)$ is the function counting the number of elements in a set. We call it the density of Γ.

Proof. Denote $\tilde{\Gamma} = \{\tilde{x}_i\}_{i\in\mathbb{Z}}$. Then we have

$$\lim_{x\to+\infty} \frac{1}{x} \#(\tilde{\Gamma} \cap [0, x)) = \lim_{k\to+\infty} \frac{1}{\tilde{x}_k} \#(\tilde{\Gamma} \cap [0, \tilde{x}_k)) = \lim_{k\to+\infty} \frac{k}{\sum_{i=1}^k \Delta\tilde{x}_i},$$

if the limit exists. By Lemma 2.15 and Lemma 2.7, we obtain that the limit exists and that we have the desired equality $\mathrm{M}(\Delta\Gamma)[\Gamma] = 1$. □

Remark 2.17. We may define a new metric $\hat{\mathrm{dist}}$ on $\mathcal{L}^0_{m,M,ap}$ by

$$\hat{\mathrm{dist}}(\Gamma, \hat{\Gamma}) := \sup_{j\in\mathbb{Z}} \mathrm{dist}(\Gamma \cdot j, \hat{\Gamma} \cdot j), \qquad \Gamma, \hat{\Gamma} \in \mathcal{L}^0_{m,M,ap}.$$

Then the shift under this metric yields an isometric dynamical system. When Γ and $\hat{\Gamma}$ are close, it follows from (2.10) that

$$\hat{\mathrm{dist}}(\Gamma, \hat{\Gamma}) = \sup_{j\in\mathbb{Z}} \sup_{i\in\mathbb{Z}} |(x_i - x_j) - (\hat{x}_i - \hat{x}_j)|.$$

One would then need to check whether $(\mathcal{L}^0_{m,M,ap}, \hat{\mathrm{dist}})$ is complete and then introduce a compact hull under the new metric. We leave the consideration of these issues to the reader and continue to use the metric $\mathrm{dist}(\cdot, \cdot)$ by (2.8) in the remainder of this paper.

3. Joint Hull

To study the long-time behavior of solutions of (1.2), we need embed it in a family of systems. The joint hull is introduced as follows. First, we equip the product space $\mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}$ with a new metric as follows:

$$\begin{aligned} &\operatorname{dist}((q^1, V^1, \Gamma^1), (q^2, V^2, \Gamma^2)) \\ &:= \max\{\|q^1 - q^2\|_\infty, \|V^1 - V^2\|_\infty, \operatorname{dist}(\Gamma^1, \Gamma^2)\}, \end{aligned} \tag{3.1}$$

where $(q^i, V^i, \Gamma^i) \in \mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}$, $i = 1, 2$. The shift on the whole product space is introduced by

$$(q, V, \Gamma) \cdot k := (q \cdot x_k, V \cdot k, \Gamma \cdot k), \qquad (q, V, \Gamma) \in \mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}. \tag{3.2}$$

By (2.12), the family of shifts $\{(q, V, \Gamma) \cdot k\}_{k\in\mathbb{Z}}$ is a dynamical system with no isometry property on $\mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}$; see Lemma 2.9. Note that the shift is a skew-product that is different from the shift of each single element, because $q \cdot x_k$ depends on both q and Γ.

Let $(q, V, \Gamma) \in \mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}$. Then the product space $\mathrm{E}_q \times \mathrm{E}_V \times \mathrm{E}_\Gamma$ is compact since each single space is compact. Denote the orbit of a triple (q, V, Γ) by

$$\mathrm{O}_{q,V,\Gamma} := \{(q, V, \Gamma) \cdot n : n \in \mathbb{Z}\}.$$

By (2.16) and uniform continuity of almost periodic functions, we have the following:

Lemma 3.1. *Let $(q, V, \Gamma) \in \mathcal{C}_{ap}(\mathbb{R}) \times \ell^\infty_{ap}(\mathbb{R}) \times \mathcal{L}^0_{m,M,ap}$. The skew-product shifts $\{(q, V, \Gamma) \cdot k\}_{k\in\mathbb{Z}}$ are equicontinuous homeomorphisms on $\mathrm{O}_{q,V,\Gamma}$.*

Proof. Let $\Gamma = \{x_n\}_{n\in\mathbb{Z}}$ and $(q^i, V^i, \Gamma^i) \in \mathrm{O}_{q,V,\Gamma}$, $i = 1, 2$. Then there exist $n_i \in \mathbb{Z}$, $i = 1, 2$ such that

$$(q^i, V^i, \Gamma^i) = (q, V, \Gamma) \cdot n_i = (q \cdot x_{n_i}, V \cdot n_i, \Gamma \cdot n_i). \tag{3.3}$$

If we denote $\Gamma^i = \{x^i_n\}_{n\in\mathbb{Z}}$, $i = 1, 2$, then it follows from (3.3) that

$$x^i_n = x_{n+n_i} - x_{n_i}. \tag{3.4}$$

By the uniform continuity of $q \in \mathcal{C}_{ap}(\mathbb{R})$, we have that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\|q \cdot \tau - q\|_\infty < \varepsilon/2 \qquad \forall\ |\tau| < \delta_1. \tag{3.5}$$

For such $\varepsilon > 0$, denote

$$\delta := \min\{\varepsilon/2, m/2, \delta_1\}.$$

We will show that

$$\operatorname{dist}((q^1, V^1, \Gamma^1) \cdot k, (q^2, V^2, \Gamma^2) \cdot k) < \varepsilon, \tag{3.6}$$

uniformly for all $k \in \mathbb{Z}$, provided that

$$\operatorname{dist}((q^1, V^1, \Gamma^1), (q^2, V^2, \Gamma^2)) < \delta. \tag{3.7}$$

Indeed, by (3.1) and (3.7), we have

$$\operatorname{dist}(\Gamma \cdot n_1, \Gamma \cdot n_2) = \operatorname{dist}(\Gamma^1, \Gamma^2) < \delta < m/2. \tag{3.8}$$

It follows from (2.10) that for all $n \in \mathbb{Z}$, we have

$$|(x_{n+n_1} - x_{n_1}) - (x_{n+n_2} - x_{n_2})| < \delta < \delta_1. \tag{3.9}$$

By (2.16), (3.8) and (3.9), we obtain

$$\begin{aligned} &\operatorname{dist}(\Gamma^1\cdot k, \Gamma^2\cdot k) = \operatorname{dist}(\Gamma\cdot(n_1+k), \Gamma\cdot(n_2+k)) \\ &\le \operatorname{dist}(\Gamma\cdot n_1, \Gamma\cdot n_2) + |(x_{n_1+k}-x_{n_1})-(x_{n_2+k}-x_{n_2})| < 2\delta < \varepsilon, \end{aligned} \tag{3.10}$$

uniformly for all $k\in\mathbb{Z}$. It is obvious that

$$\|V^1\cdot k - V^2\cdot k\|_\infty = \|V^1-V^2\|_\infty < \delta < \varepsilon, \tag{3.11}$$

uniformly for all $k\in\mathbb{Z}$. Now we consider the distance between $q^1\cdot x_k^1$ and $q^2\cdot x_k^2$. Then uniformly for all $k\in\mathbb{Z}$, we have

$$\begin{aligned} &\|q^1\cdot x_k^1 - q^2\cdot x_k^2\|_\infty \\ =& \|q\cdot x_{n_1}\cdot x_k^1 - q\cdot x_{n_2}\cdot x_k^2\|_\infty \\ =& \|q\cdot x_{n_1+k} - q\cdot x_{n_2+k}\|_\infty \\ \le& \|q\cdot x_{n_1+k} - q\cdot(x_{n_1+k}-x_{n_1}+x_{n_2})\|_\infty \\ &+\|q\cdot(x_{n_1+k}-x_{n_1}+x_{n_2}) - q\cdot x_{n_2+k}\|_\infty \\ =& \|q\cdot x_{n_1} - q\cdot x_{n_2}\|_\infty + \|q\cdot((x_{n_1+k}-x_{n_1})-(x_{n_2+k}-x_{n_2}))-q\|_\infty \\ <& 2\delta<\varepsilon, \end{aligned} \tag{3.12}$$

where (3.4), (2.2), (3.5) and (3.9) are used. The desired result (3.6) is obtained from (3.10), (3.11) and (3.12). □

Definition 3.2. *The joint hull of a triple (q,V,Γ) is defined by*

$$\mathrm{E}_{q,V,\Gamma} := \overline{\mathrm{O}_{q,V,\Gamma}}^{(\mathcal{C}_{ap}(\mathbb{R})\times\ell^\infty_{ap}(\mathbb{R})\times\mathcal{L}^0_{m,M,ap},\operatorname{dist})}.$$

Obviously, we have

$$\mathrm{E}_{q,V,\Gamma}\subset \mathrm{E}_q\times\mathrm{E}_V\times\mathrm{E}_\Gamma\subset\mathcal{C}_{ap}(\mathbb{R})\times\ell^\infty_{ap}(\mathbb{R})\times\mathcal{L}^0_{m,M,ap}.$$

Moreover, $\mathrm{E}_{q,V,\Gamma}$ is compact in $(\mathcal{C}_{ap}(\mathbb{R})\times\ell^\infty_{ap}(\mathbb{R})\times\mathcal{L}^0_{m,M,ap},\operatorname{dist})$. As before, we may equip $\mathrm{E}_{q,V,\Gamma}$ with a group structure as

$$(q^1,V^1,\Gamma^1)\cdot(q^2,V^2,\Gamma^2) := \lim_{k\to+\infty}(q,V,\Gamma)\cdot(n_k^1+n_k^2), \tag{3.13}$$

$$(q^1,V^1,\Gamma^1)^{-1} := \lim_{k\to+\infty}(q,V,\Gamma)\cdot(-n_k^1), \tag{3.14}$$

where

$$(q^i,V^i,\Gamma^i) = \lim_{k\to+\infty}(q,V,\Gamma)\cdot n_k^i\in\mathrm{E}_{q,V,\Gamma},\qquad i=1,2. \tag{3.15}$$

By Lemma 3.1, the operations of both multiplication and inverse are well defined, that is, the limits in (3.13) and (3.14) do exist and are independent of the choice of sequences $\{n_k^i\}_{k\in\mathbb{N}}$ in (3.15). We state the following result on unique ergodicity due to [17], and then apply it to the compact metric group $\mathrm{E}_{q,V,\Gamma}$.

Lemma 3.3. [17] *Let $T(g)=ag$ be a rotation on the compact metric group G. Then T is uniquely ergodic if and only if T is minimal. In this case Haar measure is the only invariant measure.*

Lemma 3.4. *Equipped with the operations of both multiplication and inverse above, the hull $(\mathrm{E}_{q,V,\Gamma},\operatorname{dist})$ is a compact abelian topological group. In particular, $\mathrm{E}_{q,V,\Gamma}$ admits the Haar measure $\nu=\nu_{q,V,\Gamma}$ that is invariant under $\{(q,V,\Gamma)\cdot k\}_{k\in\mathbb{Z}}$.*

Proof. Let $G := \mathrm{E}_{q,V,\Gamma}$ and $a := (q, V, \Gamma) \cdot 1$. By (3.13), we introduce

$$T^k((q, V, \Gamma)) := (q, V, \Gamma) \cdot k = a^k \cdot (q, V, \Gamma), \qquad k \in \mathbb{Z}.$$

It is obvious that T is minimal. Thus we obtain the desired result. □

4. Arguments and Homotopy in the Space of Symplectic Matrices

We recall the symplectic matrix in this section [13]. Denote by $\mathrm{M}(k, k)$ the space of all $k \times k$ real matrices. Let J_{2n} be the standard symplectic matrix which is represented by

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We say that $D \in \mathrm{M}(2n, 2n)$ is symplectic if and only if we have

$$D^T J_{2n} D = J_{2n},$$

where D^T is the transpose matrix of D. It is well known that the collection of all $2n\times 2n$ real symplectic matrices forms a group with respect to matrix multiplication. Let us denote this group by $\mathrm{Sp}(2n, \mathbb{R})$. Then we have:

Lemma 4.1. [13] *For any $D \in \mathrm{Sp}(2n, \mathbb{R})$, there exists a unique decomposition such that $D = AU$, where $A \in \mathrm{Sp}(2n, \mathbb{R})$ is a positive-definite matrix and $U \in \mathrm{Sp}(2n, \mathbb{R})$ is an orthogonal matrix.*

For our purpose, we just consider the special case $\mathrm{Sp}(2, \mathbb{R})$. It is well known that

$$\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) = \{D \in \mathrm{M}(2, 2) : \det(D) = 1\}.$$

Without loss of generality we assume that $\Gamma \in \mathcal{L}^0_{m,M,ap}$ in system (1.2) from now on. For definiteness, the solution of (1.2) is understood to be right continuous with respect to $x \in \mathbb{R}$, that is, $(\psi'(x+), \psi(x+))^T \equiv (\psi'(x), \psi(x))^T$. In this sense, $\psi'(x)$ and $\psi(x)$ are well defined on $\mathbb{R}$. Suppose that $\Psi(x) := \Psi_\lambda(x; q, V, \Gamma)$ is the fundamental matrix solution of (1.2) with the initial value $\Psi(0) = I_2$. Then we have the following:

Lemma 4.2. *For any $x \in \mathbb{R}$, $\Psi(x) \in \mathrm{Sp}(2, \mathbb{R})$.*

Proof. Consider the system (1.2) on $[x_n, x_{n+1})$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x) = \begin{pmatrix} 0 & q(x) - \lambda \\ 1 & 0 \end{pmatrix} \Psi(x).$$

It follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\Psi(x)^T J_2 \Psi(x)\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x)\right)^T J_2\Psi(x) + \Psi(x)^T J_2 \left(\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x)\right) \\
&= \Psi(x)^T \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix} J_2\Psi(x) + \Psi(x)^T J_2 \begin{pmatrix} 0 & q(x) - \lambda \\ 1 & 0 \end{pmatrix} \Psi(x) \\
&\equiv 0
\end{aligned}$$

Since $\Psi(0)^T J_2 \Psi(0) = J_2$, we have

$$\Psi(x)^T J_2 \Psi(x) \equiv J_2, \qquad x \in [0, x_1).$$

By the group property of $\mathrm{Sp}(2, \mathbb{R})$, we obtain the desired result for any $x \in \mathbb{R}$. □

If $(\psi'(x), \psi(x))^T$ has the initial value $(\psi'(0), \psi(0))^T = (\alpha, \beta)^T$, we have

$$(\psi'(x), \psi(x))^T = \Psi(x)(\alpha, \beta)^T.$$

Introduce the so-called Prüfer transformation as

$$\psi' + \sqrt{-1}\,\psi = r\ \mathrm{e}^{\sqrt{-1}\,\theta}. \tag{4.1}$$

Then the argument $\theta = \theta(x)$ may be denoted by

$$\theta(x) := \arg(\psi'(x) + \sqrt{-1}\,\psi(x)),$$

where $(\psi'(x), \psi(x))^T$ is any non-trivial solution of (1.2). Consider that the system (1.2) is restricted on $\mathbb{R} \setminus \Gamma$. We understand $\arg(\cdot)$ as a continuous branch on $[x_n, x_{n+1})$. It is easy to obtain that the differential equation for θ is found to be

$$\theta'(x) = \cos^2\theta - (q(x) - \lambda)\sin^2\theta, \qquad x \in \mathbb{R} \setminus \Gamma.$$

But it is important to deal with the jump of arguments on Γ via a reasonable approach, because the vector field of (1.2) on Γ is singular. To overcome this difficulty, we use homology as follows.

Let $D \in \mathrm{Sp}(2, \mathbb{R})$. The corresponding result of Lemma 4.1 for $\mathrm{Sp}(2, \mathbb{R})$ is stated as

$$D = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, \tag{4.2}$$

where $(r, \vartheta, z) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}$ is uniquely determined by D, and $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z} - \pi)$. This implies the following:

Lemma 4.3. [13, Theorem 1, p.52] *There exists a one-to-one correspondence from* $\mathrm{Sp}(2, \mathbb{R})$ *to* $\{(x, y, z) \in \mathbb{R}^3 \setminus \{z\text{-}axis\}\}$ *as*

$$g : D \mapsto (r\cos\vartheta, r\sin\vartheta, z),$$

where (r, ϑ, z) *is defined above. Moreover,* g *is a homeomorphism.*

Under the representation (4.2), the two eigenvalues of D are

$$\lambda_\pm = \frac{1}{2r}\left\{\left(r^2 + z^2 + 1\right)\cos\vartheta \pm \sqrt{\left(1 + r^2 + z^2\right)^2\cos^2\vartheta - 4r^2}\right\}.$$

Then we have

$$\mathrm{Sp}(2, \mathbb{R}) = \mathrm{Sp}^h(2, \mathbb{R}) \cup \mathrm{Sp}^e(2, \mathbb{R}) \cup \mathrm{Sp}^p(2, \mathbb{R}),$$

where

$$\begin{aligned}
\mathrm{Sp}^h(2, \mathbb{R}) &:= \left\{(r, \vartheta, z) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R} : \ \left(1 + r^2 + z^2\right)\cos\vartheta > 2r\right\}, \\
\mathrm{Sp}^e(2, \mathbb{R}) &:= \left\{(r, \vartheta, z) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R} : \ \left(1 + r^2 + z^2\right)\cos\vartheta < 2r\right\}, \\
\mathrm{Sp}^p(2, \mathbb{R}) &:= \left\{(r, \vartheta, z) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R} : \ \left(1 + r^2 + z^2\right)\cos\vartheta = 2r\right\}.
\end{aligned}$$

Due to the expression of (1.2), we only consider the following group denoted by

$$\mathrm{Trig}(2, \mathbb{R}) := \left\{R_c := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in \mathbb{R}\right\} \subset \mathrm{Sp}(2, \mathbb{R}).$$

For $R_c \in \mathrm{Trig}(2, \mathbb{R})$, the unique decomposition can be calculated as

$$R_c = \begin{pmatrix} \frac{c^2+2}{\sqrt{c^2+4}} & \frac{c}{\sqrt{c^2+4}} \\ \frac{c}{\sqrt{c^2+4}} & \frac{2}{\sqrt{c^2+4}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{c^2+4}} & \frac{c}{\sqrt{c^2+4}} \\ -\frac{c}{\sqrt{c^2+4}} & \frac{2}{\sqrt{c^2+4}} \end{pmatrix}.$$

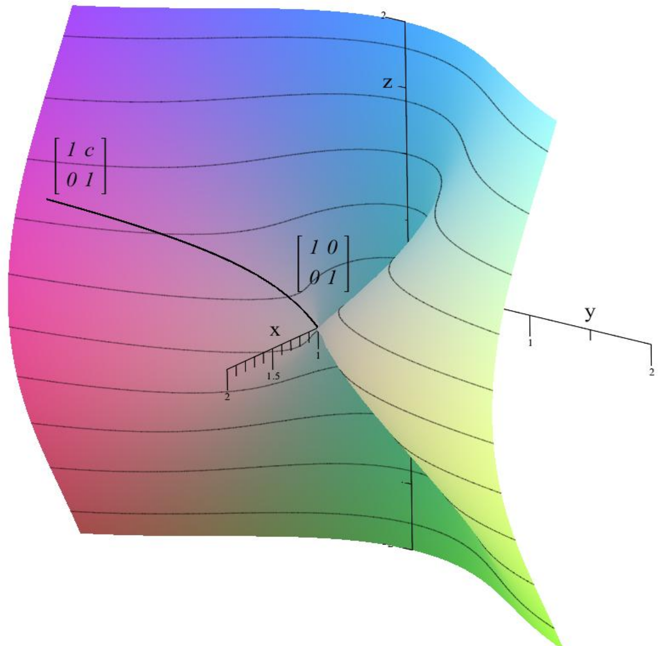

FIGURE 1. $x(x^2+y^2+z^2+1)=2x^2+2y^2$

Construct a continuous path $P_c(\cdot):[0,1]\to \mathrm{Sp}(2,\mathbb{R})$ as

$$P_c(\tau)=\begin{pmatrix} \frac{(\tau c)^2+2}{\sqrt{(\tau c)^2+4}} & \frac{\tau c}{\sqrt{(\tau c)^2+4}} \\ \frac{\tau c}{\sqrt{(\tau c)^2+4}} & \frac{2}{\sqrt{(\tau c)^2+4}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{(\tau c)^2+4}} & \frac{\tau c}{\sqrt{(\tau c)^2+4}} \\ -\frac{\tau c}{\sqrt{(\tau c)^2+4}} & \frac{2}{\sqrt{(\tau c)^2+4}} \end{pmatrix}=\begin{pmatrix} 1 & \tau c \\ 0 & 1 \end{pmatrix}.$$

$P_c(\cdot)$ connects I_2 and R_c, and lies on the hypersurface shown in Figure 1. Note that Figure 1 shows the parabolic region of $\mathrm{Sp}(2,\mathbb{R})$ with eigenvalues $\lambda_\pm=1$. The homotopy class of $P_c(\cdot)$ is denoted by $[P_c]$. Then the jump of arguments on Γ can be well defined when the homotopy class is fixed as the construction $[P_c]$.

In detail, denote by $\mathrm{V}(\mathbb{R}^2)$ the set of all vectors starting from the origin in $\mathbb{R}^2$. The equivalence $\sim$ on $\mathrm{V}(\mathbb{R}^2)$ is defined by

$$\vec{v}_1\sim\vec{v}_2 \Longleftrightarrow \vec{v}_1=k\vec{v}_2,\quad \text{for some } k\in\mathbb{R}^+.$$

It is well known that

$$\mathrm{L}(\mathbb{R}):=\mathrm{V}(\mathbb{R}^2)/\sim$$

is an orientable compact manifold of dimension one, and may be regarded as a two-covering of the real projective line $\mathbb{RP}^1$. Topologically, $\mathrm{L}(\mathbb{R})$ is homeomorphic to $\mathbb{S}_{2\pi}:=\mathbb{R}/2\pi\mathbb{Z}$.

Let $\Xi\in\mathbb{R}$. Then we have

$$P_c(\tau)(\cos\Xi,\sin\Xi)^T=(\cos\Xi+\tau c\sin\Xi,\sin\Xi)^T.$$

Since the homotopy class of $P_c(\cdot)$ is fixed and $\arg(\cdot)$ is understood as a continuous branch, the argument function

$$F(c,\tau,\Xi)=\arg(\cos\Xi+\tau c\sin\Xi+\sqrt{-1}\sin\Xi)$$

is continuous with respect to $(c,\tau,\Xi)\in\mathbb{R}\times[0,1]\times\mathbb{R}$. In particular, we may choose one continuous branch of $F(c,\tau,\Xi)$ such that when $\tau=0$, we have

$$\arg(\cos\Xi+\sqrt{-1}\sin\Xi)=\Xi.$$

Then we define the jump of arguments by

$$\begin{aligned} J(c,\Xi) &= F(c,1,\Xi)-F(c,0,\Xi)\\ &= \arg(\cos\Xi+c\sin\Xi+\sqrt{-1}\sin\Xi)-\arg(\cos\Xi+\sqrt{-1}\sin\Xi)\\ &= \arg(\cos\Xi+c\sin\Xi+\sqrt{-1}\sin\Xi)-\Xi. \end{aligned} \tag{4.3}$$

Lemma 4.4. *$J:\mathbb{R}^2\to\mathbb{R}$ is continuous with respect to $(c,\Xi)\in\mathbb{R}^2$. Moreover, $J(c,\Xi+2\pi)=J(c,\Xi)$.*

Proof. This is obvious from the continuity of $F(c,\tau,\Xi)$ and (4.3). □

Remark 4.5. Instead of the special case R_c, we have an extended version of Lemma 4.4 for a general jump matrix $A\in\mathrm{Sp}(2,\mathbb{R})$ with the help of Lemma 4.3 and (4.2); see [3]. The key point is to represent A as in (4.2) and construct a continuous path $P_A(\cdot):[0,1]\to\mathrm{Sp}(2,\mathbb{R})$ between I_2 and A as follows:

$$P_A(\tau)=\begin{pmatrix}\tau r+1-\tau & \tau z\\ \tau z & \frac{1+(\tau z)^2}{\tau r+1-\tau}\end{pmatrix}\begin{pmatrix}\cos\tau\vartheta & -\sin\tau\vartheta\\ \sin\tau\vartheta & \cos\tau\vartheta\end{pmatrix}.$$

Then the homotopy class $[P_A]$ yields a transfer function on $\mathrm{L}(\mathbb{R})$.

5. Reduction to Skew-Product Dynamical Systems

Let us revisit the system (1.2). Let $(q,V,\Gamma)\in\mathcal{C}_{ap}(\mathbb{R})\times\ell^\infty_{ap}(\mathbb{R})\times\mathcal{L}^0_{m,M,ap}$. We need to embed it in a family of systems as follows:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi'\\ \psi\end{pmatrix}=\begin{pmatrix}0 & \tilde q(x)-\lambda\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi'\\ \psi\end{pmatrix}, & x\in\mathbb{R}\setminus\tilde\Gamma,\\ \begin{pmatrix}\psi'(\tilde x_n+)\\ \psi(\tilde x_n+)\end{pmatrix}=\begin{pmatrix}1 & \tilde v_n\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi'(\tilde x_n-)\\ \psi(\tilde x_n-)\end{pmatrix}, & x=\tilde x_n\in\tilde\Gamma,\end{cases}$$

where $(\tilde q,\tilde V,\tilde\Gamma)\in\mathrm{E}_{q,V,\Gamma}$. By the Prüfer transformation (4.1), the evolution of the arguments is found to be

$$\begin{cases}\theta'(x)=\cos^2\theta(x)-(\tilde q(x)-\lambda)\sin^2\theta(x), & x\in\mathbb{R}\setminus\tilde\Gamma,\\ \theta(\tilde x_n+)-\theta(\tilde x_n-)=J(\tilde v_n,\theta(\tilde x_n-)), & x=\tilde x_n\in\tilde\Gamma.\end{cases} \tag{5.1}$$

Denote by $\theta(x)=\theta_\lambda(x+;(\tilde q,\tilde V,\tilde\Gamma),\Xi)$ the solution of (5.1) with the initial value $\theta(0)=\Xi\in\mathbb{R}$. By the boundedness of both almost periodic objects and jump of arguments, we obtain:

Lemma 5.1. *We have the following relation:*

$$\lim_{x\to+\infty}\frac{\theta_\lambda(x;(\tilde q,\tilde V,\tilde\Gamma),\Xi)-\Xi}{x}=\lim_{n\to+\infty}\frac{\theta_\lambda(\tilde x_n;(\tilde q,\tilde V,\tilde\Gamma),\Xi)-\Xi}{\tilde x_n}.$$

That is, if one of limits exists, then the other one exists as well and they are equal.

Proof. For $x \in \mathbb{R}$ and $\tilde{\Gamma} = \{\tilde{x}_n\}_{n\in\mathbb{Z}} \in \mathcal{L}^0_{m,M,ap}$, there exists $n_0 \in \mathbb{Z}$ such that $x \in [\tilde{x}_{n_0}, \tilde{x}_{n_0+1})$. Then by (5.1) and (2.9), we have

$$\begin{aligned}
&|\theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \theta_\lambda(\tilde{x}_{n_0};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)| \\
= &\left|\int_{[\tilde{x}_{n_0},x]} (\cos^2\theta(\tau) - (\tilde{q}(\tau)-\lambda)\sin^2\theta(\tau))\,\mathrm{d}\tau\right| \\
\le &\int_{[\tilde{x}_{n_0},\tilde{x}_{n_0}+M]} |\cos^2\theta(\tau) - (\tilde{q}(\tau)-\lambda)\sin^2\theta(\tau)|\,\mathrm{d}\tau \\
\le &M(1+|\lambda|+\|\tilde{q}\|_\infty) < +\infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{\theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \Xi}{x} \\
= &\lim_{x\to+\infty} \frac{\tilde{x}_{n_0}}{x} \frac{\theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \theta_\lambda(\tilde{x}_{n_0};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) + \theta_\lambda(\tilde{x}_{n_0};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \Xi}{\tilde{x}_{n_0}} \\
= &\lim_{n\to+\infty} \frac{\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \Xi}{\tilde{x}_n},
\end{aligned}$$

provided one of limits exists. □

Lemma 5.2. *For $k \in \mathbb{Z}$, we have*

$$\theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi+2k\pi) - (\Xi+2k\pi) = \theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) - \Xi.$$

Proof. By Lemma 4.4, we know that the vector field of (5.1) is 2π-periodic with respect to θ. Then both $\check{\theta}_1(x) := \theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi+2k\pi)$ and $\check{\theta}_2(x) := \theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)+2k\pi$ satisfy (5.1) with the initial value $\check{\theta}_i(0) = \Xi + 2k\pi,\ i = 1,2$. By the uniqueness of solutions of (5.1), we have

$$\theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi+2k\pi) = \theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) + 2k\pi,$$

finishing the proof. □

Lemma 5.3. *For $k_1, k_2 \in \mathbb{Z}$, we have*

$$\theta_\lambda(\tilde{x}_{k_1+k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) = \theta_\lambda(\tilde{x}_{k_1+k_2} - \tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2, \theta_\lambda(\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)),$$

where $(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2$ is defined by (3.2).

Proof. Denote

$$\bar{\theta}_1(x) := \theta_\lambda(x;(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2, \theta(\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)),$$

and

$$\bar{\theta}_2(x) := \theta_\lambda(x+\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi).$$

Then $\bar{\theta}_1(x)$ satisfies the following equation,

(5.2)
$$\begin{cases}
\theta'(x) = \cos^2\theta(x) - (\tilde{q}\cdot\tilde{x}_{k_2}(x) - \lambda)\sin^2\theta(x), \\
x \in \mathbb{R}\setminus\tilde{\Gamma}\cdot k_2, \\
\theta((\tilde{x}_{n+k_2} - \tilde{x}_{k_2})+) - \theta((\tilde{x}_{n+k_2} - \tilde{x}_{k_2})-) = J(\tilde{v}_{n+k_2}, \theta((\tilde{x}_{n+k_2} - \tilde{x}_{k_2})-)), \\
x \in \tilde{\Gamma}\cdot k_2 = \{\tilde{x}_{n+k_2} - \tilde{x}_{k_2}\}_{n\in\mathbb{Z}},
\end{cases}$$

with the initial value $\bar{\theta}_1(0) = \theta(\tilde{x}_{k_2}; (\tilde{q}, \tilde{V}, \tilde{\Gamma}), \Xi)$, and $\bar{\theta}_2(x)$ satisfies the following equation,

$$
\left\{\begin{array}{l}
\theta'(x + \tilde{x}_{k_2}) = \cos^2\theta(x + \tilde{x}_{k_2}) - (\tilde{q}(x + \tilde{x}_{k_2}) - \lambda)\sin^2\theta(x + \tilde{x}_{k_2}),\\
x + \tilde{x}_{k_2} \in \mathbb{R} \setminus \tilde{\Gamma},\\
\theta(\tilde{x}_n+) - \theta(\tilde{x}_n-) = J(\tilde{v}_n, \theta(\tilde{x}_n-)),\\
x + \tilde{x}_{k_2} \in \tilde{\Gamma} = \{\tilde{x}_n\}_{n\in\mathbb{Z}},
\end{array}\right. \tag{5.3}
$$

with the initial value $\bar{\theta}_2(0) = \theta_\lambda(\tilde{x}_{k_2}; (\tilde{q}, \tilde{V}, \tilde{\Gamma}), \Xi)$. It is easy to check that (5.2) is equivalent to (5.3). Note that this is the reason that we choose the direction of the translation of point sets as the reverse of that of functions and define it by (2.11). Due to the uniqueness of solutions of ODEs, we conclude that $\bar{\theta}_1(x) = \bar{\theta}_2(x)$ for all $x \in \mathbb{R}$. Taking $x = \tilde{x}_{k_1+k_2} - \tilde{x}_{k_2}$, we have the desired result. □

By the continuity of solutions of ODEs with respect to parameters and initial values and Lemma 4.4, we have the following:

Lemma 5.4. *When $k \in \mathbb{Z}$ is fixed, $\theta_\lambda(\tilde{x}_k; (\tilde{q}, \tilde{V}, \tilde{\Gamma}), \Xi) : \mathrm{E}_{q,V,\Gamma} \times \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof. Without loss of generality we assume that $k = 1$. Let

$$\theta_i(x) := \theta_\lambda(\tilde{x}^i_1; (\tilde{q}^i, \tilde{V}^i, \tilde{\Gamma}^i), \Xi_i), \quad i = 1, 2,$$

where $\tilde{\Gamma}^i = \{\tilde{x}^i_n\}_{n\in\mathbb{Z}}$. Then we have

$$
\left\{\begin{array}{ll}
\theta_i'(x) = \cos^2\theta_i(x) - (\tilde{q}^i(x) - \lambda)\sin^2\theta_i(x), & x \in (0, \tilde{x}^i_1),\\
\theta_i(0) = \Xi_i. &
\end{array}\right.
$$

This implies that

$$
\begin{aligned}
&\theta_2(\tilde{x}^2_1-) - \theta_1(\tilde{x}^1_1-)\\
&= (\Xi_2 - \Xi_1) + \left(\int_0^{\tilde{x}^1_1} + \int_{\tilde{x}^1_1}^{\tilde{x}^2_1}\right) \cos^2\theta_2(\tau) - (\tilde{q}^2(\tau) - \lambda)\sin^2\theta_2(\tau)\,\mathrm{d}\tau\\
&\quad - \int_0^{\tilde{x}^1_1} \cos^2\theta_1(\tau) - (\tilde{q}^1(\tau) - \lambda)\sin^2\theta_1(\tau)\,\mathrm{d}\tau
\end{aligned} \tag{5.4}
$$

Denote

$$D_1 := \int_{\tilde{x}^1_1}^{\tilde{x}^2_1} \cos^2\theta_2(\tau) - (\tilde{q}^2(\tau) - \lambda)\sin^2\theta_2(\tau)\,\mathrm{d}\tau,$$

and

$$D_2 := \int_0^{\tilde{x}^1_1} (\cos^2\theta_2(\tau) - (\tilde{q}^2(\tau) - \lambda)\sin^2\theta_2(\tau)) - (\cos^2\theta_1(\tau) - (\tilde{q}^1(\tau) - \lambda)\sin^2\theta_1(\tau))\,\mathrm{d}\tau.$$

Then by (5.4) we have

$$\theta_2(\tilde{x}^2_1-) - \theta_1(\tilde{x}^1_1-) = (\Xi_2 - \Xi_1) + D_1 + D_2. \tag{5.5}$$

By the boundedness of $\tilde{q}^2 \in \mathcal{C}_{ap}(\mathbb{R})$, we know that there exists $C_1 > 0$ such that

$$|D_1| \le C_1|\tilde{x}^2_1 - \tilde{x}^1_1|. \tag{5.6}$$

Note that $\tilde{x}^1_1 = \Delta\tilde{x}^1_1 \le M$. By a similar argument as in [20, Lemma 3.2.], we know that there exists $C_2 > 0$ such that

$$|D_2| \le C_2\|\tilde{q}^2 - \tilde{q}^1\|_\infty. \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that $\theta(\tilde{x}_1-;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi):\mathrm{E}_{q,V,\Gamma}\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous. Furthermore, we have

$$\theta(\tilde{x}_1)=\theta(\tilde{x}_1-)+J(\tilde{v}_1,\theta(\tilde{x}_1-)).$$

By Lemma 4.4, we know that $\theta_\lambda(\tilde{x}_1;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi):\mathrm{E}_{q,V,\Gamma}\times\mathbb{R}\to\mathbb{R}$ is continuous. □

Lemma 5.5. *When $(\tilde{q},\tilde{V},\tilde{\Gamma})\in\mathrm{E}_{q,V,\Gamma}$ and $k\in\mathbb{Z}$ are fixed, $\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi):\mathbb{R}\to\mathbb{R}$ is a strictly increasing homeomorphism.*

Proof. Due to the uniqueness of solutions of (5.1), we know that $\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi):\mathbb{R}\to\mathbb{R}$ is strictly increasing. By Lemma 5.3, we have

$$\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\theta_\lambda(-\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\Xi))=\Xi=\theta_\lambda(-\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)).$$

This implies that the inverse of $\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)$ is $\theta_\lambda(-\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\Xi)$. By Lemma 5.4, we obtain that $\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi):\mathbb{R}\to\mathbb{R}$ is a strictly increasing homeomorphism. □

Let $\mathbb{S}_{2\pi}:=\mathbb{R}/2\pi\mathbb{Z}$ and $\mathrm{Z}:=\mathrm{E}_{q,V,\Gamma}\times\mathbb{S}_{2\pi}$. We introduce the distance on the product space Z as

$$\begin{aligned}&\mathrm{dist}(((\tilde{q}^1,\tilde{V}^1,\tilde{\Gamma}^1),\vartheta_1),((\tilde{q}^2,\tilde{V}^2,\tilde{\Gamma}^2),\vartheta_2))\\ :=\ &\max\{\mathrm{dist}((\tilde{q}^1,\tilde{V}^1,\tilde{\Gamma}^1),(\tilde{q}^2,\tilde{V}^2,\tilde{\Gamma}^2)),|\vartheta_1-\vartheta_2|_{\mathbb{S}_{2\pi}}\}\end{aligned}$$

where $((\tilde{q}^i,\tilde{V}^i,\tilde{\Gamma}^i),\vartheta_i)\in\mathrm{Z},\ i=1,2$. We know that $(\mathrm{Z},\mathrm{dist})$ is a compact metric space.

For each $k\in\mathbb{Z}$, the skew-product transformation Φ^k on Z is defined by

$$\Phi_\lambda^k((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta):=((\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)\mod 2\pi),\tag{5.8}$$

where $((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)\in\mathrm{Z}$, and there exists $\Xi\in\mathbb{R}$ satisfying $\vartheta=\Xi\mod 2\pi$. By Lemma 5.2, Φ_λ^k is well defined for each $k\in\mathbb{Z}$. Moreover, by Lemma 5.3 and Lemma 5.5, we have:

Lemma 5.6. *$\{\Phi_\lambda^k\}_{k\in\mathbb{Z}}$ is a skew-product continuous dynamical system on the compact space* Z.

Proof. First, we show that $\{\Phi_\lambda^k\}_{k\in\mathbb{Z}}$ is continuous. Indeed, we have

$$(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k:=(\tilde{q}\cdot\tilde{x}_k,\tilde{V}\cdot k,\tilde{\Gamma}\cdot k),\qquad(\tilde{q},\tilde{V},\tilde{\Gamma})\in\mathrm{E}_{q,V,\Gamma}.$$

By Lemma 2.9, we know that $\tilde{\Gamma}\cdot k$ is continuous on $\mathrm{E}_{q,V,\Gamma}$. By (2.2), $\tilde{V}\cdot k$ is continuous on $\mathrm{E}_{q,V,\Gamma}$. Since

$$\|\tilde{q}_1\cdot\tilde{x}_k^1-\tilde{q}_2\cdot\tilde{x}_k^2\|\le\|\tilde{q}_1\cdot\tilde{x}_k^1-\tilde{q}_2\cdot\tilde{x}_k^1\|+\|\tilde{q}_2\cdot\tilde{x}_k^1-\tilde{q}_2\cdot\tilde{x}_k^2\|$$

It follows from Lemma 2.8, (2.2) and (2.3) that $\tilde{q}\cdot\tilde{x}_k$ is continuous on $\mathrm{E}_{q,V,\Gamma}$. Combining this with Lemma 5.4, we have the desired assertion.

Now we aim to prove that

$$\Phi_\lambda^{k_1+k_2}=\Phi_\lambda^{k_1}\circ\Phi_\lambda^{k_2}\quad\text{for } k_1,\ k_2\in\mathbb{Z}.$$

In fact, assume that $((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)\in \mathrm{Z}$ and there exists $\Xi\in\mathbb{R}$ satisfying $\vartheta=\Xi \mod 2\pi$. By (5.8) and Lemma 5.3, we have

$$\begin{aligned}
&\Phi_\lambda^{k_1}\circ\Phi_\lambda^{k_2}((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)\\
=\ &\Phi_\lambda^{k_1}((\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2,\theta_\lambda(\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) \mod 2\pi)\\
=\ &((\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2\cdot k_1,\theta_\lambda(\tilde{x}_{k_1+k_2}-\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k_2,\theta_\lambda(\tilde{x}_{k_2};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)) \mod 2\pi)\\
=\ &\Phi_\lambda^{k_1+k_2}((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta).
\end{aligned}$$

The proof is complete. □

Introduce the observation F_λ from Z to $\mathbb{R}$ as

$$F_\lambda((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta):=\theta_\lambda(\tilde{x}_1;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi,\quad ((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)\in \mathrm{Z}, \tag{5.9}$$

where $\Xi\in\mathbb{R}$ satisfies $\vartheta=\Xi \mod 2\pi$. By Lemma 5.2, $F_\lambda((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)$ is well defined on Z. Furthermore, by Lemma 5.4, we have:

Lemma 5.7. *$F_\lambda((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)$ is continuous on* Z.

By (4.3), we have

$$F_\lambda((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)=\theta_\lambda(\tilde{x}_1-;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi+J(\tilde{c}_1,\theta(\tilde{x}_1-)).$$

where $\Xi\in\mathbb{R}$ satisfies $\vartheta=\Xi \mod 2\pi$. By the construction above and Lemma 2.16, we reduce the existence of rotation numbers to that of the following ergodic limit with respect to the skew-product dynamical system $\{\Phi_\lambda^k\}_{k\in\mathbb{Z}}$.

Lemma 5.8. *Assume that $((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)\in \mathrm{E}_{q,V,\Gamma}\times\mathbb{S}_{2\pi}$ and $\Xi\in\mathbb{R}$ satisfies $\vartheta=\Xi$ mod 2π. Then we have the following relation:*

$$\lim_{n\to+\infty}\frac{\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi}{\tilde{x}_n}=[\Gamma]\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}F_\lambda(\Phi_\lambda^k((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)).$$

That is, if one of the limits exists, then the other one exists as well and they are equal. When $\Gamma=\mathbb{Z}$, then $[\Gamma]=1$.

Proof. By Lemma 2.16, we have

$$\lim_{n\to+\infty}\frac{\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi}{\tilde{x}_n}=[\Gamma]\lim_{n\to+\infty}\frac{\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi}{n},$$

provided one of limits exists. Furthermore,

$$\begin{aligned}
&\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi\\
=\ &\sum_{k=0}^{n-1}(\theta_\lambda(\tilde{x}_{k+1};(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi))\\
=\ &\sum_{k=0}^{n-1}(\theta_\lambda(\tilde{x}_{k+1}-\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi))-\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi))\\
=\ &\sum_{k=0}^{n-1}F_\lambda((\tilde{q},\tilde{V},\tilde{\Gamma})\cdot k,\theta_\lambda(\tilde{x}_k;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) \mod 2\pi)\\
=\ &\sum_{k=0}^{n-1}F_\lambda(\Phi_\lambda^k((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)).
\end{aligned}$$

The proof is complete. □

6. The Rotation Number

The following uniform ergodic theorem is due to Johnson and Moser [8].

Lemma 6.1. [8] *Let $\{\varphi^k\}_{k\in\mathbb{Z}}$ be a continuous discrete-time dynamical system on a compact metric space X. Then, for any $f \in \mathcal{C}(X,\mathbb{R})$ satisfying*

$$\int_X f \ \mathrm{d}\mu = 0$$

for all invariant Borel probability measures μ under $\{\varphi^k\}$, one has

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(x)) = 0$$

uniformly for all $x \in X$.

To show the existence of rotation numbers, inspired by Lemma 5.8, we introduce the following notation.

$$F_\lambda^*((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta) := \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} F_\lambda(\Phi_\lambda^k((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta)), \quad ((\tilde{q},\tilde{V},\tilde{\Gamma}),\vartheta) \in \mathrm{Z},$$

whenever the limit exists. For $(\tilde{q},\tilde{V},\tilde{\Gamma}) \in \mathrm{E}_{q,V,\Gamma}$ and $\Xi \in \mathbb{R}$, denote

$$F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) := \lim_{n\to+\infty}\frac{\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)-\Xi}{\tilde{x}_n}, \tag{6.1}$$

provided the limit exists. By Lemma 5.2, we obtain the following:

Lemma 6.2. *If $F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi_0)$ exists for $\Xi_0 \in \mathbb{R}$, then $F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi)$ exists for all $\Xi \in \mathbb{R}$ and is independent of the choice of $\Xi \in \mathbb{R}$.*

Proof. By Lemma 5.2, we know that $F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi_0+2k\pi)$ exists for all $k \in \mathbb{Z}$. Then for any $\Xi \in \mathbb{R}$, there exists $k_\Xi \in \mathbb{Z}$ such that

$$\Xi_0 + 2k_\Xi\pi \le \Xi < \Xi_0 + 2(k_\Xi+1)\pi.$$

By Lemma 5.2 and Lemma 5.5, for all $n \in \mathbb{N}$, we have

$$\theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi_0+2k_\Xi\pi) \le \theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) < \theta_\lambda(\tilde{x}_n;(\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi_0+2k_\Xi\pi)+2\pi.$$

This implies that for all $\Xi \in \mathbb{R}$, we have

$$F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi) \equiv F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi_0).$$

The proof is complete. □

We obtain the main result as follows.

Theorem 6.3. *For any $\Xi \in \mathbb{R}$, the limit $F_\lambda^\diamond((\tilde{q},\tilde{V},\tilde{\Gamma}),\Xi))$ exists and is independent of the choice of the $\Xi \in \mathbb{R}$ and $(\tilde{q},\tilde{V},\tilde{\Gamma}) \in \mathrm{E}_{q,V,\Gamma}$. Thus we denote it by $\rho_\lambda(q,V,\Gamma)$ and call it the rotation number of* (1.2).

Proof. By the Krylov-Bogoliubov theorem and Lemma 5.6, there exists an invariant Borel probability measure under $\{\Phi_\lambda^k\}$, denoted by μ. Then by the Birkhoff ergodic theorem, there exists a Borel set $\mathrm{Z}_\mu \subset \mathrm{Z}$, which depends on the measure μ, such that $\mu(\mathrm{Z}_\mu) = 1$ and $F^*_\lambda((\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta))$ exists for all $(\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta) \in \mathrm{Z}_\mu$. Furthermore, F^* is integrable and satisfies

$$\int_{\mathrm{Z}} F^*_\lambda \, \mathrm{d}\mu = \int_{\mathrm{Z}} F_\lambda \, \mathrm{d}\mu =: \rho_{\lambda,\mu}. \tag{6.2}$$

Due to Lemma 6.2, Z_μ can be written in the form $\mathrm{Z}_\mu = \mathrm{E}_\mu \times \mathbb{S}_{2\pi}$, where E_μ is a Borel set in $\mathrm{E}_{q,V,\Gamma}$. Let ν be the Haar measure on $\mathrm{E}_{q,V,\Gamma}$. Then we have $\nu(\mathrm{E}_\mu) = 1$. By the unique ergodicity of the Haar measure, there exists a set $\hat{\mathrm{E}}_\mu \subset \mathrm{E}_\mu$ such that $\nu(\hat{\mathrm{E}}_\mu) = \mu(\hat{\mathrm{E}}_\mu \times \mathbb{S}_{2\pi}) = 1$ and $F^*_\lambda((\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta))$ is a constant function on $\hat{\mathrm{E}}_\mu \times \mathbb{S}_{2\pi}$. It follows from (6.2) that the constant must be $\rho_{\lambda,\mu}$.

By (6.1), we know that $\rho_{\lambda,\mu}$ in (6.2) is independent of the choice of the measure μ. Set $\hat{F}_\lambda := F_\lambda - \rho_\lambda$. By Lemma 5.7, $\hat{F}_\lambda$ is continuous on Z. By (6.2), $\hat{F}_\lambda$ satisfies the requirement of Theorem 6.1. Thus, as $k \nearrow +\infty$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{F}_\lambda(\Phi^k_\lambda((\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta)) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{F}_\lambda(\Phi^k_\lambda((\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta)) - \rho_\lambda \to 0 \tag{6.3}$$

uniformly for all $(\tilde{q}, \tilde{V}, \tilde{\Gamma}), \vartheta) \in \mathrm{Z}$.

At last, taking $(\tilde{q}, \tilde{V}, \tilde{\Gamma}) = (q, V, \Gamma)$ in (6.3), then by Lemma 5.1 and Lemma 5.8, we obtain the existence of the desired limit (6.1). □

Remark 6.4. As we said in the Introduction, the rotation number has key connections to the spectral analysis of $H_{q,V,\Gamma}$. We numerically compute the rotation number in three cases, in which the black plot is created when $v_i \equiv -1$, the red one when $v_i \equiv 0$, and the blue one when $v_i \equiv 1$. Numerical exploration in other models can be founded in [14, 6]; see [14] for the discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ and [6] for the quasi-periodic Schrödinger operator on $L^2(\mathbb{R})$. For our model, two things should be mentioned. One is that when v_i increases, the rotation number decreases. The other is that, it is evident that the rotation number takes a constant value in a sub-interval in the λ-axis which is a gap of the spectrum of $H_{q,V,\Gamma}$. The detail on the spectrum will be discussed in a future publication.

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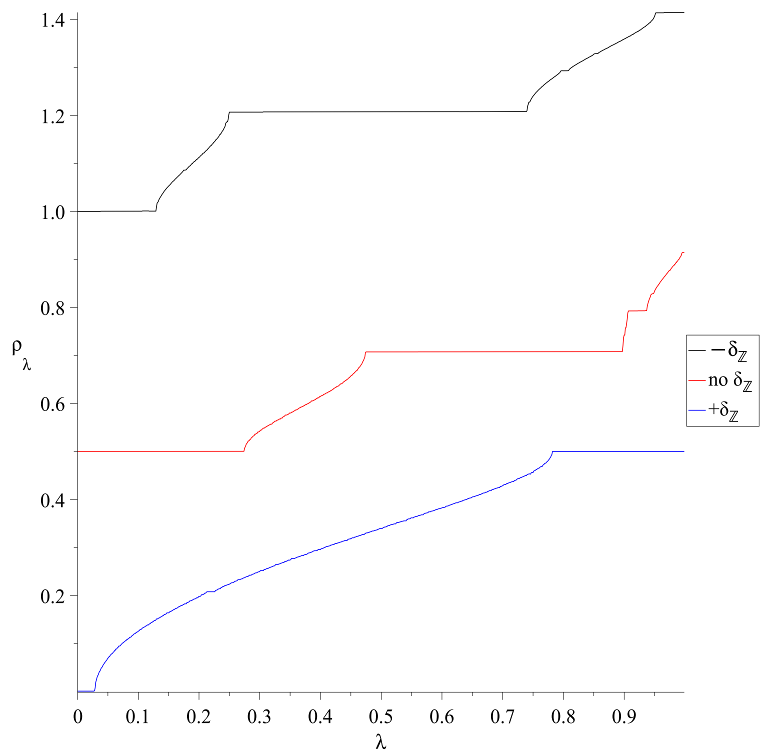

FIGURE 2. $q(x) = \sin(x) + \sin(\sqrt{2}x)$, $\Gamma = \mathbb{Z}$, $v_i \equiv -1,\ 0,\ 1$.

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