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Subordinacy theory for extended CMV matrices

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Abstract We develop subordinacy theory for extended Cantero-Moral-Velázquez (CMV) matrices, i.e., we provide explicit supports for the singular and absolutely continuous parts of the canonical spectral measure associated with a given extended CMV matrix in terms of the presence or absence of subordinate solutions to the generalized eigenvalue equation. Some corollaries and applications of this result are described as well.

Keywords spectral theory, subordinacy theory, CMV matrix, unitary operator, Carathéodory function

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1 Introduction

Subordinacy theory was first developed in the setting of continuum half-line Schrödinger operators by Gilbert and Pearson [11]. Its primary aim is to relate the spectral decomposition of the operator in question to the behavior of the solutions to the associated generalized eigenvalue equation. The following correspondence is obvious: a value E of the spectral parameter is an eigenvalue of the operator H in question if and only if the equation $Hu = Eu$ admits a non-zero solution u that belongs to the domain of the operator. Modulo a suitable regularity property, this means that u satisfies the designated boundary condition at the origin and is square-integrable at $+\infty$. Since the pure point part of any spectral measure of H is supported by the set of eigenvalues, it follows that we can extract the pure point part of any spectral measure by restricting this measure to the set of E's for which the solution that obeys the boundary condition at the origin is square-integrable. Similarly, we extract the continuous part by restriction to the set of E's for which the solution that obeys the boundary condition at the origin is not square-integrable. Gilbert-Pearson's subordinacy theory provides a similar partition related to the split between the singular part and the absolutely continuous part of a spectral measure: the crucial question is now whether the solution that obeys the boundary condition at the origin is subordinate.

A follow-up paper by Gilbert [9] developed subordinacy theory for continuum Schrödinger operators on the whole line, and the resulting theory is completely analogous if one replaces "obeying the boundary condition at the origin" by "being square-integrable/subordinate at $-\infty$" in the discussion above.

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Subsequently, subordinacy theory was developed in other settings as well: for Jacobi matrices by Khan and Pearson [18] and for CMV matrices by Simon [26]. Furthermore, there were simplifications and extensions of subordinacy theory by Remling [22], Jitomirskaya and Last [14, 15], Damanik et al. [5] and Killip et al. [19].

The papers mentioned above establish subordinacy theory for half-line and whole-line Schrödinger operators, for half-line and whole-line Jacobi matrices, and for standard (i.e., half-line) CMV matrices. There is no subordinacy theory yet for extended (i.e., whole-line) CMV matrices, and it is the purpose of this paper to fill this gap in the literature.

Thus we are naturally motivated by the fact that subordinacy theory is a fundamental result to be established for any operator family for which such a theory exists. It is usually the most convenient way to perform a spectral analysis of a given operator, precisely because the behavior of generalized eigenfunctions is easier to study than other properties of the operator in question that are relevant to the identification of its spectral type. We expect our work to be useful in the study of spectral properties for many classes of extended CMV matrices.

The rest of this paper is structured as follows. We describe the setting, the main result, and some consequences of it in Section 2. Some known results that will be used in the proofs are presented in Section 3. Section 4 develops the version of the Jitomirskaya-Last inequalities from [26] that we need to analyze the left half-line of a given extended CMV matrix. The main subordinacy result is then proved in Section 5 and its applications are discussed in Section 6.

2 The setting and the main result

In this section, we describe the setting in which we work and state the main result, a description of supports of the parts of spectral measures of extended CMV matrices in terms of solutions, along with some corollaries. We refer the reader to [25, 26] for the general background, and we follow largely the notation from these monographs.

Let μ be a non-trivial probability measure on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, which means the support of μ contains infinitely many points. By the non-triviality assumption, the functions $1, z, z^2, \ldots$ are linearly independent in the Hilbert space $\mathcal{H} = L^2(\partial\mathbb{D}, d\mu)$, and hence one can form, by the Gram-Schmidt procedure, the *monic orthogonal polynomials* $\Phi_n(z)$, whose Szegő dual is defined by $\Phi_n^* = z^n\overline{\Phi_n(1/\overline{z})}$. There are constants $\{\alpha_n\}_{n\in\mathbb{N}_0}$ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, called the *Verblunsky coefficients*, so that

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n\Phi_n^*(z) \quad \text{for } n \in \mathbb{N}_0, \tag{2.1}$$

which is the so-called *Szegő recurrence*. Conversely, every sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ in $\mathbb{D}$ arises as a sequence of recurrence coefficients corresponding to a Gram-Schmidt procedure on a non-trivial probability measure on $\partial\mathbb{D}$.

In fact, if we normalize the monic orthogonal polynomials $\Phi_n(z)$ by

$$\varphi(z, n) = \frac{\Phi_n(z)}{\|\Phi_n(z)\|_\mu},$$

where $\|\cdot\|_\mu$ is the norm of $\mathcal{H}$, it is easy to see that (2.1) is equivalent to

$$\rho_n(x)\varphi(z, n+1) = z\varphi(z, n) - \overline{\alpha}_n\varphi^*(z, n),$$

where $\rho_n = (1 - |\alpha_n|^2)^{1/2}$.

Define

$$S(\alpha, z) = \frac{1}{\rho}\begin{pmatrix} z & -\overline{\alpha} \\ -\alpha z & 1 \end{pmatrix}, \tag{2.2}$$

where $\rho = (1 - |\alpha|^2)^{1/2}$.

The Szegő recursion can be written in a matrix form as follows:

$$\begin{pmatrix} \varphi(z,n+1) \\ \varphi^*(z,n+1) \end{pmatrix} = S(\alpha_n, z) \begin{pmatrix} \varphi(z,n) \\ \varphi^*(z,n) \end{pmatrix}. \tag{2.3}$$

Alternatively, one can consider a different initial condition and derive the *orthogonal polynomials of the second kind*, by setting $\psi(z,0) = 1$ and then

$$\begin{pmatrix} \psi(z,n+1) \\ -\psi^*(z,n+1) \end{pmatrix} = S(\alpha_n, z) \begin{pmatrix} \psi(z,n) \\ -\psi^*(z,n) \end{pmatrix}.$$

The orthogonal polynomials may or may not form a basis of $\mathcal{H}$. However, if we apply the Gram-Schmidt procedure to $1, z, z^{-1}, z^2, z^{-2}, \dots$, we will obtain a basis—called the *CMV basis*. In this basis, multiplication by the independent variable z in $\mathcal{H}$ has the matrix representation

$$\mathcal{C} = \begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \rho_0 & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.4}$$

where $\alpha = \{\alpha_n\}_{n\in\mathbb{N}_0} \subset \mathbb{D}$ and $\rho_n = \sqrt{1-|\alpha_n|^2}$ for $n \in \mathbb{N}_0$. A matrix of this form is called a *CMV matrix*.

Furthermore, an *extended CMV matrix* is a special five-diagonal doubly infinite matrix in the standard basis of $\ell^2(\mathbb{Z})$ according to [25, Subsection 4.5] and [26, Subsection 10.5], written as

$$\mathcal{E} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & -\overline{\alpha}_0\alpha_{-1} & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \cdots & -\rho_0\alpha_{-1} & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ \cdots & 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ \cdots & 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ \cdots & 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.5}$$

where $\alpha = \{\alpha_n\}_{n\in\mathbb{Z}} \subset \mathbb{D}$ and $\rho_n = \sqrt{1-|\alpha_n|^2}$ for $n \in \mathbb{Z}$. In some settings, it is more natural to consider extended CMV matrices, rather than standard CMV matrices. This is the case, for example, where the Verblunsky coefficients are generated by an invertible ergodic dynamical system. This class of coefficients contains the important special cases of almost periodic and random coefficients and some important parts of the theory for ergodic coefficients, for example Kotani theory [6,7,26], require the consideration of the two-sided case.

The main goal of this paper is to provide a general approach to the study of the spectral properties of a given extended CMV matrix $\mathcal{E}$ via the properties of the solutions to the associated generalized eigenvalue equation. To this end, let us first discuss the canonical spectral measure and then the generalized eigenvalue equation.

Given an extended CMV matrix $\mathcal{E}$, the canonical spectral measure Λ is given by the sum of the spectral measures of $\mathcal{E}$ relative to the vectors δ_0 and δ_1. It is well known that $\{\delta_0, \delta_1\}$ forms a spectral basis for the operator $\mathcal{E}$ (see, e.g., [20, Lemma 3]) and hence for every $\psi \in \ell^2(\mathbb{Z})$, the spectral measure corresponding to $\mathcal{E}$ and ψ is absolutely continuous with respect to Λ.

Consider the Lebesgue decomposition of Λ into its pure point, singular continuous, and absolutely continuous parts,

$$\Lambda = \Lambda_{\mathrm{pp}} + \Lambda_{\mathrm{sc}} + \Lambda_{\mathrm{ac}},$$

i.e., Λ_{pp} is supported by a countable set, Λ_{sc} gives no weight to countable sets but is supported by some set of zero Lebesgue measure, and Λ_{ac} gives no weight to sets of zero Lebesgue measure. Here, we refer to the standard arc length measure on $\partial\mathbb{D}$ as the Lebesgue measure on $\partial\mathbb{D}$.

We also consider the singular part of Λ, $\Lambda_{\mathrm{s}} = \Lambda_{\mathrm{pp}} + \Lambda_{\mathrm{sc}}$ and the continuous part of Λ, $\Lambda_{\mathrm{c}} = \Lambda_{\mathrm{sc}} + \Lambda_{\mathrm{ac}}$.

Consider the corresponding eigenvalue equation

$$\mathcal{E}u = zu \tag{2.6}$$

with boundary conditions

$$\begin{pmatrix} \varphi_\omega(0) & \psi_\omega(0) \\ \varphi_\omega^*(0) & -\psi_\omega^*(0) \end{pmatrix} = \begin{pmatrix} \cos\omega + \mathrm{i}\sin\omega & \cos\omega + \mathrm{i}\sin\omega \\ \cos\omega - \mathrm{i}\sin\omega & -\cos\omega + \mathrm{i}\sin\omega \end{pmatrix}. \tag{2.7}$$

Here is the fundamental definition of subordinacy, introduced by Gilbert and Pearson [11] in the Schrödinger case, adapted to the CMV setting.

Definition 2.1. (a) Define for a sequence $a_0, a_1, \ldots$ and $x \in (0, \infty)$,

$$\|a\|_x^2 = \sum_{j=0}^{[x]} |a_j|^2 + (x - [x])|a_{[x]+1}|^2,$$

where $[x]$ denotes the greatest integer less than or equal to x. An analogous expression defines $\|a\|_x^2$ for $a_{-1}, a_{-2}, \ldots$ and $x \in (-\infty, -1)$. Then

$$\|a\|_x^2 = \sum_{j=\lceil x\rceil}^{-1} |a_j|^2 + (\lceil x\rceil - x)|a_{\lceil x\rceil - 1}|^2,$$

where $\lceil x\rceil$ denotes the least integer greater than or equal to x.

(b) Let $z \in \partial\mathbb{D}$. A solution u to (2.6) is called *subordinate at* $+\infty$ if it does not vanish identically and obeys

$$\lim_{x\to+\infty} \frac{\|u\|_x}{\|p\|_x} = 0$$

for any linearly independent solution p to (2.6).

Similarly, a solution u to (2.6) is called *subordinate at* $-\infty$ if it does not vanish identically and obeys

$$\lim_{x\to-\infty} \frac{\|u\|_x}{\|p\|_x} = 0$$

for any linearly independent solution p to (2.6).

We are now ready to state the main result of this paper.

Theorem 2.2. *Let* $\mathcal{E}$ *be an extended CMV matrix in* $\ell^2(\mathbb{Z})$ *and denote by* Λ *its canonical spectral measure. Then, the three parts of the canonical spectral measure have the following supports defined in terms of the behavior of the solutions to* (2.6):

(a) *Let*

$$\mathcal{P} = \{z \in \partial\mathbb{D} : (2.6) \text{ has a solution that is square-summable at } \pm\infty\}.$$

Then $\Lambda_{\mathrm{pp}}(\partial\mathbb{D} \setminus \mathcal{P}) = 0$ *and* $\Lambda_{\mathrm{c}}(\mathcal{P}) = 0$.

(b) *Let*

$$\mathcal{S} = \{z \in \partial\mathbb{D} : (2.6) \text{ has a solution that is subordinate at } \pm\infty\}.$$

Then $\Lambda_{\rm s}(\partial\mathbb{D}\setminus\mathcal{S})=0$ *and* $\Lambda_{\rm ac}(\mathcal{S})=0$. *In particular, we have*

$$\Lambda_{\rm sc}(\partial\mathbb{D}\setminus(\mathcal{S}\setminus\mathcal{P}))=0 \quad \textit{and} \quad (\Lambda_{\rm pp}+\Lambda_{\rm ac})(\mathcal{S}\setminus\mathcal{P})=0.$$

(c) *Let*

$$\mathcal{A}_\pm=\{z\in\partial\mathbb{D}:(2.6)\ \textit{has no solution that is subordinate at}\ \pm\infty\} \tag{2.8}$$

and

$$\mathcal{A}=\mathcal{A}_+\cup\mathcal{A}_-.$$

Then $\Lambda_{\rm ac}(\partial\mathbb{D}\backslash\mathcal{A})=0$ *and* $\Lambda_{\rm s}(\mathcal{A})=0$. *Moreover,* $\mathcal{A}$ *is an essential support of* $\Lambda_{\rm ac}$, *i.e., for any measurable set* $\mathcal{A}'$ *with* $\Lambda_{\rm ac}(\partial\mathbb{D}\backslash\mathcal{A}')=0$, *we have* $\mathrm{Leb}(\mathcal{A}\backslash\mathcal{A}')=0$.

Remark 2.3. Theorem 2.2(a) is well known and stated here for completeness. The statement follows quickly from the spectral theorem; see, for example, the proof of [28, Theorem 7.27(a)] for the derivation in the self-adjoint case—the argument is analogous in the unitary case. Indeed, as discussed in Section 1, the philosophy behind subordinacy theory is to identify a type of solution behavior that discriminates between the absolutely continuous and singular parts of spectral measures, just as square-summability discriminates between the continuous and pure point parts of spectral measures.

Typical applications of this result rely on sufficient conditions for the absence or presence of subordinate solutions. For example, the absence of subordinate solutions follows from the boundedness of the transfer matrices, which are defined as follows:

$$A(n,z)=\begin{cases} S(\alpha_n,z)\times\cdots\times S(\alpha_0,z), & n\geqslant 0,\\ S(-\overline{\alpha}_{n-2},z)\times S(-\overline{\alpha}_{n-1},z)\times\cdots\times S(-\overline{\alpha}_{-2},z), & n\leqslant -1,\end{cases}$$

where $S(\cdot,z)$ is given by (2.2). We will give more details in Section 4. Specifically, we have the following statement.

Corollary 2.4. *Let*

$$\mathcal{B}_\pm=\Big\{z\in\partial\mathbb{D}:\sup_{n\in\mathbb{Z}_\pm}\|A(n,z)\|<\infty\Big\}.$$

Then, $\mathcal{B}_\pm\subseteq\mathcal{A}_\pm$ *with* $\mathcal{A}_\pm$ *as defined in* (2.8). *In particular, the restriction of* Λ *to each of* $\mathcal{B}_\pm$ *is purely absolutely continuous.*

In many cases of interest, the Verblunsky coefficients are dynamically defined. As a result, the associated Szegő recursion can be expressed in terms of $\mathrm{SU}(1,1)$-valued cocycles over the base dynamical system in question. The boundedness property that feeds into Corollary 2.4 is then often established via a suitable reducibility result. Let us state another corollary in the dynamically defined setting that implements this connection.

Corollary 2.5. *Suppose* $T:\Omega\to\Omega$ *is invertible and* $f:\Omega\to\mathbb{D}$. *This gives rise to* ω*-dependent Verblunsky coefficients*

$$\alpha_n(\omega)=f(T^n\omega),\quad \omega\in\Omega,\quad n\in\mathbb{Z}$$

and ω*-dependent extended CMV matrices* $\mathcal{E}(\omega)=\mathcal{E}(\{\alpha_n(\omega)\})$. *Moreover, for each* $z\in\partial\mathbb{D}$, *consider the map* $A_z:\Omega\to\mathrm{SU}(1,1)$ *given by* $A_z(\omega)=z^{-1/2}S(f(\omega),z)$, *where* $S(\cdot,z)$ *is given by* (2.2).

Denote by $\mathcal{R}$ *the set of* $z\in\partial\mathbb{D}$ *for which there exist* $B_z:\Omega\to\mathrm{SU}(1,1)$ *bounded and* $A_z^{(0)}\in\mathrm{SU}(1,1)$ *elliptic such that for every* $\omega\in\Omega$, *we have* $A_z(\omega)=B_z(T\omega)A_z^{(0)}B_z(\omega)^{-1}$.

Then, for every $\omega\in\Omega$, *the canonical spectral measure associated with* $\mathcal{E}(\omega)$ *and* $\Lambda(\omega)$, *is purely absolutely continuous on* $\mathcal{R}$.

Recall that an $\mathrm{SU}(1,1)$ matrix is called elliptic if its trace belongs to the real interval $(-2,2)$ (in this context it is useful to remind the reader that $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$ are canonically conjugate [26, Equation (10.4.27)]). The assumptions of Corollary 2.5 can be verified in a variety of situations, in analogy to the extensive literature on reducibility for quasi-periodic $\mathrm{SL}(2,\mathbb{R})$ cocycles of sufficient regularity (see, for example, [1, 2, 12] and the references therein). This connection is presently being worked out by Long Li[1)].

We conclude this section with two applications of the description of the singular part of an extended CMV matrix in terms of subordinate solutions. Both of these applications are known via different methods, but the approach via subordinacy theory provides an interesting additional angle. Since these results are not new, we will not state them as formal corollaries.

The first application is the following statement: any extended CMV matrix $\mathcal{E}$ has the simple singular spectrum. In the setting of extended CMV matrices, this result was first proved by Simon [24]. However, a statement of this kind had been obtained earlier for second-order differential operators, originally proved by Kac [16, 17] and then proved via subordinacy theory by Gilbert [10]. The present paper provides the basis for Gilbert's approach to this result in the setting of extended CMV matrices.

Another application concerns a version of the Ishii-Pastur theorem for ergodic extended CMV matrices, proved via subordinacy theory, an approach proposed in the setting of Schrödinger operators by Buschmann [3]. Again we will not state this as a formal corollary since the result is already known (see [6, Theorem B.2]), and merely point out that the Ishii-Pastur theorem is the inclusion $\subseteq$ in the identity stated in [6, Theorem B.2], and that this inclusion can be proved along similar lines to those in [3] by using Theorem 2.2 above.

3 Preliminaries

3.1 Carathéodory functions

A *Carathéodory function* is a holomorphic map from $\mathbb{D}$ to the right half plane $\{z : \operatorname{Re} z > 0\}$. We also say a function is an *anti-Carathéodory function* when its negative is a Carathéodory function. If we modify $\alpha(n_0) = -1$, then (2.5) becomes the direct sum of matrices acting on $\ell^2([n_0+1,\infty)\cap\mathbb{Z})$ and $\ell^2((-\infty,n_0]\cap\mathbb{Z})$ of the form (2.4). We label the halves as $\mathcal{C}_+^{(n_0+1)}$ and $\mathcal{C}_-^{(n_0)}$, respectively. We consider the case where $n_0 = -1$. Concretely, (2.5) becomes the direct sum of matrices acting on $\ell^2(\mathbb{Z}_+)$ and $\ell^2(\mathbb{Z}_-)$ of the form (2.4), where we write $\mathbb{Z}_+ := [0,\infty)\cap\mathbb{Z}$ and $\mathbb{Z}_- := [-1,-\infty)\cap\mathbb{Z}$. One can find the correspondence between a given CMV matrix and its Carathéodory function in [25, Subsection 1.3]. Specifically, a Carathéodory function is the CMV analog of the m-function in the theory of Jacobi matrices, and is connected to the spectral theory of the CMV matrices.

Denote the Carathéodory function corresponding to $\mathcal{C}_+^{(0)}$ by

$$F_+(z,0) = \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\Lambda_+(\zeta,0)$$

and $\mathcal{C}_-^{(-1)}$ by

$$F_-(z,-1) = -\int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\Lambda_-(\zeta,-1),$$

where $\Lambda_+(\zeta,0)$ and $\Lambda_-(\zeta,-1)$ are the spectral measures of $\mathcal{C}_+^{(0)}$ and $\mathcal{C}_-^{(-1)}$, respectively.

The Carathéodory function for $\mathcal{E}$ is given by the formula

$$F(z) = \int \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\, d\Lambda(\theta),$$

where as above Λ is the sum of the spectral measures of $\mathcal{E}$ relative to the vectors δ_0 and δ_1.

3.2 Gesztesy-Zinchenko description

The Gesztesy-Zinchenko (GZ) matrix from [8] is a key tool to encode the behavior of solutions to (2.6). As we follow the conventions from [25, 26], let us point out that there are some differences between the notations in [8] and ours, which are as follows: $\alpha_n = -\overline{\alpha}_{n-1}$, $U(\{\alpha_n\}) = \mathcal{E}(\{-\overline{\alpha}_{n-1}\})$, $U_{+,0}(\{\alpha_n\}) =$

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$\mathcal{C}_+^{(0)\top}(\{-\overline{\alpha}_{n-1}\})$, $u_\pm(z,n)=s_\pm(z,n)$ and $v_\pm(z,n)=s_\pm(z,n)$, where the left-hand sides are their notations and the right-hand sides are our notations.

First, recall that any extended CMV matrix $\mathcal{E}$ can be factorized into direct sums of 2×2 matrices of the form

$$\Theta=\begin{pmatrix}\overline{\alpha} & \rho\\ \rho & -\alpha\end{pmatrix}.$$

Let

$$\mathcal{L}:=\bigoplus_{j\in\mathbb{Z}}\Theta(\alpha_{2j})\quad\text{and}\quad\mathcal{M}:=\bigoplus_{j\in\mathbb{Z}}\Theta(\alpha_{2j+1}).$$

Then $\mathcal{E}=\mathcal{L}\mathcal{M}$.

Set

$$P(\alpha,z):=\frac{1}{\rho}\begin{pmatrix}-\alpha & z^{-1}\\ z & -\overline{\alpha}\end{pmatrix}\quad\text{and}\quad Q(\alpha,z):=\frac{1}{\rho}\begin{pmatrix}-\overline{\alpha} & 1\\ 1 & -\alpha\end{pmatrix}\quad\text{for }\ z\in\mathbb{C}\setminus\{0\}.$$

Now, if u is a complex sequence such that $\mathcal{E}u=zu$ and $v=\mathcal{M}u$, one can easily see that $\mathcal{E}^{\mathrm{T}}v=zv$ holds. By [4, Proposition 2.1], the following equation holds for $n\in\mathbb{N}_0$, which can be extended to $n\in\mathbb{Z}$,

$$\begin{pmatrix}u(n+1)\\ v(n+1)\end{pmatrix}=T(n,z)\begin{pmatrix}u(n)\\ v(n)\end{pmatrix},\tag{3.1}$$

where

$$T(n,z)=\begin{cases}P(\alpha_n,z), & n\text{ is even},\\ Q(\alpha_n,z), & n\text{ is odd}.\end{cases}$$

Definition 3.1. We denote by

$$\begin{pmatrix}u_+(z,n,n_0)\\ v_+(z,n,n_0)\end{pmatrix}_{n\geqslant n_0}\quad\text{and}\quad\begin{pmatrix}p_+(z,n,n_0)\\ q_+(z,n,n_0)\end{pmatrix}_{n\geqslant n_0}$$

for $z\in\mathbb{C}\setminus\{0\}$, two linearly independent solutions to (3.1) for $n\geqslant 0$ with the following initial conditions:

$$\begin{pmatrix}u_+(z,n_0,n_0)\\ v_+(z,n_0,n_0)\end{pmatrix}=\begin{cases}\begin{pmatrix}1\\ 1\end{pmatrix}, & n_0\text{ is even},\\ \begin{pmatrix}1\\ z\end{pmatrix}, & n_0\text{ is odd},\end{cases}\tag{3.2}$$

$$\begin{pmatrix}p_+(z,n_0,n_0)\\ q_+(z,n_0,n_0)\end{pmatrix}=\begin{cases}\begin{pmatrix}1\\ -1\end{pmatrix}, & n_0\text{ is even},\\ \begin{pmatrix}-1\\ z\end{pmatrix}, & n_0\text{ is odd}.\end{cases}$$

Similarly, we denote by

$$\begin{pmatrix}u_-(z,n,n_0)\\ v_-(z,n,n_0)\end{pmatrix}_{n\leqslant n_0}\quad\text{and}\quad\begin{pmatrix}p_-(z,n,n_0)\\ q_-(z,n,n_0)\end{pmatrix}_{n\leqslant n_0}$$

for $z\in\mathbb{C}\setminus\{0\}$, two linearly independent solutions to (3.1) for $n\leqslant -1$ with the following initial conditions:

$$\begin{pmatrix}u_-(z,n_0,n_0)\\ v_-(z,n_0,n_0)\end{pmatrix}=\begin{cases}\begin{pmatrix}1\\ -z\end{pmatrix}, & n_0\text{ is even},\\ \begin{pmatrix}-1\\ 1\end{pmatrix}, & n_0\text{ is odd},\end{cases}\tag{3.3}$$

$$\begin{pmatrix} p_-(z,n_0,n_0) \\ q_-(z,n_0,n_0) \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ z \end{pmatrix}, & n_0 \text{ is even}, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & n_0 \text{ is odd}. \end{cases}$$

Remark 3.2. The above definition is from [8, Definition 2.4]. Here, our $u_\pm$, $v_\pm$, $p_\pm$ and $q_\pm$ are their $r_\pm$, $p_\pm$, $s_\pm$ and $q_\pm$, respectively. By (2.3) and (3.1), we have

$$u_+(z,n) = \begin{cases} z^{\frac{-(n+1)}{2}}\varphi^*(z,n), & n \text{ is odd}, \\ z^{\frac{-n}{2}}\varphi(z,n), & n \text{ is even}, \end{cases} \tag{3.4}$$

$$v_+(z,n) = \begin{cases} z^{\frac{-(n-1)}{2}}\varphi(z,n), & n \text{ is odd}, \\ z^{\frac{-n}{2}}\varphi^*(z,n), & n \text{ is even}, \end{cases} \tag{3.5}$$

$$p_+(z,n) = \begin{cases} -z^{\frac{-(n+1)}{2}}\psi^*(z,n), & n \text{ is odd}, \\ z^{\frac{-n}{2}}\psi(z,n), & n \text{ is even}, \end{cases} \tag{3.6}$$

$$q_+(z,n) = \begin{cases} z^{\frac{-(n-1)}{2}}\psi(z,n), & n \text{ is odd}, \\ -z^{\frac{-n}{2}}\psi^*(z,n), & n \text{ is even} \end{cases} \tag{3.7}$$

for $z \in \partial\mathbb{D}$.

For simplicity to check (3.4)–(3.7), we rewrite the equation for $\{\varphi(z,n)\}_{n\in\mathbb{N}_0}$, $\{\psi(z,n)\}_{n\in\mathbb{N}_0}$, $\{u_+(z,n)\}_{n\in\mathbb{N}_0}$ and $\{v_+(z,n)\}_{n\in\mathbb{N}_0}$. Once we have (3.4) and (3.5), (3.6) and (3.7) hold immediately. Indeed, for $\{\varphi(z,n)\}_{n\in\mathbb{N}_0}$ and $\{\psi(z,n)\}_{n\in\mathbb{N}_0}$, we have

$$\begin{aligned} \rho_n\varphi(z,n+1) &= z\varphi(z,n) - \overline{\alpha}_n\varphi^*(z,n), \\ \rho_n\varphi^*(z,n+1) &= -\alpha_n z\varphi(z,n) + \varphi^*(z,n). \end{aligned}$$

For $\{u_+(z,n)\}_{n\in\mathbb{N}_0}$ and $\{v_+(z,n)\}_{n\in\mathbb{N}_0}$, when n is even,

$$\begin{aligned} \rho_n u_+(z,n+1) &= -\alpha_n u_+(z,n) + z^{-1}v_+(z,n), \\ \rho_n v_+(z,n+1) &= zu_+(z,n) - \overline{\alpha}_n v(z,n); \end{aligned}$$

when n is odd,

$$\begin{aligned} \rho_n u_+(z,n+1) &= -\overline{\alpha}_n u_+(z,n) + v_+(z,n), \\ \rho_n v_+(z,n+1) &= u_+(z,n) - \alpha_n v_+(z,n). \end{aligned}$$

It follows that (3.4) and (3.5) hold.

Lemma 3.3 (See [8, Corollary 2.16]). *There are solutions* $\binom{s_\pm(z,\cdot)}{t_\pm(z,\cdot)}_{n\in\mathbb{Z}}$ *to* (3.1), *unique up to constant multiples so that for* $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$,

$$\begin{pmatrix} s_+(z,\cdot) \\ t_+(z,\cdot) \end{pmatrix} \in \ell^2(\mathbb{Z}_+)^2, \quad \begin{pmatrix} s_-(z,\cdot) \\ t_-(z,\cdot) \end{pmatrix} \in \ell^2(\mathbb{Z}_-)^2.$$

Lemma 3.4 (See [23, Theorem 5.3]). *Let* $z \in \mathbb{D}$. *Then*

$$\sum_{n=0}^{\infty} \left| \begin{pmatrix} \psi(z,n) \\ -\psi^*(z,n) \end{pmatrix} + \beta \begin{pmatrix} \varphi(z,n) \\ \varphi^*(z,n) \end{pmatrix} \right|^2 < \infty$$

if and only if

$$\beta = F_+(z,0).$$

Let

$$\eta_+(z,n) = \psi(z,n) + F_+(z,0)\varphi(z,n)$$

and

$$\eta_+^{\circledast}(z,n) = -\psi^*(z,n) + F_+(z,0)\varphi^*(z,n).$$

Consider the equation

$$\Xi_n = S_n(z)\Xi_0 \tag{3.8}$$

with the boundary condition

$$\Xi_0 = \begin{pmatrix} 1 + F_+(z,0) \\ -1 + F_+(z,0) \end{pmatrix},$$

where $S_n(z) = S(\alpha_{n-1}, z) \cdots S(\alpha_0, z)$. Then $\left(\begin{smallmatrix} \eta_+(z,n) \\ \eta_+^{\circledast}(z,n) \end{smallmatrix}\right)$ is the unique ℓ^2 solution to (3.8).

The Green function G (or the resolvent function $(\mathcal{E}-z)^{-1}$) for $\mathcal{E}$ is computed by using formal eigenvalues of $\mathcal{C}_\pm$ and $\mathcal{C}_\pm^{\mathrm{T}}$. For $k \in \mathbb{Z}$, define

$$a_k = 1 - \overline{\alpha}_k, \quad b_k = 1 + \overline{\alpha}_k,$$

where α_k's are Verblunsky coefficients associated with the CMV matrix $\mathcal{E}$.

Lemma 3.5 (See [8, Lemma 3.1]). *For $z \in \mathbb{C}\backslash(\partial\mathbb{D}\cup\{0\})$, let $M_-(z,0)$ be an anti-Carathéodory function in* [8, (2.139)]*, which is related to $F_-(z,-1)$ by*

$$M_-(z,0) = \frac{\operatorname{Re}(a_{-1}) + \mathrm{i}\operatorname{Im}(b_{-1})F_-(z,-1)}{\mathrm{i}\operatorname{Im}(a_{-1}) + \operatorname{Re}(b_{-1})F_-(z,-1)}. \tag{3.9}$$

Let $s_\pm$ be ℓ^2 solutions to $(\mathcal{C}_\pm - z)s = 0$, and let $t_\pm$ be ℓ^2 solutions to $(\mathcal{C}_\pm^{\mathrm{T}} - z)t = 0$, normalized by

$$\begin{aligned} s_+(z,0) &= 1 + F_+(z,0), \quad s_-(z,0) = 1 + M_-(z,0), \\ t_+(z,0) &= -1 + F_+(z,0), \quad t_-(z,0) = 1 - M_-(z,0). \end{aligned}$$

These $s_\pm$ and $t_\pm$ are equivalent to the ones in Lemma 3.3*. We may extend these solutions to solutions to $(\mathcal{E} - z)w = 0$ and $(\mathcal{E}^{\mathrm{T}} - z)w = 0$.*

Then the resolvent function $(\mathcal{E} - z)^{-1}(x,y)$ can be expressed as

$$\frac{-1}{2z(F_+(z,0) - M_-(z,0))} = \begin{cases} t_-(z,m)s_+(z,n), & \text{if } m < n \text{ or } m = n \text{ and } m \text{ is odd}, \\ t_+(z,m)s_-(z,n), & \text{if } m > n \text{ or } m = n \text{ and } m \text{ is even}. \end{cases} \tag{3.10}$$

From [8, p. 181, the table], we are in the case of $k_0 = 0$ and obtain

$$\begin{aligned} t_-(z,1) &= \frac{1}{\rho_0}(z + \overline{\alpha}_0) + \frac{1}{\rho_0}(z - \overline{\alpha}_0)M_-(z,0), \\ s_+(z,1) &= \frac{1}{\rho_0}\left(-\frac{1}{z} - \alpha_0\right) + \frac{1}{\rho_0}\left(\frac{1}{z} - \alpha_0\right)F_+(z,0). \end{aligned}$$

Notice that our $s_\pm$ and $t_\pm$ are their $v_\pm$ and $\tilde{u}_\pm$, respectively.

3.3 Green and Carathéodory functions

One can write $G_{00} + G_{11}$ as

$$\begin{aligned} G_{00} + G_{11} &= -\frac{(-1 + F_+(z,0))(1 + M_-(z,0))}{2z(F_+(z,0) - M_-(z,0))} \\ &\quad - \frac{[z + \overline{\alpha}_0 + M_-(z,0)(z - \overline{\alpha}_0)][-1 - \alpha_0 z + F_+(z,0)(1 - \alpha_0 z)]}{2\rho_0^2 z^2(F_+(z,0) - M_-(z,0))} \\ &= \frac{\rho_0^2 z(1 - F_+(z,0) + M_-(z,0) - F_+(z,0)M_-(z,0)) + (z + \overline{\alpha}_0)(1 + \alpha_0 z)}{2\rho_0^2 z^2(F_+(z,0) - M_-(z,0))} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(z-\overline{\alpha}_0)(1+\alpha_0 z)M_-(z,0)-(z+\overline{\alpha}_0)(1-\alpha_0 z)F_+(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
&-\frac{(z-\overline{\alpha}_0)(1-\alpha_0 z)F_+(z,0)M_-(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
=&\frac{(\rho_0^2 z+z+\overline{\alpha}_0+\alpha_0 z^2+|\alpha_0|^2 z)+(\rho_0^2 z+z-\overline{\alpha}_0+\alpha_0 z^2-|\alpha_0|^2 z)M_-(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
&+\frac{(-\rho_0^2 z+\alpha_0 z^2-z+|\alpha_0|^2 z-\overline{\alpha}_0)F_+(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
&+\frac{(-\rho_0^2 z+\alpha_0 z^2-z-|\alpha_0|^2 z+\overline{\alpha}_0)F_+(z,0)M_-(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
=&\frac{(2z+\overline{\alpha}_0+\alpha_0 z^2)+(2\rho_0^2 z-\overline{\alpha}_0+\alpha_0 z^2)M_-(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
&+\frac{(\alpha_0 z^2-\overline{\alpha}_0-2\rho_0^2 z)F_+(z,0)+(\alpha_0 z^2+\overline{\alpha}_0-2z)M_-(z,0)F_+(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
=&\frac{(2z+\overline{\alpha}_0+\alpha_0 z^2)+(\alpha_0 z^2-\overline{\alpha}_0)(F_+(z,0)+M_-(z,0))}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}\\
&+\frac{2\rho_0^2 z(M_-(z,0)-F_+(z,0))+(\alpha_0 z^2+\overline{\alpha}_0-2z)M_-(z,0)F_+(z,0)}{2\rho_0^2 z^2(F_+(z,0)-M_-(z,0))}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
2z(G_{00}(z)+G_{11}(z))=-2&+\frac{(\overline{\alpha}_0+2z+\alpha_0 z^2)+(\alpha_0 z^2-\overline{\alpha}_0)(M_-(z,0)+F_+(z,0))}{\rho_0^2 z(F_+(z,0)-M_-(z,0))}\\
&+\frac{(\overline{\alpha}_0-2z+\alpha_0 z^2)M_-(z,0)F_+(z,0)}{\rho_0^2 z(F_+(z,0)-M_-(z,0))}.
\end{aligned}\tag{3.11}
$$

Finally we note the connection between $G_{00}+G_{11}$ and the Carathéodory function F corresponding to $\mathcal{E}$ and $d\Lambda$. We have by definition

$$
F(z)=\int\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}d\Lambda(\theta).
$$

Define

$$
d\Lambda_r(\theta)=\operatorname{Re}F(r\mathrm{e}^{\mathrm{i}\theta})\frac{d\theta}{2\pi}.
$$

It is well known that $d\Lambda_r$ converges to $d\Lambda$ weakly as $r\uparrow 1$. Moreover,

$$
\begin{aligned}
F(z)&=\int\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}d\Lambda(\theta)\\
&=1+2z\int\frac{1}{\mathrm{e}^{\mathrm{i}\theta}-z}d\Lambda(\theta)\\
&=1+2z(G_{00}(z)+G_{11}(z)).
\end{aligned}
$$

4 Jitomirskaya-Last inequalities

In this section, we obtain a suitable version of the Jitomirskaya-Last inequality for the left half-line CMV matrix

$$
\mathcal{C}_-=\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \overline{\alpha}_{-4}\rho_{-5} & -\overline{\alpha}_{-4}\alpha_{-5} & \overline{\alpha}_{-3}\rho_{-4} & \rho_{-3}\rho_{-4} & 0\\
\cdots & \rho_{-4}\rho_{-5} & -\rho_{-4}\alpha_{-5} & -\overline{\alpha}_{-3}\alpha_{-4} & -\rho_{-3}\alpha_{-4} & 0\\
\cdots & 0 & 0 & \overline{\alpha}_{-2}\rho_{-3} & -\overline{\alpha}_{-2}\alpha_{-3} & \overline{\alpha}_{-1}\rho_{-2}\\
\cdots & 0 & 0 & \rho_{-2}\rho_{-3} & -\rho_{-2}\alpha_{-3} & -\overline{\alpha}_{-1}\alpha_{-2}
\end{pmatrix}
$$

via a relation between the eigenfunctions of $\mathcal{C}_-$ and the associated right half-line CMV matrix $\tilde{\mathcal{C}}_+ = \mathcal{C}_+^{(0)}(\{\tilde{\alpha}_n\}_{n\geqslant 0})$, where $\tilde{\alpha}_n = -\overline{\alpha}_{-(n+2)}$ and $\tilde{\alpha}_{-1} = -1$.

First, recall the Jitomirskaya-Last inequality for a right half-line CMV matrix. With the solutions to (2.6) obeying (2.7) with $\omega = 0$ and the local ℓ^2 norms from Definition 2.1, we have the following lemma.

Lemma 4.1 (See [25, Theorem 10.8.2]). *For $z \in \partial\mathbb{D}$ and $r \in [0,1)$, define $x(r) \in (0,\infty)$ to be the unique solution to*

$$(1-r)\|\varphi.(z)\|_{x(r)}\|\psi.(z)\|_{x(r)} = \sqrt{2}.$$

Then

$$A^{-1}|F_+(rz,0)| \leqslant \frac{\|\psi.(z)\|_{x(r)}}{\|\varphi.(z)\|_{x(r)}} \leqslant A|F_+(rz,0)|, \tag{4.1}$$

where A is a universal constant in $(1,\infty)$.

Remark 4.2. By Remark 3.2, (4.1) is equivalent to

$$A^{-1}|F_+(rz,0)| \leqslant \frac{\|p_+(z)\|_{x(r)}}{\|u_+(z)\|_{x(r)}} \leqslant A|F_+(rz,0)|, \tag{4.2}$$

where $A \in (1,\infty)$ is a universal constant.

Next, we address the relation between $\mathcal{C}_-$ and $\tilde{\mathcal{C}}_+$. Let J be the matrix with elements

$$J_{i,j} = \begin{cases} 1, & \text{if } i = -j-1,\\ 0, & \text{otherwise}\end{cases}$$

for $i = -1,-2,-3,\dots$ and $j = 0,1,2,\dots$ Let $\tilde{J}$ be the matrix with elements

$$\tilde{J}_{i,j} = \begin{cases} 1, & \text{if } i = -j-1,\\ 0, & \text{otherwise}\end{cases}$$

for $i = 0,1,2,\dots$ and $j = -1,-2,-3,\dots$

Define the operator $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ that maps $\ell^2(\mathbb{Z}_-) \to \ell^2(\mathbb{Z}_+)$ as follows:

$$U = \begin{pmatrix} \mathbf{0} & J \\ \tilde{J} & \mathbf{0}\end{pmatrix},$$

where $\mathbf{0}$ is the zero matrix, i.e., $U\delta_n = \delta_{-n-1}$ for $n \in \mathbb{N}_0$.

A direct calculation implies

$$U\mathcal{C}_-U^* = \begin{pmatrix} -\overline{\alpha}_{-1}\alpha_{-2} & -\rho_{-2}\alpha_{-3} & \rho_{-2}\rho_{-3} & 0 & 0 & \cdots \\ \overline{\alpha}_{-1}\rho_{-2} & -\overline{\alpha}_{-2}\alpha_{-3} & \overline{\alpha}_{-2}\rho_{-3} & 0 & 0 & \cdots \\ 0 & -\rho_{-3}\alpha_{-4} & -\overline{\alpha}_{-3}\alpha_{-4} & -\rho_{-4}\alpha_{-5} & \rho_{-4}\rho_{-5} & \cdots \\ 0 & \rho_{-3}\rho_{-4} & \overline{\alpha}_{-3}\rho_{-4} & -\overline{\alpha}_{-4}\alpha_{-5} & \overline{\alpha}_{-4}\rho_{-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Set $\tilde{\alpha}_n = -\overline{\alpha}_{-(n+2)}$ and $\tilde{\alpha}_{-1} = -1$. Then $\tilde{\rho}_n = \rho_{-(n+2)}$ and $\tilde{\mathcal{C}}_+ = U\mathcal{C}_-U^*$.

For $\tilde{\mathcal{C}}_+$, denote $\tilde{\varphi}$ and $\tilde{\psi}$ to be the orthogonal polynomials and the orthogonal polynomials of the second kind, respectively. Denote u_- and $\tilde{u}_+$ (p_- and $\tilde{p}_+$) to be the eigenfunctions for $\mathcal{C}_-$ and $\tilde{\mathcal{C}}_+$, respectively, and v_- and $\tilde{v}_+$ (q_- and $\tilde{q}_+$) to be the eigenfunctions for $\mathcal{C}_-^{\mathrm{T}}$ and $\tilde{\mathcal{C}}_+^{\mathrm{T}}$, respectively. Since $\tilde{\mathcal{C}}_+ = U\mathcal{C}_-U^*$, $\tilde{u}_+(n) = u_-(-(n+1))$ for $n \in \mathbb{N}_0$.

We have

$$u_-(z,-n-1) = \tilde{u}_+(z,n) = \begin{cases} z^{\frac{-(n+1)}{2}}\tilde{\varphi}^*(z,n), & n \text{ is odd},\\ -z^{\frac{-n}{2}}\tilde{\varphi}(z,n), & n \text{ is even},\end{cases}$$

$$v_-(z,-n-1)=\tilde{v}_+(z,n)=\begin{cases}-z^{\frac{-(n-1)}{2}}\tilde{\varphi}(z,n), & n \text{ is odd},\\ z^{\frac{-n}{2}}\tilde{\varphi}^*(z,n), & n \text{ is even},\end{cases}$$

$$p_-(z,-n-1)=\tilde{p}_+(z,n)=\begin{cases}z^{\frac{-(n+1)}{2}}\tilde{\psi}^*(z,n), & n \text{ is odd},\\ z^{\frac{-n}{2}}\tilde{\psi}(z,n), & n \text{ is even},\end{cases}$$

$$q_-(z,-n-1)=\tilde{q}_+(z,n)=\begin{cases}z^{\frac{-(n-1)}{2}}\tilde{\psi}(z,n), & n \text{ is odd},\\ z^{\frac{-n}{2}}\tilde{\psi}^*(z,n), & n \text{ is even}\end{cases}$$

for $z\in\partial\mathbb{D}$. Thus the initial conditions (3.3) with $n_0=-1$ are equivalent to

$$\begin{pmatrix}-\tilde{\varphi}(z,0)\\ \tilde{\varphi}^*(z,0)\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix},\quad \begin{pmatrix}\tilde{\psi}(z,0)\\ \tilde{\psi}^*(z,0)\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{4.3}$$

Since $z\in\partial\mathbb{D}$, $\|u_-(z)\|_{-x(r)-1}=\|\tilde{\varphi}(z)\|_{x(r)}$ and $\|p_-(z)\|_{-x(r)-1}=\|\tilde{\psi}(z)\|_{x(r)}$, where $x(r)$ is as in Lemma 4.1. Due to Lemma 3.4, there must then be a unique $\tilde{F}_+(z)$ such that

$$\begin{pmatrix}-\tilde{\varphi}(z,n)+\tilde{F}_+(z)\tilde{\psi}(z,n)\\ \tilde{\varphi}^*(z,n)+\tilde{F}_+(z)\tilde{\psi}^*(z,n)\end{pmatrix}\in\ell^2(\mathbb{Z}_+).$$

Due to the unitarity of U, $F_-(z,-1)=-\tilde{F}_+(z,0)$, where $\tilde{F}_+(z,0)$ is the Carathéodory function for $\tilde{\mathcal{C}}_+$. Hence, the Jitomirskaya-Last inequality holds for $\mathcal{C}_-$. For $z\in\partial\mathbb{D}$ and $r\in[0,1)$, define $x_1(r)\in(-\infty,-1)$ to be the unique solution to

$$(1-r)\|u_-(z)\|_{x_1(r)}\|p_-(z)\|_{x_1(r)}=\sqrt{2}.$$

Then

$$A^{-1}|F_-(rz,-1)|\leqslant\frac{\|u_-(z)\|_{x_1(r)}}{\|p_-(z)\|_{x_1(r)}}\leqslant A|F_-(rz,-1)|, \tag{4.4}$$

where A is a universal constant in $(1,\infty)$.

Next, we extend the Jitomirskaya-Last inequality, which holds for the boundary condition $\varphi(z,0)=1$, to a general boundary condition of the form

$$\varphi(z,0)(\cos\omega-\mathrm{i}\sin\omega)-\varphi^*(z,0)(\cos\omega+\mathrm{i}\sin\omega)=0. \tag{4.5}$$

Given $z\in\mathbb{D}$ and $\omega\in[0,\pi)$, let $\binom{\varphi_\omega}{\varphi_\omega^*}$ and $\binom{\psi_\omega}{-\psi_\omega^*}$ denote the solutions to (3.8) obeying (2.7). Thus, $\binom{\varphi_\omega}{\varphi_\omega^*}$ obeys the boundary condition (4.5) and $\binom{\psi_\omega}{-\psi_\omega^*}$ obeys the orthogonal boundary condition.

Define $u_\omega(z,n)$ and $p_\omega(z,n)$ to be the solutions to (2.6), subject to the boundary conditions (2.7). For $r\in[0,1)$, define $x(r)$ to be the unique solution to

$$(1-r)\|u_\omega(z)\|_{x(r)}\|p_\omega(z)\|_{x(r)}=\sqrt{2}. \tag{4.6}$$

By [8, Theorem 2.18], there are a unique $F_+^\omega(z,0)$ such that

$$\xi_+^\omega(z,n,0)=p_\omega(z,n)+F_+^\omega(z,0)u_\omega(z,n)$$

is ℓ^2 at infinity and a unique $M_-^\omega(z,0)$ such that

$$\xi_-^\omega(z,n,0)=p_\omega(z,n)+M_-^\omega(z,0)u_\omega(z,n)$$

is ℓ^2 at $-\infty$. By (3.4) and (3.6), we have

$$\xi_+^\omega(z,0,0)=\psi_\omega(z,0)+F_+(z,0)\varphi_\omega(z,0),$$
$$\xi_-^\omega(z,0,0)=\psi_\omega(z,0)+M_-(z,0)\varphi_\omega(z,0).$$

Define

$$\xi_+^{\omega,*}(z,0,0)=-\psi_\omega^*(z,0)+F_+(z,0)\varphi_\omega(z,0),$$
$$\xi_-^{\omega,*}(z,0,0)=-\psi_\omega^*(z,0)+M_-(z,0)\varphi_\omega^*(z,0).$$

With these definitions the following generalization of Lemma 3.4 holds.

Lemma 4.3. *For $z \in \partial\mathbb{D}$, define $x(r) \in (0,\infty)$ to be the unique solution to*

$$(1-r)\|u_\omega(z)\|_{x(r)}\|p_\omega(z)\|_{x(r)} = \sqrt{2}.$$

Then we have

$$A_1^{-1}|F_+^\omega(rz,0)| \leqslant \frac{\|p_\omega(z)\|_{x(r)}}{\|u_\omega(z)\|_{x(r)}} \leqslant A_1|F_+^\omega(rz,0)|, \tag{4.7}$$

where A_1 is a universal constant in $(1,\infty)$. Similarly, define $x_1(r) \in (-\infty,-1)$ to be the unique solution to

$$(1-r)\|u_\omega(z)\|_{x_1(r)}\|p_\omega(z)\|_{x_1(r)} = \sqrt{2}.$$

Then we have

$$A_2^{-1}|F_-^\omega(rz,0)| \leqslant \frac{\|p_\omega(z)\|_{x_1(r)}}{\|u_\omega(z)\|_{x_1(r)}} \leqslant A_2|F_-^\omega(rz,0)|, \tag{4.8}$$

where A_2 is a universal constant in $(1,\infty)$.

Proof. In order to follow the proof of [26, Theorem 10.8.2], define T_n as

$$T_n = \frac{1}{2}\begin{pmatrix} e^{-i\omega}(\varphi_\omega(n)+\psi_\omega(n)) & e^{i\omega}(\varphi_\omega(n)-\psi_\omega(n)) \\ e^{-i\omega}(\varphi_\omega^*(n)-\psi_\omega^*(n)) & e^{i\omega}(\varphi_\omega^*(n)+\psi_\omega^*(n)) \end{pmatrix}.$$

This is from [25, (3.2.27)] with the boundary conditions $\binom{e^{i\omega}}{e^{-i\omega}}$ and $\binom{e^{i\omega}}{-e^{-i\omega}}$. By [25, (3.2.28)], $\det T_n = z^n$. We have

$$T_l^{-1} = \frac{1}{2z^l}\begin{pmatrix} e^{i\omega}(\varphi_\omega^*(l)+\psi_\omega^*(l)) & -e^{i\omega}(\varphi_\omega(l)-\psi_\omega(l)) \\ -e^{-i\omega}(\varphi_\omega^*(l)-\psi_\omega^*(l)) & e^{-i\omega}(\varphi_\omega(l)+\psi_\omega(l)) \end{pmatrix}.$$

A direct calculation shows that $T_{n\leftarrow l} = T_nT_l^{-1}$ which is the same as $T_{n\leftarrow l}$ in [26, (10.8.8)]. The remaining proof is the same as the proof of [26, Theorem 10.8.2]. We can then conclude that (4.7) holds. □

Lemma 4.4. *Let $\theta \in [0,2\pi)$ be given.*

(1) *One has $\lim_{r\uparrow 1} F_+(re^{i\theta},0) = -i\cot\omega$ for some $\omega \in [0,\pi)$ if and only if u_ω is subordinate at $+\infty$.*

(2) *One has $\lim_{r\uparrow 1} M_-(re^{i\theta},0) = -i\cot\omega$ for some $\omega \in [0,\pi)$ if and only if u_ω is subordinate at $-\infty$.*

(3) *The difference equation (2.6) enjoys a subordinate solution at $+\infty$ if and only if*

$$\lim_{r\uparrow 1} F_+(re^{i\theta},0) \in i(\mathbb{R}\cup\{\infty\}).$$

(4) *The difference equation (2.6) enjoys a subordinate solution at $-\infty$ if and only if*

$$\lim_{r\uparrow 1} M_-(re^{i\theta},0) \in i(\mathbb{R}\cup\{\infty\}).$$

(5) *The difference equation (2.6) enjoys a solution that is subordinate at $\pm\infty$ if and only if*

$$\lim_{r\uparrow 1} F_+(re^{i\theta},0) = \lim_{r\uparrow 1} M_-(re^{i\theta},0).$$

Proof. (1) Consider the m-function for orthogonal polynomials on the unit circle (OPUC) of the form

$$m_0^+(re^{i\theta}) = \frac{\xi_+^{0,*}(re^{i\theta},0,0)}{\xi_+^0(re^{i\theta},0,0)}.$$

By Lemma 3.3,

$$m_0^+(re^{i\theta}) = \frac{\xi_+^{\omega,*}(re^{i\theta},0,0)}{\xi_+^\omega(re^{i\theta},0,0)}$$

implies

$$\frac{-\psi^*(re^{i\theta},0)+F_+(re^{i\theta},0)\varphi^*(re^{i\theta},0)}{\psi(re^{i\theta},0)+F_+(re^{i\theta},0)\varphi(re^{i\theta},0)} = \frac{-\psi_\omega^*(re^{i\theta},0)+F_+^\omega(re^{i\theta},0)\varphi_\omega^*(re^{i\theta},0)}{\psi_\omega(re^{i\theta},0)+F_+^\omega(re^{i\theta},0)\varphi_\omega(re^{i\theta},0)}.$$

For simplicity, write $\varphi_\omega(re^{i\theta},0)$ and $\psi_\omega(re^{i\theta},0)$ as φ_ω and ψ_ω, respectively. It follows that

$$\frac{-1+F_+(re^{i\theta},0)}{1+F_+(re^{i\theta},0)}=\frac{-\psi_\omega^*+F_+^\omega(re^{i\theta},0)\varphi_\omega^*}{\psi_\omega+F_+^\omega(re^{i\theta},0)\varphi_\omega},$$

and then

$$F_+^\omega(re^{i\theta},0)=\frac{-\psi_\omega^*+\psi_\omega-F_+(re^{i\theta},0)(\psi_\omega^*+\psi_\omega)}{-(\varphi_\omega+\varphi_\omega^*)+F_+(re^{i\theta},0)(\varphi_\omega-\varphi_\omega^*)}.$$

By (2.7), we have

$$F_+^\omega(re^{i\theta},0)=\frac{i\sin\omega-F_+(re^{i\theta},0)\cos\omega}{-\cos\omega+iF_+(re^{i\theta},0)\sin\omega}. \tag{4.9}$$

We see that $\lim_{r\uparrow 1}|F_+^\omega(re^{i\theta},0)|=\infty$ if and only if

$$\lim_{r\uparrow 1}F_+(re^{i\theta},0)=-i\cot\omega.$$

By Lemma 4.3, $\lim_{r\uparrow 1}|F_+^\omega(re^{i\theta},0)|=\infty$ if and only if u_ω is subordinate at $+\infty$. By putting the two equivalences together, (1) follows.

(2) Consider the m-function for the left half-line CMV matrix $\tilde{\mathcal{C}}_+$ as

$$\tilde{m}^-(re^{i\theta})=\frac{\xi_-^{0,*}(re^{i\theta},0,0)}{\xi_-^0(re^{i\theta},0,0)}.$$

It implies

$$\frac{-\psi^*(re^{i\theta},0)+M_-(re^{i\theta},0)\varphi^*(re^{i\theta},0)}{\psi(re^{i\theta},0)+M_-(re^{i\theta},0)\varphi(re^{i\theta},0)}=\frac{-\psi_\omega^*(re^{i\theta},0)+M_-^\omega(re^{i\theta},0)\varphi_\omega^*(re^{i\theta},0)}{\psi_\omega(re^{i\theta},0)+M_-^\omega(re^{i\theta},0)\varphi_\omega(re^{i\theta},0)}.$$

A direct calculation gives

$$M_-^\omega(re^{i\theta},0)=\frac{i\sin\omega-M_-(re^{i\theta},0)\cos\omega}{-\cos\omega+iM_-(re^{i\theta},0)\sin\omega}. \tag{4.10}$$

Since the ℓ^2 solution is unique up to a non-zero constant C, by [8, Remark 2.19], we have

$$C\xi_-(z,0,0)=p_-(z,0,0)+F_-(z,0)u_-(z,0,0),$$

which implies that

$$C(1+M_-(z,0))=1+F_-(z,0), \tag{4.11}$$

where $F_-(z,0)$ is the Carathéodory function for C_- acting on $\ell^2([0,-\infty)\cap\mathbb{Z})$ with $|\alpha_0|=1$. Thus,

$$-M_-(z,0)=1-C^{-1}(1+F_-(z,0)),$$

which means $\lim_{r\uparrow 1}|M_-(re^{i\theta},0)|=\infty$ if and only if $\lim_{r\uparrow 1}|F_-(re^{i\theta},0)|=\infty$. We see that

$$\lim_{r\uparrow 1}|M_-^\omega(re^{i\theta},0)|=\infty$$

if and only if $\lim_{r\uparrow 1}M_-(re^{i\theta},0)=-i\cot\omega$. By the Jitomirskaya-Last inequalities, $\lim_{r\uparrow 1}|F_-^\omega(re^{i\theta},0)|=\infty$ if and only if u_ω is subordinate at $-\infty$. By putting the three equivalences together, (2) follows.

We observe that (3) and (4) follow immediately from (1) and (2). Recall now that $F_+(re^{i\theta},0)$ is a Carathéodory function and $M_-(re^{i\theta},0)$ is an anti-Carathéodory function. Thus (5) follows immediately from (3) and (4) by noting that $\lim_{r\uparrow 1}F_+(re^{i\theta},0)=\lim_{r\uparrow 1}M_-(re^{i\theta},0)$ forces the common limit to belong to $i(\mathbb{R}\cup\{\infty\})$. □

5 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. We will make use of the preparatory work in the previous sections, culminating in Lemmas 4.3 and 4.4.

Proof of Theorem 2.2. (a) As discussed in Remark 2.3, this statement is well known and a proof can be given by arguments similar to those in the proof of [28, Theorem 7.27(a)].

Before turning our attention to the statements (b) and (c) in Theorem 2.2, let us recall that F is the Borel transform of the canonical spectral measure Λ and it can be expressed as follows:

$$\begin{aligned} F(z) &= 1 + 2z(G_{00}(z) + G_{11}(z)) \\ &= -1 + \frac{(\overline{\alpha}_0 + 2z + \alpha_0 z^2) + (\alpha_0 z^2 - \overline{\alpha}_0)(M_-(z,0) + F_+(z,0))}{\rho_0^2 z(F_+(z,0) - M_-(z,0))} \\ &\quad + \frac{(\overline{\alpha}_0 - 2z + \alpha_0 z^2)M_-(z,0)F_+(z,0)}{\rho_0^2 z(F_+(z,0) - M_-(z,0))} \qquad (5.1a) \\ &= -1 + \frac{\frac{(\overline{\alpha}_0 + 2z + \alpha_0 z^2)}{F_+(z,0)M_-(z,0)} + (\alpha_0 z^2 - \overline{\alpha}_0)(\frac{1}{F_+(z,0)} + \frac{1}{M_-(z,0)}) + (\overline{\alpha}_0 - 2z + \alpha_0 z^2)}{\rho_0^2 z(\frac{1}{M_-(z,0)} - \frac{1}{F_+(z,0)})}. \qquad (5.1b) \end{aligned}$$

From [25, Subsection 1.3.5], recall that $\Lambda_{\rm s}$ is supported on

$$S_\Lambda = \Big\{\theta : \lim_{r\uparrow 1} \operatorname{Re} F(r\mathrm{e}^{\mathrm{i}\theta}) = \infty\Big\},$$

and an essential support of $\Lambda_{\rm ac}$ is given by

$$\mathcal{A}_\Lambda = \Big\{\theta : 0 < \lim_{r\uparrow 1} \operatorname{Re} F(r\mathrm{e}^{\mathrm{i}\theta}) < \infty\Big\}.$$

With these preliminaries out of the way, we can now address Theorems 2.2(b) and 2.2(c).

(b) It suffices to show that $\Lambda_{\rm s}(S_\Lambda \backslash S) = 0$. Consider $\theta \in S_\Lambda$, i.e., $\lim_{r\uparrow 1} \operatorname{Re} F(r\mathrm{e}^{\mathrm{i}\theta})$ exists and is infinite. In particular, we also have $\lim_{r\uparrow 1} |F(r\mathrm{e}^{\mathrm{i}\theta})| = \infty$. There are the following three cases:

(1) $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta},0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta},0)$ both exist (with ∞ being an admissible limit).

(2) Exactly one of $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta},0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta},0)$ exists.

(3) Neither of them exists.

We will show that θ's that are in Case (1) belong to S, Case (2) is impossible, and θ's in Case (3) have the zero measure with respect to Λ_s. Combining these three statements, we obtain the desired $\Lambda_{\rm s}(S_\Lambda \backslash S) = 0$.

Case (1) Suppose the limits $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta},0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta},0)$ both exist (with ∞ being an admissible limit). By (5.1a), as $r \uparrow 1$, we must have

$$|F_+(r\mathrm{e}^{\mathrm{i}\theta},0) - M_-(r\mathrm{e}^{\mathrm{i}\theta},0)| \to 0 \qquad (5.2)$$

or

$$\begin{aligned} &|(\overline{\alpha}_0 + 2z + \alpha_0 z^2) + (\alpha_0 z^2 - \overline{\alpha}_0)(M_-(r\mathrm{e}^{\mathrm{i}\theta},0) + F_+(r\mathrm{e}^{\mathrm{i}\theta},0)) \\ &\quad + (\overline{\alpha}_0 - 2z + \alpha_0 z^2)M_-(r\mathrm{e}^{\mathrm{i}\theta},0)F_+(r\mathrm{e}^{\mathrm{i}\theta},0)| \to \infty. \qquad (5.3) \end{aligned}$$

Let us consider the first case where $|F_+(r\mathrm{e}^{\mathrm{i}\theta},0) - M_-(r\mathrm{e}^{\mathrm{i}\theta},0)| \to 0$, which implies that

$$\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta},0) = \lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta},0).$$

If $\lim_{r\uparrow 1} |F_+(r\mathrm{e}^{\mathrm{i}\theta},0)| = \lim_{r\uparrow 1} |M_-(r\mathrm{e}^{\mathrm{i}\theta},0)|$ is finite, by Lemma 4.4(5), there is a solution u_ω that is subordinate at $\pm\infty$ with non-zero ω. If $\lim_{r\uparrow 1} |F_+(r\mathrm{e}^{\mathrm{i}\theta},0)| = \lim_{r\uparrow 1} |M_-(r\mathrm{e}^{\mathrm{i}\theta},0)|$ is infinite, $\lim_{r\uparrow 1} |F_-(r\mathrm{e}^{\mathrm{i}\theta},0)|$ is infinite by (4.11). Thus, $u_+(\mathrm{e}^{\mathrm{i}\theta})$ is a subordinate solution at ∞ and $u_-(\mathrm{e}^{\mathrm{i}\theta})$ is a subordinate solution at $-\infty$. From Lemmas 4.4(1) and 4.4(2), we have $F_+(re^{i\theta},0)=-i\cot\omega_1$ and $M_-(re^{i\theta},0)=-i\cot\omega_2$, so $\omega_1=\omega_2=0$, which means that there is a subordinate solution $u_0(e^{i\theta})$ at $\pm\infty$.

Now, we consider the second case, which means $|M_-(re^{i\theta},0)F_+(re^{i\theta},0)|\to\infty$, $|F_+(re^{i\theta},0)|\to\infty$ or $|M_-(re^{i\theta},0)|\to\infty$. In this case, if one of $|M_-(re^{i\theta},0)|$ and $|F_+(re^{i\theta},0)|$ goes to infinity while the other one goes to a constant, then $|F(re^{i\theta})|$ cannot go to infinity.

Therefore, both $|M_-(re^{i\theta},0)|$ and $|F_+(re^{i\theta},0)|$ go to infinity. As in the first case, there is a subordinate solution $u_0(e^{i\theta})$ at $\pm\infty$.

Case (2) Suppose exactly one of $\lim_{r\uparrow1}F_+(re^{i\theta},0)$ and $\lim_{r\uparrow1}M_-(re^{i\theta},0)$ exists, i.e., $\lim_{r\uparrow1}F_+(re^{i\theta},0)$. As in Case (1), one of (5.2) and (5.3) must hold true. First, consider the case where (5.2) holds. It implies that if $\lim_{r\uparrow1}|F_+(re^{i\theta},0)|$ exists, then $\lim_{r\uparrow1}|M_-(re^{i\theta},0)|$ also exists, which leads to a contradiction. Now, consider the case where (5.3) holds, which implies $\lim_{r\uparrow1}|F_+(re^{i\theta},0)|=\infty$. By (5.1b), $|M_-(re^{i\theta},0)|$ must go to infinity as $r\uparrow1$, which leads to a contradiction. Thus, Case (2) is impossible.

Case (3) Suppose that neither $\lim_{r\uparrow1}F_+(re^{i\theta},0)$ nor $\lim_{r\uparrow1}M_-(re^{i\theta},0)$ exists. Therefore,

$$\liminf_{r\uparrow1}F_+(re^{i\theta},0)\quad\text{and}\quad\liminf_{r\uparrow1}M_-(re^{i\theta},0)$$

are finite. We can thus choose a sequence $r_n\uparrow1$ in such a way that the limits

$$\ell_+:=\lim_{r_n\uparrow1}F_+(r_ne^{i\theta},0)\quad\text{and}\quad\ell_-=\lim_{r_n\uparrow1}M_-(r_ne^{i\theta},0)$$

exist, and ℓ_+ is finite. Following a similar argument to that in Case (1), since $\lim_{r\uparrow1}|F(re^{i\theta})|=\infty$, $\ell_-=\ell_+\in i\mathbb{R}$, which means that all accumulation points of $\{F_+(re^{i\theta},0)\}_{0<r<1}$ and $\{M_-(re^{i\theta},0)\}_{0<r<1}$ are either 0 or purely imaginary numbers. Thus $\operatorname{Re}\ell_+=0$.

Since $M_-(re^{i\theta},0)$ is an anti-Carathéodory function, it is analytic in $\mathbb{D}$. If $ia,ib\in i\mathbb{R}$ are two different accumulation points of $M_-(re^{i\theta},0)$, by the intermediate value theorem and the fact that all accumulation points are imaginary, we can find a sequence of $\{r_n'\}$ such that $M_-(r_n'e^{i\theta},0)$ goes to ic for any $c\in(a,b)$ as $r_n'\uparrow1$. We thus observe that since the limits do not exist and the real parts of accumulation points are zero, $M_-(re^{i\theta},0)$ has uncountably many accumulation points.

So we can choose r_n and t_n in such a way that

$$\begin{aligned}&\lim_{r_n\uparrow1}M_-(r_ne^{i\theta},0)=\ell_1,\quad \lim_{t_n\uparrow1}M_-(t_ne^{i\theta},0)=\ell_2,\\&\lim_{r_n\uparrow1}F_+(r_ne^{i\theta},0)=\ell_1,\quad \lim_{t_n\uparrow1}F_+(t_ne^{i\theta},0)=\ell_2,\end{aligned}$$

where $\ell_1\neq\ell_2$ and $\ell_1,\ell_2\in i\mathbb{R}$. Let Λ_0 denote the δ_0-spectral measure of $\mathcal{E}$, given by

$$\langle\delta_0,(\mathcal{E}+zI)(\mathcal{E}-zI)^{-1}\delta_0\rangle=\int_{\partial\mathbb{D}}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\Lambda_0=:M_{00},\quad z\in\mathbb{C}\backslash\partial\mathbb{D}.$$

By [8, (3.25)], M_{00} is of the form

$$M_{00}(z)=1+\frac{[1-\alpha_0-(1+\alpha_0)F_+(z,0)][1-\overline{\alpha}_0+(1+\overline{\alpha}_0M_-(z,0))]}{\rho_0^2(F_+(z,0)-M_-(z,0))}.$$

By defining Λ_1 and M_{11} similarly, the canonical spectral measure is $\Lambda=\Lambda_0+\Lambda_1$. Since $\Lambda_0\ll\Lambda$ and $F(z)=\int\frac{e^{i\theta}+z}{e^{i\theta}-z}d\Lambda(\theta)$, the corresponding Radon-Nikodym derivative satisfies

$$\frac{d\Lambda_0}{d\Lambda}(\theta)=\lim_{r\uparrow1}\frac{M_{00}(re^{i\theta})}{F(re^{i\theta})}\tag{5.4}$$

for Λ_s-almost every θ.

This follows from Poltoratskii's theorem [13, 21]. We are grateful to Jake Fillman for the idea to use Poltoratskii's theorem to simplify this part of the proof. The version we use can be found in [27, Remark after Proposition 3.1], which says that for any complex Borel measure μ on $\partial\mathbb{D}$ and any $g \in L^1(\partial\mathbb{D}, d\mu)$ we have that for almost any $\mathrm{e}^{\mathrm{i}\theta'}$ with respect to $d\mu_{\mathrm{s}}$ (but not for $d\mu_{\mathrm{ac}}$),

$$\lim_{r\uparrow 1} \frac{\int_{\partial\mathbb{D}} \frac{\mathrm{e}^{\mathrm{i}\theta}+r\mathrm{e}^{\mathrm{i}\theta'}}{\mathrm{e}^{\mathrm{i}\theta}-r\mathrm{e}^{\mathrm{i}\theta'}} f(\theta) d\mu(\theta)}{\int_{\partial\mathbb{D}} \frac{\mathrm{e}^{\mathrm{i}\theta}+r\mathrm{e}^{\mathrm{i}\theta'}}{\mathrm{e}^{\mathrm{i}\theta}-r\mathrm{e}^{\mathrm{i}\theta'}} d\mu(\theta)} = f(\theta').$$

Due to our choices of r_n, t_n and $\ell_1 \neq \ell_2$, we obtain

$$\lim_{n\to\infty} \frac{M_{00}(r_n\mathrm{e}^{\mathrm{i}\theta})}{F(r_n\mathrm{e}^{\mathrm{i}\theta})} \neq \lim_{n\to\infty} \frac{M_{00}(t_n\mathrm{e}^{\mathrm{i}\theta})}{F(t_n\mathrm{e}^{\mathrm{i}\theta})}.$$

Thus, the θ's in Case (3) are contained in the set where (5.4) fails, which has the zero Λ_s-measure.

(c) Since Λ_{ac} gives the zero weight to sets of zero Lebesgue measure and N_Λ is an essential support, it suffices to show that

$$\mathrm{Leb}(\mathcal{A}\backslash\mathcal{A}_\Lambda) = 0 \tag{5.5}$$

and

$$\mathrm{Leb}(\mathcal{A}_\Lambda\backslash\mathcal{A}) = 0. \tag{5.6}$$

Assume that $\lim_{r\uparrow 1} F(r\mathrm{e}^{\mathrm{i}\theta})$, $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ all have finite boundary values as $r \uparrow 1$ for all $z = \mathrm{e}^{\mathrm{i}\theta}$ in question.

To prove (5.5), we consider $\theta \in \mathcal{A}_\Lambda$ for which $\lim_{r\uparrow 1} |F(r\mathrm{e}^{\mathrm{i}\theta})|$, $\lim_{r\uparrow 1} |F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)|$ and $\lim_{r\uparrow 1} |M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)|$ exist and are finite, and show that $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathcal{A}$.

Let us first consider the possibility that $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ are both purely imaginary. Specifically, let $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0) = \mathrm{i}a$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0) = \mathrm{i}b$, where $a \neq b \in \mathbb{R}$. Since $F(r\mathrm{e}^{\mathrm{i}\theta}) = 1 + 2z(G_{00} + G_{11})$, we firstly consider $1 + 2zG_{00}$. We have

$$\begin{aligned} 1 + 2zG_{00} &= \frac{\mathrm{i}(a-b) - (-1+\mathrm{i}a)(1+\mathrm{i}b)}{\mathrm{i}(a-b)} \\ &= \frac{1+ab}{\mathrm{i}(a-b)}, \end{aligned}$$

which is a purely imaginary number. Consider $2zG_{11}$, which is

$$\begin{aligned} 2zG_{11} &= -\frac{[z+\overline{\alpha}_0 + \mathrm{i}b(z-\overline{\alpha}_0)][-1-\alpha_0 z + \mathrm{i}a(1-\alpha_0 z)]}{\mathrm{i}\rho_0^2 z(a-b)} \\ &= \frac{(z+\overline{\alpha}_0)(1+\alpha_0 z) + ab(z-\overline{\alpha}_0)(1-\alpha_0 z)}{\mathrm{i}\rho_0^2 z(a-b)} \\ &\quad + \frac{\mathrm{i}[b(z-\overline{\alpha}_0)(1+\alpha_0 z) - a(z+\overline{\alpha}_0)(1-\alpha_0 z)]}{\mathrm{i}\rho_0^2 z(a-b)} \\ &= \frac{z + \overline{\alpha}_0 + \alpha_0 z^2 + |\alpha_0|^2 z + abz - \overline{\alpha}_0 ab - \alpha_0 abz^2 + ab|\alpha_0|^2 z}{\mathrm{i}\rho_0^2 z(a-b)} \\ &\quad + \frac{\mathrm{i}[bz - b\overline{\alpha}_0 + b\alpha_0 z^2 - b|\alpha_0|^2 z - az - a\overline{\alpha}_0 + a\alpha_0 z^2 + a|\alpha_0|^2 z]}{\mathrm{i}\rho_0^2 z(a-b)} \\ &= \frac{1 + \overline{\alpha}_0 z^{-1} + \alpha_0 z + |\alpha_0|^2 + ab - \overline{\alpha}_0 abz^{-1} - \alpha_0 abz + ab|\alpha_0|^2}{\mathrm{i}\rho_0^2(a-b)} \\ &\quad + \frac{\mathrm{i}[b - b\overline{\alpha}_0 z^{-1} + b\alpha_0 z - b|\alpha_0|^2 - a - a\overline{\alpha}_0 z^{-1} + a\alpha_0 z + a|\alpha_0|^2]}{\mathrm{i}\rho_0^2(a-b)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Re}\,(-2\rho_0^2(a-b)zG_{11}) &= \mathrm{Re}\,(\mathrm{i}\overline{\alpha}_0 z^{-1} + \mathrm{i}\alpha_0 z - \mathrm{i}ab\overline{\alpha}_0 z^{-1} - \mathrm{i}ab\alpha_0 z + (a-b)\rho_0^2 \\ &\quad + b\overline{\alpha}_0 z^{-1} - b\alpha_0 z + a\overline{\alpha}_0 z^{-1} - a\alpha_0 z) \\ &= (1-ab)\mathrm{Re}\,(\mathrm{i}\overline{\alpha}_0 z^{-1} + \mathrm{i}\alpha_0 z) + (a+b)\mathrm{Re}\,(\overline{\alpha}_0 z^{-1} - \alpha_0 z) \end{aligned}$$

$$+ (a-b)\rho_0^2.$$

With $z = r\mathrm{e}^{\mathrm{i}\theta}$, we have

$$\mathrm{Re}\,(\mathrm{i}\overline{\alpha}_0 r^{-1}\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{i}\alpha_0 r\mathrm{e}^{\mathrm{i}\theta}) \to \mathrm{Re}\,(\mathrm{i}(\overline{\alpha_0\mathrm{e}^{\mathrm{i}\theta}} + \alpha_0\mathrm{e}^{\mathrm{i}\theta})) = 0$$

and

$$\mathrm{Re}\,(\overline{\alpha}_0 r^{-1}\mathrm{e}^{-\mathrm{i}\theta} - \alpha_0 r\mathrm{e}^{\mathrm{i}\theta}) \to \mathrm{Re}\,(\overline{\alpha_0\mathrm{e}^{\mathrm{i}\theta}} - \alpha_0\mathrm{e}^{\mathrm{i}\theta}) = 0$$

as $r \uparrow 1$.

Putting these together, we have $\lim_{r\uparrow 1}\mathrm{Re}\,F(r\mathrm{e}^{\mathrm{i}\theta}) = -1$, which is impossible. If $a = b$, $|\lim_{r\uparrow 1} F(r\mathrm{e}^{\mathrm{i}\theta})| = \infty$, which is also impossible. We conclude that $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ and $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ cannot both be purely imaginary.

Without loss of generality, we assume that $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ is not a purely imaginary number, which implies that there is no subordinate solution at $+\infty$ by Lemma 4.4(3). Thus, $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathcal{A}_+$. Therefore, $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathcal{A}$.

To prove (5.6), we consider $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathcal{A}$ for which

$$\lim_{r\uparrow 1}|F(r\mathrm{e}^{\mathrm{i}\theta})|, \quad \lim_{r\uparrow 1}|F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)| \quad \text{and} \quad \lim_{r\uparrow 1}|M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)|$$

exist and are finite, and show that $\theta \in \mathcal{A}_\Lambda$. Without loss of generality, assume that $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathcal{A}_+$, so (2.6) has no solution that is subordinate at $+\infty$. Then Lemma 4.4 implies that $\lim_{r\uparrow 1} F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ cannot be purely imaginary, which implies that $0 < \lim_{r\uparrow 1}\mathrm{Re}\,F_+(r\mathrm{e}^{\mathrm{i}\theta}, 0) < \infty$. Since $\lim_{r\uparrow 1} M_-(r\mathrm{e}^{\mathrm{i}\theta}, 0)$ also exists and must belong to $\mathbb{C}_\ell = \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) < 0\}$, we have $0 < \lim_{r\uparrow 1}\mathrm{Re}\,F(r\mathrm{e}^{\mathrm{i}\theta}) < \infty$. Therefore, $\theta \in \mathcal{A}_\Lambda$. □

6 Proofs of the corollaries

In this section, we prove Corollaries 2.4 and 2.5.

Proof of Corollary 2.4. Suppose that $z \in \mathcal{B}_+$, i.e., $z \in \partial\mathbb{D}$ is such that

$$\sup_{n\in\mathbb{Z}_\pm}\|A(n,z)\| < \infty.$$

By an extension of the discussion in the proof of [26, Corollary 10.8.4], it follows from the Jitomirskaya-Last inequality that (2.6) has no solution that is subordinate at $+\infty$, and hence $z \in \mathcal{A}_+$. Specifically, due to (3.4)–(3.7), we have

$$|u_+(z,n)| = |\varphi_n(z)| \quad \text{and} \quad |p_+(z,n)| = |\psi_n(z)|, \quad \text{when } n \text{ is even},$$
$$|v_+(z,n)| = |\varphi_n(z)| \quad \text{and} \quad |q_+(z,n)| = |\psi_n(z)|, \quad \text{when } n \text{ is odd}.$$

Let $c = \sup_{n\in\mathbb{Z}_+}\|A(n,z)\|$. Then $|u_+(z,n)| \leqslant c$ if n is even and $|v_+(z,n)| \leqslant c$ if n is odd. By [26, (3.2.23)], $|p_+(z,n)| \geqslant c^{-1}$ if n is even and $|q_+(z,n)| \geqslant c^{-1}$ if n is odd. Thus,

$$c^{-2} \leqslant \frac{\|u_+(z,n)\|_{x(r)}}{\|p_+(z,n)\|_{x(r)}} \leqslant c^2, \quad \text{when } n \text{ is even},$$
$$c^{-2} \leqslant \frac{\|v_+(z,n)\|_{x(r)}}{\|q_+(z,n)\|_{x(r)}} \leqslant c^2, \quad \text{when } n \text{ is odd}.$$

It follows that $\mathcal{B}_+ \subseteq \mathcal{A}_+$.

The inclusion $\mathcal{B}_- \subseteq \mathcal{A}_-$ is proved in a similar way. In Section 4, we have proved $\mathcal{C}_-$ conjugates to $\tilde{\mathcal{C}}_+$ with $-\overline{\alpha}_{-(n+2)} = \tilde{\alpha}_n$ for $n \in \mathbb{N}_0$. Hence, we can rewrite the representation of $A(n,z)$ for $n \leqslant -1$ as

$$S(\tilde{\alpha}_{-n}, z) \times S(\tilde{\alpha}_{-n+1}, z) \times \cdots \times S(\tilde{\alpha}_0, z).$$

From the above statements, $\tilde{\mathcal{C}}_+$ has no solution that is subordinate at $+\infty$. Due to the conjugation, (2.6) has no solution that is subordinate at $-\infty$. It follows that $\mathcal{B}_- \subseteq \mathcal{A}_-$.

The fact that the restriction of Λ to each of $\mathcal{B}_\pm$ is purely absolutely continuous now follows from Theorem 2.2(c). □

Proof of Corollary 2.5. If $z \in \mathcal{R}$, then it follows from the definition of $\mathcal{R}$ that there exist $B_z : \Omega \to \mathrm{SU}(1,1)$ bounded and $A_z^{(0)} \in \mathrm{SU}(1,1)$ elliptic such that for every $\omega \in \Omega$, we have $A_z(\omega) = B_z(T\omega)A_z^{(0)}B_z(\omega)^{-1}$. This in turn shows that for $n \geqslant 2$, we have

$$\begin{aligned} &A_z(T^{n-1}\omega) \times \cdots \times A_z(T\omega)A_z(\omega) \\ &\quad = B_z(T^n\omega)A_z^{(0)}B_z(T^{n-1}\omega)^{-1} \times \cdots \times B_z(T^2\omega)A_z^{(0)}B_z(T\omega)^{-1}B_z(T\omega)A_z^{(0)}B_z(\omega)^{-1} \\ &\quad = B_z(T^n\omega)(A_z^{(0)})^n B_z(\omega)^{-1}. \end{aligned}$$

Given that the matrix on the left-hand side has the same norm as $A(n,z;\omega)$ (it is obtained from that matrix by multiplication with the unimodular number $z^{-n/2}$) and the right-hand side remains bounded as $n \to \infty$ since $A_z^{(0)}$ is elliptic and $B_z : \Omega \to \mathrm{SU}(1,1)$ is bounded, we find that

$$\mathcal{R} \subseteq \mathcal{B}_+(\omega). \tag{6.1}$$

We note in passing that a similar analysis can be applied to the left half-line and yield $\mathcal{R} \subseteq \mathcal{B}_-(\omega)$.

Here, we denote the matrices $A(n,z)$ and the sets $\mathcal{B}_\pm$ introduced earlier for a fixed extended CMV matrix $\mathcal{E}$ by $A(n,z;\omega)$ and $\mathcal{B}_\pm(\omega)$ if they are associated with the dynamically defined extended CMV matrix $\mathcal{E}(\omega)$.

The assertion of Corollary 2.5 now follows from (6.1), which as discussed above holds for every $\omega \in \Omega$, and Corollary 2.4. □

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