JOURNAL OF MODERN DYNAMICS
VOLUME 16, 2020, 351–371

doi: 10.3934/jmd.2020014

THE WORK OF SÉBASTIEN GOUËZEL ON LIMIT THEOREMS AND ON WEIGHTED BANACH SPACES

DMITRY DOLGOPYAT

(Communicated by Giovanni Forni)

ABSTRACT. We review recent advances in the spectral approach to studying statistical properties of dynamical systems highlighting, in particular, the role played by Sébastien Gouëzel.

1. BACKGROUND

An important discovery of the last century is that deterministic systems may display stochastic behavior. In particular, ergodic sums along the orbits of the deterministic systems often obey the same limit theorems as sums of independent identically distributed (i.i.d) random variables.

Let us review the basic results about summations of i.i.d. random variables. Let $S_N = \sum_{n=0}^{N-1} \xi_n$, where $\{\xi_n\}$ are i.i.d. bounded random variables. Then $\frac{S_N - N\mathbb{E}(\xi)}{\sqrt{n\text{Var}(\xi)}}$ converges in law as $N \to \infty$ to a standard Gaussian. That is, for each $z \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{S_N - N\mathbb{E}(\xi)}{\sqrt{n\text{Var}(\xi)}} \le z\right) = \Phi(z) := \int_{-\infty}^{z} e^{-s^2/2} ds.$$

Stronger results include Berry-Esseen Theorem saying that

$$\left|\mathbb{P}\left(\frac{S_N - N\mathbb{E}(\xi)}{\sqrt{n\text{Var}(\xi)}} \le z\right) - \Phi(z)\right| \le \frac{C(\xi)}{\sqrt{N}}$$

and the Local Limit Theorem saying that, unless $\xi \in a + h\mathbb{Z}$ with probability one, we have that for each $z \in \mathbb{R}$ and each interval I

$$\lim_{N\to\infty} \sqrt{N}\mathbb{P}(S_N - N\mathbb{E}(\xi) - \sqrt{N}z \in I) = \frac{1}{\sqrt{2\pi\text{Var}(\xi)}} e^{-z^2/(2\text{Var}(\xi))}|I|.$$

Note that the normalization constants appearing in these results, namely $\mathbb{E}(\xi)$ and $\text{Var}(\xi)$, depend smoothly on ξ.

One could also consider unbounded observables. The case where

$$\mathbb{P}(\xi > t) \approx \frac{c_+\ell(t)}{t^s}, \quad \mathbb{P}(\xi < -t) \approx \frac{c_-\ell(t)}{t^s}$$

Received November 20, 2020.

2010 *Mathematics Subject Classification:* Primary: 37A50, 37C30; Secondary: 37D20, 37D25, 37D30, 60F05.

Key words and phrases: Hyperbolicity, limit theorems, quasi compactness, Lasota-Yorke inequality, Pollicott-Ruelle resonances, induced map.

for some constants $c_+, c_- \in \mathbb{R}^+, s \in (0,2]$ and a slowly varying function ℓ is of particular interest. In this case S_N converges, after a proper normalization, to a stable random variable of index s.

A lot of work in probability theory is devoted to extending these basic results to *weakly dependent* random variables. There are many methods for proving limit theorems in the weakly dependent case. One of the most powerful is the spectral method developed by Nagaev (see [48] for an excellent review and [56, 69] for the detailed exposition of the results obtained in the last century).

A classical approach to limit theorems and related results is the asymptotic analysis of the characteristic function. Namely, the CLT amounts to proving that for each $u \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{E}\left(\exp \frac{iu(S_N - N\mathbb{E}(\xi))}{\sqrt{N\mathrm{Var}(\xi)}}\right) = e^{-u^2/2}. \tag{1.1}$$

In the i.i.d. case the key fact is that $\mathbb{E}\left(e^{iuS_N}\right) = \mathbb{E}\left(e^{iu\xi}\right)^N$, so the Taylor expansion of the characteristic function of ξ near 0 gives (1.1).

To illustrate the spectral method, consider a Markov chain $\{x_n\}$ on state space $\mathscr{S}$ defined by the transition kernel $K(x, dy)$ (this includes dynamical systems $x_n = f^n(x_0)$ as a degenerate case) and let $S_N = \sum_{n=0}^{N-1} A(x_n)$ where A is a function on $\mathscr{S}$. Using the first step analysis we get

$$\mathbb{E}\left(e^{iuS_N}\right) = \mathbb{E}\left(e^{iuA(x_0)}\mathbb{E}_{x_1}\left(e^{iuS_{N-1}}\right)\right). \tag{1.2}$$

Introducing the operator $(\mathscr{U}_u h)(x_0) = \mathbb{E}_{x_0}\left(e^{iuA(x_0)}h(x_1)\right)$ and iterating (1.2) we get

$$\mathbb{E}\left(e^{iuS_N}\right) = \int \mathscr{U}_u^N(1)d\nu(x_0)$$

where ν denotes the initial distribution of our Markov chain. One could also consider the adjoint operator $\mathscr{L}_u = \mathscr{U}_u^*$ and write

$$\mathbb{E}\left(e^{iuS_N}\right) = \langle \mathscr{L}_u^N 1, 1\rangle,$$

where $\langle \dots \rangle$ is the scalar product in $L^2(\nu)$.

In the case our Markov chain has a bounded density with respect to some background measure $K(x_n, dx_{n+1}) = k(x_n, x_{n+1})d\nu(x_{n+1})$, both $\mathscr{U}_u$ and $\mathscr{L}_u$ are compact operators. Moreover, under natural mixing assumptions (e.g., if k is bounded from below) 1 is an isolated eigenvalue of $\mathscr{L}_0$, and there are no other eigenvalues on the unit circle. Hence, by a classical perturbation theory (see e.g., [57]) for small u

$$\mathscr{L}_u^N(h) = \lambda_u^N \ell_u(h)\phi_u + O(\theta^N)$$

for some $\theta < 1$, where $\lambda_u \in \mathbb{C}$, $\phi_u \in L^\infty(\mathscr{S}, \mathbb{C})$, $\ell_u \in L^1(\mathscr{S}, \mathbb{C})$, and the map $u \mapsto (\lambda_u, \phi_u, \ell_u)$ is analytic in a small neighborhood of 0. Now one could prove the CLT (and other results) by the Taylor expansion similar to the independent case. In fact, compactness is not necessary, it suffices that $\mathscr{L}_0$ is *quasi-compact*, that is, its essential spectral radius $\rho_{ess}(\mathscr{L}_0)$ is strictly smaller than its spectral radius $\rho(\mathscr{L}_0)$. A standard condition for quasi-compactness is so called Doeblin-Fortet-Lasota-Yorke inequality [30, 60]. Namely, let an operator $\mathscr{L}$ act on a Banach spaces $(\mathbb{B}, \|\cdot\|)$ and $(\mathbb{B}_w, \|\cdot\|_w)$ so that $\mathbb{B}\subset\mathbb{B}_w$ and the inclusion is compact. Suppose that there are positive constants M, R, $k\in\mathbb{N}$ and $r<\rho(\mathscr{L},\mathbb{B})$ such that

$$\|\mathscr{L}^k h\|_w \le M\|h\|_w, \quad \|\mathscr{L}^k h\| \le r^k\|h\| + R\|h\|_w, \tag{1.3}$$

then $\rho_{ess}(\mathscr{L},\mathbb{B})\le r$.

A classical example of applicability of this criterion is Doeblin condition. Namely, consider a Markov operator K such that there is a constant $\theta\in(0,1)$ such that

$$K(x,dy) = \theta k(x,y)d\nu(y) + (1-\theta)\bar{K}(x,dy),$$

where $k(\cdot,\cdot)$ is bounded from above and below and $\bar{K}$ is an arbitrary kernel. In this case (1.3) holds with $\|\cdot\| = \|\cdot\|_{L^\infty}$, $\|\cdot\|_w = \|\cdot\|_{L^1}$. The quasi compactness may also hold in the dynamical systems setting. Namely, let $f:\mathbb{T}^1\to\mathbb{T}^1$ be an orientation preserving expanding map, that is, there exists $\Lambda>1$ such that for all $x\in\mathbb{T}^1$, $f'(x)\ge\Lambda$. Then

$$(\mathscr{L}h)(x) = \sum_{fy=x}\frac{h(y)}{f'(y)}. \tag{1.4}$$

Taking $\|\cdot\| = \|\cdot\|_{C^1}$, $\|\cdot\|_w = \|\cdot\|_{C^0}$, we get

$$\frac{\partial}{\partial x}(\mathscr{L}h)(x) = \sum_{fy=x}\frac{h'(y)}{f'(y)^2} - \sum_{fy=x}\frac{h(y)f''(y)}{|f'(y)|^2}.$$

The second term is controlled by the weak norm, while the first one is smaller than $\sum_{fy=x}\frac{|h'(y)|}{\Lambda f'(y)} \le \frac{\mathscr{L}|h'|}{\Lambda}$. One could use the above estimates to show that in this case $\rho(\mathscr{L},C^1) = \rho(\mathscr{L},C^0) = 1$ while $\rho_{ess}(\mathscr{L},C^1)\le\Lambda^{-1}$. A similar argument (see e.g. [24, 71]) shows that $\rho_{ess}(\mathscr{L},C^r)\le\Lambda^{-r}$ for each $r\in\mathbb{N}$.

We note that quasi compactness is useful not only in the study of the characteristic function. In particular, it could also be used to obtain the asymptotics of the correlation functions via the identity

$$\int A(f^n x)B(x)dx = \int(\mathscr{L}^n B)(y)A(y)dy.$$

Thus if $\rho_{ess}(\mathscr{L}_0,\mathbb{B}) = \boldsymbol{\rho}$ on some space $\mathbb{B}$, then for $B\in\mathbb{B}, A\in\mathbb{B}^*$ and any $\varepsilon>0$ one can write

$$\int A(f^n x)B(x)dx = \mu(A)\int B(x)dx + \sum_{k=1}^{N_\varepsilon}P_k(n)\lambda_k^n + O((\boldsymbol{\rho}+\varepsilon)^n). \tag{1.5}$$

The program of studying the dynamical systems using the spectral approach was pioneered in the works of Lasota-Yorke ([60]), and Guivarc'h and collaborators ([52, 53]). However, by the end of the last century this approach was only applicable to uniformly hyperbolic systems such as

- expanding maps (see above);

- piecewise expanding maps (with $\mathbb{B}_w = L^1$, $\mathbb{B} = BV$);
- subshifts of a finite type (with $\mathbb{B}_w = C^0$ and $\mathbb{B}$–Holder).

Now the spectral approach has been applied successfully to a large class of non-uniformly hyperbolic as well as some partially hyperbolic systems, and it remains very much in the forefront of the modern research in dynamics. Sébastien Gouëzel played a central role in this development proving the optimal results in many cases as well as opening new directions of research. In this survey, I will provide some highlights of his contributions, directing the readers to the original papers for additional details.

There are two natural ways for extending the spectral approach beyond the uniformly hyperbolic realm.

1. Given a map f one can define an induced map $F(x) = f^{\tau(x)}(x)$ so that F is uniformly hyperbolic.
2. One can construct a Banach space $\mathbb{B}$ adapted to the geometry of f, so that the transfer operator is quasi-compact.

Sébastien Gouëzel made fundamental contributions to both directions. This work was recognized in his 2019 Brin Prize citation. The contributions to the inducing approach are described in Section 2 while the results using adapted Banach spaces are discussed in Section 3.

2. INDUCTION

If $F(x) = f^{\tau(x)}(x)$, where τ is defined so that $f^{\tau(x)} \in Y \subset X$, then X can be conveniently represented as a tower of Y. This approach was actively pursued by Lai-Sang Young in the late 90s (see [81, 82]). She proved that many systems admit a so called *Young tower* and established the quasi compactness of the transfer operators provided that τ has exponential tail $\mathrm{mes}(x : \tau(x) > n) \le C\theta^n$ for some $C > 0, \theta < 1$. In the case of polynomial tails the quasi-compactness fails, so only partial results were available.

We note that the induction also serves as a powerful tool in probability. For example, given a Markov chain with a countable phase space, one can select a site s_0 and consider only the times when our process visits this site. Since consecutive excursions away from s_0 are independent, the process starts afresh after each return. To control the process between the visits to s_0 one uses so called *renewal theory*. The central object in this theory is the *renewal equation* which I will explain now in the context of mixing, following [70, 72].

Let A and B be the functions supported on Y. Denote

$$\rho_n(A,B) = \langle A, B \circ f^n \rangle$$

and consider the generating function

$$t(z) = \sum_n \rho_n(A,B) z^n =: \langle \mathcal{T}(z) A, B \rangle = \langle A, \mathcal{U}(z) B \rangle,$$

where

$$\mathcal{U}(z) B = \sum_n \bar{z}^n B \circ f^n = B + \bar{z}^\tau \mathcal{U}(B \circ F).$$

Hence

$$\langle \mathcal{T}(z)A, B\rangle = \langle A, B\rangle + \langle \mathcal{R}(z)\mathcal{T}(z)A, B\rangle,$$

where $\mathcal{R}(z)$ is the adjoint to $B \mapsto \bar{z}^{\tau} B \circ f^{\tau}$. Thus we have

$$\mathcal{T}(z) = I + \mathcal{R}(z)\mathcal{T}(z), \quad \text{that is,} \quad \mathcal{T}(z) = (I - \mathcal{R}(z))^{-1}.$$

Our goal is to understand the coefficients of $\mathcal{T}(z)$ in terms of the coefficients of $\mathcal{R}$ which are given explicitly in terms the hitting time τ. This is achieved by the following result.

Let $\mathbb{D}$ denote the unit disc $\mathbb{D} = \{|z| < 1\}$.

THEOREM 2.1 ([72, 44]). *Suppose that $\mathcal{T}(z)$ and $\mathcal{R}(z)$ are holomorphic families of operators acting on some Banach space $\mathbb{B}$ so that on $\mathbb{D}$*

$$\mathcal{T}(z) = I + \sum_{n=1}^{\infty} z^n T_n, \quad \mathcal{R}(z) = \sum_{n=1}^{\infty} z^n R_n.$$

where T_n and R_n are bounded operators on $\mathbb{B}$ and $\sum_n \|R_n\| \le \infty$. Assume

(a) **Renewal equation:** $\mathcal{T}(z) = (I - \mathcal{R}(z))^{-1}$ *in* $\mathbb{D}$;
(b) **Spectral gap:** *1 is a simple isolated eigenvalue of $\mathcal{R}(1)$ on unit circle, with eigenprojection P;*
(c) **Aperiodicity:** $I - \mathcal{R}(z)$ *is invertible on* $\overline{\mathbb{D}} - \{1\}$;
(d) **Polynomial tails:** $\sum_{k>n} \|R_k\| = O(1/n^{\beta})$, $\beta > 1$;
(e) **Transversality**[1] **:** $P\mathcal{R}'(1)P \neq 0$.

Then

$$T_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k>n} P_k + E_n, \tag{2.1}$$

where μ is defined by the condition $P\mathcal{R}'(1)P = \mu P$, $P_n = \sum_{k>n} PR_kP$, and $\|E_n\| =$

$$O(\gamma_{\beta}(n)), \text{ where } \gamma_{\beta}(n) = \begin{cases} 1/n^{\beta} & \text{if } \beta > 2, \\ (\ln n)/n^2 & \text{if } \beta = 2, \\ 1/n^{2\beta-2} & \text{if } 2 > \beta > 1. \end{cases}$$

Note that the error in the above theorem is smaller than the second term in (2.1) which is typically of order $n^{1-\beta}$.

Theorem 2.1 was proved in [72] under the assumption that $\beta > 2$ and with a weaker error bound $\gamma_{\beta}(n) = \frac{1}{n^{[\beta]}}$ (for $\beta > 2$). The optimal result above is due to [44].

Combining (2.1) with property (d) we obtain.

COROLLARY 2.2 ([44]). *If $PA = 0$ then $T_n A = O(1/n^{\beta})$.*

[1]The differentiability of $\mathcal{R}$ at 1 follows from (d) since $\sum_{n=1}^{\infty} n\|R_n\| = \sum_{n=1}^{\infty} \sum_{k \ge n} \|R_k\|$.

In the theorem below, we say that F on Y is uniformly hyperbolic if every point $y \in Y$ possesses strong stable and strong unstable manifolds $W^s(y)$ and $W^u(y)$, respectively, Y has a local product structure (that is, for y_1, y_2 nearby, $W^u(y_1)$ intersects $W^s(y_2)$ transversally), F uniformly contracts stable manifolds, F^{-1} uniformly contracts unstable manifolds and F has bounded distortion (see [81, Section 1] for the precise conditions).

THEOREM 2.3 ([44])**.** *If F is uniformly hyperbolic, and A, B are Hölder and supported on Y, then*

(a) $\rho_n(A,B) - \mu(A)\mu(B) = \mu(A)\mu(B)\sum_{k>n} m(\tau > k) + O(\gamma_\beta(n))$.

(b) *If* $\mu(A) = 0$, *then* $\rho_n(A,B) = O(1/n^\beta)$.

(c) *Without assuming that A, B are supported on Y, we get*

$$\rho_n(A,B) - \mu(A)\mu(B) = O(1/n^{\beta-1}).$$

Let $A_N(x) = \sum_{n=0}^{N-1} A(f^n x)$ denote the ergodic sum of A.

THEOREM 2.4 ([44])**.** *Ergodic sums $A_N(x)$ satisfy CLT if either $\beta > 2$ or $\beta > 1$, A is supported on Y and $\mu(A) = 0$.*

To derive this result from Theorem 2.1, consider first the case when the functions are constant on the stable manifolds. For such function the operator $\mathscr{R}$ behaves similarly to the transfer operator (1.4) for the expanding map, so the asymptotics of the correlation function follows from Theorem 2.1. Arbitrary observables are approximated by functions $A_{(k)}$, $B_{(k)}$ such that $A_{(k)} \circ F^{-k}$ and $B_{(k)} \circ F^{-k}$ are constant on the stable manifolds and use the invariance of the correlation functions by the dynamics.

The last theorem gives CLT for slow mixing systems, where it was not expected before. This result has several important applications, my favorite being semidispersing billiards.

We consider two classical examples: Bunimovich stadium and a billiard with one cusp (see Figure 1).

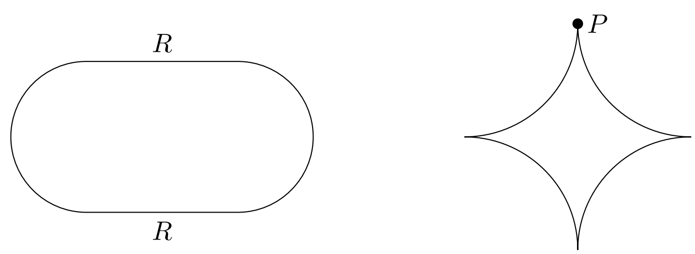

FIGURE 1. Semidispersing billiards

In the first case, let $I_B = \int_R A(s,0)\,ds$.

In the second case, let $I_C = \int_{-\pi/2}^{\pi/2} [A(p_-,\phi) + A(p_+,\phi)]\,d\phi$.

THEOREM 2.5. (a)([15]) *For stadium there is a constant c such that* $\frac{A_N}{\sqrt{N\ln N}}$ *converges as $N\to\infty$ to the normal distribution with zero mean and variance* cI_B^2. [2] *If $I_B=0$, then standard CLT holds.*
(b)([14]) *the same result holds for cusp with with I_B replaced by I_C.*

In the early 2000s, Sébastien Gouëzel made a systematic studies of dynamical systems admitting uniformly hyperbolic inducing schemes. This includes

- stable laws [45];
- local limit theorem [46];
- Berry–Esseen theorem for the rate of convergence (it turns out that the optimal rate is $n^{-\delta/2}$ where $\delta=\min(\beta-1,1)$), see [46];
- almost sure CLT [23];
- almost sure invariance principle [47];
- large and moderate deviations [25].

Currently, the dynamical renewal theory is an important tool in statistical properties of dynamical systems. Active research topics include continuous time systems ([1, 68, 31]), infinite measure systems ([64, 67]) as well as the renewal theory for sequential systems ([2, 55]).

3. WEIGHTED BANACH SPACES

3.1. **First steps.** The work described in Section 2 constitute a significant advance in understanding of stochastic properties of non-uniformly hyperbolic systems. However, there were still a few shortcomings.

(a) While the restriction of F to Y could be made much better than the original map $f: X\to X$, if we want to study F using symbolic dynamics we need $F(x)$ to have a Markov return and it may be difficult to control the tail of the return time.

[2] In order to understand an anomalous $\sqrt{N\ln N}$ scaling for the stadium we note that the orbits with poor hyperbolicity properties stay for a long time on the same boundary component. It turns out that the main problem appears when the orbit moves for a long time almost parallel to the straight boundary. Therefore, one can take F to be the induced map to the circular part of the boundary and let τ be the corresponding return time. One can see that if $\tau>t$ then the orbit should make an angle $O(1/t)$ with the normal to the straight part and the corresponding orbit hits a circle for a last last time in a $O(1/t)$ neighborhood of an endpoint of the circular arc. It follows that $P(\tau>t)=\bar{c}t^{-2}(1+o(1))$ as $t\to\infty$ where $\bar{c}$ is some constant. Next if A is a smooth function then $\bar{A}(x):=\sum_{n=0}^{\tau(x)-1}A(x)=\bar{c}I_B\tau(1+o(1))$. It follows that the distribution function of $\bar{A}$ belongs domain of attraction of stable law with index 2 (which is normal distribution) and moreover $\mathbb{E}(\bar{A}^2 1_{|\bar{A}|<T})=\hat{c}\ln T(1+o(1))$ which explains the extra log factor in Theorem 2.5. We also not that in contrast with i.i.d. case the prefactor in front of $\sqrt{N\ln N}$ is determined not only by the variance $\mathbb{E}(\bar{A}^2 1_{|\bar{A}|<T})$ but also by the covariances $\mathbb{E}(\bar{A}(x)\bar{A}(F^n x)1_{|\bar{A}(x)|<T})$, see Section 4 of [15], in particular, Remark 4.11 where. The justification for $\sqrt{N\ln N}$ scaling for the cusp is similar but the computation is more involved, we refer to [65] for details. In fact a similar reasoning explains the superdiffusion in infinite horizon Lorentz gas, see [16] for the heuristic argument and [77] for the rigorous treatment based on the method of [15].

(b) The dependence of the coefficients of the limit laws on parameters was unclear.

(c) The essential spectrum obtained by the standard approach was large (cf. [24]) making it impossible to obtain the complete asymptotics of the correlation function.

One way to address these issues is to work directly with the transfer operator of the smooth system. To this end, one needs to design Banach spaces $\mathbb{B}_\alpha$ so that the transfer operator has good spectral properties on $\mathbb{B}_\alpha$. There are two approaches to this problem.

(I) *Geometric approach.* Recall that if f is a contraction then $\mathscr{U}A = A\circ f$ improves smoothness since

$$D^k(\mathscr{U}A) = (D^k A)df^k + L.O.T.,$$

where $L.O.T.$ includes the derivatives of order less than k (which could be controlled using the weak norms). Accordingly if f is expanding, then $\mathscr{U}$ improves the regularity in the dual space $(C^k)^*$. So one needs to consider the functions which "are smooth in the stable directions and are distributions in the unstable ones." The same reasoning applies to the transfer operators, that is, the adjoint to $\mathscr{U}$ given by

$$\mathscr{L}(A) = \frac{A\circ f^{-1}}{\det(df)\circ f^{-1}}, \tag{3.1}$$

however, since one composes with f^{-1} rather than f, then the roles of stable and unstable directions are interchanged.[3]

(II) *Microlocal approach.* Let $L:\mathbb{T}^2\to\mathbb{T}^2$ be a linear Anosov map. Let $e_k(x) = e^{2\pi i\langle k,x\rangle}$. Then

$$(e_k\circ L)(x) = e^{2\pi i\langle k,Lx\rangle} = e_{L^*k}(x).$$

Since the orbits of L^* are hyperbolas, one can easily construct Lyapunov functions decaying along the orbits. Namely, if $e = e_u + e_s$, where e_u and e_s denote the unstable and stable components of e and $V(e) = \|e_s\|^{\alpha_s}/\|e_u\|^{\alpha_u}$. Then $V(L^*(e)) \le \theta V(e)$ with $\theta = \lambda_s^{\alpha_s}/\lambda_u^{\alpha_u}$, where λ_s and λ_u are weakest expansion and contraction rates of L^*. Thus, denoting $\|A\|^2 = \sum_k V(k)|c_k|^2$, where $A = \sum_k c_k e_k$ has zero mean, we get

$$\|A\circ L\|^2 = \sum_k V(L^*(k))|c_k|^2 \le \theta\sum_k V(k)|c_k|^2 \le \theta\|A\|^2.$$

For nonlinear maps the idea of construction is similar, however, since the Fourier harmonics are no longer preserved one needs to use microlocal analysis to describe high frequency dynamics of wave packets.

[3]In dynamics literature one usually deals with operators $\mathscr{L}$ involving backward compositions, because in the case of expanding maps whose operators behave well on the space of smooth functions. In probability literature one more often considers forward looking operators $(\mathscr{U}A)(x_0) = \mathbb{E}_{x_0}(A(x_1))$. In the case of hyperbolic maps which are considered in this section both forward and backward operators lead to similar results.

In §3.2 I concentrate mostly on geometric method, since it was chronologically the first and so it made stronger impact on the field. Moreover, most of the work of Sébastien Gouëzel relies on this method. On the other hand, in §3.3 describing the current research I include papers using both methods. Indeed, many problems could be approached by both methods with similar results. An advantage of microlocal method is that one use Banach spaces from the PDE literature whose properties are well established. On the other hand, the geometric method works better for systems of low regularity such as piecewise smooth maps, since it is easy to adapt the Banach spaces to the geometry of the singularities.

Since the early work on Banach spaces adapted to dynamics dealt with uniformly hyperbolic systems let me recall a few definitions.

Let f be a smooth diffeo of a manifold X. A set Λ is called a *basic hyperbolic set* if there is a neighborhood U of Λ such that $\bigcap_{n=-\infty}^{\infty} f^n U = \Lambda$ and there is a continuous df invariant splitting $T_\Lambda X = E^u \oplus E^s$, a Riemannian metric $\|\cdot\|$ and constants $\lambda_s < 1, \lambda_u > 1$ such that

(a) f uniformly contracts E^s, i.e. for all $v \in E^s$ $\|dfv\| \le \lambda_s \|v\|$;

(b) f uniformly expands E^u, i.e. for all $v \in E^u$ $\|dfv\| \ge \lambda_u \|v\|$.

If f is hyperbolic, then the unstable cones

$$\mathscr{C}^u = \{v = v^u + v^s, v^* \in E^* : \|v^s\| \le \delta \|v^u\|\}$$

are forward invariant while the stable cones

$$\mathscr{C}^s = \{v = v^u + v^s, v^* \in E^* : \|v^u\| \le \delta \|v^s\|\}$$

are backward invariant.

If, additionally, U could be chosen to be forward invariant, then we say that Λ is a *hyperbolic attractor* and if $\Lambda = X$ we say that f is an *Anosov* diffeomorphism. For topologically transitive hyperbolic attractors, a classical result of Sinai ([74]) says that there is an invariant measure, called *SRB measure*, μ_{SRB} such that if x is distributed according to a smooth density and A is Hölder then $\frac{A_N(x)-N\mu_{SRB}(A)}{\sqrt{N}}$ converges in Law as $N \to \infty$ to the normal distribution with zero mean and some variance $\sigma^2(A)$. Moreover, μ_{SRB} enjoys exponential decay of correlations [74]. The first task in the Banach space approach was to reprove and precise the results cited above.

The first construction of a Banach space adapted to a smooth system was achieved in [20]. Let f be an Anosov diffeomorphism whose invariant distributions are of class C^τ. Fix constants β and γ such that $0 < \beta < \gamma < \tau$, $\beta < \tau\gamma$, and consider

$$\|A\|_s = \sup_{\|\phi\|_{C^\beta(W^s)} \le 1} \int \phi A dm, \quad \|A\|_u = \sup_{v \in E^u, \|v\|_{C^\beta(W^s)} \le 1} V(A) dm,$$

$$\|A\| = \|A\|_u + \|A\|_s, \quad \|A\|_w = \sup_{\|\phi\|_{C^\gamma(W^s)} \le 1} \int \phi A dm.$$

Here $C^\beta(W^s)$ denote functions (or vector fields) which are bounded and such that their restrictions to stable leaves are uniformly Hölder and $V(A)$ is the Lie derivative of A along V.

[20] proved that the essential spectrum of $\mathscr{L}$ defined by (3.1) is smaller than $\max\left(\frac{1}{\lambda_u}, \lambda_s^{\tau+\varepsilon}\right)$. While [20] showed that one can go beyond the symbolic setting and get the quasi compactness, this paper did not solve the issues (a)–(c) mentioned at the beginning of this section. In particular, the Banach spaces constructed in [20] depend heavily on f, so that nearby maps lead to completely different spaces, making it impossible to develop a perturbation theory in those spaces.

3.2. **The contribution of Gouëzel.** The next breakthrough was achieved in the work of Gouëzel-Liverani. They introduced the following norms:

$$\|A\|^-_{p,q} = \sup_{W\in\mathfrak{A}} \sup_{\substack{v_1,\dots v_p\\ \|v_j\|_{C^r}\le 1}} \sup_{\phi\in C^q_0(W)} \int_W (v_1\dots v_p A)\phi dm,$$

$$\|A\|_{p,q} = \max_{0\le k\le p} \|A\|^-_{k,q+k}.$$

Here $\mathfrak{A}$ denotes the class of admissible manifolds, i.e., those with tangent spaces in the stable cones and having bounded geometry in the sense that their sizes are neither too small nor too large and their C^r-norms with $r > p+q$ are controlled (see [50, Section 3] for the precise definition), $v_1,\dots v_p$ are (C^r)-smooth vector fields defined near W and ϕ are (C^q)-smooth functions on W with compact support. Let $\mathbb{B}_{p,q}$ denote the closure of $C^\infty(M)$ with respect to $\|\cdot\|_{p,q}$. Then $\mathbb{B}_{p,q}$ is good for all maps near f.

THEOREM 3.1 ([50])**.** $\rho_{ess}(\mathscr{L},\mathbb{B}_{p,q}) \le \max(\lambda_u^{-p},\lambda_s^q)$.

In particular, if $f\in C^\infty$, then one can make $\rho_{ess}(\mathscr{L},\mathbb{B}_{p,q})$ as small as one wishes by taking p and q large.

COROLLARY 3.2. $\int A(x)B(f^n x)d\mathrm{Vol}(x)$ *admits complete asymptotic expansion (that is,* $\boldsymbol{\rho}$ *in* (1.5) *can be made arbitrarily small).*

COROLLARY 3.3. *The maps* $(A,f)\mapsto\mu_{SRB}(A,f)$, $(A,f)\mapsto\sigma_{SRB}(A,f)$ *are smooth.*

Thus [50] addressed issues (b) and (c) mentioned at the beginning of Section 3. Issue (a) was not addressed, because it required an ability to handle discontinuous maps such as $F(x) = f(x)^{\tau(x)}$, where τ is piecewise constant but not constant. However, following the ideas of [50], the papers [13, 11, 12, 27, 28, 29] designed Banach spaces adapted to piecewise smooth maps addressing (a).

To convey the flavor of [50] let me discuss the Lasota-Yorke inequality. We need to estimate

$$\int_W (v_1\dots v_p)\left(\frac{A}{\det(df^n)}\circ f^{-n}\right)\phi dm.$$

If E^u and E^s were smooth, then we could split $v_j = v_j^u + v_j^s$, get rid of v_j^s using the integration by parts (paying by a weaker norm!) and reduce the problem to estimating

$$\int_W (v_1^u \dots v_p^u)\left(\frac{A}{\det(df^n)} \circ f^{-n}\right)\phi dm. \tag{3.2}$$

Covering $f^{-n}W \subset \bigcup_{j=1}^{s} W_j$, where W_j are admissible, the cover has bounded intersection multiplicity, and, taking the partition of unity $1 = \sum_j \rho_j$ with ρ_j supported on W_j, we get

$$(3.2) = \sum_j \int_{W_j} \left[((df^{-n}v_1^u)\dots(df^{-n}v_p^u))A\right]\left[(\phi\circ f^n)\frac{\det(df^n|TW_j)}{\det(df^n)}\right]dm.$$

Since $\|df^{-n}|E^u\| \le C\lambda_u^{-n}$, the bounded distortion property of the determinants in the last formula allows us to estimate (3.2) by

$$\lambda_u^{-np}\|A\|_{p,q}^{-}\sum_j \int_{W_j}\left[\frac{\det(df^n|TW_j)}{\det(df^n)}\right]dm = \lambda_u^{-np}\|A\|_{p,q}^{-}\int_W \det(df^{-n})dm.$$

Since the last integral is controlled by $\|\mathcal{L}^n 1\|$ which is bounded (see footnote 4 below) we conclude that the contribution of (3.2) is $O\left(\lambda_u^{-p}\right)$.

Unfortunately, E^u and E^s are not smooth and, as a result, v_j^u and v_j^s are not smooth. However, the following estimate holds.

LEMMA 3.4 ([50]). *In a small neighbourhood U of $f^n(W_j)$ one can split $v = \tilde{w}^u + \tilde{w}^s$ so that*

(a) *for $x \in f^n(W_j)$ we have $\tilde{w}^s(x) \in T(f^n W_j)$,*
(b) *$\|\tilde{w}^s\|_{C^{r+1}(U)} \le C_n$ and $\|\tilde{w}^u\|_{C^{r+1}(U)} \le C_n$, where C_n is a constant which may depend on n;*
(c) *$\|\tilde{w}^s \circ f^n\|_{C^r(W_j)} \le C$ and $\|(df^{-n}\tilde{w}^u)\circ f^n\|_{C^{p+q}(f^{-n}U)} \le C\lambda_u^{-n}$.*

Thus one can prove Theorem 3.1 by repeating the above argument using the splitting $v = \tilde{w}^u + \tilde{w}^s$ instead of $v = v^u + v^s$.

The next advance came in the paper [51], where the authors extended results of [50] to the Gibbs states on basic hyperbolic sets. Recall that, given a map f and a function (potential) ϕ, an equilibrium state (Gibbs measure) μ_ϕ is an argmax of the pressure functional

$$P(\mu) = \sup_{\mu - f \text{ invariant}} \left[h_\mu + \mu(\phi)\right].$$

For topologically transitive hyperbolic basic sets, Gibbs states for Hölder potentials exists and are unique [19, 69, 74]. It is also clear from the definition that homologous functions have the same Gibbs states. In particular, the Gibbs states for ϕ, $\phi\circ f$ and $\phi\circ f^{-1}$ are the same. Moreover, the SRB measure is a Gibbs state with the potential

$$\psi_u(x) = -\ln\det(df|E^u)(x). \tag{3.3}$$

A natural idea to handle more general Gibbs states is to consider weighted transfer operators which are adjoint to

$$\mathscr{U}_\phi(A) = e^\phi (A\circ f) \tag{3.4}$$

(note that the weighted operators were also used in Section 1 to prove the Central Limit Theorem). In fact, one can show that (see e.g. [59])

$$\int (\mathscr{U}_\phi^n A)\, dm(x) = \ell(A) e^{P(\phi+\psi_u)}(1+o_{N\to\infty}(1)).$$

The same results apply to f^{-1} with ψ_u being replaced by the SRB potential for f^{-1} given by[4]

$$\psi_s = \ln\det(df|E^s). \tag{3.5}$$

Accordingly, the arguments of [50] could be extended with some additional work to handle the Gibbs measure with potentials $\psi(x) = \psi_u(x) + \phi(x)$, where ϕ is a smooth function. However, since ψ_u is not smooth, such an extension would be insufficient to cover smooth potentials such as $\psi \equiv 0$. The authors point out that one could overcome this difficulty by considering the actions on the forms of dimension d_u (or dimension d_s for backward composition). Indeed, if W is a piece of an unstable manifold and ω is a d_u form whose restriction on unstable distribution equals $A\, dm_u$, then

$$\int_{fW} e^{\phi\circ f^{-1}}\omega = \int_W (A\circ f)e^\phi \det(df|E_u)\, dm_u = \int_W (A\circ f)e^{\phi-\psi_u}\, dm_u,$$

so the spectral radius of the resulting operator is equal to

$$e^{P((\phi-\psi_u)+\psi_u)} = e^{P(\phi)},$$

as needed. In fact, in order to handle both smooth potentials such as $\phi = 0$ and non-smooth ones such as ψ_u and ψ_s it is convenient to consider more general objects. Let F be the induced action on Grassmannians of dimension d_s. Then, if Λ is a basic hyperbolic set for f, then F^{-1} possesses a basic hyperbolic set $\bar\Lambda$ which is a graph of E^s. [51] consider a space $\mathbb{B}$ whose elements are maps which associate to each subspace E of dimension d_s a volume form on E. Thus, if W is a submanifold of dimension d_s, and $\alpha\in\mathbb{B}$ then one can integrate α over W by integrating $\alpha(x, T_xW)$ over W. Given a smooth potential $\phi(x,E)$ on the Grassmann bundle, consider the operator

$$\mathscr{L}_\phi(\alpha)(x,E) = \pi(f^{-1}x)e^{\phi(F^{-1}(x,E))} f^*(\alpha\circ F^{-1}),$$

[4] In particular, the spectral radius of the operator $\mathscr{L}$ in (3.1) equals $e^{P(-\ln\det(df)+\psi_s)} = e^{P(\psi_u)} = 1$, since $P(\psi_u) = 0$ by the Pesin formula. The same result could also be deduced from the asymptotics

$$\int A(x)B(f^n x)\, dm(x) = \int B(y)(\mathscr{L}^n A)(y)\, dy = m(A)\mu_{SRB}(B) + o(1)$$

valid for continuous A and B.

where π is a cutoff function supported in a small neighborhood of Λ and equal to 1 on Λ. In other words,

$$\int_W B\mathcal{L}_\phi(\alpha) = \int_{f^{-1}W} (B\circ f)\pi e^\phi \alpha.$$

Set $\bar\phi(x) = \phi(x, E^s(x))$. Note that, if ϕ depends only on x, then $\bar\phi = \phi$, while for $\phi(x,E) = \ln\det(df|E)$, we get $\bar\phi = \psi_s$ given by (3.5), and for

$$\phi(x,E) = \ln\det(df|E)(x) - \ln\det(df)(x),$$

we get $\bar\phi = \psi_u$ given by (3.3).

The results of [51] could be summarized as follows

THEOREM 3.5. *There are Banach spaces $\mathbb{B}^{p,q}$ (defined similarly to [50] but with f replaced by F) such that*

(a) *The spectral radius of $\mathcal{L}_\phi$ is $e^{P(\bar\phi)}$, while the essential spectral radius is at most $\max(\lambda_u^{-p}, \lambda_s^q)e^{P(\bar\phi)}$.*
(b) *$\mathcal{L}$ has a unique eigenvector α_0 with eigenvalue $e^{P(\phi)}$ and its adjoint has unique eigenvector ℓ_0 with eigenvalue $e^{P(\phi)}$. The functional $\mu_\phi(A) = \ell_0(A\alpha_0)$ is the Gibbs measure with potential $\bar\phi$.*
(c) *If A, B are smooth, then the correlation function $\mu_\phi((A\circ f^n)B)$ admits asymptotic expansion (1.5).*

COROLLARY 3.6. *The maps $(f, A, \phi) \mapsto \mu_{\bar\phi}(f, A)$ and $(f, A, \phi) \mapsto \sigma^2(f, A, \bar\phi)$ are smooth.*

This result extends significantly the previous results in this area ([58]).

The papers discussed above provided a new look at uniformly hyperbolic system but the method of weighted Banach spaces proved useful far beyond this setting. Some of the areas where it has been applied successfully are described in §3.3 below.

3.3. **Current research.** The rules of Brin prize stipulate that the prize is given for work which has a lasting impact on the field. For this reason, in this subsection I describe some of the spectacular advances which rely on the theory of weighted Banach spaces pioneered by Sébastien Gouëzel. As it was mentioned in Section 1, the most straightforward application of the quasi compactness is to obtain the refined asymptotics of the correlation function

$$\rho_n(A,B) = \int A(x)B(f^n x)\,d\mu(x)$$

for smooth A and B and natural invariant measures μ. In fact, more general integrals of the form

$$\int H(x, f^n x)\,d\mu(x) \tag{3.6}$$

can be handled provided that H is sufficiently smooth, by expanding H as the sum of products $H(x,y) = \sum_k A_k(x)B_k(y)$. Moreover, the requirement that H is smooth is not optimal, some singularities are allowed provided that the resulting observables belong to appropriate Banach spaces. The current developments of the theory proceeds mostly in the following three directions

(i) Extending the theory to more general dynamical systems;
(ii) Extending the theory to more singular observables to fit the needs of applications;
(iii) Describing the exponents λ_k in (1.5) (called *Pollicott-Ruelle resonances*) and relating them to geometry or physics of the system at hand.

While there are many open questions pertaining to each of the above mentioned directions, the results obtained so far convincingly demonstrate the power of the theory.

3.3.1. *Decay of correlations.* The spectral approach is very useful for obtaining precise asymptotics of the correlation function using the identity (1.5). A promising direction of current research is to use the spectral approach to understand mixing properties of partially hyperbolic systems. Notable partial results on this subject include [5, 18, 21, 78].

One particular important class of partially hyperbolic systems are transversely hyperbolic system with symmetries, where the central direction is spanned by the symmetry group of the system. Perhaps the most studied example is provided by hyperbolic flows. In this setting one could combine in a nice way the techniques of Banach spaces with the renewal theoretic approach described in Section 2. Recent important developments on this subject include exponential mixing for Sinai billiard flows [10], and for mixing Anosov flows on three dimensional manifolds [80] (extending a earlier work [79]).

Another major development is the paper [4] proving exponential mixing of the Teichmüller geodesic flow with respect to all $SL_2(\mathbb{R})$ invariant measures. This result combines the classification of the invariant measures obtained in the work of Eskin–Mirzakhani with representation theory and the theory of weighted Banach spaces. [4] extends an earlier work of [6] which treats the mixing with respect to the absolutely continuous measure on strata. We refer the readers to [61, Section 3] for a more detailed discussion of this topic.

Finally I would like to mention paper [9] where weighted Banach spaces are used to study mixing properties of Gibbs measures for piecewise hyperbolic systems with singularities.

3.3.2. *Periodic orbits.* One of the classical questions in dynamics is growth of the number of periodic points. This can be considered as a special case of the mixing problem (3.6) where H is a product of a smooth function and δ-function on the diagonal (cf. see [66])[5]. More generally one can consider weighted counting problems where a periodic orbit p of period n has weight $\prod_{k=0}^{n-1} e^{\phi(f^k p)}$. Similarly, for a flow g^t one can assign to a periodic point p of weight T weight $e^{\phi_T(p)}$,

[5] In fact, a simpler treatment could be made if one considers a product map $f \times f^{-1}$ acting on $X \times X$, [63].

where $\phi_T(p) = \int_0^T \phi(g^t p)\,dt$. While weighted periodic orbits could be counted using mixing properties of appropriate transfer operators there is an alternative approach based on the trace formulae. The idea is to regard the transfer operator as an integral operator with kernel involving the delta function $\delta(y - f(x))$. Since the trace of an integral operator equals to the integral of the kernel on the diagonal, in the present setting we get a weighted sum of periodic orbits. On the other hand, if the transfer operator is quasi compact, then it can be split as $\mathscr{L} = \mathscr{K} + \mathscr{R}$, where $\mathscr{K}$ has finite rank and the spectral radius of $\mathscr{R}$ is smaller than the essential spectral radius of $\mathscr{L}$ plus an an arbitrary small error. However, while the trace of $\mathscr{K}$ is well understood and is related to the resonances of $\mathscr{L}$, the trace of $\mathscr{R}$ could be infinite. Therefore, a significant work involving additional approximations is required to related the poles of the zeta function (the generating function of orbit counting) to the spectrum of $\mathscr{L}$ (cf. [3, 7, 62]).

Below I discuss some of the spectacular recent advances in this area which rely heavily on the techniques of weighted Banach spaces. I will limit myself to the flow case where the most significant progress have been achieved. Given an Anosov flow g^t and a smooth function ϕ, consider the generalized Ruelle ζ-function

$$\zeta_R^\phi(z) = \prod_\tau (1 - e^{\phi_{\ell(\tau)} - z\ell(\tau)})^{-1} = \exp\left[-\sum_\tau \sum_m \frac{1}{m} e^{m\phi_{\ell(\tau)} - m\ell(\tau)}\right],$$

where the product is over the prime periodic orbits and $\ell(\tau)$ is the period of τ. This includes the classical Ruelle zeta function $\zeta_R(z) = \prod_\tau (1 - e^{-z\ell(\tau)})^{-1}$ corresponding to $\phi = 0$.

It was shown in the work of Fried [39] that if g^t is analytic then $\zeta_R^\phi(z)$ admits a meromorphic extension to the whole complex plane. This result was extended to C^∞ flows in [43] and [32]. For example, [43] utilizes Gouëzel–Liverani type Banach spaces to show that if g^t is C^r then ζ_R^ϕ admits a meromorphic extension to a strip

$$\Re(z) > P(\phi, g^1) - \mathfrak{c}\left[\frac{r-1}{2}\right]$$

where the constant $\mathfrak{c}$ is determined by the Anosov splitting and $[\cdot]$ denotes the integer part. Moreover, poles and zeroes of ζ_R^ϕ have spectral interpretation as the eigenvalues of appropriate transfer operators acting on forms in even and odd dimensions.

Once meromorphic continuation is established, one can ask about the properties of zeta functions and and their relations to the topology of the phase space. In this direction it is shown in [33] that if g^t is a geodesic flow on a negatively curved surface Q then Ruelle zeta function ζ_R vanishes at zero to the order given by the absolute value of the Euler characteristic. Previously such result was only known in constant curvature (see [39]) where one can use the relation between Ruelle and Selberg zeta functions as well as Selberg Trace Formula.

This result implies that for a negatively curved connected oriented Riemannian surface, its length spectrum (that is, lengths of closed geodesics counted with multiplicity) determines its genus. Remarkably, for hyperbolic three manifolds, the order of zero is not a topological invariant. Namely ([22]), this order is equal to $4-2b_1(Q)$ for constant curvature metrics and it is equal to $4-b_1(Q)$ for generic conformal perturbations of such metrics (here $b_1(Q)$ is the first Betti number of Q).

The results mentioned above discuss the behavior of the zeta function near zero. The behavior for large $\Im(z)$ including the numbers of zeroes and poles in growing regions is also of considerable interest. In particular, the case where g^t is the geodesic flow on a unit tangent bundle over a negatively curved manifold Q was discussed in the physics literature, since this system corresponds to the motion of the classical particle on Q and one would like to see how this motion reflects the behavior of the quantum counterpart. A significant progress in this area has been achieved in the work of Faure–Tsujii (see [35]–[38]). Those papers discuss contact Anosov flows (which includes geodesic flows on the unit tangent bundles of negatively curved manifolds). The authors show that the spectra of operators $X+\phi$ (which are infinitesimal analogues of (3.4)) have a band structure and the number of resonances in each band satisfies the (averaged) Weyl Law. The strongest results pertain to the semiclassical zeta function

$$\zeta_Q(z) = \exp\Big[-\sum_\tau \sum_m \frac{e^{-zm\ell(\tau)}}{m\sqrt{|\det(I-D_\tau^m)|}}\Big],$$

where D_τ is the transversal Jacobian matrix along a prime periodic orbit τ (this function plays an important role in quantum chaos, see [54]). [37] shows that there exists $\kappa>0$ such that all zeroes of ζ_Q in the region $\Re(z) > -\kappa$ converge to the imaginary axis. The authors also obtain a constructive estimate for the spectral gap for Liouville measure. We refer the reader to [49] for an excellent introduction to the field of dynamical resonances and the role played by weighted Banach spaces in the recent developments.

We note that the precise description of the Ruelle spectrum is available in only few special cases, including geodesic flows on manifolds of constant negative curvature where one can use Selberg trace formula (see [73]), linear pseudo Anosov maps analyzed in [34], as well as some hyperbolic maps coming from Blaschke products [75, 76].

3.3.3. *Deviations of ergodic averages.* The results of Section 2 provide a precise control of ergodic sums and integrals for hyperbolic systems. In contrast, the behavior of ergodic integrals for parabolic systems where the deviations of nearby orbits is only polynomial is understood only a few cases. One case where a lot of information can be obtained is renormalizable systems. Namely, suppose that a flow h_s and a diffeomorphism f satisfy $f^n h_s = h_{a^n} f^n$ for some $a>1$. Assume further the distributions $\phi=\phi_{x,t}$ given by

$$\langle \phi, A\rangle = \int_0^t A(h_s x)\,ds \tag{3.7}$$

belongs to a suitable weighted Banach space for the renormalizing map f and that correlation functions for observables in that space satisfy the asymptotic expansion (1.5). Since

$$\langle \phi, A\circ f^n\rangle = \frac{1}{a^n}\int_0^{a^n t} A(h_s f^n x)\,ds,$$

choosing $t\in[1,a]$ and n such that $ta^n = T$, we obtain that

$$\int_0^T A(h_s x)\,ds = \mu(A) + \sum_{k=1}^{N_\varepsilon} P_k(\ln T)\,T^{\alpha_k} + O(T^{\beta+\delta}),$$

where $\alpha_k = \frac{\lambda_k}{\ln a}$ and $\beta = \min\left(\frac{\boldsymbol{\rho}}{\ln a}, 1\right)$, where $\boldsymbol{\rho}$ is the essential spectral radius of the associated transfer operator. One can also handle the case where f expands the orbit foliation of $\{h_s\}$ with different rates at different points by considering a weighted transfer operator.

However, in order to get a small essential spectral radius one usually needs to impose a smoothness assumption in the unstable direction which rules out the observables given by (3.7). Instead one can accommodate observables of the form

$$\langle \psi, A\rangle = \int_{-\infty}^{\infty} \Theta(s) A(h_s x)\,ds, \tag{3.8}$$

where Θ is a smooth compactly supported function. A method to overcome this difficulty was recently developed in [42], where the authors decompose the observable (3.7) into a infinite sum of observables satisfying (3.8). It turns out that the resonances λ_k in (1.5) leading to $\alpha_k > 0$ correspond to deviations of ergodic sums while the resonances leading to $\alpha_k \le 0$ are responsible for obstructions for having smooth solutions to the cohomological equation.

This provides a correspondence between the resonances of the renormalizing map and the deviations exponents of the renormalized flow, which opens a fruitful interaction between hyperbolic and parabolic dynamics. A sample of papers exploring this connection includes [8, 17, 40, 41, 34], but this is a very active research area and many new results could be expected soon.

4. CONCLUSION

Sebastien Gouëzel made significant contribution to dynamics by developing new tools to put probabilistic and geometric intuition into a firm analytic framework. His methods already brought spectacular applications to geometry and group theory and they are playing a central role in several current investigations. The present survey provides a very brief overview of some of the ideas behind those developments and I urge the readers to study the original papers for deeper insights.

Acknowledgments. The author thanks Giovanni Forni for careful reading of the manuscript and numerous suggestions for improving the exposition.

REFERENCES

[1] J. Aaronson and D. Terhesiu, Local limit theorems for suspended semiflows, *Discrete Contin. Dyn. Syst.*, **40** (2020), 6575–6609.

[2] J. F. Alves, W. Bahsoun and M. Ruziboev, Almost sure rates of mixing for partially hyperbolic attractors, preprint, arXiv:1904.12844.

[3] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes. I., *Ann. of Math. (2)*, **86** (1967), 374–407.

[4] A. Avila and S. Gouëzel, Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow, *Ann. of Math. (2)*, **178** (2013), 385–442.

[5] A. Avila, S. Gouëzel and M. Tsujii, Smoothness of solenoidal attractors, *Discrete Contin. Dyn. Syst.*, **15** (2006), 21–35.

[6] A. Avila, S. Gouëzel and J.-C. Yoccoz, Exponential mixing for the Teichmüller flow, *Publ. Math Inst. Hautes Études Sci.*, **104** (2006), 143–211.

[7] V. Baladi, Dynamical zeta functions and dynamical determinants for hyperbolic maps. A functional approach, in *Ergebnisse der Mathematik und ihrer Grenzgebiete*, A Series of Modern Surveys in Mathematics, 68, Springer, Cham, 2018.

[8] V. Baladi, There are no deviations for the ergodic averages of the Giulietti-Liverani horocycle flows on the two-torus, preprint, to appear in *Ergodic Theory Dynam. Systems.*

[9] V. Baladi and M. F. Demers, Thermodynamic formalism for dispersing billiards, preprint, arXiv:2009.10936.

[10] V. Baladi, M. F. Demers and C. Liverani, Exponential decay of correlations for finite horizon Sinai billiard flows, *Invent. Math.*, **211** (2018), 39–177.

[11] V. Baladi and S. Gouëzel, Good Banach spaces for piecewise hyperbolic maps via interpolation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009), 1453–1481.

[12] V. Baladi and S. Gouëzel, Banach spaces for piecewise cone-hyperbolic maps, *J. Mod. Dyn.*, **4** (2010), 91–137.

[13] V. Baladi and M. Tsujii, Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms, *Ann. Inst. Fourier (Grenoble)*, **57** (2007), 127–154.

[14] P. Bálint, N. Chernov and D. Dolgopyat, Limit theorems for dispersing billiards with cusps, *Comm. Math. Phys.*, **308** (2011), 479–510.

[15] P. Bálint and S. Gouëzel, Limit theorems in the stadium billiard, *Comm. Math. Phys.*, **263** (2006), 461–512.

[16] P. M. Bleher, Statistical properties of two-dimensional periodic Lorentz gas with infinite horizon, *J. Stat. Phys.*, **66** (1992), 315–373.

[17] O. Butterley and L. D. Simonelli, Parabolic flows renormalized by partially hyperbolic maps, *Boll. Unione Mat. Ital.*, **13** (2020), 341–360.

[18] R. T. Bortolotti, Physical measures for certain partially hyperbolic attractors on 3-manifolds, *Ergodic Theory Dynam. Systems*, **39** (2019), 74–104.

[19] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, 2^{nd} edition, Lecture Notes in Mathematics, 470, Springer-Verlag, Berlin, 2008.

[20] M. Blank, G. Keller and C. Liverani, Ruelle-Perron-Frobenius spectrum for Anosov maps, *Nonlinearity*, **15** (2002), 1905–1973.

[21] R. Castorrini and C. Liverani, Quantitative statistical properties of two-dimensional partially hyperbolic systems, preprint, arXiv:2007.05602.

[22] M. Cekić, S. Dyatlov, B. Kuster and G. P. Paternain, The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds, preprint, arXiv:2009.08558.

[23] J.-R. Chazottes and S. Gouëzel, On almost-sure versions of classical limit theorems for dynamical systems, *Probab. Theory Related Fields*, **138** (2007), 195–234.

[24] P. Collet and S. Isola, On the essential spectrum of the transfer operator for expanding Markov maps, *Comm. Math. Phys.*, **139** (1991), 551–557.

[25] J. Dedecker, S. Gouëzel and F. Merlevède, Large and moderate deviations for bounded functions of slowly mixing Markov chains, *Stoch. Dyn.*, **18** (2018), 1850017, 38 pp.

[26] M. F. Demers and C. Liverani, Stability of statistical properties in two-dimensional piecewise hyperbolic maps, *Trans. Amer. Math. Soc.*, **360** (2008), 4777–4814.
[27] M. F. Demers and H.-K. Zhang, Spectral analysis of the transfer operator for the Lorentz gas, *J. Mod. Dyn.*, **5** (2011), 665–709.
[28] M. F. Demers and H.-K. Zhang, A functional analytic approach to perturbations of the Lorentz gas, *Comm. Math. Phys.*, **324** (2013), 767–830.
[29] M. F. Demers and H.-K. Zhang, Spectral analysis of hyperbolic systems with singularities, *Nonlinearity*, **27** (2014), 379–433.
[30] W. Doeblin and R. Fortet, Sur des chaînes à liaisons complètes, *Bull. Soc. Math. France*, **65** (1937), 132–148.
[31] D. Dolgopyat and P. Nándori, On mixing and the local central limit theorem for hyperbolic flows, *Ergodic Theory Dynam. Systems*, **40** (2020), 142–174.
[32] S. Dyatlov and M. Zworski, Dynamical zeta functions for Anosov flows via microlocal analysis, *Ann. Sci. Éc. Norm. Supér (4)*, **49** (2016), 543–577.
[33] S. Dyatlov and M. Zworski, Ruelle zeta function at zero for surfaces, *Invent. Math.*, **210** (2017), 211–229.
[34] F. Faure, S. Gouëzel and E. Lanneau, Ruelle spectrum of linear pseudo-Anosov maps, *J. Éc. Polytech. Math.*, **6** (2019), 811–877.
[35] F. Faure and M. Tsujii, Band structure of the Ruelle spectrum of contact Anosov flows, *C. R. Math. Acad. Sci. Paris*, **351** (2013), 9–10, 385–391.
[36] F. Faure and M. Tsujii, Resonances for geodesic flows on negatively curved manifolds, *Proceedings of the International Congress of Mathematicians – Seoul 2014*, **3** (2014), 683–697.
[37] F. Faure and M. Tsujii, The semiclassical zeta function for geodesic flows on negatively curved manifolds, *Invent. Math.*, **208** (2017), 851–998.
[38] F. Faure and M. Tsujii, Fractal Weyl law for the Ruelle spectrum of Anosov flows, preprint, arXiv:1706.09307.
[39] D. Fried, Meromorphic zeta functions for analytic flows, *Comm. Math. Phys.*, **174** (1995), 161–190.
[40] G. Forni, Ruelle resonances from cohomological equations, arXiv:2007.03116.
[41] G. Forni, On the equidistribution of unstable curves for pseudo-Anosov diffeomorphisms of compact surfaces, arXiv:2007.03144.
[42] P. Giulietti and C. Liverani, Parabolic dynamics and anisotropic Banach spaces, *J. Eur. Math. Soc. (JEMS)*, **21** (2019), 2793–2858.
[43] P. Giulietti, C. Liverani and M. Pollicott, Anosov flows and dynamical zeta functions, *Ann. of Math. (2)*, **178** (2013), 687–773.
[44] S. Gouëzel, Sharp polynomial estimates for the decay of correlations, *Israel J. Math.*, **139** (2004), 29–65.
[45] S. Gouëzel, Central limit theorem and stable laws for intermittent maps, *Probab. Theory Related Fields*, **128** (2004), 82–122.
[46] S. Gouëzel, Berry-Esseen theorem and local limit theorem for non uniformly expanding maps, *Ann. Inst. H. Poincaré Probab. Statist.*, **41** (2005), 997–1024.
[47] S. Gouëzel, Almost sure invariance principle for dynamical systems by spectral methods, *Ann. Probab.*, **38** (2010), 1639–1671.
[48] S. Gouëzel, Limit theorems in dynamical systems using the spectral method, *Proc. Sympos. Pure Math.*, **89** (2015) 161–193.
[49] S. Gouëzel, Spectre du flot géodésique en courbure négative, *Séminaire Bourbaki, volume 2014/2015 exposés 1089–1103 (exposé 1098)*, **380** (2016), 325–353.
[50] S. Gouëzel and C. Liverani, Banach spaces adapted to Anosov systems, *Ergodic Theory Dynam. Systems*, **26** (2006), 189–217.
[51] S. Gouëzel and C. Liverani, Compact locally maximal hyperbolic sets for smooth maps: Fine statistical properties, *J. Differential Geom.*, **79** (2008), 433–477.

[52] Y. Guivarc'h and J. Hardy, Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov, *Ann. Inst. H. Poincaré Probab. Statist.*, **24** (1988), 73–98.
[53] Y. Guivarc'h and A. Raugi, Products of random matrices: Convergence theorems, *Contemp. Math.*, **50** (1986), 31–54.
[54] M. C. Gutzwiller, Chaos in classical and quantum mechanics, *Interdisciplinary Applied Mathematics*, 1, Springer-Verlag, New York, 1990.
[55] Y. Hafouta, Limit theorems for random non-uniformly expanding or hyperbolic maps, preprint, arXiv:2008.06024.
[56] H. Hennion and L. Hervé, Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness, Lecture Notes in Mathematics, 1766, Springer-Verlag, Berlin, 2001.
[57] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[58] A. Katok, G. Knieper, M. Pollicott and H. Weiss, Differentiability of entropy for Anosov and geodesic flows, *Bull. Amer. Math. Soc. (N. S.)*, **22** (1990), 285–293.
[59] Y. Kifer, Large deviations in dynamical systems and stochastic processes, *Trans. Amer. Math. Soc.*, **321** (1990), 505–524.
[60] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, *Trans. Amer. Math. Soc.* **186** (1973), 481–488.
[61] F. Ledrappier, Three problems solved by Sébastien Gouëzel, *J. Mod. Dyn.*, **16** (2020), 373–387.
[62] C. Liverani, Fredholm determinants, Anosov maps and Ruelle resonances, *Discrete Contin. Dyn. Syst.*, **13** (2005), 1203–1215.
[63] C. Liverani and M. Tsujii, Zeta functions and dynamical systems, *Nonlinearity*, **19** (2006), 2467–2473.
[64] C. Liverani and D. Terhesiu, Mixing for some non-uniformly hyperbolic systems, *Ann. Henri Poincaré*, **17** (2016), 179–226.
[65] J. Machta, Power law decay of correlations in a billiard problem, *J. Stat. Phys.*, **32** (1983), 555–564.
[66] G. A. Margulis, On some aspects of the theory of Anosov systems, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.
[67] I. Melbourne and D. Terhesiu, Operator renewal theory and mixing rates for dynamical systems with infinite measure, *Invent. Math.*, **189** (2012), 61–110.
[68] I. Melbourne and D. Terhesiu, Renewal theorems and mixing for non Markov flows with infinite measure, *Ann. Inst. Henri. Poincaré Probab. Stat.*, **56** (2020), 449–476.
[69] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque*, **187–188** (1990), 268 pp.
[70] M. Pollicott, On the rate of mixing of Axiom A flows, *Invent. Math.*, **81** (1985), 413–426.
[71] D. Ruelle, The thermodynamic formalism for expanding maps, *Comm. Math. Phys.*, **125** (1989), 239–262.
[72] O. Sarig, Subexponential decay of correlations, *Invent. Math.*, **150** (2002), 629–653.
[73] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc. (N. S.)*, **20** (1956), 47–87.
[74] J. G. Sinai, Gibbs measures in ergodic theory, *Uspehi Mat. Nauk*, **27** (1972), 21–64.
[75] J. Slipantschuk, O. F. Bandtlow and W. Just, Spectral structure of transfer operators for expanding circle maps, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **34** (2017), 31–43.
[76] J. Slipantschuk, O. F. Bandtlow and W. Just, Complete spectral data for analytic Anosov maps of the torus, *Nonlinearity*, **30** (2017), 2667–2686.
[77] D. Szasz and T. Varju, Limit laws and recurrence for the planar Lorentz process with infinite horizon, *J. Stat. Phys.*, **129** (2007), 59–80.
[78] M. Tsujii, Quasi-compactness of transfer operators for contact Anosov flows, *Nonlinearity*, **23** (2010), 1495–1545.

[79] M. Tsujii, Exponential mixing for generic volume-preserving Anosov flows in dimension three, *J. Math. Soc. Japan*, **70** (2018), 757–821.

[80] M. Tsujii and Z. Zhang, Smooth mixing Anosov flows in dimension three are exponential mixing, arXiv:2006.04293.

[81] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. of Math. (2)*, **147** (1998), 585–650.

[82] L.-S. Young, Recurrence times and rates of mixing, *Israel J. Math.*, **110** (1999), 153–188.

DMITRY DOLGOPYAT <dmitry@math.umd.edu>: Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA