

THREE PROBLEMS SOLVED BY SÉBASTIEN GOUËZEL

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ABSTRACT. We present three results of Sébastien Gouëzel's: the local limit theorem for random walks on hyperbolic groups, a multiplicative ergodic theorem for non-expansive mappings (joint work with Anders Karlsson), and the description of the essential spectrum of the Laplacian on $SL(2, \mathbb{R})$ orbits in the moduli space (joint work with Artur Avila).

INTRODUCTION

Sébastien Gouëzel was awarded the 2019 Brin prize for his *groundbreaking and influential work* on statistical properties of dynamical systems and random walks. Dima Dolgopyat presents in [7] the main thread of this work. I chose to concentrate on three other examples of groundbreaking and influential contributions from Sébastien Gouëzel. They are the topic of the following papers

1. [11] Local limit theorems for symmetric random walks in Gromov hyperbolic groups, *J. Amer. Math. Soc.*, **27** (2014), 893–928.
2. [13] Subadditive and multiplicative ergodic theorems (with A. Karlsson), *J. Euro. Math. Soc. (JEMS)*, **22** (2020), 1893–1915.
3. [2] Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow (with A. Avila), *Ann. of Math. (2)*, **178** (2013), 385–442.

We discuss these three results in the three sections of this paper.

1. LOCAL LIMIT THEOREMS FOR RANDOM WALKS ON HYPERBOLIC GROUPS

1.1. Statements and background.

THEOREM 1.1 ([11]). *Let Γ be a non-elementary finitely generated Gromov hyperbolic group, μ a symmetric probability measure on Γ . Assume that the support of μ is finite, that it generates Γ as a group and is aperiodic. Consider the n -fold convolution $\mu^{(*n)}$. Then there are a number $R > 1$ and a positive function C on Γ such that, as $n \rightarrow \infty$,*

$$\mu^{(*n)}(x) \sim C(x)R^{-n}n^{-3/2}.$$

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The number R^{-1} is the spectral radius of the Markov operator

$$PF(x) := \sum_{y \in \Gamma} F(y^{-1}x)\mu(y)$$

on $\ell^2(\Gamma)$. By Kesten's classical result, since Γ is non-amenable, $R > 1$. Moreover, by the spectral theory of the operator P , for all $x \in \Gamma$,

$$-\frac{1}{n} \log \mu^{(*n)}(x) \rightarrow \log R.$$

The *Local Limit Theorem 1.1* gives the precise asymptotics of $\mu^{(*n)}(x)$, in particular with the universal term $n^{-3/2}$. One sees directly that the function C is a positive eigenfunction for the Markov operator with eigenvalue R^{-1} . The most general result so far is in [12]: Theorem 1.1 holds even if the measure μ has infinite support, as soon as it satisfies a strong exponential moment condition, namely $\sum_g e^{\alpha|g|}\mu(g) < +\infty$ for all $\alpha > 0$, where $|g|$ is the word length of an element g in some finite system of generators of Γ .

Theorem 1.1 was proven by S. Lalley ([23]) when G is a free group with $d \geq 2$ generators (by [16] when μ is supported by the free generators). The exponent $3/2$ appears in the Local Limit Theorem proved by P. Bougerol ([6]) for rank one semi-simple groups when μ satisfies an isotropy condition (see also [26] for the p -adic case); in that case, C is the Harish-Chandra function. Theorem 1.1 was first proven by S. Gouëzel and S. Lalley ([14]) in the case of surface groups. The strategy in [23], [14] and [11] uses the behavior of the resolvent G_r , $G_r = (\mathbb{I} - rP)^{-1}$ as $r \rightarrow R$. Namely, set, for $r < R$,

$$G_r(x) := \sum_{n \geq 0} r^n \mu^{(*n)}(x) < +\infty.$$

It is known that $G_R(x) := \sum_{n \geq 0} R^n \mu^{(*n)}(x)$ is finite as well.

THEOREM 1.2 ([11]). *Same setting as Theorem 1. Then, there exists a positive function C' on Γ , such that, as $r \nearrow R$,*

$$\frac{dG_r(x)}{dr} \sim \frac{C'(x)}{\sqrt{R-r}}.$$

Theorem 1.1 follows from Theorem 1.2 using Karamata theorem and some a priori regularity of $\mu^{(*n)}(x)$ in n . The symmetry of μ is used in the Tauberian argument. Sébastien Gouëzel conjectures that it is not necessary. See the Appendix of [11] for results in the case of surface groups without assuming μ symmetric.

It can be seen that $\frac{dG_r(x)}{dr}$ is comparable to $(R-r)^{-1/2}$. The point of Theorem 1.2 is that there is a precise equivalent. S. Lalley proved Theorem 1.2 for the free groups by observing that, as functions of r , $G_r(x)$ satisfy algebraic equations and then applying Puiseux Theorem. There is no similar argument for other hyperbolic groups.

1.2. A geometric proof in the free group case. In this section, we present the ingredients of the proofs of Theorem 1.2 in [14] and [11], in the case of a symmetric measure with finite support on the free group \mathbb{F}_d with d generators. We assume that $\bigcup_n \text{support} \mu^{(*n)} = \mathbb{F}_d$.

(a) Assume y is a point on the segment $[e, x]$. Since its steps are bounded, a trajectory from e to x has to pass near y . In particular, there is a constant D such that, uniformly in $r, 1 \leq r \leq R$,

$$G_r(x) \leq DG_r(y)G_r(y^{-1}x).$$

The other inequality follows from the support condition: there is a constant (also denoted D) such that uniformly in $r, 1 \leq r \leq R$,

$$D^{-1}G_r(y)G_r(y^{-1}x) \leq G_r(x).$$

(b) The boundary $\partial\mathbb{F}_d$ of \mathbb{F}_d is the one-sided subshift of finite type Σ_+ with alphabet $\{a_1^{\pm 1}, \dots, a_d^{\pm 1}\}$ and forbidden words $a_i a_i^{-1}, a_i^{-1} a_i, i = 1, \dots, d$. Inequalities in (a) and Ancona theory of the Martin boundary (cf. [5], [20]) yield

PROPOSITION 1.3. *For all $r, 1 \leq r \leq R$, there are functions $K_r(x, \xi)$ on $\mathbb{F}_d \times \partial\mathbb{F}_d$ and $\theta_r(\eta, \xi)$ on $\partial\mathbb{F}_d \times \partial\mathbb{F}_d$ such that, as $y \rightarrow \xi, x \rightarrow \eta$,*

$$\frac{G_r(x^{-1}y)}{G_r(y)} \rightarrow K_r(x, \xi), \quad \frac{G_r(x^{-1}y)}{G_r(x^{-1})G_r(y)} \rightarrow \theta_r(\eta, \xi).$$

Moreover, there is α such that, for any r , the function $\xi \mapsto K_r(x, \xi)$ is α -Hölder continuous on $\partial\mathbb{F}_d$ and the function $(\xi, \eta) \mapsto \theta_r(\eta, \xi)$ is α -Hölder continuous outside of the diagonal of $\partial\mathbb{F}_d \times \partial\mathbb{F}_d$. The functions $r \mapsto K_r$ and $r \mapsto \theta_r$ are continuous into the spaces of Hölder continuous functions.

(c) We identify the two-sided shift Σ with alphabet $\{a_1^{\pm 1}, \dots, a_d^{\pm 1}\}$ and forbidden words $a_i a_i^{-1}, a_i^{-1} a_i, i = 1, \dots, d$ with a subset of the product $\partial\mathbb{F}_d \times \partial\mathbb{F}_d$ by defining $\pi(\xi)$, for $\underline{\xi} = \{\xi_n, n \in \mathbb{Z}\}$, as $\pi(\underline{\xi}) := (\lim_{m \rightarrow \infty} \xi_{-1}^{-1} \cdots \xi_m^{-1}, \lim_{n \rightarrow \infty} \xi_0 \cdots \xi_n)$.

For $\underline{\xi} = \{\xi_n, n \in \mathbb{Z}\} \in \Sigma$, set $\underline{\xi}_+ = \xi_0, \xi_1, \dots \in \Sigma_+$, let $\varphi_r(\underline{\xi}) := -2\log K_r(\xi_0, \underline{\xi}_+)$ and let $P(r)$ be the pressure of the function φ_r :

$$P(r) := \max_m \left\{ h_m(\sigma) + \int \varphi_r dm \right\},$$

where m varies over all shift invariant probability measures on Σ and $h_m(\sigma)$ denotes the measure entropy of m for the shift σ . Let m_r be the unique invariant probability measure that realizes the maximum.

For two sequences of positive numbers $\{a_n, b_n\}$, write

$$a_n \underset{n \rightarrow \infty}{\sim} b_n \text{ if } \lim_{n \rightarrow \infty} a_n/b_n = 1.$$

Thermodynamical formalism (see, e.g., [27]) yields a Hölder continuous function h_r on Σ_+ such that, for a fixed z of word length n , $z = g_1 \cdots g_n$, if $g_n \neq \xi_0$,

uniformly on r and $\underline{\xi}$,

$$\begin{aligned} m_r \left[\{\underline{\xi}, \underline{\xi}_{-n} \cdots \underline{\xi}_{-1} = z\} \mid \underline{\xi}_+ \right] &= e^{-nP(r)} K_r^2(z^{-1}, \underline{\xi}_+) \frac{h_r(g_1^{-1} \cdots g_n^{-1}, \xi_0, \xi_1, \dots)}{h_r(\underline{\xi}_+)} \\ &\underset{n \rightarrow \infty}{\sim} e^{-nP(r)} G_r^2(z) \theta_r^2(\eta, \underline{\xi}_+) \frac{h_r(g_1^{-1} \cdots g_n^{-1}, \xi_0, \xi_1, \dots)}{h_r(\underline{\xi}_+)}, \end{aligned}$$

where η is any point in Σ_+ beginning as z^{-1} . Uniformity on r comes from the continuity of $r \mapsto \varphi_r$ (respectively $r \mapsto \theta_r$) in the space of Hölder continuous functions on Σ_+ (respectively Σ) (Step (b)). So, we get, uniformly on r ,

$$e^{-nP(r)} G_r^2(z) \underset{n \rightarrow \infty}{\sim} \int_{\{\underline{\xi}, \underline{\xi}_{-n} \cdots \underline{\xi}_{-1} = z\}} \theta_r^{-2} \left(\pi(\sigma^{-n} \underline{\xi}) \right) \frac{h_r(\underline{\xi}_+)}{h_r((\sigma^{-n} \underline{\xi})_+)} dm_r(\underline{\xi}).$$

PROPOSITION 1.4. *For $1 \leq r \leq R$, there is a positive constant C_r such that*

$$(1.1) \quad e^{-nP(r)} \sum_{z, |z|=n} G_r^2(z) \underset{n \rightarrow \infty}{\sim} C_r.$$

Moreover, the mapping $r \mapsto C_r$ is continuous on $[1, R]$.

Proof. By the above computation, $e^{-nP(r)} \sum_{z, |z|=n} G_r^2(z)$ has the same limit, as $n \rightarrow \infty$, as $\int_{\Sigma} \theta_r^{-2} \left(\pi(\sigma^{-n} \underline{\xi}) \right) \frac{h_r(\underline{\xi}_+)}{h_r((\sigma^{-n} \underline{\xi})_+)} dm_r(\underline{\xi})$. Assume that h_r is normalised so that $\int h_r dm_r = 1$. Since the measure m_r is shift-mixing, the limit is

$$C_r := \int_{\Sigma} \theta_r^{-2} \circ \pi h_r^{-1} dm_r.$$

The continuity and the positivity follow from the corresponding properties of θ_r and h_r , the continuity of $r \mapsto m_r$ and the uniformity on r of the limit as $n \rightarrow \infty$. \square

(d) The following observation is easy to prove and very general:

PROPOSITION 1.5 ([14], Proposition 1.9). *For any $r, 1 \leq r < R$, any $x, y \in G$, we have*

$$(1.2) \quad \frac{d}{dr} G_r(x^{-1} y) = \frac{1}{r} \sum_z G_r(x^{-1} z) G_r(z^{-1} y) - \frac{1}{r} G_r(x^{-1} y).$$

So, the quantity of interest in Theorem 1.2 is

$$\frac{d}{dr} G_r(e) = \frac{1}{r} \sum_z G_r^2(z) - \frac{1}{r} G_r(e).$$

COROLLARY 1.6. *For $1 \leq r < R$, $P(r) < 0$. $P(R) \leq 0$.*

Proof. Set $F(r) := \sum_z G_r^2(z)$. By (1.2), $F(r) < +\infty$ for $r < R$. Reporting in (1.1), this is possible only if $P(r) < 0$. $P(R) \leq 0$ follows by continuity. \square

Another consequence of (1.2) is the following

$$(1.3) \quad \frac{d}{dr} F(r) = \frac{2}{r} \sum_{z, y} G_r(y) G_r(y^{-1} z) G_r(z) - \frac{2}{r} F(r).$$

(e) $P(R) = 0$. Indeed, we know that $P(R) \leq 0$ by Corollary 1.6. By (1.2), if $P(R) < 0$, there is a $\delta > 0$ such that $\sum_z e^{\delta|z|} G_r^2(z) < +\infty$. From this, it follows that there exists $\varepsilon > 0$ such that $G_{R+\varepsilon}(e) < +\infty$, a contradiction with the definition of R . A clever qualitative way of showing $G_{R+\varepsilon}(e) < +\infty$ is presented in [14], based on the interpretation of G_r as the Green function of the branching random walk with distribution of steps given by μ and distribution of the number of offsprings given by a Poisson law with parameter r .

COROLLARY 1.7. $\lim_{r \rightarrow R} -P(r) \frac{d}{dr} G_r(e) = \frac{C_R}{R}$. In particular, $\frac{d}{dr} G_r(e) \rightarrow +\infty$ as $r \rightarrow R$.

Proof. By (1.2), it suffices to show that $\lim_{r \rightarrow R} -P(r)F(r) = C_R$. We have, for n large and r close to R ,

$$\begin{aligned} -P(r) \sum_z G_r^2(z) &= -P(r) \sum_n e^{nP(r)} \sum_{z:|z|=n} e^{-nP(r)} G_r^2(z) \\ &\underset{n \rightarrow \infty}{\sim} -P(r) \sum_n e^{nP(r)} C_r \rightarrow C_R. \end{aligned} \quad \square$$

(f) The final step goes through a further derivative estimate.

PROPOSITION 1.8. *There is a positive number C such that*

$$\lim_{r \rightarrow R} -P(r)^3 \frac{d}{dr} F(r) = C.$$

Proof. $\frac{d}{dr} F(r)$ is given by (1.3). By Corollary 1.7, as $r \rightarrow R$, the second term goes to 0 when multiplied by $P(r)^3$. We have to consider

$$\lim_{r \rightarrow R} -P(r)^3 \sum_{z,y} G_r(y) G_r(y^{-1}z) G_r(z) = \lim_{r \rightarrow R} -P(r)^3 \sum_w \sum_{z,y:z \wedge y=w} G_r(y) G_r(y^{-1}z) G_r(z),$$

where $z \wedge y$ is the initial common part of the shortest writing of z and y . We write:

$$\begin{aligned} G_r(y) &= G_r(w) G_r(w^{-1}y) \frac{G_r(y)}{G_r(w) G_r(w^{-1}y)}, \\ G_r(z) &= G_r(w) G_r(w^{-1}z) \frac{G_r(z)}{G_r(w) G_r(w^{-1}z)} \end{aligned}$$

and

$$G_r(y^{-1}z) = G_r(y^{-1}w) G_r(w^{-1}z) \frac{G_r(y^{-1}z)}{G_r(y^{-1}w) G_r(w^{-1}z)},$$

so that we have to consider

$$\lim_{r \rightarrow R} -P(r)^3 \sum_w \sum_{z,y:z \wedge y=w} G_r^2(w) G_r^2(w^{-1}z) G_r^2(w^{-1}y) \phi(w, y, z),$$

where $\phi(w, y, z)$ is bounded and converges to a Hölder continuous function on the set of points $(\zeta, \eta, \xi) \in \partial F_d \times \partial F_d \times \partial F_d$ with pairwise distinct first letters when $w^{-1} \rightarrow \zeta$, $w^{-1}y \rightarrow \eta$ and $w^{-1}z \rightarrow \xi$. It follows from Corollary 1.7 that the terms with $|w|, |w^{-1}y|$ or $|w^{-1}yz|$ bounded do not contribute to the limit.

The proof of Proposition 1.8 goes through a similar computation to the one in Step (c) and in Corollary 1.7, done successively with respect to $z' := w^{-1}z$, $y' = w^{-1}y$ and w .¹ \square

We can now prove Theorem 1.2. The function $F(r) = r \frac{d}{dr} G_r(e) + G_r(e)$ satisfies

$$\lim_{r \rightarrow R} -P(r)F(r) = C_R, \quad \lim_{r \rightarrow R} -P(r)^3 \frac{d}{dr} F(r) = C \quad \text{and} \quad \lim_{r \rightarrow R} F(r) = +\infty.$$

It follows that $\frac{d}{dr}(1/F^2(r))$ converges as $r \rightarrow R$ to a positive constant. Since $\lim_{r \rightarrow R} F(r) = +\infty$, Theorem 1.2 follows for $x = e$. The general case for $\frac{dG_r(x)}{dr}$ is proven in the same way. The careful reader can follow the proofs and find an expression for the function $C'(x)$.

1.3. The hyperbolic case. The proof in the general case follows the above scheme. The most important step is the analog of (a).

THEOREM 1.9. *Let μ be a symmetric probability measure with finite support on a finitely generated hyperbolic group Γ . Assume the the group generated by the support of μ is the whole Γ . Consider y is a point on the segment $[e, x]$. There is a constant D such that, uniformly in $r, 1 \leq r \leq R$,*

$$G_r(x) \leq D G_r(y) G_r(y^{-1}x).$$

For a fixed $r < R$, Theorem 1.9 is the central result of [4], but the constant D obtained in the proof depends on r and goes to infinity as $r \rightarrow R$. The new uniform argument considers first a superexponential estimate as follows.

For $\mathcal{D} \subset G, x, y \in \mathcal{D}$, write $G_r(x, y : \mathcal{D})$ for the r -Green function of the random walk with the constraint that the trajectories stay in \mathcal{D} at all times. Then

LEMMA 1.10. *There exist $n_0 > 0$ and $\varepsilon > 0$ such that for $n \geq n_0$, whenever x, y, z are three points in that order on a geodesic segment with $d(x, y), d(x, z) \geq 10n$,*

$$G_R(x, z : (B(y, n))^c) \leq 2^{-e^{\varepsilon n}}.$$

Theorem 1.9 follows from Lemma 1.10 by a nested induction. The proof of Lemma 1.10 uses a family of $N := e^{\varepsilon n}$ barriers that a trajectory that goes from x to z avoiding $B(y, n)$ should cross in succession. By hyperbolicity, it is possible to choose the barriers far enough from one another that each barrier makes the Green function outside of $B(y, n)$ drop by a factor $1/2$. The computation of this drop is quite clever. Among others, a trick is to make the barriers slightly random to be able to choose a suitable family. Another difference with the free group case is that one has to prove the a priori estimate $\sup_n \sum_{z, |z|=n} G_R^2(z) < +\infty$ which is already used in the proof of Lemma 1.10.

Step (b) is classical in hyperbolic groups (see e.g. [18]). An important feature is the reinforcement of Theorem 1.9 to a qualitative approximation of the Cayley graph of Γ by trees. Say the pairs of pairs of points $\{(x, x'), (y, y')\}$ are n -apart if there exists a geodesic segment of length n common to $[x, y], [x, y'], [x', y]$ and

¹The detailed proof uses uniform 3-mixing of the measures m_r , for $1 \leq r \leq R$.

$[x', y']$. Under our hypotheses, there exist constants $C > 0$ and $\rho > 0$ such that, if the pairs of pairs of points $\{(x, x'), (y, y')\}$ are n -apart, then, for any $1 \leq r \leq R$,

$$\left| \frac{G_r(x^{-1}y)/G_r(x'^{-1}y)}{G_r(x^{-1}y')/G_r(x'^{-1}y')} - 1 \right| \leq Ce^{-\rho n}.$$

Step (c) presents another challenge: a priori, there is no mixing Markov coding of the boundary or of the group. If one wants to imitate what was done for the free group, the Markov codings one can use are not even transitive. The computation of Step c) hold for each transitive component of the Markov coding (with a possible complication due to non-mixing). Special care has to be taken in putting together these estimates for proving the analog of Proposition 1.4.

Step (d) is the same.

Steps (e) and (f) follow the same scheme as in the free group case. Again, the limits in the proof of Proposition 1.8 are taken on each transitive component of the Markov coding and have to be compared.

2. SUBADDITIVE AND MULTIPLICATIVE THEOREMS

The celebrated Oseledets theorem can be extended to saying that the composition of random isometries of a symmetric space with nonpositive curvature sublinearly follow a geodesic (see [19]). The question arises of a similar result for the composition of random contractions of a metric space. An extension has been proven by A. Karlsson and G. Margulis ([22]). A further extension, the most general statement today, is the topic of the paper [13]. Remarkably, these extensions rest on refinements of Kingman subadditive ergodic theorem. In this section, we review these three pairs of results. In all the section, $(\Omega, \mathcal{A}, m, T)$ is an ergodic measurable dynamical system; an integrable *real subadditive cocycle* is a family of measurable functions $a(n, \omega)_{n \in \mathbb{N}}$ such that

$$a(n + n', \omega) \leq a(n, \omega) + a(T^n \omega, n') \quad \text{and} \quad \int a^+(1, \cdot) dm < +\infty.$$

2.1. Oseledets and Kingman ergodic theorems.

THEOREM 2.1 (Kingman Subadditive Ergodic Theorem [21]). *Let $a(n, \omega)_{n \in \mathbb{N}}$ be an integrable real subadditive cocycle. Then, for m -a.e. ω , as $n \rightarrow \infty$,*

$$\frac{1}{n} a(n, \omega) \rightarrow \inf_n \frac{1}{n} \int a(n, \omega) dm.$$

Let A be a measurable mapping $A: \Omega \rightarrow GL(d, \mathbb{R})$ such that

$$\int \log^+ \|A(\omega)\| dm(\omega) < +\infty$$

and form

$$A(n, \omega) := A(\omega) \circ A(T\omega) \circ \dots \circ A(T^{n-1}\omega).$$

Let $GL(d, \mathbb{R})$ act on the space X of symmetric matrices by $A(S) = A^* S A$. Then, for d the natural invariant metric, $a(n, \omega) := d(\mathbb{I}, A(n, \omega)(\mathbb{I}))$ is an integrable real subadditive cocycle and set $\alpha := \lim_{n \rightarrow \infty} \frac{1}{n} d(\mathbb{I}, A(n, \omega)(\mathbb{I}))$. Then,

PROPOSITION 2.2 ([19]). *Assume $\alpha > 0$. Then, for a.e. ω , any $y \in X$, there is a unique geodesic γ_ω in X such that*

$$\gamma_\omega(0) = y \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} d(A(n, \omega)(y), \gamma_\omega(n\alpha)) = 0.$$

The proof in [19] examines what it means for a sequence of symmetric matrices $S_n, n \in N$ that $\lim_n \frac{1}{n} d(S_n, \gamma(n\alpha)) = 0$ for some geodesic γ and that if $\alpha > 0$ and $\gamma(0)$ is given, this geodesic is unique. Then, Kingman subadditive ergodic theorem applied to the integrable real subadditive cocycles $\log \|\wedge^p A(n, \omega)\|, p = 1, \dots, d$ implies that for all $y \in S$, almost all $\omega \in \Omega$, the sequence $A(n, \omega)^* y A(n, \omega)$ follows a geodesic.

THEOREM 2.3 (Oseledets Multiplicative Ergodic Theorem [25]). *Let A be a measurable mapping $A : \Omega \rightarrow GL(d, \mathbb{R})$ such that $\int \log^+ \|A(\omega)\| dm(\omega) < +\infty$. With the above notations, for a.e. $\omega \in \Omega$, there exists a symmetric matrix $\Lambda(\omega)$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \| (A(n, \omega)) \Lambda(\omega)^{-n} \| = 0.$$

Indeed, a geodesic in the space of symmetric matrices is of the form $\gamma_\omega(t) = \exp(tH_\omega)$, where H_ω is a nonzero symmetric matrix. We can set $\Lambda(\omega) = \exp(\alpha H_\omega)$. The Lyapunov exponents $\lambda_1 \geq \dots \geq \lambda_d$ are the eigenvalues (with multiplicities) of the matrix $\Lambda(\omega)$, the Oseledets spaces $V_1(\omega) \supset \dots \supset V_k(\omega)$ the spaces generated by the eigenspaces of $\Lambda(\omega)$ with decreasing maximum eigenvalue.

2.2. Karlsson-Margulis.

THEOREM 2.4 ([22]). *Let $a(n, \omega)$ be an integrable subadditive cocycle and assume that*

$$\alpha := \inf_n \frac{1}{n} \int a(n, \omega) dm > -\infty.$$

Then, for a.e. ω , every $\varepsilon > 0$, there exist $K(\omega, \varepsilon)$ and infinitely many n such that for all $k, K \leq k \leq n$,

$$a(n, \omega) - a(n - k, T^k \omega) \geq (\alpha - \varepsilon)k.$$

Observe that Theorem 2.4 implies Kingman Theorem 2.1: firstly, by subadditivity, for all $J \in \mathbb{N}$,

$$a(nJ, \omega) \leq \frac{1}{J} \left(\sum_{j=1}^J a^+(j, \omega) + \sum_{k=0}^{n-1} a(J, T^{kj+j} \omega) + \sum_{j=1}^J a^+(J-j, T^{j+(N-1)J} \omega) \right).$$

By the ergodic theorem applied to T^J , it follows that, for a.e. ω ,

$$\limsup_n \frac{1}{n} a(n, \omega) \leq \frac{1}{J} \int a(J, \omega) dm(\omega)$$

and so $\limsup_n \frac{1}{n} a(n, \omega) \leq \alpha$ m -a.e.. This proves Theorem 2.1 if $\alpha = -\infty$.

In the case when $\alpha > -\infty$, Theorem 2.4 and subadditivity imply that for all $\varepsilon > 0$, almost all ω , there is $K(\omega, \varepsilon)$ such that, for all $k \geq K(\omega, \varepsilon)$, $a(k, \omega) \geq (\alpha - \varepsilon)k$. Theorem 2.1 follows by letting $\varepsilon \rightarrow 0$.

Observe that the difference between the two Theorems lies exactly in this last statement: not only $a(k, \omega) \geq (\alpha - \varepsilon)k$ for k large enough, but this is also true for all k large enough for the corresponding increments $a(n, \omega) - a(n - k, T^k \omega)$ at special times n , similar to the *hyperbolic times* of the additive ergodic theorem (see, e.g., [1]).

Theorem 2.4 allows A. Karlsson and G. Margulis to prove a generalization of Proposition 2.2 to random products of contractions of spaces satisfying some geometric conditions, namely uniformly convex, complete metric spaces satisfying the Busemann non-positive curvature condition (see [22] for the precise definitions); $CAT(0)$ spaces, uniformly convex Banach spaces such as $L^p, 1 < p < +\infty$ satisfy these conditions. Let (Y, d) be a uniformly convex, complete metric space satisfying the Busemann non-positive curvature condition. A non-expanding mapping $A : Y \rightarrow Y$ is a mapping satisfying $d(Ay, Ay') \leq d(y, y')$ for all $y, y' \in Y$. The set of non-expanding mappings is equipped with the Borel structure associated to the compact-open topology. For $A(\omega)$ a random non-expanding mapping, set

$$A(n, \omega) := A(\omega) \circ A(T\omega) \circ \cdots \circ A(T^{n-1}\omega).$$

THEOREM 2.5 ([22]). *Let $A(\omega)$ a random non-expanding mapping of a uniformly convex, complete metric space satisfying the Busemann non-positive curvature condition. Assume for some $y \in Y$, $\int d(y, A(\omega)y) dm(\omega) < +\infty$ and*

$$\alpha := \lim_{n \rightarrow \infty} \frac{1}{n} d(y, A(n, \omega)y)$$

is positive. Then, for a.e. ω , there exists a unique geodesic ray γ_ω in Y starting at y and such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} d(A(n, \omega)(y), \gamma_\omega(n\alpha)) = 0.$$

2.3. Gouëzel-Karlsson.

THEOREM 2.6 ([13]). *Let $a(n, \omega)$ be an integrable subadditive cocycle and assume that*

$$\alpha := \inf_n \frac{1}{n} \int a(n, \omega) dm > -\infty.$$

Then, for a.e. ω , there exist $n_i := n_i(\omega) \rightarrow \infty$ and positive real numbers $\delta_\ell := \delta_\ell(\omega) \rightarrow 0$ such that for all i and all $\ell \leq n_i$,

$$|a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) - \ell\alpha| \leq \ell\delta_\ell(\omega).$$

Moreover, on a set of large measure, one can take δ_ℓ uniform and have a large density of good times n_i .

Saying that $a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) \leq \alpha\ell + \ell\delta'_\ell(\omega)$, with $\delta'_\ell \rightarrow 0$, is a direct consequence of Theorem 2.1, since $a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) \leq a(\ell, \omega)$. The other inequality yields that the cocycle is almost additive in the sense that there are (many) good times $n_i(\omega) \rightarrow \infty$ such that, for all $\ell \leq n_i$,

$$a(n_i - \ell, T^\ell \omega) + a(\ell, \omega) - a(n_i, \omega) \leq \ell(\delta_\ell + \delta'_\ell).$$

The information from Theorem 2.6 is more precise than the one from Theorem 2.4. Indeed, let ω be such that Theorems 2.6 holds, with the associated $\delta_\ell(\omega)$ and $n_i(\omega)$. Given $\varepsilon > 0$, choose $K(\omega, \varepsilon)$ large enough that $\delta_\ell(\omega) \leq \varepsilon$ for $\ell \geq K(\omega, \varepsilon)$. Then, for all ℓ , $K(\omega, \varepsilon) \leq \ell \leq n_i(\omega)$,

$$a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) - \ell \alpha \geq -\ell \delta_\ell(\omega) \geq -\ell \varepsilon.$$

For the proof of Theorem 2.6, we may assume that T is invertible. One then sets

$$b(n, \omega) := a(n, T^{-n} \omega).$$

The family $\{b(n, \omega)\}$ is an integrable real cocycle for the transformation $\tau := T^{-1}$, with the same a.e. limit α . One key lemma is the following

LEMMA 2.7. *Let $b(n, \omega)$ be an integrable subadditive cocycle with*

$$\alpha := \inf_n \frac{1}{n} \int b(n, \omega) dm$$

finite. Let $\varepsilon > 0, \delta > 0$. There exists $k \geq 1$ such that, for a.e. ω , the set V has superior density in \mathbb{N} at most δ , where

$$V := \{n \in \mathbb{N}; b(n, \omega) - b(n - \ell, \omega) \leq (\alpha - \varepsilon)\ell \text{ for some } \ell \in [k, n]\}.$$

Fix $\delta > 0$. There exists $c > 0$ such that the set V' has superior density in \mathbb{N} at most δ , where

$$V' := \{n \in \mathbb{N}; b(n, \omega) - b(n - \ell, \omega) \leq (\alpha - c)\ell \text{ for some } \ell \in [1, n]\}.$$

The proofs of both Lemma 2.7 and Theorem 2.6 use subtle combinatorics along the orbits.²

The extension of Theorem 2.5 concerns random non-expanding mappings of a general metric space (X, d) . The result will be weaker, following a geodesic being replaced by having the right growth rate along a *metric functional*. Fix $x_0 \in X$ and for $x \in X$, define a function h_x on X by

$$h_x(y) := d(x, y) - d(x, x_0).$$

DEFINITION 2.8. A metric functional is a pointwise limit of functions h_x .

Limits exist since the set of 1-Lipschitz functions is compact in \mathbb{R}^X endowed with the product topology. Moreover, metric functionals are 1-Lipschitz as well. Metric functionals are called *horofunctions* if X is proper and the convergence is uniform on compact sets, *Busemann functions* if X is a geodesic space and x goes to infinity along a geodesic.

THEOREM 2.9 ([13]). *Let $A(\omega)$ a random non-expanding mapping of a metric space (X, d) . Assume for some $x \in X$, $\int d(x, A(\omega)x) dm(\omega) < +\infty$ and set $\alpha := \lim_{n \rightarrow \infty} \frac{1}{n} d(x, A(n, \omega)x)$. Then, for a.e. ω , there exists a metric functional h_ω such that, for all $x \in X$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} h_\omega(A(n, \omega)(x)) = -\alpha.$$

² One could mention that these proofs have been formalized and verified by a computer. S. Gouëzel is promoting these formal validations, see the discussion in the introduction of [15].

Proof. By Theorem 2.6, there is, for a.e. ω , $n_i := n_i(\omega) \rightarrow \infty$ and positive real numbers $\delta_\ell := \delta_\ell(\omega) \rightarrow 0$ such that for all i and all $\ell \leq n_i$,

$$d(A(n_i, \omega)x_0, x_0) - d(A(n_i - \ell, T^\ell \omega)x_0, x_0) \geq \ell(\alpha - \delta_\ell(\omega)).$$

Consider such an ω . Write h_i for $h_i(y) := d(A(n_i, \omega)x_0, y) - d(A(n_i, \omega)x_0, x_0)$. So, for all i and all $\ell \leq n_i$,

$$\begin{aligned} h_i(A(\ell, \omega)x_0)) &= d(A(n_i, \omega)x_0, A(\ell, \omega)x_0) - d(A(n_i, \omega)x_0, x_0) \\ &\leq d(A(n_i - \ell, T^\ell \omega)x_0, x_0) - d(A(n_i, \omega)x_0, x_0) \leq -(\alpha - \delta_\ell)\ell. \end{aligned}$$

By compactness, there is a metric functional h_ω such that, for all ℓ ,

$$h_\omega(A(\ell, \omega)x_0)) \leq -(\alpha - \delta_\ell)\ell$$

and therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} h_\omega(A(n, \omega)(x_0)) \leq -\alpha.$$

By the 1-Lipschitz property of h_ω , $\liminf_{n \rightarrow \infty} \frac{1}{n} h_\omega(A(n, \omega)(x_0)) \geq -\alpha$ and thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} h_\omega(A(n, \omega)(x_0)) = -\alpha.$$

The property follows for all $x \in X$ by the 1-Lipschitz property of h_ω . \square

In the case of Banach spaces, this gives

COROLLARY 2.10. *Let $A(\omega)$ a random non-expanding mapping of a Banach space X . Assume for some $y \in X$, $\int \|A(\omega)y\| dm(\omega) < +\infty$. Then, for a.e. ω , there exists a linear functional f_ω of norm 1 such that, for all x ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} f_\omega(A(n, \omega)(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \|A(n, \omega)(x)\|.$$

3. EIGENVALUES OF THE LAPLACIAN

3.1. Statement and background. Let S be a surface of genus g , $\Sigma = \{\sigma_1, \dots, \sigma_j\}$ a set of punctures with multiplicities $K := \{\kappa_1, \dots, \kappa_j\}$ such that $\sum(\kappa_i - 1) = 2g - 2$, $\text{Teich}(S, \Sigma, K)$ (resp. $\text{Teich}_1(S, \Sigma, K)$) the corresponding space of abelian differentials (resp. of area 1), Γ the mapping class group, X the quotient space $\text{Teich}_1(S, \Sigma, K)/\Gamma$. The Teichmüller spaces are endowed with a natural affine structure given by the periods of the abelian differentials. There is a natural action of $SL(2, \mathbb{R})$ on X and it defines a foliation of X into orbits. On the quotient by $SO(2, \mathbb{R})$, this action defines a foliation of $SO(2, \mathbb{R}) \backslash X$ into (possibly quotients of) hyperbolic planes. Let μ be a probability measure on X that is invariant ergodic under the action of $SL(2, \mathbb{R})$. The foliated Laplacian Δ defines an essentially self adjoint operator on $L_0^2(SO(2, \mathbb{R}) \backslash X, \mu)$. The result of [2] concerns the spectrum Σ_μ of this operator.

THEOREM 3.1 ([2]). *With the above notations, for any $\delta > 0$, $\Sigma_\mu \cap (0, 1/4 - \delta)$ is made of finitely many eigenvalues of finite multiplicities.*

In other words, Theorem 3.1 says that *the essential spectrum of Δ on $L_0^2(SO(2, \mathbb{R}) \backslash X, \mu)$ is contained in $[1/4, +\infty)$.* For the Masur-Veech Lebesgue measure on $SO(2, \mathbb{R}) \backslash X$, it was known that 0 is isolated in the spectrum ([3]).³ In [2], Theorem 3.1 was shown under the hypothesis that μ is related to the affine structure of X . A famous result of A. Eskin and M. Mirzakhani ([9]) shows that all $SL(2, \mathbb{R})$ -invariant ergodic probability measures on X satisfy these algebraic conditions. In particular:

- μ is invariant by the diagonal Teichmüller flow $\varphi_t, t \in \mathbb{R}$,
- μ is locally the product of the Lebesgue measure on the orbits of the Teichmüller flow and the Lebesgue measures on affine subspaces of its stable and unstable manifolds,
- the derivative of the restriction of the Teichmüller flow to (un)stable manifolds is locally constant.

One can distinguish three steps in the proof:

- Control the essential spectrum of some resolvent of the Teichmüller flow on some space of distributions (à la Gouëzel-Liverani, see D. Dolgopyat's review [7]).
- Relate the eigenvalues of that resolvent to the poles of a meromorphic extension of the Laplace transforms of the correlation functions.
- Do a *reverse Ratner argument*: describe the representations of $SL(2, \mathbb{R})$ compatible with these properties of the correlation functions and read the eigenvalues on the corresponding representations.

3.2. Spectral gap; exponential mixing of the Teichmüller flow. Let D be the space of C^∞ functions on X with compact support and, for $\delta > 0$, let \mathcal{L} be the operator

$$\mathcal{L}f := \int_0^\infty e^{-4\delta t} f \circ \varphi_t dt.$$

THEOREM 3.2 ([2]). *There exists two norms $\|\cdot\|, \|\cdot\|'$ on D such that*

- *there is C s. t., for $f \in D, t \geq 0$, $\|f \circ \varphi_t\| \leq C \|f\|$,*
- *the unit ball in the $\|\cdot\|$ completion is relatively compact in the $\|\cdot\|'$ norm and*
- *there is a Doeblin-Fortet inequality: there exists n such that*

$$\|\mathcal{L}^n f\| \leq (1 + \delta)^n \|f\| + \|f\|'.$$

It follows from Theorem 3.2 that the essential spectral radius of \mathcal{L} on the $\|\cdot\|$ closure of D is smaller than $(1 + \delta)$ ([17]). Observe that $\mathcal{L}1 = 1/4\delta$, so the spectral gap between the spectral radius and the essential spectral radius is huge.

The proof of Theorem 3.2 is the more involved part of the paper. It follows the scheme of the proof of exponential mixing for contact Anosov flows ([24]). The affine structure brings some simplifications, since the stable and unstable foliations are smooth, flat and the expansion is locally constant, but one has to control excursions of the flow outside of compact regions. This was already

³[3] was part of the citation of A. Avila for the 2011 Brin prize; see [10] for details.

done to get a spectral gap in [3] by using a quantitative recurrence estimate of Eskin-Masur ([8]) and introducing a special Finsler metric. To get a big spectral gap, one knows from [24] that the special norms to be constructed have to take into account a finite large number of derivatives of functions in D . Putting everything together is quite convoluted.

By Theorem 3.2, we know that the operator \mathcal{L} has a finite number $\lambda_i, i = 1, \dots, \lambda_\ell$, of eigenvalues of modulus larger than $(1 + \delta)$ and that they all have finite multiplicity. We set $\Pi_j, j = 1 \dots \ell$, for the finite dimensional associated projectors.

3.3. End of the proof: correlation functions and reverse Ratner argument. Let $f_1, f_2 \in D$ and μ a $SL(2, \mathbb{R})$ -invariant ergodic probability measure on X . The Laplace transform of the correlation function

$$F(z) := \int_0^\infty e^{-zt} \left(\int_X f_1 \cdot f_2 \circ \varphi_t d\mu \right) dt$$

is holomorphic on $\Re z > 0$.

PROPOSITION 3.3. *The function $F(z)$ has a meromorphic extension to the union of $\Re z > 0$ and a neighborhood of size δ of the segment $[-1 + 8\delta, 0]$. Its poles are at $4\delta - 1/\lambda_i, i = 1, \dots, \ell$. Moreover, the residue at the pole $4\delta - 1/\lambda_i$ is $\int_X f_1 \Pi_i f_2 d\mu$.*

Sketch of proof. Set $G(z) = \int_X f_1 S(z) f_2 d\mu$, where

$$S(z)(f) := \frac{1}{4\delta - z} \mathcal{L} \left(\frac{1}{4\delta - z} - \mathcal{L} \right)^{-1} (f).$$

Since μ is a $SL(2, \mathbb{R})$ -invariant ergodic probability measure, Theorem 3.2 applies and $G(z)$ has the properties we want. Formally, $G(z) = F(z)$. \square

Consider now the decomposition $\int^\oplus \mathcal{H}_\xi dm(\xi)$ into irreducible representations of the action of $SL(2, \mathbb{R})$ on the Hilbert space $L_0^2(X, \mu)$. We are interested in the part of the spectrum of the operator Δ on $SO(2, \mathbb{R})$ -invariant function that intersects $(0, 1/4)$. By direct examination, it can only come from the complementary series elements of the decomposition and correspond to Dirac components of the measure m supported by these complementary series representations. On the other hand, poles of the Laplace transform of the correlation function also come from the complementary series part of the decomposition, and any eigenvalue (with multiplicity m_i) of Δ in $(0, 1/4)$ corresponds to a pole of F (with order at least m_i). This correspondence was established in the other direction by M. Ratner in order to study the speed of mixing of the geodesic flow on finite volume hyperbolic surfaces ([28]).

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