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The Maximal Rank Conjecture for sections of curves

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ABSTRACT

Let $C \subset \mathbb{P}^r$ be a general curve of genus g embedded via a general linear series of degree d. The *Maximal Rank Conjecture* asserts that the restriction maps $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$ are of maximal rank; this determines the Hilbert function of C. In this paper, we prove an analogous statement for the union of hyperplane sections of general curves. More specifically, if $H \subset \mathbb{P}^r$ is a general hyperplane, and $C_1, C_2, \ldots, C_n$ are general curves, we show $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{(C_1 \cup C_2 \cup \cdots \cup C_n) \cap H}(m))$ is of maximal rank, except for some counterexamples when $m = 2$.

As explained in [5], this result plays a key role in the author's proof of the Maximal Rank Conjecture [7].

1. Introduction

Let $\mathcal{H}_{d,g,r}$ denote the Hilbert scheme classifying subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(x) = dx + 1 - g$. We have a natural rational map from any component of $\mathcal{H}_{d,g,r}$ whose general member is a smooth curve to the moduli space M_g of curves. The Brill–Noether theorem asserts that there exists such a component whose general member is nondegenerate and that dominates M_g if and only if

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$$\rho(d, g, r) := (r+1)d - rg - r(r+1) \geq 0.$$

Moreover, it is known that when $\rho(d, g, r) \geq 0$, there exists a unique such component that dominates M_g. We shall refer to a curve $C \subset \mathbb{P}^r$ lying in this component as a *Brill–Noether Curve (BN-curve).*

A natural first step in understanding the extrinsic geometry of general curves is to understand their Hilbert function. Here we have the *Maximal Rank Conjecture*:

Conjecture 1.1 *(Maximal Rank Conjecture).* If C is a general BN-curve and m is a positive integer, then the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

is of maximal rank.

Remark 1.2. Since $H^1(\mathcal{O}_C(m)) = 0$ for $m \geq 2$ when C is a general BN-curve, the Maximal Rank Conjecture completely determines the Hilbert function of C.

In this paper, we study a related question for hyperplane sections. Namely, we prove that the general hyperplane section of a general union of BN-curves imposes the expected number of conditions on hypersurfaces of every degree, apart from a few counterexamples that occur for quadric hypersurfaces. Both the results and the techniques developed here play a critical role in the author's proof of the Maximal Rank Conjecture [7], as explained in [5]. More precisely, we prove:

Theorem 1.3 *(Hyperplane Maximal Rank Theorem). If $C_1, C_2, \ldots, C_n$ are independently general BN-curves of degrees d_i and genera g_i, and $H \subset \mathbb{P}^r$ is a general hyperplane, and m is a positive integer, then the restriction map*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{(C_1 \cup C_2 \cup \cdots \cup C_n) \cap H}(m))$$

is of maximal rank, except possibly when $m = 2$ and $d_i < g_i + r$ for some i.

The conclusion that this restriction map is of maximal rank can be reformulated in terms of the cohomology of the twists of the ideal sheaf as follows:

$$H^0(\mathcal{I}_{((C_1 \cup C_2 \cup \cdots \cup C_n) \cap H)/H}(m)) = 0 \quad \text{when } \sum_{i=1}^{n} d_i \geq \binom{m+r-1}{r-1},$$

$$H^1(\mathcal{I}_{((C_1 \cup C_2 \cup \cdots \cup C_n) \cap H)/H}(m)) = 0 \quad \text{when } \sum_{i=1}^{n} d_i \leq \binom{m+r-1}{r-1}.$$

In the course of proving Theorem 1.3, we will also prove stronger results for $r = 3$ and for $r = 4$. Namely:

Theorem 1.4. *Let $X \subset H \simeq \mathbb{P}^2 \subset \mathbb{P}^3$ be a subscheme, and $C \subset \mathbb{P}^3$ be a general BN-curve.*

- *If C is a canonical curve and $m = 2$, suppose that X is nonempty.*
- *If C is a canonical curve and $m \neq 2$, write $\Lambda \subset H$ for a general line, and suppose that the restriction maps*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$$
$$H^0(\mathcal{O}_H(m-1)) \to H^0(\mathcal{O}_X(m-1))$$
$$H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X\cap\Lambda}(m))$$

  are of maximal rank, with either the second one an injection, or the third one a surjection with kernel of dimension at least 4.
- *Otherwise, suppose the map*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$$

  is of maximal rank.

Then the map

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X\cup(C\cap H)}(m))$$

is of maximal rank.

Theorem 1.5. *Let $X \subset H \simeq \mathbb{P}^3 \subset \mathbb{P}^4$ be a subscheme, and $C \subset \mathbb{P}^4$ be a general BN-curve of degree d and genus g.*

- *If $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$ and $m = 2$, suppose that X is either positive dimensional or of degree at least $11 - d$.*
- *If $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$ and $m \neq 2$, write $\Lambda \subset H$ for a general plane, and suppose that the restriction maps*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$$
$$H^0(\mathcal{O}_H(m-1)) \to H^0(\mathcal{O}_X(m-1))$$
$$H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X\cap\Lambda}(m))$$

  are of maximal rank, with either the second one an injection, or the third one a surjection with kernel of dimension at least 8.
- *Otherwise, suppose the map*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$$

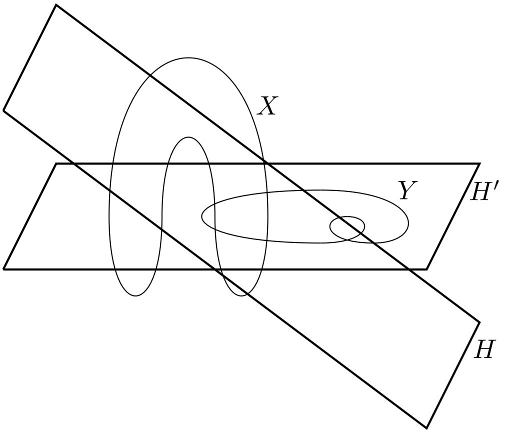
is of maximal rank.

Then the map

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X\cup(C\cap H)}(m))$$

is of maximal rank.

We shall prove Theorem 1.3 using an inductive approach due originally to Hirschowitz [3]. In its simplest form, suppose that $C = X \cup Y$ is a reducible curve such that Y is contained in some hyperplane H':



Then we have the exact sequence of sheaves

$$0 \to \mathcal{I}_{(X\cap H)/H}(m-1) \to \mathcal{I}_{(C\cap H)/H}(m) \to \mathcal{I}_{(Y\cap H)/(H\cap H')}(m) \to 0,$$

which gives rise to a long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) \to H^i(\mathcal{I}_{(C\cap H)/H}(m)) \to H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) \to \cdots .$$

Consequently, we can deduce the hyperplane maximal rank theorem for the general hyperplane section of C from the hyperplane maximal rank theorem for the general hyperplane sections of X and Y.

The structure of this paper is as follows. First, in Section 2, we give several methods of constructing reducible BN-curves that will be useful for specialization arguments later on. In Sections 3 and 4, we prove the hyperplane maximal rank theorem in the special cases $r = 3$ and $m = 2$ respectively. We then deduce the general case in Sections 5 and 6 via the above inductive argument, by finding appropriate BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H' \subset \mathbb{P}^r$ satisfying the hyperplane maximal rank theorem for $(m-1, r)$ and $(m, r-1)$ respectively.

Notational Convention: We say a BN-curve $X \subset \mathbb{P}^r$ is *nonspecial* if $d \geq g + r$, i.e. if X is a *limit* of curves with nonspecial hyperplane section.

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2. Some gluing lemmas

In this section, we will give some lemmas that let us construct examples of BN-curves.

Lemma 2.1. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and D be a rational normal curve of degree $d \leq r$ that is k-secant to X, where*

$$k \leq \begin{cases} d+1 & \text{if } d < r; \\ r+2 & \text{if } d = r. \end{cases}$$

Then $X \cup D$ is smoothable and $H^1(N_{X \cup D}) = 0$. Moreover, if X is a BN-curve, then $X \cup D$ is a BN-curve.

Proof. The vanishing of $H^1(N_{X \cup D})$ and smoothability of $X \cup D$ are consequences of Theorem 4.1 of [2] (via the same argument as Corollary 4.2 of [2]), together with the fact that

$$N_D = \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus(r-d)} \oplus \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus(d-1)}.$$

Now assume X is a BN-curve. To show that $X \cup D$ is a BN-curve, we just need to count the dimension of the space of embeddings of $X \cup D$ into projective space (this suffices because there is a unique component of the Hilbert scheme that dominates M_g). In order to do this, first note that

$$\rho(X \cup D) = \rho(X) + (r+1)d - r(k-1).$$

Consequently, the verification that $X \cup D$ is a BN-curve boils down to the following two assertions, both of which are straight-forward to check:

1. Given a $\mathbb{P}^1$ with $k \leq d+1$ marked points, the family of degree d embeddings of $\mathbb{P}^1$ as a rational normal curve with given values at the marked points has dimension

$$(r-d)(d-k+1) + d(d+2-k) = (r+1)d - r(k-1).$$

2. Given a $\mathbb{P}^1$ with $r+2$ marked points, there is a unique embedding of $\mathbb{P}^1$ as a rational normal curve of degree r with given values at all marked points.

This completes the proof. □

Lemma 2.2. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and R be a rational normal curve of degree $r-1$ that is $(r+1)$-secant to X, and L be a line that is 1-secant to both X and R. Then $H^1(N_{X\cup R\cup L}) = 0$.*

Proof. Note that for curves A and B,

$$H^1(N_{A\cup B}|_A) = 0 \quad \text{and} \quad H^1(N_{A\cup B}|_B(-A\cap B)) = 0 \quad \Rightarrow \quad H^1(N_{A\cup B}) = 0;$$

indeed, this holds for $N_{A\cup B}$ replaced by any vector bundle.

In particular, since N_A is a subbundle of full rank in $N_{A\cup B}|_A$, we can conclude that $H^1(N_{A\cup B}) = 0$ provided that

$$\begin{aligned} H^1(N_A) = 0 \quad &\text{and} \quad H^1(N_{A\cup B}|_B(-A\cap B)) = 0, \\ \text{or respectively} \quad H^1(N_{A\cup B}|_A) = 0 \quad &\text{and} \quad H^1(N_B(-A\cap B)) = 0. \end{aligned}$$

Thus, the vanishing of $H^1(N_{X\cup R\cup L})$ follows from the following facts:

$$\begin{aligned} H^1(N_X) &= 0 \\ H^1(N_{R\cup L}|_R(-X\cap R)) &= H^1(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 0. \\ H_1(N_L(-L\cap(X\cup R))) &= H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0. \quad \square \end{aligned}$$

Lemma 2.3. *Let $X \subset \mathbb{P}^r$ be a curve with $H^1(N_X) = 0$, and L be a line 3-secant to X. Assume that the tangent lines to X at the three points of intersection do not all lie in a plane. Then $X \cup D$ is smoothable and $H^1(N_{X\cup D}) = 0$.*

Proof. See Remark 4.2.2 of [2]. □

We end this section with two simple observations, that will be used several times in the remainder of the paper and will therefore be useful to spell out.

Lemma 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be irreducible families of curves in $\mathbb{P}^r$, sweeping out subvarieties $\overline{\mathcal{X}}, \overline{\mathcal{Y}} \subset \mathbb{P}^r$ of codimension at most one. Let X and Y be specializations of $\mathcal{X}$ and $\mathcal{Y}$ respectively, such that $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$, and $X \cap Y$ is quasi-transverse and general in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$.*

Then there are simultaneous generalizations X' and Y' of X and Y respectively such that $X' \cup Y'$ is a BN-curve with $\#(X\cap Y) = \#(X'\cap Y')$. Equivalently, in more precise language, write B_1 and B_2 for the bases of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then we are asserting *the existence of an irreducible $B \subset B_1 \times B_2$ dominating both B_1 and B_2, such that any fiber (X', Y') of $(\mathcal{X} \times \mathcal{Y}) \times_{(B_1 \times B_2)} B$ satisfies the given conclusion.*

Proof. As $\overline{\mathcal{Y}}$ has codimension at most one, the intersection of any generalization X' of X with $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$ contains a generalization of $X \cap Y$. Similarly, the intersection of any generalization Y' of Y with $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$ contains a generalization of $X \cap Y$. The existence of simultaneous generalizations X' and Y' of X and Y respectively with $\#(X \cap Y) = \#(X' \cap Y')$ thus follows from the generality of $X \cap Y$ in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$.

Moreover, since $H^1(N_{X \cup Y}) = 0$, the curve $X \cup Y$ is a smooth point of the corresponding Hilbert scheme; consequently, any generalization $X' \cup Y'$ of $X \cup Y$ is a BN-curve. $\square$

Lemma 2.5. *Let $S \subset \mathbb{P}^r$ and $T \subset \mathbb{P}^r$ be sets of points such that the restriction maps*

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_S(m)) \quad \text{and} \quad H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T}(m))$$

are of maximal rank. Then, for every integer $0 \leq n \leq \#T$, there exists a subset $T' \subset T$ of cardinality n such that

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T'}(m))$$

is of maximal rank.

In particular, taking $T = \mathbb{P}^r(\mathbb{C}) \smallsetminus S$, if $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_S(m))$ is of maximal rank, then for n general points $T' \subset \mathbb{P}^r$, the map $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T'}(m))$ is also of maximal rank.

Proof. We argue by induction on n. When $n = 0$, the conclusion holds by assumption. When $n = 1$, we note that the conclusion is obvious if $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_S(m))$ is injective or if $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T}(m))$ is surjective. We may therefore suppose that the map $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_S(m))$ is surjective but not injective, whose kernel contains a nonzero polynomial f; and that $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T}(m))$ is injective. In particular, there is a point $p \in T$ with $f|_p \neq 0$. Taking $T' = \{p\}$, the map $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T'}(m))$ is surjective by construction.

For the inductive step, let $T'' \subset T$ be of size $n - 1$ such that $H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_{S \cup T''}(m))$ is of maximal rank. Applying our inductive hypothesis with $(S, T) = (S \cup T'', T \smallsetminus T'')$ completes the proof. $\square$

3. The case $r = 3$

In this section, we will prove Theorems 1.4 and 1.5. As a consequence of Theorem 1.4, we will deduce that if $C_1, C_2, \ldots, C_n \subset \mathbb{P}^3$ are independently general BN-curves, then

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{(C_1 \cup C_2 \cup \cdots \cup C_n) \cap H}(m))$$

is of maximal rank, unless $n = 1$, and C_1 is a canonically embedded curve of genus 4, and $m = 2$. (In which case by inspection the above map fails to be of maximal rank.)

Proof of Theorem 1.4. If C is not a canonical curve, Theorem 1.5 of [6] states that $C \cap H$ is a general set of points, and so Lemma 2.5 yields the desired result.

If C is a canonical curve, Theorem 1.5 of [6] states that $C \cap H$ is a set of 6 points which are general subject to the constraint that they lie on a conic. In particular, $C \cap H$ imposes independent conditions on $H^0(\mathcal{O}_H(1))$ and on any fixed proper subspace of $H^0(\mathcal{O}_H(2))$. Since X is nonempty by assumption if $m = 2$, the kernel of $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$ is a proper subspace of $H^0(\mathcal{O}_H(m))$ if $m = 2$. If $m \leq 2$, we therefore conclude that

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup (C \cap H)}(m))$$

is injective (so in particular of maximal rank as desired).

If $m \geq 3$, we specialize the conic to the union of two lines, and the points of $C \cap H$ to consist of 2 points on one line (which is just a set of 2 general points), and 4 points on the other. Using our assumption that $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$ is of maximal rank and applying Lemma 2.5 twice, it suffices to show

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup Y}(m))$$

is of maximal rank, where Y is a set of $\max(4, \dim\ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)))$ points which are general subject to the condition that they lie on a line Λ. For this, we use the exact sequence

$$0 \to \mathcal{I}_X(m-1) \to \mathcal{I}_{X \cup Y}(m) \to \mathcal{I}_{Y/\Lambda}(m) \to 0.$$

Note that $H^0(\mathcal{I}_{Y/\Lambda}(m)) = 0$, and if $\dim\ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) \geq 4$, then we have $H^1(\mathcal{I}_{Y/\Lambda}(m)) = 0$ too. In particular, the associated long exact sequence in cohomology implies $H^0(\mathcal{I}_{X \cup Y}(m)) = 0$ provided that $H^0(\mathcal{I}_X(m-1)) = 0$, and similarly for H^1 if $\dim\ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) \geq 4$.

Our assumption that $H^0(\mathcal{O}_H(m-1)) \to H^0(\mathcal{O}_X(m-1))$ is of maximal rank and injective if $\dim\ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) < 4$ thus implies that $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup Y}(m))$ is of maximal rank, as desired. □

Proof of Theorem 1.5. If $(d, g) \notin \{(8, 5), (9, 6), (10, 7)\}$, Theorem 1.6 of [6] states that $C \cap H$ is a general set of points, and so Lemma 2.5 yields the desired result.

If $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$, then Theorem 1.5 of [6] states that $C \cap H$ is a general complete intersection of 3 quadrics, a general set of 9 points on a complete intersection of 2 quadrics, or a general set of 10 points on a quadric, respectively. In particular, we may specialize $C \cap H$ to consist of 8 points which are a general complete intersection of 3 quadrics, together with $d - 8$ independently general points. Applying Lemma 2.5, it suffices to show the result when $(d, g) = (8, 5)$ and $C \cap H$ is a general complete intersection of 3 quadrics.

In particular $C \cap H$ imposes independent conditions on $H^0(\mathcal{O}_H(1))$ and on any fixed subspace of $H^0(\mathcal{O}_H(2))$ of codimension at least 3. Since any subscheme of $\mathbb{P}^3$ of positive dimension or of degree at least 3 imposes at least 3 conditions on quadrics, if $m \leq 2$ we therefore conclude that

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup (C \cap H)}(m))$$

is injective (so in particular of maximal rank as desired).

If $m \geq 3$, we claim we may further specialize $C \cap H$ to 8 general points in a plane. To see this, take a general set Γ of 8 points in a plane. Then there is a smooth plane cubic curve E containing Γ. Let $p \in E$ be a point so that $\mathcal{O}_E(2)(2p) \simeq \mathcal{O}_E(\Gamma)$. Choose a basis $\langle f_1, f_2, f_3 \rangle$ for $H^0(\mathcal{O}_E(1))$, so that $E \subset \mathbb{P}^2$ is embedded via $[f_1 : f_2 : f_3]$, and let f_0 be an extension to a basis of $H^0(\mathcal{O}_E(1)(p))$. Then for λ generic, the image of Γ in $\mathbb{P}^3$ under $[\lambda f_0 : f_1; f_2 : f_3]$ is a set of 8 points on the image of E, with class twice the pullback to E under this embedding of the hyperplane class in $\mathbb{P}^3$ — in particular, as E is the complete intersection of two quadrics and is projectively normal, is a complete intersection of 3 quadrics in $\mathbb{P}^3$. Specializing $\lambda \to 0$, we obtain the set Γ of 8 general points in the plane that we started with.

Using our assumption that $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_X(m))$ is of maximal rank and applying Lemma 2.5, it suffices to show

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup Y}(m))$$

is of maximal rank, where Y is a set of $\max(8, \dim \ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)))$ points which are general subject to the condition that they lie on a plane Λ.

Note that $H^0(\mathcal{I}_{Y/\Lambda}(m)) = 0$, and if $\dim \ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) \geq 8$, then we have $H^1(\mathcal{I}_{Y/\Lambda}(m)) = 0$ too. In particular, the associated long exact sequence in cohomology implies $H^0(\mathcal{I}_{X \cup Y}(m)) = 0$ provided that $H^0(\mathcal{I}_X(m-1)) = 0$, and similarly for H^1 if $\dim \ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) \geq 8$.

Our assumption that $H^0(\mathcal{O}_H(m-1)) \to H^0(\mathcal{O}_X(m-1))$ is of maximal rank and injective if $\dim \ker H^0(\mathcal{O}_\Lambda(m)) \to H^0(\mathcal{O}_{X \cap \Lambda}(m)) < 8$ thus implies that $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{X \cup Y}(m))$ is of maximal rank, as desired. $\square$

Corollary 3.1. *If $C_1, C_2, \ldots, C_n$ are independently general space BN-curves, $H \subset \mathbb{P}^3$ is a general hyperplane, and m is a positive integer, then the restriction map*

$$H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{(C_1 \cup C_2 \cup \cdots \cup C_n) \cap H}(m))$$

is of maximal rank, except if $m = 2$ and $n = 1$ and C_1 is a canonical curve.

Proof. Applying Theorem 1.4, we immediately see all cases of this statement by induction (starting with $n = 0$ as our base case), provided we check the case when $m = 3$ and $n = 2$ and C_1 and C_2 are both canonical curves. In this case, by Theorem 1.5 of [6] $(C_1 \cup C_2) \cap H$ is a collection of 12 points which are general subject to the condition that 6 of them lie on conic Q_1 and the other 6 lie on a conic Q_2; we want to show such the general such subscheme does not lie on any cubics.

For this, we specialize one of the points on Q_1 to one of the points of intersection $Q_1 \cap Q_2$, and one the points on Q_2 to a different point of intersection $Q_1 \cap Q_2$. The resulting subscheme of degree 12 meets Q_1 in 7 points, but a cubic not containing Q_1 can only meet Q_1 in 6 points by Bezout's theorem. Any such cubic must therefore contain Q_1, and symmetrically Q_2. But $Q_1 \cup Q_2$ is of degree 4, so is contained in no cubics, as desired. □

4. The case $m = 2$

In this section, we will prove the hyperplane maximal rank theorem when $m = 2$, and the curves C_i are all nonspecial. We will begin by constructing reducible curves with the following lemma, to which we will apply the method of Hirschowitz outlined in the introduction.

Lemma 4.1. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $d \geq g + r$ and $g \geq 0$. Assume d_1 and d_2 are nonnegative integers with $d = d_1 + d_2$. Then there exist curves $X \subset \mathbb{P}^r$ and $Y \subset H'$, of degrees d_1 and d_2 respectively, both of which are either nonspecial BN-curves, rational normal curves, or empty; with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a nondegenerate BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$.*

Proof. We argue by induction on d (which satisfies $d \geq r$). For the base case, we take $d = r$, which forces $g = 0$. We may then let X and Y be rational normal curves of degrees d_1 and d_2 respectively, meeting at one point; this gives a BN-curve with $H^1(N_{X\cup Y}) = 0$ by Lemma 2.1.

For the inductive step, we assume $d \geq r + 1$; in particular, if $d_1 \leq 1$, then $d_2 \geq r$. Define $g' = \max(0, g - 1)$ and

$$(d_1', d_2') = \begin{cases} (d_1 - 1, d_2) & \text{if } d_1 \geq 2; \\ (d_1, d_2 - 1) & \text{else.} \end{cases}$$

By our inductive hypothesis, there exists curves $X' \subset \mathbb{P}^r$ and $Y' \subset H'$, of degrees d_1' and d_2' respectively, both of which are either nonspecial BN-curves, rational normal curves, or empty; with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a nondegenerate BN-curve of genus g' with $H^1(N_{X'\cup Y'}) = 0$.

If $d_1 \geq 2$ and $g = 0$, we take $X = X' \cup L$ for L a general 1-secant line to X', and $Y = Y'$; by Lemma 2.1, both X and $X \cup Y$ are BN-curves, and $H^1(N_{X\cup Y}) = 0$.

Similarly if $d_1 \leq 1$ and $g = 0$ (respectively $g \geq 1$), we take $X = X'$, and $Y = Y' \cup L$ for L a general 1-secant (respectively 2-secant) line to Y'; by Lemma 2.1, both Y and $X \cup Y$ are BN-curves, and $H^1(N_{X\cup Y}) = 0$.

Finally, we consider the case $d_1 \geq 2$ and $g \geq 1$. If X' is nondegenerate, we take $X = X' \cup L$ for L a general 2-secant line to X, and $Y = Y'$; by Lemma 2.1, both X and $X \cup Y$ are BN-curves, and $H^1(N_{X\cup Y}) = 0$. If X' is degenerate, then since $X' \cup Y'$ is nondegenerate by assumption, the general line L meeting X' and Y' each once intersects Y' in a point which is independently general from $X' \cap Y'$. We then take we take $X = X' \cup L$, and $Y = Y'$; again by Lemma 2.1, both X and $X \cup Y$ are BN-curves, and $H^1(N_{X\cup Y}) = 0$. $\square$

Combining this with Lemma 2.4, we obtain:

Corollary 4.2. *Let $C_1, C_2, \ldots, C_n \subset \mathbb{P}^r$ be independently general nonspecial BN-curves, and $H' \subset \mathbb{P}^r$ be a hyperplane. Then we may specialize the C_i to curves $X_i \cup Y_i$ such that $\sum \deg X_i$ and $\sum \deg Y_i$ are any two nonnegative integers adding up to $\sum \deg C_i$; and such that $X_1, X_2, \ldots, X_n \subset \mathbb{P}^r$ and $Y_1, Y_2, \ldots, Y_n \subset H'$ are each sets of independently general BN-curves or rational normal curves.*

Proposition 4.3. *Let $C_1, C_2, \ldots, C_n \subset \mathbb{P}^r$ be independently general BN-curves, and $H \subset \mathbb{P}^r$ be a general hyperplane. Assume that C_i is nonspecial for all i. Then*

$$H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{(C_1\cup C_2\cup\cdots\cup C_n)\cap H}(2))$$

is of maximal rank.

Proof. We use induction on r; when $r = 3$, this is a consequence of Corollary 3.1. For the inductive step, write $d = \sum \deg C_i$, and let (d_1, d_2) be nonnegative integers with $d = d_1 + d_2$, such that

$$d_1 \geq r \quad \text{and} \quad d_2 \geq \binom{r}{2} \qquad \text{if} \quad d \geq \binom{r+1}{2},$$
$$d_1 \leq r \quad \text{and} \quad d_2 \leq \binom{r}{2} \qquad \text{if} \quad d \leq \binom{r+1}{2}.$$

Pick a hyperplane H' transverse to H. By Corollary 4.2, we may specialize the C_i to curves $X_i \cup Y_i$ such that $\sum \deg X_i = d_1$ and $\sum \deg Y_i = d_2$; and such that

$$X := X_1 \cup X_2 \cup \cdots \cup X_n \subset \mathbb{P}^r \quad \text{and} \quad Y := Y_1 \cup Y_2 \cup \cdots \cup Y_n \subset H'$$

are each unions of independently general BN-curves or rational normal curves. Since the hyperplane section of a rational normal curve is a general set of points, our inductive hypothesis in combination with Lemma 2.5 implies

$$H^0(\mathcal{O}_{H\cap H'}(2)) \to H^0(\mathcal{O}_{Y\cap H}(2))$$

is of maximal rank. Define

$$i = \begin{cases} 0 & \text{if } d \geq \binom{r+1}{2}, \\ 1 & \text{if } d \leq \binom{r+1}{2}; \end{cases}$$

so we want to show

$$H^i(\mathcal{I}_{(C_1\cup C_2\cup\cdots\cup C_n)\cap H/H}(2)) = 0,$$

and know by induction that

$$H^i(\mathcal{I}_{Y\cap H/(H\cap H')}(2)) = 0$$

By direct examination, $H^i(\mathcal{I}_{X\cap H/H}(1)) = 0$. Consequently, we may use the exact sequence of sheaves

$$0 \to \mathcal{I}_{X\cap H/H}(1) \to \mathcal{I}_{(C_1\cup C_2\cup\cdots\cup C_n)\cap H/H}(2) \to \mathcal{I}_{Y\cap H/(H\cap H')}(2) \to 0,$$

which gives rise to the long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{X\cap H/H}(1)) \to H^i(\mathcal{I}_{(C_1\cup C_2\cup\cdots\cup C_n)\cap H/H}(2)) \to H^i(\mathcal{I}_{Y\cap H/(H\cap H')}(2)) \to \cdots,$$

to conclude that $H^i(\mathcal{I}_{(C_1\cup C_2\cup\cdots\cup C_n)\cap H/H}(2)) = 0$ as desired. □

4.1. The condition $d \geq g + r$

The condition $d \geq r$ is necessary; indeed when $d < g + r$, the map will sometimes fail to be of maximal rank, as shown by the following proposition:

Proposition 4.4. *Let $C \subset \mathbb{P}^r$ be any curve of degree d and genus g, with $d < g + r$ and $4d - 2g < r(r + 3)$. Then the restriction map*

$$H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$$

fails to be of maximal rank.

Proof. We compute

$$\dim H^0(\mathcal{O}_{\mathbb{P}^r}(2)) - \dim H^0(\mathcal{O}_C(2)) = \binom{r+2}{2} - (2d+1-g) = \frac{r(r+3) - (4d-2g)}{2} > 0,$$

and so C lies on a quadric. Moreover, we have

$$\begin{aligned}\dim H^0(\mathcal{O}_{\mathbb{P}^r}(2)) - \dim H^0(\mathcal{O}_C(2)) &= \frac{r(r+3)-(4d-2g)}{2} \\ &= \binom{r+1}{2} - d + (g+r-d) \\ &= \dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2)) + (g+r-d) \\ &> \dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2)).\end{aligned}$$

Now every quadric containing C restricts to a quadric in H containing $H \cap C$; as C is nondegenerate, this restriction has no kernel. Consequently, there is a subspace of $H^0(\mathcal{O}_H(2))$ in the kernel of $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ which is of positive dimension that exceeds $\dim H^0(\mathcal{O}_H(2)) - \dim H^0(\mathcal{O}_{C\cap H}(2))$. In other words, $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ is not of maximal rank. $\square$

When $n = 1$, the cases in Proposition 4.4 are the only cases in which the restriction map $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C\cap H}(2))$ fails to be of maximal rank.

Indeed, if C is a general BN-curve with $d < g + r$, then C is linearly normal, i.e. $H^1(\mathcal{I}_C(1))$ vanishes. Now consider the exact sequence of sheaves

$$0 \to \mathcal{I}_C(1) \to \mathcal{O}_{\mathbb{P}^r}(1) \oplus \mathcal{I}_C(2) \to \mathcal{I}_{C\cap H}(2) \to 0;$$

this induces a long exact sequence of cohomology groups:

$$\cdots \to H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \oplus H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_{C\cap H}(2)) \to H^1(\mathcal{I}_C(1)) \to \cdots$$

It follows that $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \oplus H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_{C\cap H}(2))$ is surjective, i.e. every quadric $Q \subset H$ containing $C \cap H$ is the intersection with H of a quadric $\widetilde{Q} \subset \mathbb{P}^r$ containing C. For $4d - 2g \geq r(r + 3)$, the maximal rank conjecture for quadrics (see [1] or [4]) implies that C is not contained in any quadric, and consequently that $C \cap H$ is not contained in any quadric.

5. Construction of reducible curves

In this section, which is the heart of the proof, we will construct examples of reducible BN-curves $X \cup Y$ where $Y \subset H'$. These reducible curves will be the essential ingredient in applying the inductive method of Hirschowitz in the following section to deduce the hyperplane maximal rank theorem.

Lemma 5.1. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $\rho(d, g, r) \geq 0$ and $d \geq g + r - 2$. Assume d_1 and d_2 are positive integers with $d = d_1 + d_2$, that additionally satisfy:*

$$d_1 \geq r + \max(0, g + r - d) \quad \textit{and} \quad d_2 \geq r - 1.$$

Then there exist nonspecial BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$.

Proof. We will argue by induction on d and $\rho(d, g, r)$. Notice that our inequalities for d_1 and d_2 imply $d \geq 2r - 1$; for the base case, we consider when $d = 2r - 1$ or $\rho(d, g, r) = 0$.

If $d = 2r - 1$, we take X to be a rational normal curve of degree r, and $Y \subset H$ to be a rational normal of degree $r - 1$ that meets $X \cap H$ in $g + 1$ points. (Note that as $\rho(2r - 1, g, r) \geq 0$, we have $g + 1 \leq r$.) By inspection, $X \cup Y$ is of genus g; as $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $X \cap Y$ is general. Moreover, $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$ by Lemma 2.1.

If $\rho(d, g, r) = 0$ and $d \geq g + r - 2$, then either $(d, g) = (2r, r + 1)$ or $(d, g) = (3r, 2r + 2)$. In the case $(d, g) = (2r, r + 1)$, we take X to be the union of a rational normal curve R of degree r with a 2-secant line L, and Y to be a rational normal curve of degree $r - 1$ passing through $X \cap H$. Again, by inspection $X \cup Y$ is of genus $r + 1$; as $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $X \cap Y$ is general. To see that $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$, we apply Lemma 2.1 to the decomposition $X \cup Y = (Y \cup L) \cup R$.

Now suppose that $(d, g) = (3r, 2r + 2)$. If $d_2 = r - 1$, then we take $X = C \cup L$ to be the union of a canonical curve C with a general 1-secant line L. We take Y to be the rational normal curve of degree $r - 1$ passing through $L \cap H'$ and through $r + 1$ points of $C \cap H'$. By inspection $X \cup Y$ is of genus $2r + 2$. To see that $X \cap Y$ is general, first note that since $\operatorname{Aut} H$ acts $(r + 1)$-transitively on points in linear general position, $C \cap Y$ is general; moreover, $L \cap H$ is general with respect to C. To see that $X \cup Y$ is a BN-curve, we apply Lemma 2.1 to the decomposition $X \cup Y = C \cup (L \cup Y)$, while noting that $L \cup Y$ is the specialization of a rational normal curve of degree r. Moreover, by Lemma 2.2, we have $H^1(N_{X\cup Y}) = 0$.

Otherwise, we have $d_2 \geq r$ and $d_1 \geq r + 2$; in this case we take $X = R_1 \cup L_0 \cup L_1 \cup N_1$ and $Y = R_2 \cup L_2 \cup N_2$, where:

1. R_1 is a general rational normal curve of degree r.
2. L_0 is a general 2-secant line to R_1.
3. R_2 is a general rational normal curve of degree $r - 1$ passing through all $r + 1$ points of $(R_1 \cup L_0) \cap H$.
4. L_1 is a general line meeting R_1 once and L_0 once.
5. L_2 is a general 2-secant line to R_2, passing through $L_1 \cap H$.
6. N_1 is a general rational normal curve of degree $d_1 - r - 2$ meeting L_1 once and R_1 in $d_1 - r - 2$ points (we take $N_1 = \emptyset$ if $d_1 = r + 2$).
7. N_2 is a general rational normal curve of degree $d_2 - r$ meeting L_2 once and R_2 in $d_2 - r$ points (we take $N_2 = \emptyset$ if $d_2 = r$).

In order for this to make sense, we need conditions 4 and 5 to be consistent. The consistency of 4 and 5, as well as the assertion that $X \cap Y$ is general, both follow from the following two claims:

- $L_1 \cap H$ is general relative to $(R_1 \cup L_0) \cap H$. This follows from $L_1 \cap R_1$ being general relative to L_0 and $R_1 \cap H$, which in turn follows from the existence of a rational normal curve of degree r through a general collection of $r + 3$ points.
- The 2-secant lines to R_2 sweep out H as we vary R_2 over all rational normal curves of degree $r - 1$ passing through all $r + 1$ points of $(R_1 \cup L_0) \cap H$. This follows from the observation that R_2 sweeps out H, which again follows from the existence of a rational normal curve of degree $r - 1$ through a general collection of $r + 2$ points in H'.

By inspection, $X \cup Y$ is a curve of genus g and X and Y are nonspecial. To show that $X \cup Y$ is a BN-curve, we apply Lemma 2.1 to the decomposition

$$X \cup Y = (L_0 \cup R_2) \cup R_1 \cup (L_1 \cup L_2 \cup N_1 \cup N_2).$$

Similarly, to show $H^1(N_{X \cup Y}) = 0$, we apply Lemma 2.1 and then Lemma 2.3 to the decomposition

$$X \cup Y = (L_0 \cup R_2) \cup R_1 \cup L_2 \cup N_1 \cup N_2 \cup L_1.$$

To apply Lemma 2.3, we need to check that the tangent lines to $(L_0 \cup R_2) \cup R_1 \cup L_2 \cup N_1 \cup N_2$ at the points of intersection with L_1 do not all lie in a plane. Since L_1 intersects L_0, the only possible plane that could contain all 3 tangents is $\overline{L_0 L_1}$. But as this plane contains the two points of intersection of L_0 with R_1 and a plane can only intersect a rational normal curve at 3 points with multiplicity, the tangent line to R_1 at $L_1 \cap R_1$ cannot be contained in this plane. Consequently, we may apply Lemma 2.3 as claimed.

For the inductive step, we have $d \geq 2r$ and $\rho(d, g, r) > 0$. We claim that these inequalities imply that

$$r + \max(0, g + r - d) + r - 1 < d = d_1 + d_2. \tag{1}$$

Of course,

$$r + \max(0, g + r - d) + r - 1 = \max(2r - 1, 3r - 1 + g - d);$$

consequently, as $2r - 1 < 2r \leq d$, it suffices to show $3r - 1 + g - d < d$, or equivalently $g < 2d + 1 - 3r$. To see this, note that if $g \geq 2d + 1 - 3r$, then we would have

$$-(r - 1)(d - 2r) = (r + 1)d - r(2d + 1 - 3r) - r(r + 1) \geq (r + 1)d - rg - r(r + 1) > 0,$$

which is a contradiction; thus, $g < 2d+1-3r$, and so (1) holds. Consequently, there exists (d_1', d_2') either equal to $(d_1 - 1, d_2)$ or to $(d_1, d_2 - 1)$, such that $d_1' \geq r + \max(0, g + r - d)$ and $d_2' \geq r - 1$. (Otherwise $d_1 - 1 < r + \max(0, g + r - d)$ and $d_2 - 1 < r - 1$, i.e. $d_1 \leq r + \max(0, g + r - d)$ and $d_2 \leq r - 1$; adding these contradicts (1).)

If we define $g' = \max(0, g - 1)$, then $\max(0, g + r - d) = \max(0, g' + r - (d - 1))$. Thus by the inductive hypothesis, there are BN-curves $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ of degrees d_1' and d_2' respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of genus g' with $H^1(N_{X' \cup Y'}) = 0$. To complete the inductive step, we take

$$(X, Y) = \begin{cases} (X', Y' \cup L) & \text{if } d_1' = d_1; \\ (X' \cup L, Y') & \text{if } d_2' = d_2; \end{cases} \quad \text{where} \quad L = \begin{cases} \text{a 1-secant line} & \text{if } g' = g; \\ \text{a 2-secant line} & \text{if } g' \neq g. \end{cases}$$

This satisfies the desired conclusion by Lemma 2.1. □

Lemma 5.2. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $\rho(d, g, r) \geq 0$. Assume d_1 and d_2 are positive integers with $d = d_1 + d_2$, that additionally satisfy:*

$$d_1 \geq r + \max(0, g + r - d) \quad \textit{and} \quad d_2 \geq r - 1.$$

Then there exists BN-curves $X \subset \mathbb{P}^r$ and $Y \subset H'$ of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X \cup Y}) = 0$. Moreover, we can take X to be nonspecial if

$$d_2 \geq (r - 1) \cdot \left\lceil \frac{\max(0, g + r - d)}{2} \right\rceil. \tag{2}$$

Proof. We will argue by induction on d. When $d \geq g + r - 2$, we are done by Lemma 5.1. Thus we may assume that $d < g + r - 2$. In particular, this implies that $d \geq 4r$, and that $\max(0, g + r - d) = g + r - d$. We claim that

$$r + \max(0, g + r - d) + r - 1 = 3r - 1 + g - d < d - 2(r - 2) = d_1 + d_2 - 2(r - 2). \tag{3}$$

This is equivalent to $g < 2d + 5 - 5r$; to see this, note that if $g \geq 2d + 5 - 5r$, then

$$-(r-1)(d-4r) - 2r = (r+1)d - r(2d+5-5r) - r(r+1) \geq (r+1)d - rg - r(r+1) = 0,$$

which is a contradiction; thus, $g < 2d + 5 - 5r$, and so (3) holds. Consequently, there exists (d_1', d_2') either equal to $(d_1 - 1, d_2 - r + 1)$ or to $(d_1 - r, d_2)$, such that

$$d_1' \geq r + \max(0, g + r - d) - 1 = r + \max(0, (g - r - 1) + r - (d - r))$$

and $d_2' \geq r - 1$. (Otherwise $d_1 - r < r + \max(0, g + r - d) - 1$ and $d_2 - r + 1 < r - 1$, i.e. $d_1 - (r - 2) \leq r + \max(0, g + r - d)$ and $d_2 - (r - 2) \leq r - 1$; adding these contradicts (3).)

Thus by the inductive hypothesis, there are BN-curves $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ of degrees d_1' and d_2' respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of genus $g - r - 1$ with $H^1(N_{X' \cup Y'}) = 0$. To complete the inductive step, we take

$$(X, Y) = \begin{cases} (X' \cup L, Y' \cup R_2) & \text{if } d_1' = d_1 - 1; \\ (X' \cup R_1, Y') & \text{if } d_2' = d_2. \end{cases}$$

Here, R_1 is a rational normal curve of degree r that is $(r+2)$-secant to X', and L is a 1-secant line to X', and R_2 is a rational normal curve of degree $r - 1$ intersecting Y' in $r + 1$ points and passing through $L \cap H$.

Tracing through the proof, we notice then when (2) is satisfied, we add a 1-secant line to X at least as many times as we add an $(r+2)$-secant rational normal curve of degree r. In particular, when (2) holds, the curve X we constructed is nonspecial. □

Lemma 5.3. *Let $H' \subset \mathbb{P}^r$ be a hyperplane, and (d, g) be integers with $\rho(d, g, r) \geq 0$. Write*

$$d_1 = 1 + \max(0, g + r - d) \quad \textit{and} \quad d_2 = d - d_1.$$

Then there exists a rational curve $X \subset \mathbb{P}^r$, and a BN-curve $Y \subset H'$, of degrees d_1 and d_2 respectively, with $X \cap Y$ general, such that $X \cup Y \subset \mathbb{P}^r$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$.

Proof. We argue by induction on $\rho(d, g, r)$.

When $\rho(d, g, r) = 0$, then $(d, g, r) = (r(t+1), (r+1)t, r)$ for some nonnegative integer t; so for $\rho(d, g, r) = 0$ we may argue by induction on t. When $t = 0$, we let X be a line, and Y be a rational normal curve of degree $r - 1$ in H', passing through $X \cap H'$; by Lemma 2.1, the union $X \cup Y$ is a BN-curve with $H^1(N_{X\cup Y}) = 0$. For the inductive step, we let $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ be of degrees $d_1 - 1$ and $d_2 - r + 1$ respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of degree $d - r$ and genus $g - r - 1$ with $H^1(N_{X'\cup Y'}) = 0$. We then pick a general point $p \in H'$, and let

$$X = X' \cup L \quad \text{and} \quad Y = Y' \cup R,$$

where L is a 1-secant line to X' through p, and R is a rational normal curve of degree $r - 1$ which is $(r + 1)$-secant to Y' and passes through p. Applying Lemmas 2.1 and 2.2, we conclude that the union $X \cup Y$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$ as desired.

For the inductive step, we let $X' \subset \mathbb{P}^r$ and $Y' \subset H'$ be of degrees d_1 and $d_2 - 1$ respectively, with $X' \cap Y'$ general, such that $X' \cup Y' \subset \mathbb{P}^r$ is a BN-curve of degree $d - 1$ and genus $g' := \max(0, g - 1)$ with $H^1(N_{X'\cup Y'}) = 0$. We then take

$$X = X' \quad \text{and} \quad Y = Y' \cup L,$$

where L is a line which is 1-secant to Y' if $g = g'$ and 2-secant otherwise. Applying Lemma 2.1, we conclude that the union $X \cup Y$ is a BN-curve of genus g with $H^1(N_{X\cup Y}) = 0$ as desired. □

Lemma 5.4. *Let $C_1, C_2, \ldots, C_n \subset \mathbb{P}^r$ be independently general BN-curves, of degrees d_i and genera g_i, and $H, H' \subset \mathbb{P}^r$ be transverse hyperplanes. Let d' and d'' be nonnegative integers with*

$$d' + d'' = \sum d_i \quad \text{and} \quad d' \geq r - 1 + \sum [1 + \max(0, g_i + r - d_i)].$$

Then we may specialize the C_i to curves C_i° with $\sum \#(C_i^\circ \cap H \cap H') = d''$, so that

$$C_1^\circ \cap H \cap H', C_2^\circ \cap H \cap H', \ldots, C_n^\circ \cap H \cap H' \subset H \cap H' \quad \text{and}$$
$$C_1^\circ \cap H \smallsetminus H', C_2^\circ \cap H \smallsetminus H', \ldots, C_n^\circ \cap H \smallsetminus H' \subset H$$

are sets of subsets of hyperplane sections of independently general BN-curves.

Moreover, we can assume the second of these sets is a set of subsets of hyperplane sections of independently general nonspecial BN-curves if

$$d'' \geq (r-1) \cdot \sum \left\lceil \frac{\max(0, g_i + r - d_i)}{2} \right\rceil. \tag{4}$$

Proof. We first note that it suffices to consider the case where all C_i are special. Indeed, if $C_1, C_2, \ldots, C_m$ are special, and $C_{m+1}, C_{m+2}, \ldots, C_n$ are nonspecial, then the result for $C_1, C_2, \ldots, C_n$ follows from the result for $C_1, C_2, \ldots, C_m$ combined with Corollary 4.2 for $C_{m+1}, C_{m+2}, \ldots, C_n$.

We may thus suppose C_i is special for all i. In particular, for all i,

$$d_i \geq d_i - [(r+1)d_i - rg_i - r(r+1)] = r + r(g_i + r - d_i). \tag{5}$$

We now argue by induction on n. When $n = 1$, this follows from Lemmas 5.2 and 2.4 if $d'' \geq r - 1$. If $d'' \leq r - 2$, then by the uniform position principle, the points of $C_1 \cap H$ are in linear general position. We may therefore apply an automorphism of H so that exactly d'' of these points lie in H'.

For the inductive step, note that Equation (5) gives, in combination with $g_n + r - d_n \geq 1$,

$$d_n - r - \max(0, g_n + r - d_n) \geq (r-1) \cdot \left\lceil \frac{\max(0, g_n + r - d_n)}{2} \right\rceil \geq r - 1.$$

In particular, so long as

$$2r - 2 + \sum [1 + \max(0, g_i + r - d_i)] \leq d' \leq d - r + 1,$$

we may combine our inductive hypothesis (for $C_1, \ldots, C_{n-1}$) with Lemmas 5.2 and 2.4 (for C_n) to deduce the result.

If $d' \geq d - r + 2$, the result follows from our inductive hypothesis (for $C_1, \ldots, C_{n-1}$); we do not specialize C_n.

Finally, if $r - 1 + \sum[1 + \max(0, g_i + r - d_i)] \leq d' \leq 2r - 2 + \sum[1 + \max(0, g_i + r - d_i)]$, then upon rearrangement,

$$d' - [1 + \max(0, g_n + r - d_n)] \leq 2r - 2 + \sum_{i<n}[1 + \max(0, g_n + r - d_n)],$$

so by combining Lemmas 5.3 and 2.4(for C_n) with our inductive hypothesis (for $C_1, \ldots, C_{n-1}$), it suffices to show $2r - 2 + \sum_{i<n}[1 + \max(0, g_n + r - d_n)] \leq \sum_{i<n} d_i$, which follows in turn from

$$d_1 \geq 2r - 1 + \max(0, g_1 + r - d_1),$$

which in turn follows from Equation (5) together with $g_n + r - d_n \geq 1$. $\square$

6. The inductive argument

In this section, we combine the results of the previous three sections to inductively prove the hyperplane maximal rank theorem. This essentially boils down to manipulating inequalities to show that we can choose the integers (d', d'') appearing in the previous section in the appropriate fashion.

We begin by giving some bounds on the expressions appearing in Lemma 5.2 that are easier to manipulate.

Lemma 6.1. *Let d, g, and r be integers with $\rho(d, g, r) \geq 0$. Then*

$$1 + \max(0, g + r - d) \leq \frac{d}{r} \quad \textit{and} \quad (r - 1) \cdot \left\lceil \frac{\max(0, g + r - d)}{2} \right\rceil \leq \frac{r - 1}{2r} \cdot d.$$

Proof. By assumption,

$$r \cdot (g + r - d) \leq r \cdot (g + r - d) + (r + 1)d - rg - r(r + 1) = d - r$$
$$\Rightarrow \max(0, g + r - d) \leq \frac{d - r}{r}.$$

Substituting this in, we find

$$1 + \max(0, g + r - d) \leq 1 + \frac{d - r}{r} = \frac{d}{r}$$
$$\left\lceil \frac{\max(0, g + r - d)}{2} \right\rceil \leq \frac{\frac{d-r}{r} + 1}{2} = \frac{r - 1}{2r} \cdot d. \qquad \square$$

Lemma 6.2. *Let d, r, and m be integers with*

$$r \geq 4, \quad m \geq 3, \quad \textit{and} \quad d \geq 2r + 2.$$

Assume that

$$d \geq \binom{m+r-1}{m}, \quad \textit{respectively} \quad d \leq \binom{m+r-1}{m}.$$

Then there are integers d' and d'' such that $d = d' + d''$ and

$$d' \geq \binom{m+r-2}{m-1} \quad \textit{and} \quad d'' \geq \binom{m+r-2}{m},$$
$$\textit{respectively} \quad d' \leq \binom{m+r-2}{m-1} \quad \textit{and} \quad d'' \leq \binom{m+r-2}{m},$$

which moreover satisfy

$$d' \geq r - 1 + \frac{d}{r} \quad \textit{and} \quad d'' \geq r - 1.$$

Additionally, if $m = 3$, we can replace $d'' \geq r - 1$ by the stronger assumption that

$$d'' \geq \frac{r-1}{2r} \cdot d.$$

Proof. First we consider the case where

$$d = \binom{m+r-1}{m} \geq \binom{r+2}{3} \geq 2r + 2.$$

In this case, we take

$$d' = \binom{m+r-2}{m-1} \quad \text{and} \quad d'' = \binom{m+r-2}{m}.$$

To see that these satisfy the given conditions, first note that

$$d'' \geq \binom{r+1}{3} \geq r - 1.$$

Next note that

$$\binom{m+r-1}{m} \geq \frac{r-1}{\frac{m}{m+r-1} - \frac{1}{r}};$$

indeed, the LHS is an increasing function of m, the RHS is a decreasing function of m, and the inequality is obvious for $m = 3$. Rearranging, we get

$$d' = \binom{m+r-2}{m-1} = \frac{m}{m+r-1} \cdot \binom{m+r-1}{m} \geq r - 1 + \frac{1}{r} \cdot \binom{m+r-1}{m} = r - 1 + \frac{d}{r}.$$

If $m = 3$, then

$$d'' = \binom{r+1}{3} \geq \frac{r-1}{2r} \cdot \binom{r+2}{3} = \frac{r-1}{2r} \cdot d.$$

In general, we induct upwards on d in the $\geq$ case and downwards on d in the $\leq$ case. To do this, we want to show that if d' and d'' satisfy

$$d' \geq r - 1 + \frac{d}{r} \quad \text{and} \quad d'' \geq \frac{r-1}{2r} \cdot d \quad \text{where} \quad d = d' + d'' \geq 2r + 2,$$

then either $(d' - 1, d'')$ or $(d', d'' - 1)$, as well as either $(d' + 1, d'')$ or $(d', d'' + 1)$, satisfy the above two conditions. We note that

$$\begin{aligned} d' \geq r - 1 + \frac{d}{r} = r - 1 + \frac{d' + d''}{r} \quad &\Leftrightarrow \quad (r-1)d' \geq r(r-1) + d''. \\ d'' \geq \frac{r-1}{2r} \cdot d = \frac{r-1}{2r} \cdot (d' + d'') \quad &\Leftrightarrow \quad (r+1)d'' \geq (r-1)d'. \end{aligned}$$

Assume (to the contrary) that neither $(d'-1, d'')$ nor $(d', d''-1)$ satisfy the conditions, respectively that neither $(d'+1, d'')$ nor $(d', d''+1)$ satisfy the conditions. Then we must have

$$\begin{aligned} (r-1)(d'-1) < r(r-1) + d'' \quad &\text{and} \quad (r+1)(d''-1) < (r-1)d', \\ \text{respectively} \quad (r-1)d' < r(r-1) + d'' + 1 \quad &\text{and} \quad (r+1)d'' < (r-1)(d'+1). \end{aligned}$$

Equivalently, we must have

$$\begin{aligned} (r-1)(d'-1) + 1 \leq r(r-1) + d'' \quad &\text{and} \quad (r+1)(d''-1) + 1 \leq (r-1)d', \\ \text{respectively} \quad (r-1)d' \leq r(r-1) + d'' \quad &\text{and} \quad (r+1)d'' + 1 \leq (r-1)(d'+1). \end{aligned}$$

Adding twice the first equation to the second, we must have

$$\begin{aligned} 2(r-1)(d'-1) + 2 + (r+1)(d''-1) + 1 &\leq 2r(r-1) + 2d'' + (r-1)d', \\ \text{respectively} \quad 2(r-1)d' + (r+1)d'' + 1 &\leq 2r(r-1) + 2d'' + (r-1)(d'+1). \end{aligned}$$

Simplifying yields

$$(r-1)(d' + d'') \leq 2r^2 + r - 4, \quad \text{respectively} \quad (r-1)(d' + d'') \leq 2r^2 - r - 2.$$

In particular,

$$d = d' + d'' \leq \frac{2r^2 + r - 4}{r - 1} = 2r + 3 - \frac{1}{r-1} \quad \Rightarrow \quad d \leq 2r + 2.$$

Consequently, we can reach via upward and downward induction every value of d that is at least $2r + 2$. $\square$

Proof of the Hyperplane Maximal Rank Theorem. We use induction on m and r. For $m = 2$, this is a consequence of Proposition 4.3; for $r = 3$, this is a consequence of Corollary 3.1. Note that if $\sum d_i \leq 2r - 1$, then all the C_i are nonspecial and so $H^0(\mathcal{O}_H(2)) \to H^0(\mathcal{O}_{C \cap H}(2))$ is surjective; consequently, $H^0(\mathcal{O}_H(m)) \to H^0(\mathcal{O}_{C\cap H}(m))$ is surjective for all $m \geq 2$. Thus, we may suppose $\sum d_i \geq 2r$.

For the inductive step, we define integers (d', d'') as follows. If $\sum d_i \in \{2r, 2r+1\}$, we take $(d', d'') = (r+1, \sum d_i - r - 1)$. Otherwise, for $\sum d_i \geq 2r+2$, we let (d', d'') be as in Lemma 6.2. Fix another hyperplane H' transverse to H. By Lemma 5.4, plus Lemma 6.1 when $d \geq 2r + 2$, we may specialize the C_i to curves C_i° with $\sum \#(C_i^\circ \cap H \cap H') = d''$, so that

$$X := (C_1^\circ \cap H \cap H') \cup (C_2^\circ \cap H \cap H') \cup \cdots \cup (C_n^\circ \cap H \cap H') \subset H \cap H' \quad \text{and}$$
$$Y := (C_1^\circ \cap H \smallsetminus H') \cup (C_2^\circ \cap H \smallsetminus H') \cup \cdots \cup (C_n^\circ \cap H \smallsetminus H') \subset H$$

are unions of subsets of hyperplane sections of independently general BN-curves. Moreover, if $m = 3$, then we can arrange for X to be a union of subsets of hyperplane sections of independently general nonspecial BN-curves. By our inductive hypothesis, Lemma 2.5, and the uniform position principle, we know that the restriction maps

$$H^0(\mathcal{O}_H(m-1)) \to H^0(\mathcal{O}_Y(m-1)) \quad \text{and} \quad H^0(\mathcal{O}_{H\cap H'}(m)) \to H^0(\mathcal{O}_X(m))$$

are of maximal rank.

Define

$$i = \begin{cases} 0 & \text{if } \sum d_i \geq \binom{r+m-1}{m}, \\ 1 & \text{if } \sum d_i \leq \binom{r+m-1}{m}; \end{cases}$$

so we want to show $H^i(\mathcal{I}_{(X\cup Y)\cap H}(m)) = 0$. The exact sequence of sheaves

$$0 \to \mathcal{I}_{(X\cap H)/H}(m-1) \to \mathcal{I}_{(X\cup Y)\cap H/H}(m) \to \mathcal{I}_{(Y\cap H)/(H\cap H')}(m) \to 0,$$

gives rise to a long exact sequence in cohomology

$$\cdots \to H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) \to H^i(\mathcal{I}_{(X\cup Y)\cap H/H}(m)) \to H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) \to \cdots.$$

By the inductive hypothesis, we have $H^i(\mathcal{I}_{(X\cap H)/H}(m-1)) = H^i(\mathcal{I}_{(Y\cap H)/(H\cap H')}(m)) = 0$. Consequently, $H^i(\mathcal{I}_{(X\cup Y)\cap H/H}(m)) = 0$, as desired. $\square$

References

[1] E. Ballico, Embeddings of general curves in projective spaces: the range of the quadrics, Lith. Math. J. 52 (2) (2012) 134–137.
[2] R. Hartshorne, A. Hirschowitz, Smoothing algebraic space curves, in: Algebraic Geometry, Sitges (Barcelona), 1983, in: Lecture Notes in Math., vol. 1124, Springer, Berlin, 1985, pp. 98–131.
[3] A. Hirschowitz, Sur la postulation générique des courbes rationnelles, Acta Math. 146 (3–4) (1981) 209–230.
[4] David Jensen, Sam Payne, Tropical independence II: the maximal rank conjecture for quadrics, Algebra Number Theory 10 (8) (2016) 1601–1640.
[5] Eric Larson, Degenerations of curves in projective space and the maximal rank conjecture, https://arxiv.org/abs/1809.05980.
[6] Eric Larson, The generality of a section of a curve, http://arxiv.org/abs/1605.06185.
[7] Eric Larson, The maximal rank conjecture, https://arxiv.org/abs/1711.04906.