



Interpolation for Brill–Noether curves in $\mathbb{P}^4$

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Received: 21 December 2018 / Revised: 20 September 2019 / Accepted: 27 March 2020 /
Published online: 26 June 2020
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Abstract

We compute the number of general points through which a general Brill–Noether curve in $\mathbb{P}^4$ passes. We also prove an analogous theorem when some points are constrained to lie in a transverse hyperplane. As explained in [10], these results play an essential role in the first author’s proof of the Maximal Rank Conjecture [9].

1 Introduction

Interpolation of curves through points is a fundamental problem in algebraic geometry whose study dates back to antiquity. The first cases that were studied historically were rational plane curves. Some of these appear in Euclid’s *Elements*, such as the first postulate that there exists a line through two points, and Proposition 5 from book IV, which shows that every triangle may be inscribed in a circle (i.e., that there exists a circle through any three non-collinear points). Other classical results include the formula—often attributed to Lagrange, but originally discovered by Waring in 1779 [14]—for the unique degree d polynomial with specified values at $d + 1$ distinct inputs.

The modern study of interpolation has focused on extensions of these results to curves of higher genus in higher-dimensional projective spaces. The natural setting for this problem is to consider *Brill–Noether curves*. By the Brill–Noether Theorem, when $\rho(d, g, r) = (r + 1)d - rg - r(r + 1) \geq 0$, there is a unique component $\overline{M}_g(\mathbb{P}^r, d)^\circ$ of Kontsevich’s space $\overline{M}_g(\mathbb{P}^r, d)$ of stable maps dominating $\overline{M}_g$, which

The first author acknowledges the generous support both of the Fannie and John Hertz Foundation, and of the Department of Defense (NDSEG fellowship). The second author acknowledges the generous support of the National Science Foundation Graduate Research Fellowship Program under DGE-1122374.

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is of dimension

$$(r+1)d - (r-3)(g-1).$$

We call curves parameterized by this component *Brill–Noether curves* (BN-curves). Since passing through a point in $\mathbb{P}^r$ is expected to impose $r-1$ independent conditions, BN-curves are expected to interpolate through

$$f(d, g, r) := \left\lfloor \frac{(r+1)d - (r-3)(g-1)}{r-1} \right\rfloor$$

general points. While there are known exceptions in low-dimensional projective spaces, we expect:

Conjecture 1.1 *For all but finitely many tuples (d, g, r) , a Brill–Noether curve of degree d and genus g in $\mathbb{P}^r$ interpolates through $f(d, g, r)$ general points (and no more).*

This conjecture has historically attracted significant attention. A related question for plane curves (i.e., $r = 2$) was investigated using degeneration by Hirschowitz in [5]. When $r = 3$, Perrin [11] investigated Conjecture 1.1 using liaison, and Atanasov [2] proved it when $d \geq g + 3$. The conjecture for $r = 3$ was finally settled by the second author in [13]. In higher dimensions, asymptotic results when d is large compared with g and r have been obtained by Ballico [3]. The most complete result for general r is [1, Theorem 1.3], where the case $d \geq g + r$ of the conjecture was settled.

The interpolation problem is hardest when the degree is small relative to the genus and dimension of the projective space. This is for two reasons:

- All known counterexamples occur when the degree is as small as possible given g and r . Consequently, any techniques which can probe curves where d is small relative to g and r , must be complicated enough to work in exactly the cases where the conjecture is true.
- The dimension of the space of maps from a curve of genus g to $\mathbb{P}^r$ is an increasing function of the degree. Therefore the map may only be degenerated in limited ways when the degree is small. In the range $d \geq g + r$, a BN-curve can be degenerated to a union of lines, where each line is attached inductively to general points on the curve. Such degenerations are *only* available in this range, so methods of [1] cannot hope to probe the range $d < g + r$ without new ideas.

Outside of the results of Vogt [13] when $r = 3$ and Stevens [12] when $d = 2r$ and $g = r + 1$, Conjecture 1.1 has remained open in the range $d < g + r$ in higher-dimensional projective spaces.

Additionally, one can study a variant of the above conjecture where we constrain points to lie on a hypersurface. In low dimensions ($r \leq 4$), the intersection of a general Brill–Noether curve of degree d and genus g with a transverse hypersurface S of degree n in $\mathbb{P}^r$ exhibits behavior that does not appear in higher-dimensional projective spaces ($r \geq 5$). Namely, for exactly five values of (r, n) — $(r, n) \in \{(2, 1), (2, 2), (3, 1), (3, 2), (4, 1)\}$ — this intersection is a general set of dn points

on S for all but finitely many (d, g) [8]. In these five cases, it is therefore natural to ask a stronger question: can one pass a general Brill–Noether curve of degree d and genus g through $f(d, g, r)$ points which are general subject to the constraint that dn of them lie on a transverse hypersurface of degree n ? This question is straightforward for $(r, n) \in \{(2, 1), (2, 2)\}$, and work of [8, 13] answer this stronger question for $(r, n) = (3, 1)$ and $(r, n) = (3, 2)$, so the only remaining case is $(r, n) = (4, 1)$.

In this paper, we focus on the case $r = 4$. This is a natural question because it is both the smallest unsolved dimension for the interpolation problem and the largest dimension in which the above phenomena occur. From our main Theorem 1.4, we deduce a complete answer to all of the above questions when $r = 4$:

Corollary 1.2 *A general Brill–Noether curve of degree d and genus g in $\mathbb{P}^4$ passes through a set of $f(d, g, 4)$ points which are general subject to the constraint that d of them lie on a transverse hyperplane if and only if*

$$(d, g) \notin \{(8, 5), (9, 6), (10, 7)\}.$$

Remark 1.3 The geometry of these three cases is as follows:

- In these three cases, the statement fails because the section by a transverse hyperplane of the general curve is not a general set of d points in the hyperplane; instead it is a set of d points which are general subject to the condition that they are distinct points which lie on a complete intersection of $11 - d$ quadrics.
- Nevertheless, for $(d, g) \in \{(8, 5), (10, 7)\}$ —but not for $(d, g) = (9, 6)$ —a general Brill–Noether curve of this degree and genus still passes through $f(d, g, 4) + 1$ points which are general subject to the constraint that d of them lie on a transverse hyperplane and are distinct points which lie on a complete intersection of $11 - d$ quadrics.
- This fails for $(d, g) = (9, 6)$ because a general Brill–Noether curve of this degree and genus lies on a quartic del Pezzo surface in $\mathbb{P}^4$. However, such a curve still passes through $f(d, g, 4)$ points which are general subject to either the constraint that d of them lie on a transverse hyperplane and are distinct points which lie on a complete intersection of $11 - d$ quadrics, or the constraint that $d - 1$ of them lie on a transverse hyperplane.

In addition to being interesting in their own right, these results have already found applications in other work. As outlined in [10], the proof of the Maximal Rank Conjecture revolves around constructing curves through sets of points, some of which are constrained to lie in hypersurfaces, and the results of the present paper play a key role.

We deduce these results by studying the normal bundle N_C (respectively the twisted normal bundle $N_C(-1)$) for C a general Brill–Noether curve. We say that a vector bundle E on a curve C satisfies interpolation if it is nonspecial ($h^1(E) = 0$) and for a general effective divisor D of any degree, either

$$h^0(E(-D)) = 0, \quad \text{or} \quad h^1(E(-D)) = 0.$$

The main result is then:

Theorem 1.4 *Let C be a general Brill–Noether curve of degree d and genus g in $\mathbb{P}^4$. Then $N_C(-1)$ satisfies interpolation if and only if*

$$(d, g) \notin \{(6, 2), (8, 5), (9, 6), (10, 7)\}.$$

Furthermore, if D is an effective divisor of degree $d - 1$ supported in a general hyperplane section, then $N_C(-D)$ satisfies interpolation if and only if

$$(d, g) \neq (6, 2).$$

Using Theorem 1.4, some additional work in the cases $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$ yields:

Theorem 1.5 *Let C be a general Brill–Noether curve of degree d and genus g in $\mathbb{P}^4$. Then N_C satisfies interpolation if and only if*

$$(d, g) \neq (6, 2).$$

In turn, we deduce an answer to the first question posed earlier for $r = 4$:

Corollary 1.6 *A general Brill–Noether curve of degree d and genus g in $\mathbb{P}^4$ passes through a set of $f(d, g, 4)$ general points, with no exceptions.*

The main techniques in this paper are related to those in [1, 8], namely we degenerate to reducible curves. As interpolation is an open condition, it suffices to find one BN-curve of each relevant degree and genus whose (twisted) normal bundle satisfies interpolation. We achieve this by specializing to nodal curves in the boundary of $\overline{M}_g(\mathbb{P}^4, d)^\circ$, where we can relate interpolation for the normal bundle of the union to interpolation for some modifications of the normal bundles of the components. The degenerations here, though, are necessarily more complicated than those in [1], as degenerating to the union of a curve and a 1- or 2-secant line alone cannot hope to apply to special curves where $d < g + r$. We employ three primary classes of degenerations.

First, as developed in [8], we degenerate to a BN-curve which is the union of a curve $X \subset H \simeq \mathbb{P}^3$ in a hyperplane and a nonspecial curve Y , transverse to H , meeting X transversely at a finite set of specified points. This allows us to leverage the known interpolation results in the nonspecial range [1] and in $\mathbb{P}^3$ [13] to inductively prove interpolation for all BN-curves from a finite set of base cases. Resolving these base cases consumes the remainder of the paper.

The second type of degeneration is to a connected nodal curve containing a canonical curve $D \subset \mathbb{P}^4$ as well as three 2-secant lines to D . Using the geometry of the canonical curve, in particular the fact that $N_D(-1) \simeq K_D^{\oplus 3}$, we prove a delicate degeneration lemma to reduce to the problem of interpolation for the normal bundle of the union to interpolation for a simple modification of the normal bundle of the remainder.

The final specialization is to the union of a degenerate rational normal quartic curve meeting the other component in six points. The rational normal quartic curve is the union of three 2-secant lines, and the unique fourth line meeting all three. We hope that this technique is robust, and that some variant may be useful for related problems.

The paper is organized as follows. In Sect. 2 we briefly recall basic facts about modifications of vector bundles on curves and the technique of degenerating to reducible curves to prove interpolation. In Sect. 3 we employ the first degeneration to the union of a curve in a hyperplane and a curve transverse to the hyperplane to reduce to checking a finite set of cases. The canonical curve degeneration lemma is in Sect. 4 and the degeneration with the rational quartic curve is in Sect. 5. At this point all of the difficult work has been done, and we collect loose ends in Sect. 6 to prove the main theorems.

2 Preliminaries

2.1 Notation

In this paper, we deal with the Kontsevich space compactification of Brill–Noether curves of degree d and genus g in $\mathbb{P}^r$. In particular, for the remainder of the paper, a curve $C \rightarrow \mathbb{P}^r$ is shorthand for a stable map $f: C \rightarrow \mathbb{P}^r$. In all of our computations, we will assume that the map f is unramified, so that the normal sheaf of the map, which we will denote by N_C when the map f is implicit, or N_f to draw attention to the map, is a vector bundle and its cohomology controls the deformation theory of f . A general such map is an embedding of a smooth curve, and so N_f is isomorphic to the normal bundle of the embedded smooth curve. The only subtlety arises when degenerating such curves.

Suppose that $f: C \rightarrow \mathbb{P}^r$ and $g: D \rightarrow \mathbb{P}^r$ are curves of degree and genus (d, g) and (d', g') respectively, and $\{p_i\}_{i \in I} \in C$ and $\{q_i\}_{i \in I} \in D$ are points such that

$$f(p_i) = g(q_i), \quad \text{for all } i \in I.$$

Write $\Gamma = f(\{p_i\}) = g(\{q_i\})$. Let $C \cup_\Gamma D$ denote the nodal curve obtained by gluing $p_i \in C$ to $q_i \in D$, mapped to $\mathbb{P}^r$ via the stable map

$$f \cup g: C \cup_\Gamma D \rightarrow \mathbb{P}^r.$$

If $C \rightarrow \mathbb{P}^r$ and $D \rightarrow \mathbb{P}^r$ are curves passing through a finite set of points $\Gamma \subset \mathbb{P}^r$, then we use $C \cup_\Gamma D$ to denote the same construction for some choice of $\{p_i\}$ and $\{q_j\}$ mapping isomorphically onto Γ under f and g respectively.

Note that the curve $C \cup_\Gamma D$ is of degree $d + d'$ and genus $g + g' + \# I - 1$ even if $f(C)$ and $g(D)$ meet away from Γ . This is our primary interest in using the Kontsevich space compactification.

2.2 Modifications of vector bundles

In this section we recall some basic facts about modifications of vector bundles on curves to streamline the coming computations. For a more detailed exposition, see [1, Sections 2–3].

Definition 2.1 Let E be a vector bundle on a variety X and let $D \subset X$ be an effective Cartier divisor. Suppose that $U \subset X$ is an open subset containing D and that $F \subseteq E|_U$ is a subbundle defined on U . Then the *modification of E at D towards F* , denoted $E[D \rightarrow F]$, is defined by the exact sequence

$$0 \rightarrow E[D \rightarrow F] \rightarrow E \rightarrow (E/F)|_D \rightarrow 0.$$

The reason for the terminology is that sections of the modification $E[D \rightarrow F]$ “point towards F ” when restricted to the divisor D :

$$H^0(E[D \rightarrow F]) = \{\sigma \in H^0(E) : \sigma|_D \in H^0(F|_D)\}.$$

The modified bundle $E[D \rightarrow F]$ is isomorphic to E when restricted to $X \setminus D$. Therefore given another divisor $D' \subset X \setminus D$ and another subbundle $F' \subseteq E$, we may form the multiple modification

$$E[D \rightarrow F][D' \rightarrow F']$$

taking our open set $U = X \setminus D$. In this setting

$$E[D \rightarrow F][D' \rightarrow F'] \simeq E[D' \rightarrow F'][D \rightarrow F];$$

however we will need multiple modifications in a stronger setting in which D and D' are allowed to meet. For that reason, we have the following definition:

Definition 2.2 Let $\{F_i \subseteq E|_{U_i}\}_{i \in I}$ be a finite collection of subbundles of E defined on open neighborhoods U_i of a point $x \in X$. We say that this collection is *tree-like at $x \in X$* if for all index subsets $I' \subseteq I$, either

- the fibers $\{F_i|_x\}_{i \in I'} \subset E|_x$ are linearly independent, or
- there are distinct indices $i, j \in I'$, and an open $U \subset U_i \cap U_j$ containing x such that $F_i|_U \subset F_j|_U$.

Definition 2.3 A *modification datum* M for a vector bundle E is a finite collection of tuples $\{(D_i, U_i, F_i)\}_{i \in I}$, where D_i is an effective Cartier divisor on X , and $U_i \subset X$ is an open subset containing D_i , and $F_i \subseteq E|_{U_i}$ is a subbundle. We say that a modification datum is *tree-like* if for all $x \in X$, the collection of subbundles $\{F_i \subseteq E|_{U_i} : x \in D_i\}$ is tree-like at x .

The key result is then the following:

Proposition 2.4 ([1, Proposition 2.17]) *There is a bijection φ between modification data M for E such that $(D, U, F) \cup M$ is tree-like and modification data M' for $E[D \rightarrow F]$ such that $(D, U, F) \cup M'$ is tree-like. This bijection is compatible with pullbacks and restricts to the identity when $D = \emptyset$.*

We may therefore recursively define $E[M]$, for $M = (D, U, F) \cup N$ by

$$E[M] := E[D \rightarrow F][\varphi(N)].$$

For simplicity, when $M = \{(D_i, U_i, F_i)\}_{1 \leq i \leq n}$, we will write

$$E[M] = E[D_1 \rightarrow F_1] \cdots [D_n \rightarrow F_n].$$

We have the following nice properties of tree-like modifications:

Proposition 2.5 ([1, Propositions 2.20, 2.21, 2.23]) *Let E be a vector bundle on X and let M be a tree-like modification datum. Then we have*

- (Commuting modifications) *If M' is any reordering of M , then $E[M] \simeq E[M']$.*
- (Commuting with twists) *If D is any effective Cartier divisor, we have $E[M](D) \simeq E(D)[M]$.*
- (Combining modifications to the same bundle) *If D and D' are effective Cartier divisors, we have*

$$E[D \rightarrow F][D' \rightarrow F] \simeq E[D + D' \rightarrow F].$$

- (Combining modifications at the same divisor) *If F_1 and F_2 are independent, then*

$$E[D \rightarrow F_1][D \rightarrow F_2] \simeq E[D \rightarrow F_1 \oplus F_2](-D).$$

Remark 2.6 By the curve-to-projective extension theorem, vector subbundles of a vector bundle E on a curve C are naturally in bijection with vector subbundles of $E|_{C \setminus D}$ for any divisor D disjoint from C_{sing} . Since $E|_{C \setminus D} \simeq E[D \rightarrow F]|_{C \setminus D}$, vector subbundles of E are thus in bijection with vector subbundles of $E[D \rightarrow F]$. This implies the analog of Proposition 2.4 for arbitrary collections of modifications (i.e., if we drop the tree-like assumption on both sides).

However, we still need the language of tree-like modifications for two reasons. First, the properties of Proposition 2.5 will not, in general, hold when the modifications are not tree-like.

Second, we need the more general formalism of modifications on arbitrary varieties in order to “limit” various modifications together by considering vector bundles on the total space of a family of curves, see Proposition 2.8 below for an illustrative example. Compatibility with base-change guarantees that the results agrees with the expected result on each curve.

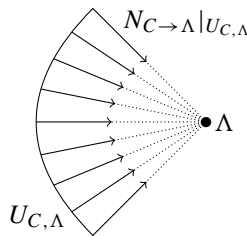
We will deal primarily with modifications of the normal bundles of curves $C \rightarrow \mathbb{P}^r$ towards pointing bundles $N_{C \rightarrow \Lambda}$, for $\Lambda \subset \mathbb{P}^r$ a linear space of dimension γ . To recall, suppose that the locus

$$U_{C, \Lambda} := \{p \in C : T_p C \cap \Lambda = \emptyset\}$$

is dense and contains the singular locus of C . Then on $U_{C,\Lambda}$, we suggestively define $N_{C \rightarrow \Lambda}|_{U_{C,\Lambda}}$ to be the kernel of the map

$$N_C|_{U_{C,\Lambda}} \rightarrow N_{\pi_\Lambda|_{U_{C,\Lambda}}},$$

where π_Λ denotes the map of projection from Λ , and $N_{\pi_\Lambda|_{U_{C,\Lambda}}}$ is the normal sheaf of $\pi_\Lambda|_{U_{C,\Lambda}}$.



By our assumptions on $U_{C,\Lambda}$, the curve-to-projective extension theorem implies that there is a unique vector subbundle $N_{C \rightarrow \Lambda} \subseteq N_C$ agreeing on $U_{C,\Lambda}$ with $N_{C \rightarrow \Lambda}|_{U_{C,\Lambda}}$.

When modifying towards a pointing bundle, we will write

$$N_C[D \rightarrow \Lambda] := N_C[D \rightarrow N_{C \rightarrow \Lambda}].$$

These modifications naturally arise when computing the restrictions of normal bundles of nodal curves to the irreducible factors, as in the following fundamental result of Hartshorne and Hirschowitz.

Lemma 2.7 ([4, Corollary 3.2]) *Let $C_1 \cup_\Gamma C_2$ be a nondegenerate nodal curve in $\mathbb{P}^r$, with $\Gamma = \{p_1, \dots, p_n\}$. For each i , let p'_i be a choice of point on the tangent line $T_{p_i} C_1$ and let p''_i be a choice of point on the tangent line $T_{p_i} C_2$. Then we have*

$$\begin{aligned} N_{C_1 \cup_\Gamma C_2}|_{C_1} &\simeq N_{C_1}(\Gamma)[p_1 \rightarrow p'_1] \cdots [p_n \rightarrow p'_n], \\ N_{C_1 \cup_\Gamma C_2}|_{C_2} &\simeq N_{C_2}(\Gamma)[p_1 \rightarrow p'_1] \cdots [p_n \rightarrow p'_n]. \end{aligned}$$

As indicated above, we will make ample use of the fact that the semicontinuity theorem guarantees that interpolation for a vector bundle E on C follows from interpolation for some specialization of E . We make this precise in the following example, but will omit these details in the future.

Proposition 2.8 *Let $C \rightarrow \mathbb{P}^r$ be a curve, and p_1 and p_2 distinct points on C . Let q_1 and q_2 be distinct points in $\mathbb{P}^r$.*

(i) *If p_2 is general and $T_{p_1} C, q_1, q_2$ span a $\mathbb{P}^3$, interpolation for*

$$N_C[p_1 \rightarrow q_1][p_1 \rightarrow q_2] \simeq N_C(-p_1)[p_1 \rightarrow q_1 + q_2]$$

implies interpolation for $N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_2]$.

(ii) If q_2 is general and $T_{p_2}C, q_1$ span a $\mathbb{P}^2$, interpolation for

$$N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_1] \simeq N_C[p_1 + p_2 \rightarrow q_1]$$

implies interpolation for $N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_2]$.

Proof For part (i), let $B = C$ and let $\Delta \subset B \times C$ denote the diagonal. Write P_i for the divisor $B \times \{p_i\} \subset B \times C$. Let $\pi_1: B \times C \rightarrow B$ and $\pi_2: B \times C \rightarrow C$ denote the two projections. Then we have inclusions of vector bundles on $B \times C$:

$$\pi_2^* N_{C \rightarrow q_1} \subseteq \pi_2^* N_C, \quad \pi_2^* N_{C \rightarrow q_2} \subseteq \pi_2^* N_C.$$

By our assumption that $T_{p_1}C, q_1, q_2$ are in linear general position, the modification datum

$$M = \{(P_1, B \times C, \pi_2^* N_{C \rightarrow q_1}), (\Delta, B \times C, \pi_2^* N_{C \rightarrow q_2})\}$$

is tree-like at $\Delta \cap P_1 = (p_1, p_1) \in B \times C$ and hence is tree-like. Therefore the multiple modification

$$\pi_2^* N_C[M] = \pi_2^* N_C[p_1 \rightarrow q_1][\Delta \rightarrow q_2]$$

is a vector bundle on $B \times C$, which restricts to $N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_2]$ on the fiber $\pi_1^{-1}(p_2)$ and $N_C[p_1 \rightarrow q_1][p_1 \rightarrow q_1]$ on the fiber $\pi_1^{-1}(p_1)$. By the semicontinuity theorem, interpolation for the second implies interpolation for the first with p_2 general, proving part (i).

For part (ii), let $B = \mathbb{P}^r$. It is shown in [1, Section 5] that as the linear space q_2 varies in B , the open subsets $U_{C, q_2} \subset C$ fit together into an open set $U_{B \times C, q_2} \subset B \times C$. Furthermore, over this open subset the pointing bundles $N_{C \rightarrow q_2}$ fit together into a vector bundle, which we will abusively also denote $N_{C \rightarrow q_2}$, on $U_{B \times C, q_2}$. By assumption, $T_{p_2}C \cap q_1 = \emptyset$, so $(q_1, p_2) \in U_{B \times C, q_2}$, and therefore after shrinking B we may assume that the divisor $P_2 \subset U_{B \times C, q_2}$. Therefore

$$(P_2, U_{C \times B, q_2}, N_{C \rightarrow q_2})$$

is a modification datum for $\pi_2^* N_C$, and the modification data

$$M = \{(P_1, B \times C, \pi_2^* N_{C \rightarrow q_1}), (P_2, U_{B \times C, q_2}, N_{C \rightarrow q_2})\}$$

is vacuously tree-like. Therefore the multiple modification $\pi_2^* N_C[M]$ is a vector bundle on $U_{B \times C, q_2}$ which restricts to

$$N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_1]$$

over the special point $q_1 \in B$ and to $N_C[p_1 \rightarrow q_1][p_2 \rightarrow q_2]$ for general q_2 in a neighborhood of q_1 in B . Interpolation for the second therefore follows from interpolation for the first by the semicontinuity theorem. $\square$

In the above proof, the generality assumptions on the points in C and linear subspaces in $\mathbb{P}^r$ guaranteed that the necessary modifications on the total space of the family of curves were well-defined and tree-like. In the remainder of the paper, when we wish to specialize either points or linear spaces to prove interpolation, we will indicate what genericity assumptions we are using, but leave the reader to fill in the details of the argument with the above example as a guide.

2.3 Interpolation for vector bundles

In the course of the proof, we will make use of the following lemma, which allows us to prove interpolation by finding a single divisor such that the twist of our vector bundle by that divisor has a vanishing cohomology group. As in [13], if the Euler characteristic $\chi(E)$ is divisible by the rank r of E , it suffices to find a single effective divisor of degree $\chi(E)/r$ such that $h^0(E(-D)) = 0$, or equivalently $h^1(E(-D)) = 0$. The following generalization covers all remaining possibilities for a vector bundle of rank 3.

Lemma 2.9 *Let E be a nonspecial vector bundle of rank r on a curve. Assume that $\chi(E) \equiv 1 \pmod{r}$ (respectively $\chi(E) \equiv -1 \pmod{r}$). If there exists an effective divisor D of degree $\lfloor \chi(E)/r \rfloor$ (respectively $\lceil \chi(E)/r \rceil$) such that $h^1(E(-D)) = 0$ (respectively $h^0(E(-D)) = 0$), then E satisfies interpolation.*

Proof By [1, Proposition 4.6] it suffices to find one effective divisor D_+ of degree $\lceil \chi(E)/r \rceil$ such that $h^0(E(-D_+)) = 0$ and one effective divisor D_- of degree $\lfloor \chi(E)/r \rfloor$ such that $h^1(E(D_-)) = 0$. By assumption we have one of these; we will show that we can construct the other by adding or subtracting a point.

Let D be the effective divisor on C specified in the statement of the lemma. As E is assumed to be nonspecial, the following sequence:

$$0 \rightarrow H^0(E(-D)) \rightarrow H^0(E) \rightarrow E|_D \rightarrow H^1(E(-D)) \rightarrow 0$$

is exact.

Suppose first that $\chi(E) \equiv 1 \pmod{r}$, so we can take $D_- = D$. We take $D_+ = D + p$ for $p \in C$ general. As $h^1(E(-D)) = 0$, we must have that $h^0(E(-D)) = 1$, and so $E(-D)$ has a unique global section. As such a section does not vanish at a general point $p \in C$, the twist $h^0(E(-D - p)) = 0$ as desired.

Suppose next that we are in the case $\chi(E) \equiv -1 \pmod{r}$, so we can take $D_+ = D$. Then

$$H^0(E) \hookrightarrow E|_D \simeq \bigoplus_{p \in D} E|_p$$

is the inclusion of a hyperplane in a vector space of dimension $\chi(E) + 1$. As the subspaces $E|_p$ for $p \in D$ generate $E|_D$, there is some $p \in D$ such that $E|_p \not\subset H^0(E)$.

If V is any vector space, $H \subset V$ any codimension 1 subspace, and $\Lambda \subset V$ any subspace not contained in H , the composition with the projection map

$$H \hookrightarrow V \rightarrow V/\Lambda$$

is surjective. Applying this with $V = E|_D$, and $H = H^0(E)$, and $\Lambda = E|_p$, we conclude that the composition with the projection map

$$H^0(E) \rightarrow E|_D \rightarrow E|_{D \setminus p}$$

is surjective, and hence $h^1(E(-D + p)) = 0$. $\square$

In the course of the proof of Theorem 1.4, we will need the following more general definition of interpolation for the space of sections of a vector bundle.

Definition 2.10 Let E be a vector bundle of rank r on a curve C and let $V \subset H^0(C, E)$ be some subspace of its space of global sections. For points $p_1, \dots, p_n \in C$, we will abbreviate

$$V(-p_1 - \dots - p_n) := \{\sigma \in V : \sigma|_{p_1} = \dots = \sigma|_{p_n} = 0\}.$$

Then we say that V satisfies interpolation if for general points $p_1, \dots, p_n \in C$,

$$\dim V(-p_1 - \dots - p_n) = \max(0, \dim V - rn).$$

Remark 2.11 With this definition, a vector bundle E satisfies interpolation if and only if $\chi(E) = h^0(E)$ and its space of global sections $H^0(E)$ satisfies interpolation.

Lemma 2.12 Let $C = X \cup_{\Gamma} Y$ be a reducible nodal curve and let E be a vector bundle on C . Suppose the restriction map

$$\text{res}_{X,\Gamma}: H^0(E|_X) \rightarrow E|_{\Gamma}$$

is injective. If the space of sections

$$V := \{\sigma \in H^0(E|_Y) : \sigma|_{\Gamma} \in \text{Im}(\text{res}_{X,\Gamma})\},$$

has dimension $\chi(E)$ and satisfies interpolation, then E satisfies interpolation as well.

Proof By assumption we have that restriction to Y gives an isomorphism

$$H^0(C, E) \xrightarrow{\cong} V. \quad (1)$$

For any divisor $D \subset Y \setminus \Gamma$, the isomorphism (1) restricts to an isomorphism $H^0(E(-D)) \rightarrow V(-D)$; therefore interpolation for E follows from interpolation for V and the fact that $\chi(E) = \dim V$. $\square$

Corollary 2.13 *With the setup of Lemma 2.12, if there exists a vector bundle F on Y with*

$$H^0(F) = V, \quad \chi(F) = \chi(E), \quad \mathrm{rk}(F) = \mathrm{rk}(E),$$

then if F satisfies interpolation as a vector bundle on Y , we have that E satisfies interpolation as a vector bundle on C .

The main techniques in Sects. 4–5 will be degeneration to nodal curves containing lines which are 1-, 2-, or 3-secant to the remainder of the curve. We give here some results on the bundle F of Corollary 2.13 which will be used there.

In order to deal with modified normal bundles we will set up the following notation. Suppose that $X \rightarrow \mathbb{P}^r$ is a nodal curve, and L is a secant line to X meeting X in a set S . Let $N'_{X \cup_S L}$ be a vector bundle on the union $X \cup_S L$ that is isomorphic to $N_{X \cup_S L}$ on some open subset $U \subset X \cup_S L$ containing all of L . Then we will write N'_X for the bundle on X which is $N'_{X \cup_S L}|_{X \setminus S}$ glued along U to $N_{X \cup_S L}|_X$.

Lemma 2.14 *Let $X \rightarrow \mathbb{P}^r$ be a curve and $p \in X$ a smooth point. Let $L \subset \mathbb{P}^r$ be a line through p distinct from the tangent line to X at p , and denote by $C = X \cup_p L$ the nodal curve which is the union. Let $x \in L$ be another point and suppose that $x' \in \mathbb{P}^r$ is a point not contained in the tangent plane to C at p . Then if the bundle*

$$N'_X(-1)(p)[p \rightarrow x][p \rightarrow x + x']$$

satisfies interpolation (as a vector bundle on X), then $N'_C(-1)[x \rightarrow x']$ satisfies interpolation on C .

Proof This follows from [1, Lemma 8.5] taking $p_1 = x$ and $\Lambda_1 = x'$; and taking $p_2 = \mathcal{O}_C(1)|_L$ and $\Lambda_2 = \emptyset$. $\square$

We now consider the case $r = 4$, and prove three different degeneration results for the normal bundle N_C and the twisted normal bundle $N_C(-1)$.

Lemma 2.15 *Let $X \rightarrow \mathbb{P}^4$ be a curve, p and q be smooth points of X , and x' be a point in $\mathbb{P}^4$. Suppose that the tangent lines to X at p and q , together with x' , span $\mathbb{P}^4$. Let $L \subset \mathbb{P}^4$ be a 2-secant line through p and q . Let x and y be points on L distinct from p and q . Denote by C the union $X \cup_{p,q} L$. If the bundle*

$$N'_X(-1)[p \rightarrow q][q \rightarrow p]$$

satisfies interpolation, then the bundle $N'_C(-1)(-y)[x \rightarrow x']$, and hence $N'_C(-1)[x \rightarrow x']$, satisfies interpolation.

Proof Write p' and q' for choices of points on $T_p C$ and $T_q C$ distinct from p and q respectively. By assumption p, q, p', q' and x' span $\mathbb{P}^4$; we may therefore choose three independent hyperplanes H_1, H_2 , and H_3 from the net defining the line L such that

$p' \in H_2 \cap H_3$, $q' \in H_1 \cap H_3$, and $x' \in H_1 \cap H_2$. With the choice of the H_i , the normal bundle of L is simply

$$N_L \simeq N_{H_1}|_L \oplus N_{H_2}|_L \oplus N_{H_3}|_L. \quad (2)$$

By Lemma 2.7, we have

$$N'_C(-1)(-y)[x \rightarrow x']|_L \simeq N_L(-1)(p + q - y)[p \rightarrow p'][q \rightarrow q']|_L[x \rightarrow x'].$$

As $p' \in H_2 \cap H_3$, with respect to the decomposition in (2), the modification at p towards p' is simply a modification towards $N_{H_1}|_L$ at p . Therefore this modification is the simple twist

$$N_L[p \rightarrow p'] \simeq N_{H_1}|_L \oplus N_{H_2}|_L(-p) \oplus N_{H_3}|_L(-p).$$

If we use the analogous expressions for the modifications at q and x , we obtain the isomorphism

$$\begin{aligned} N'_C(-1)(-y)[x \rightarrow x']|_L &\simeq N_{H_1}|_L(-1)(p + q - y - q - x) \\ &\quad \oplus N_{H_2}|_L(-1)(p + q - y - p - x) \oplus N_{H_3}|_L(-1)(p + q - y - p - q) \\ &\simeq \mathcal{O}_L(p - y - x) \oplus \mathcal{O}_L(q - y - x) \oplus \mathcal{O}_L(-y) \\ &\simeq \mathcal{O}_L(-1)^{\oplus 3}. \end{aligned}$$

As the bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ has no global sections, the restriction map to p and q is trivially injective and the desired bundle F from Corollary 2.13 is therefore

$$N'_C(-1)(-y)[x \rightarrow x']|_X(-p - q) \simeq N'_X(-1)[p \rightarrow q][q \rightarrow p],$$

which has the correct rank and Euler characteristic. $\square$

Lemma 2.16 *Let X be a curve in $\mathbb{P}^4$ and let L be a 3-secant line meeting X at smooth points o, p, q such that the tangent lines T_oX, T_pX , and T_qX span $\mathbb{P}^4$. Let $C := X \cup_{o+p+q} L$. If $N'_C|_X$ satisfies interpolation, then N'_C satisfies interpolation.*

Proof Write o', p' , and q' for points on T_oX, T_pX , and T_qX respectively, distinct from o, p , and q respectively. By our assumption on the tangent directions, we have

$$N'_C|_L \simeq N_L(o + p + q)[o \rightarrow o'][p \rightarrow p'][q \rightarrow q'] \simeq \mathcal{O}_L(2)^{\oplus 3}.$$

Restriction to the three points $o + p + q$ yields an isomorphism on global sections

$$H^0(\mathcal{O}_L(2))^{\oplus 3} \rightarrow \mathcal{O}_L(2)^{\oplus 3}|_{o+p+q}.$$

Therefore the bundle F of Corollary 2.13 is simply $N'_C|_X$, which is of the correct rank and Euler characteristic. $\square$

Lemma 2.17 *Let X be a curve in $\mathbb{P}^4$ and let L be a 3-secant line meeting X at smooth points o, p, q such that the tangent lines T_oX, T_pX , and T_qX span $\mathbb{P}^4$. Let $C := X \cup_{o+p+q} L$. Let x and y be points on L disjoint from o, p , and q . If $N'_C(-1)|_X(-o - p - q)$ satisfies interpolation, then $N'_C(-1)(-x - y)$, and hence $N'_C(-1)$, satisfies interpolation.*

Proof Analogously as in Lemma 2.16, our assumptions give

$$N'_C(-1)|_L \simeq \mathcal{O}_L(1)^{\oplus 3},$$

and so $N'_C(-1)(-x - y)|_L \simeq \mathcal{O}_L(-1)^{\oplus 3}$, which has no global sections. Therefore, the bundle F of Corollary 2.13 is simply $N'_C(-1)(-x - y)|_X(-o - p - q) \simeq N'_C(-1)|_X(-o - p - q)$, which has the correct rank and Euler characteristic. $\square$

3 Reduction to finite list

In this section we reduce the proof of Theorem 1.4 to a finite set of cases. To make the notation less burdensome, we make the following definition:

Definition 3.1 A pair of integers (d, g) is called a *Brill–Noether pair* (BN-pair) if $\rho(d, g, 4) \geq 0$. Furthermore, we say that a BN-pair (d, g) is *good* if for a general BN-curve C of degree d and genus g , the twisted normal bundle $N_C(-1)$ satisfies interpolation.

The main result is the following lemma:

Lemma 3.2 *Let (d, g) be a BN-pair which is not in $\{(6, 2), (8, 5), (9, 6), (10, 7)\}$. Then (d, g) is good provided that all pairs $\{(9, 5), (11, 8), (12, 10), (13, 10), (13, 11), (14, 12)\}$ are good.*

In order to prove this result, we will leverage the interpolation result in $\mathbb{P}^3$ [13, Theorem 1.1] and in the nonspecial range in $\mathbb{P}^4$ [1]. More specifically, we specialize to a reducible curve $X \cup_{\Gamma} Y$, where X is contained in a hyperplane $H \simeq \mathbb{P}^3$, and Y is transverse to the hyperplane and meets X in a finite set of points Γ which are general in H . The key result here is the following:

Lemma 3.3 *Let s be a positive integer, and $X \subset H$ be a BN-curve of degree $d'' \geq s + 1$ and genus $g'' := 2s + 1 - d''$ in a hyperplane $H \subset \mathbb{P}^4$. Let Γ be a finite set of s general points on X . Let Y be a BN-curve of degree and genus (d', g') passing through Γ with tangent direction transverse to the hyperplane H . Let $C = X \cup_{\Gamma} Y$ (which by construction is of degree $d = d' + d''$ and genus $g = g' + 3s - d''$). If both $N_{X/H}(-1)$ and $N_Y(-1)$ satisfy interpolation, then so does $N_C(-1)$.*

Proof Note that

$$\frac{\chi(N_C(-1))}{\operatorname{rk}(N_C(-1))} = \frac{2d + 1 - g}{3} = d'' - s + \frac{2d' + 1 - g'}{3} \geq d'' - s.$$

Let E and F be general effective divisors on Y and X respectively, of degrees e and $f = d'' - s$ respectively. We will show that if $e \leq (2d' + 1 - g')/3$, then $h^1(N_C(-1)(-E - F)) = 0$; and if $e \geq (2d' + 1 - g')/3$, then $h^0(N_C(-1)(-E - F)) = 0$.

By [8, Lemma 2.7], to show the appropriate vanishing of h^i it suffices to check the three vanishings

$$h^i(N_{X/H}(-1)(-\Gamma - F)) = 0, \quad h^i(\mathcal{O}_X(\Gamma - F)) = 0, \quad h^i(N_Y(-1)(-E)) = 0.$$

In the first case, $N_{X/H}(-1)$ satisfies interpolation by assumption, and so $N_{X/H}(-1)(-\Gamma - F)$ has no cohomology as $\Gamma + F$ is general of degree d'' . In the second case, $\mathcal{O}_X(\Gamma - F)$ is a general line bundle of degree $g'' - 1$ (since $d'' \geq s + 1$ implies $s + (d'' - s) \geq 2s + 1 - d'' = g''$), and so also has no cohomology by Riemann–Roch. Finally, the last vanishing is exactly the statement that $N_Y(-1)$ satisfies interpolation. $\square$

To find suitable X and Y as above, we use the following elementary lemma, which follows from the fact that if the degree of a line bundle L is at least $d + g$, then $L(-1)$ is effective.

Lemma 3.4 *Let E be a vector bundle of rank k satisfying interpolation on a curve $C \subset \mathbb{P}^r$ of degree d and genus g . If*

$$\chi(E) \geq k(d + g),$$

then $E(-1)$ also satisfies interpolation.

Corollary 3.5 *If C is a general BN-curve in $\mathbb{P}^3$ (respectively $\mathbb{P}^4$) of degree d and genus g with $d \geq g$ (respectively $d \geq 2g$), then*

$$N_C \text{ satisfies interpolation} \iff N_C(-1) \text{ satisfies interpolation.}$$

We will now inductively prove Lemma 3.2 by attaching curves $X \subset \mathbb{P}^3$ to curves $Y \subset \mathbb{P}^4$, for which $N_Y(-1)$ is already known to satisfy interpolation. We will only use the following curves for X :

d''	g''	s	$\rho(X \cup Y) - \rho(Y)$
9	6	7	-3
3	0	1	15
6	3	4	6
7	4	5	3
8	5	6	0

In order to conclude from Lemma 3.3 that for a general $C \rightarrow \mathbb{P}^4$ of degree $d' + d''$ and genus $g' + 3s - d''$, the bundle $N_C(-1)$ satisfies interpolation, we must show that $X \cup_\Gamma Y$ is a BN-curve. In all of our cases this ends up being easily deduced from previous results.

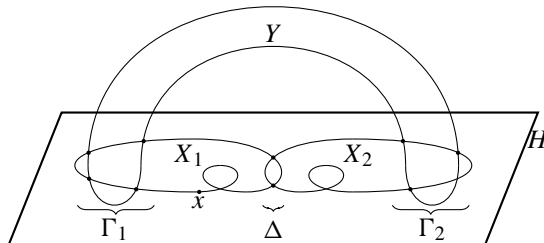
First we use this construction with X a curve of degree 9 and genus 6 to reduce to checking small genus:

Lemma 3.6 *If a BN-pair (d', g') is good and $d' \geq 7$, then the pair $(d' + 9, g' + 12)$ is good (when it is BN).*

Proof Let $Y \rightarrow \mathbb{P}^4$ be a general BN-curve of degree d' and genus g' , and H be a general hyperplane. Since (d', g') is good by assumption, the hyperplane section $Y \cap H$ is general; in particular, since $d' \geq 7$, it contains a set Γ of seven general points which satisfies $h^1(N_Y(-\Gamma)) = 0$. Let X be a general curve of degree 9 and genus 6 in H passing through Γ (again possible as X satisfies interpolation). Let C be the reducible curve $X \cup_{\Gamma} Y$, which is of degree $d = d' + 9$ and genus $g = g' + 12$. It suffices to show that C is a BN-curve.

For this, it suffices to show C admits a specialization C° with $h^1(N_{C^\circ}) = 0$ such that C° is a BN-curve; indeed, $h^1(N_{C^\circ}) = 0$ guarantees the smoothness of the Kontsevich space at $[C^\circ]$, which implies $[C^\circ]$ lies in a unique component of the Kontsevich space.

We shall degenerate C to $C^\circ = X^\circ \cup_{\Gamma} Y$ where $X^\circ = X_1 \cup_{\Delta} X_2$ is a degeneration of X to a reducible curve which still passes through Γ , with Δ and Γ disjoint; write $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \subset X_1$ and $\Gamma_2 \subset X_2$. Let $x \in X_1 \setminus \Gamma_1$ be arbitrary.



From the exact sequences

$$\begin{aligned} 0 &\rightarrow N_{C^\circ}|_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta) \rightarrow N_{C^\circ} \rightarrow N_{C^\circ}|_{X_2} \rightarrow 0, \\ 0 &\rightarrow N_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta - x) \rightarrow N_{C^\circ}|_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta) \rightarrow \mathbb{C}_x^2 \oplus \mathbb{C}_{\Gamma_2} \oplus \mathbb{C}_{\Delta} \rightarrow 0, \\ 0 &\rightarrow N_{X_2/H} \rightarrow N_{C^\circ}|_{X_2} \rightarrow \mathcal{O}_{X_2}(1)(\Gamma_2) \rightarrow 0, \end{aligned}$$

where $\mathbb{C}_x^2 \oplus \mathbb{C}_{\Gamma_2} \oplus \mathbb{C}_{\Delta}$ denotes a punctual sheaf supported at $\Delta \cup \Gamma_2 \cup \{x\}$, which in particular satisfies $h^1(\mathbb{C}_x^2 \oplus \mathbb{C}_{\Gamma_2} \oplus \mathbb{C}_{\Delta}) = 0$, we see that to check $h^1(N_{C^\circ}) = 0$, it suffices to check $h^1(N_{X_2/H}) = h^1(\mathcal{O}_{X_2}(1)(\Gamma_2)) = h^1(N_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta - x)) = 0$.

We shall consider the specialization where X_1 is a rational normal curve, and X_2 is a canonical curve of genus 4, which meet at a set Δ of three points. We do this so that $\#\Gamma_1 = 4$ and $\#\Gamma_2 = 3$; this can be done because X_1 passes through four general points, while X_2 passes through six general points, by [1, Corollary 1.4]. By inspection, $h^1(N_{X_2/H}) = h^1(\mathcal{O}_{X_2}(1)(\Gamma_2)) = 0$, so it remains to check

$h^1(N_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta - x)) = 0$. This follows from the exact sequences

$$\begin{aligned} 0 &\rightarrow N_{X_1 \cup_{\Gamma_1} Y}|_Y(-\Gamma) \rightarrow N_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_2 - \Delta - x) \rightarrow N_{X_1 \cup_{\Gamma_1} Y}|_{X_1}(-\Delta - x) \rightarrow 0, \\ 0 &\rightarrow N_{X_1/H}(-\Delta - x) \rightarrow N_{X_1 \cup_{\Gamma_1} Y}|_{X_1}(-\Delta - x) \rightarrow \mathcal{O}_{X_1}(1)(\Gamma_1 - \Delta - x) \rightarrow 0, \\ 0 &\rightarrow N_Y(-\Gamma) \rightarrow N_{X_1 \cup_{\Gamma_1} Y}|_Y(-\Gamma) \rightarrow \mathbb{C}_{\Gamma_1} \rightarrow 0, \end{aligned}$$

where $\mathbb{C}_{\Gamma_1}$ denotes a punctual sheaf supported at Γ_1 , which in particular satisfies $h^1(\mathbb{C}_{\Gamma_1}) = 0$.

It thus remains to show $C^\circ = (X_1 \cup_\Delta X_2) \cup_\Gamma Y = (X_1 \cup_{\Gamma_1} Y) \cup_{(\Gamma_2 \cup \Delta)} X_2$ is a BN-curve. For this, we can apply [7, Theorem 1.9] twice, first to conclude that $X_1 \cup_{\Gamma_1} Y$ is a BN-curve, and second to conclude that C° is a BN-curve as desired — provided that $X_1 \cup_{\Gamma_1} Y$ admits a deformation which continues to pass through $\Gamma_1 \cup \Delta$ and is transverse to H along $\Gamma_1 \cup \Delta$.

Since $X_1 \cup_{\Gamma_1} Y$ is already transverse to H along Γ_1 , this reduces by deformation theory to checking $h^1(N_{X_1 \cup_{\Gamma_1} Y}(-\Gamma_1 - \Delta - x)) = 0$ for any $x \in \Delta$, which was shown above. $\square$

Corollary 3.7 *If for all (d, g) satisfying $g \leq 14$ and $\rho(d, g, 4) \geq 3$, a general BN-curve Y of degree d and genus g satisfies interpolation for the twist $N_Y(-1)$, then a general BN-curve of any genus $g > 14$ does as well.*

Proof If (d, g) is BN and $g \geq 15$, then $d \geq 16$. Hence $d - 9 \geq 7$, and we can apply Lemma 3.6 to inductively reduce to a curve of smaller genus with $\rho \geq 3$. $\square$

Note that all the counterexamples appearing in Theorem 1.4 have $\rho \leq 2$, so it suffices to prove Theorem 1.4 for $g \leq 14$. We now deal with these remaining cases. First by attaching twisted cubics at a single point, we show that it suffices to consider three BN-pairs for each $g \leq 14$.

Lemma 3.8 *If a BN-pair (d', g') is good, then the pair $(d' + 3, g')$ is also good.*

Proof Attach a general BN-curve Y of degree d' and genus g' at a single point to a general twisted cubic $X \subset H$. The union has degree $d' + 3$ and genus g' , and lies in the correct component of $M_{g'}(\mathbb{P}^4, d' + 3)$ by [7, Theorem 1.6]. $\square$

Proof of Lemma 3.2 Corollary 3.5 provides plenty of base cases for the induction: all curves with $d \geq 2g$. If $g \leq 5$ and $(d, g) \neq (8, 5)$ or $(9, 5)$, then $d \geq 2g$. So, using the result of Corollary 3.7 we will only consider $6 \leq g \leq 14$. For each such g , let $d_{\min}$ be the minimal d such that (d, g) is BN and $(d, g) \notin \{(6, 2), (8, 5), (9, 6), (10, 7)\}$. By Lemma 3.8, it suffices to prove that $(d_{\min}, g)$, $(d_{\min} + 1, g)$, and $(d_{\min} + 2, g)$ are good. We do this using Lemma 3.3 with curves $X = (6, 3)$, $(7, 4)$, and $(8, 5)$:

In each of these cases, $X \cup_\Gamma Y$ is a BN-curve by [7, Theorem 1.9]. $\square$

In the remainder of the paper we show that the (d, g) pairs

$$\{(9, 5), (11, 8), (12, 10), (13, 10), (13, 11), (14, 12)\}$$

are good; and furthermore, we show interpolation for the untwisted normal bundle in the cases $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$.

g	$d_{\min}$	certificate for $(d_{\min}, g)$	certificate for $(d_{\min} + 1, g)$	certificate for $(d_{\min} + 2, g)$
6	10	$X = (6, 3), Y = (4, 0)$	$X = (6, 3), Y = (5, 0)$	$12 \geq 2 \cdot 6$
7	11	$X = (6, 3), Y = (5, 1)$	$X = (6, 3), Y = (6, 1)$	$X = (6, 3), Y = (7, 1)$
8	11	by assumption	$X = (7, 4), Y = (5, 0)$	$X = (6, 3), Y = (7, 2)$
9	12	$X = (7, 4), Y = (5, 1)$	$X = (6, 3), Y = (7, 3)$	$X = (6, 3), Y = (8, 3)$
10	12	by assumption	by assumption	$X = (6, 3), Y = (8, 4)$
11	13	by assumption	$X = (7, 4), Y = (7, 3)$	$X = (6, 3), Y = (9, 5)$
12	14	by assumption	$X = (7, 4), Y = (8, 4)$	$X = (6, 3), Y = (10, 6)$
13	15	$X = (8, 5), Y = (7, 3)$	$X = (7, 4), Y = (9, 5)$	$X = (6, 3), Y = (11, 7)$
14	16	$X = (8, 5), Y = (8, 4)$	$X = (6, 3), Y = (11, 8)$	$X = (6, 3), Y = (12, 8)$

4 Degeneration with a canonical curve

The key argument for $(d, g) \in \{(12, 10), (13, 10), (13, 11), (14, 12)\}$ consists of degenerating to the union of a canonical curve $D \subset \mathbb{P}^4$ of degree 8 and genus 5, and another nodal curve.

Setup. Let D be a general canonical curve $((d, g) = (8, 5))$, and let $p_1, q_1, p_2, q_2, p_3, q_3$ be six general points. Let $\Gamma = p_1 + q_1 + \cdots + q_3$ denote the sum of these points on D . For each i , let $p'_i \in T_{p_i} D$ be a choice of point on the tangent line. Denote by L_i the line $\overline{p_i, q_i}$. Suppose that $C \supset L_1 \cup L_2 \cup L_3$ is a connected nodal BN-curve containing the lines L_1, L_2 and L_3 . Let $Y = C \cup_{\Gamma} D$ be the nodal curve obtained from gluing C and D along Γ ; this is a BN-curve by [7, Theorem 1.6].

Lemma 4.1 *If $N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]$ satisfies interpolation, then $N_Y(-1)$ satisfies interpolation as well.*

Proof The canonical curve D is the complete intersection of a net of quadrics in $\mathbb{P}^4$. Note that no quadric in the net contains all 2-secant lines to D , as the secant variety is of degree $\binom{8-1}{2} - 5 = 16$. Since it is one condition on a quadric containing D to contain a 2-secant line to D , we may choose three independent quadrics from the net such that each contains exactly two of the three lines L_1, L_2 , and L_3 . This gives an isomorphism

$$N_Y(-1)|_D \simeq K_D(p_1 + q_1) \oplus K_D(p_2 + q_2) \oplus K_D(p_3 + q_3).$$

Let w be a general point on D . The line bundle $F_i := K_D(p_i + q_i - w)$ on D is of degree 9 and has a 5-dimensional space of global sections, which correspond to meromorphic differentials on D that vanish at w and have at most poles at p_i and q_i with opposite residues.

Since we may compute that $H^0(F_i(-\Gamma)) = 0$, the restriction map

$$H^0(F_i) \xrightarrow{\text{res}} F_i|_{p_1} \oplus F_i|_{q_1} \oplus F_i|_{p_2} \oplus F_i|_{q_2} \oplus F_i|_{p_3} \oplus F_i|_{q_3},$$

is an isomorphism onto a 5-dimensional subspace of $F_i|_{\Gamma} \simeq \mathbb{C}^6$, which can be described as the preimage of the antidiagonal (i.e., the two coordinates negatives of each other) in the 2-dimensional $F_i|_{p_i} \oplus F_i|_{q_i}$, under the restriction map $F_i|_{\Gamma} \rightarrow F_i|_{p_i} \oplus F_i|_{q_i}$.

The bundle $F_i(-p_1 - p_2 - p_3) = K_D(q_i - w - p_j - p_k)$ on D satisfies $h^0(F_i(-p_1 - p_2 - p_3)) = 2$ and $h^1(F_i(-p_1 - p_2 - p_3)) = 0$, as $h^0(K_D(q_i)) = 5$ and w, p_j , and p_k are general points on D . For points (x, y, z) in some neighborhood $U \subset D \times D \times D$ of the point (p_1, p_2, p_3) , the bundle $F_i(-x - y - z)$ therefore also has two global sections by semicontinuity. Let $\mathcal{F}_i$ denote the bundle on $D \times U$ which specialized to $F_i(-x - y - z)$ on the fiber $D \times \{(x, y, z)\}$. Let

$$\pi_1: D \times U \rightarrow D, \quad \pi_2: D \times U \rightarrow U$$

denote the two projections.

By the theorem on cohomology and base change, the pushforward $\pi_{2*}\mathcal{F}_i$ is a rank 2 vector subbundle of a rank 5 trivial bundle:

$$\pi_{2*}\mathcal{F}_i \subseteq \pi_{2*}(\pi_1^*F_i) = H^0(F_i) \otimes \mathcal{O}_U.$$

Composing with the restriction map above yields a map

$$r_i: \pi_{2*}\mathcal{F}_i \rightarrow H^0(F_i) \otimes \mathcal{O}_U \rightarrow (F_i|_\Gamma \simeq \mathbb{C}^6) \otimes \mathcal{O}_U,$$

whose fiber over (x, y, z) extracts the image

$$S_{xyz}^i := r_i(\pi_{2*}\mathcal{F}_i|_{(x,y,z)}) = \text{res}(H^0(F_i(-x - y - z))) \subset F_i|_\Gamma$$

for each point $(x, y, z) \in U$. Write $S_{xyz} := S_{xyz}^1 \oplus S_{xyz}^2 \oplus S_{xyz}^3$.

By Lemma 2.12, in order to prove interpolation for $N_Y(-1)(-x - y - z - w)$, with x, y, z, w general points, it suffices to prove interpolation for the space of sections

$$V_{xyz} := \{\sigma \in H^0(N_Y(-1)|_C) : \sigma|_\Gamma \in S_{xyz} \subseteq N_Y(-1)(-x - y - z - w)|_\Gamma \simeq \mathbb{C}^{18}\},$$

provided that it has the correct dimension. Furthermore as satisfying interpolation and having the correct dimension is an open condition, it suffices to prove that for the specialization (p_1, p_2, p_3) of the general points $(x, y, z) \in U$, the space of sections

$$V_{p_1 p_2 p_3} = \{\sigma \in H^0(N_Y(-1)|_C) : \sigma|_\Gamma \in S_{p_1 p_2 p_3}\},$$

is of the correct dimension and satisfies interpolation. In this case,

$$S_{p_1 p_2 p_3}^i = \text{res}(H^0(K_D(q_i - w - p_j - p_k))) = F_i|_{q_j} \oplus F_i|_{q_k} \subset F_i|_\Gamma, \quad i, j \neq k.$$

When viewed as meromorphic sections $N_C(-1)$, the sections of $N_Y(-1)|_C$ whose restrictions to Γ lie in the subspace $S_{p_1 p_2 p_3}$ therefore vanish at the p_i and are regular at the q_i . Therefore, it suffices to prove that

$$\begin{aligned} V_{p_1 p_2 p_3} &= \{\sigma \in H^0(N_Y(-1)|_C) : \sigma|_{p_i} = 0, \sigma \text{ is regular at } q_j\} \\ &= \{\sigma \in H^0(N_Y(-1)|_C(-p_1 - p_2 - p_3)) : \sigma \text{ preserves the node at } q_j\} \end{aligned}$$

is of the correct dimension and satisfies interpolation. This subspace is the space of global sections of the vector bundle

$$N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3],$$

which has the same rank and Euler characteristic as $N_Y(-1)(-x - y - z - w)$. Therefore it suffices to prove that this vector bundle satisfies interpolation. $\square$

We now deal with each of the cases $(d, g) \in \{(12, 10), (13, 10), (13, 11), (14, 12)\}$ separately.

4.1 Degree 12, genus 10

In this case we take C in the setup above to be the union of the three general lines $L_1 \cup L_2 \cup L_3$ and a line M meeting all three lines, which is a BN-curve by [7, Theorem 1.6]. Let $m_i := L_i \cap M$ be the points of intersection so that the curve C is precisely the nodal union

$$C := (L_1 \cup L_2 \cup L_3) \cup_{m_1+m_2+m_3} M.$$

By Lemma 4.1, it suffices to prove interpolation for the modified normal bundle $N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]$.

By Lemma 2.14, we may pull off each of the 1-secant lines L_i in turn to reduce to proving interpolation for the bundle

$$N_M(-1)(m_1 + m_2 + m_3)[m_1 \rightarrow p_1][m_1 \rightarrow p_1 + p'_1] \\ [m_2 \rightarrow p_2][m_2 \rightarrow p_2 + p'_2][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3],$$

or, by limiting p'_1 to p_2 , and p'_2 to p_3 , and p'_3 to p_1 (and using the properties in Proposition 2.5) for the bundle

$$N_M(-1)(2m_1 + 2m_2 + 2m_3)[2m_1 \rightarrow p_1][m_1 \rightarrow p_2] \\ [2m_2 \rightarrow p_2][m_2 \rightarrow p_3][2m_3 \rightarrow p_3][m_3 \rightarrow p_1].$$

Limiting $m_2, m_3 \rightarrow m_1$ is tree-like, as the points p_1, p_2 , and p_3 are general in $\mathbb{P}^4$. Therefore, it suffices to prove interpolation for

$$N_M(-1)(6m_1)[3m_1 \rightarrow p_1][3m_1 \rightarrow p_2][3m_1 \rightarrow p_3] \\ \simeq N_M(-1)[3m_1 \rightarrow p_1 + p_2 + p_3] \simeq N_M(-1).$$

The result now follows from the fact that $N_M(-1) \simeq \mathcal{O}^{\oplus 3}$ satisfies interpolation.

4.2 Degree 13, genus 10

We continue with the notation of the setup above. In this case we take M to be a general line meeting L_1 and L_2 and N to be a general line meeting L_2 and L_3 . Call $M \cap L_i =: m_i$ and $N \cap L_j =: n_j$. Let

$$C = (N \cup M) \cup_{m_1+m_2+n_2+n_3} (L_1 \cup L_2 \cup L_3)$$

be the union, which is a BN-curve by [7, Theorem 1.6]. So by Lemma 4.1 it suffices to prove interpolation for $N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]$. By Lemma 2.14, this follows from interpolation for the bundle

$$N_1 := N_{N \cup_{n_2} L_2 \cup_{m_2} M}(-1)(m_1 + n_3)[p_2 \rightarrow p'_2][m_1 \rightarrow p_1] \\ [m_1 \rightarrow p_1 + p'_1][n_3 \rightarrow p_3][n_3 \rightarrow p_3 + p'_3].$$

By Lemma 2.9, it suffices to show the vanishing of $h^0(N')$ where $N' := N_1(-x - y)$ where x and y are general points on N and M respectively.

Restricting to M we have

$$N'|_M \simeq N_M(-1)(-y + m_1 + m_2)[m_1 \rightarrow p_1][m_1 \rightarrow p_1 + p'_1][m_2 \rightarrow p_2] \\ \simeq N_M[m_1 \rightarrow p_1][m_1 \rightarrow p_1 + p'_1][m_2 \rightarrow p_2].$$

As the points p_1, p_2 and p'_1 are independent, we may make a choice of the three hyperplanes H_i defining $M = H_1 \cap H_2 \cap H_3$ so that $p_1 \in H_2 \cap H_3$, $p_2 \in H_1 \cap H_3$ and $p'_1 \in H_1 \cap H_2$. Then the modifications towards these points become simple twists:

$$N'|_M \simeq N_{H_1}|_M(-m_2) \oplus N_{H_2}|_M(-2m_1) \oplus N_{H_3}|_M(-m_1 - m_2) \simeq \mathcal{O}_M \oplus \mathcal{O}_M(-1)^{\oplus 2}.$$

This restriction has a unique global section, coming from the factor $N_{H_1}|_M(-m_2)$ of normal directions pointing to p_1 . Therefore, applying the argument of Lemma 2.12, it suffices to show the vanishing of h^0 of the subbundle of $N'|_{N \cup_{n_2} L_2}$ whose sections at m_2 point towards p_1 :

$$N_{N \cup_{n_2} L_2}(-1)(-x + n_3 + m_2)[p_2 \rightarrow p'_2][m_2 \rightarrow m_1][m_2 \rightarrow p_1][n_3 \rightarrow p_3][n_3 \rightarrow p_3 + p'_3].$$

We may apply the same argument to N to reduce to showing there are no sections of

$$N_{L_2}(-1)(n_2 + m_2)[p_2 \rightarrow p'_2][n_2 \rightarrow n_3][n_2 \rightarrow p_3][m_2 \rightarrow m_1][m_2 \rightarrow p_1] \\ \simeq N_{L_2}(-1)[p_2 \rightarrow p'_2][n_2 \rightarrow n_3 + p_3][m_2 \rightarrow m_1 + p_1].$$

Limiting p_2 to n_2 , which is tree-like as p'_2 is general, it suffices to show the vanishing of h^0 for

$$N_{L_2}(-1)(-n_2)[n_2 \rightarrow n_3 + p_3 + p'_2][m_2 \rightarrow m_1 + p_1] \\ \simeq N_{L_2}(-1)(-n_2)[m_2 \rightarrow m_1 + p_1],$$

and as m_1 and p_1 are in general directions, we have

$$N_{L_2}(-1)(-n_2)[m_2 \rightarrow m_1 + p_1] \simeq \mathcal{O}_M(-2) \oplus \mathcal{O}_M(-1) \oplus \mathcal{O}_M(-1),$$

which has no global sections, as desired.

4.3 Degree 13, genus 11

As in the setup above, we choose C to contain the lines L_1, L_2, L_3 as well as two further lines M and N , where M meets each L_i at a single point m_i and N meets L_1 and L_2 at points n_1 and n_2 respectively. As the lines L_1, L_2 , and L_3 are general, the lines can be assumed to satisfy various genericity conditions: $N \cup M \cup L_i$ span a $\mathbb{P}^3$ for $i = 1, 2$.

Then the resulting union

$$C := (N \cup M) \cup_{m_1+m_2+m_3+n_1+n_2} (L_1 \cup L_2 \cup L_3)$$

is a nodal connected curve of degree 5 and arithmetic genus 2, which is a BN-curve by [7, Theorem 1.6]. By Lemma 4.1, it suffices to show that

$$N'_C := N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]$$

satisfies interpolation. By Lemma 2.15, it suffices to show that

$$N_{(N \cup M) \cup_{m_2+m_3+n_2}(L_2 \cup L_3)}(-1)[p_2 \rightarrow p'_2][p_3 \rightarrow p'_3][n_1 \rightarrow m_1][m_1 \rightarrow n_1]$$

satisfies interpolation. By Lemma 2.14, this reduces in turn to showing that

$$N_{M \cup_{m_2+m_3} L_2 \cup L_3}(-1)(n_2)[p_2 \rightarrow p'_2][p_3 \rightarrow p'_3][m_1 \rightarrow n_1][n_2 \rightarrow n_1][n_2 \rightarrow n_1 + m_1]$$

satisfies interpolation. Again by Lemma 2.14, this further reduces to proving interpolation for

$$N_{M \cup_{m_2} L_2}(-1)(n_2 + m_3)[p_2 \rightarrow p'_2][m_1 \rightarrow n_1][n_2 \rightarrow n_1] \\ [n_2 \rightarrow n_1 + m_1][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3].$$

We may limit $p_2 \rightarrow n_2$, since p'_2 is disjoint from $\overline{n_1, m_1}$ and so this is tree-like. We are therefore reduced to proving interpolation for the vector bundle

$$N_{M \cup_{m_2} L_2}(-1)(n_2 + m_3)[n_2 \rightarrow p'_2][m_1 \rightarrow n_1][n_2 \rightarrow n_1] \\ [n_2 \rightarrow n_1 + m_1][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3] \\ \simeq N_{M \cup_{m_2} L_2}(-1)(m_3)[m_1 \rightarrow n_1][n_2 \rightarrow n_1] \\ [n_2 \rightarrow n_1 + m_1 + p'_2][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3].$$

As p'_2 is general and m_1, n_1 , and L_2 span a $\mathbb{P}^3$, the linear space $\overline{p'_2, m_1, n_1, L_2} = \mathbb{P}^4$. Therefore the above bundle is isomorphic to

$$N_{M \cup_{m_2} L_2}(-1)(m_3)[m_1 \rightarrow n_1][n_2 \rightarrow n_1][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3].$$

We may now apply Lemma 2.14 a final time to reduce to showing interpolation for

$$N_M(-1)(m_2 + m_3)[m_2 \rightarrow n_2][m_2 \rightarrow n_2 + n_1][m_1 \rightarrow n_1][m_3 \rightarrow p_3][m_3 \rightarrow p_3 + p'_3].$$

Limit p'_3 to n_1 . Then we may limit m_2 and m_3 to m_1 , as n_1, n_2 and p_3 are independent so this is tree-like. The resulting vector bundle is

$$\begin{aligned} N_M(-1)(2m_1)[m_1 \rightarrow n_2][m_1 \rightarrow n_2 + n_1][m_1 \rightarrow n_1][m_1 \rightarrow p_3][m_1 \rightarrow p_3 + n_1] \\ \simeq N_M(-1)(4m_1)[m_1 \rightarrow n_1][2m_1 \rightarrow n_1][2m_1 \rightarrow n_2][2m_1 \rightarrow p_3] \\ \simeq N_M(-1)[m_1 \rightarrow n_1][2m_1 \rightarrow n_1 + m_1 + p_3]. \end{aligned}$$

As n_1, m_1 and p_3 are independent, and so span all of $\mathbb{P}^4$, we have that this is isomorphic to

$$N_M(-1)[m_1 \rightarrow n_1] \simeq \mathcal{O}_M \oplus \mathcal{O}_M(-1)^{\oplus 2},$$

which satisfies interpolation by inspection.

4.4 Degree 14, genus 12

Continuing in the setup above, let M be a general line meeting each L_i at m_i and let N be a general conic meeting each L_i at n_i . Pick points $n'_i \neq n_i$ on $T_{n_i}N$, and a point $m \in M$ distinct from m_1, m_2 , and m_3 . Let C denote the union

$$C := (M \cup N) \cup_{m_1+m_2+m_3+n_1+n_2+n_3} (L_1 \cup L_2 \cup L_3),$$

which is a BN-curve by [7, Theorem 1.6]. It then suffices to prove interpolation for $N'_C := N_C(-1)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]$. This vector bundle has Euler characteristic 5. So by Lemma 2.9, it suffices to show that for two points $x + y$, the twist down satisfies

$$h^0(N_C(-1)(-x - y)[p_1 \rightarrow p'_1][p_2 \rightarrow p'_2][p_3 \rightarrow p'_3]) = 0. \quad (3)$$

We will choose x and y to be general points on M , so that $N'_C(-x - y)|_M \simeq \mathcal{O}(-1)^{\oplus 3}$. Therefore to show the vanishing of (3), it suffices to show that

$$h^0(N'_C|_{N \cup_{n_1+n_2+n_3} L_1 \cup L_2 \cup L_3}(-m_1 - m_2 - m_3)) = 0,$$

as in the proof of Lemma 2.17. This bundle restricts to each L_i to be

$$N_{L_i}(-1)(n_i)[n_i \rightarrow n'_i][p_i \rightarrow p'_i][m_i \rightarrow m] \simeq \mathcal{O}_{L_i}(-1)^{\oplus 3},$$

as L_i , n'_i , p'_i , and m are linearly independent. Therefore as in Lemma 2.12, it suffices to show that

$$h^0(N_N(-1)[n_1 \rightarrow p_1][n_2 \rightarrow p_2][n_3 \rightarrow p_3]) = 0,$$

where the p_i are in general directions. By the semicontinuity theorem it suffices to prove this vanishing after specializing p_1 , p_2 and p_3 . Fix a quadric Q and hyperplanes H_1 and H_1 such that N is the complete intersection $Q \cap H_1 \cap H_2$. We have

$$N_N(-1) \simeq N_Q(-1)|_N \oplus N_{H_1}(-1)|_N \oplus N_{H_2}(-1)|_N \simeq \mathcal{O}_N(1) \oplus \mathcal{O}_N^{\oplus 2}. \quad (4)$$

Specialize p_1 and p_2 so that they are general in $T_{n_1}(Q \cap H_2)$ and $T_{n_2}(Q \cap H_2)$ respectively, and p_3 so that it is general in $T_{n_3}(Q \cap H_1)$. Then the negative modifications are towards specific factors in the decomposition (4), and so become simple twists:

$$\begin{aligned} N_N(-1)[n_1 \rightarrow p_1][n_2 \rightarrow p_2][n_3 \rightarrow p_3] \\ \simeq \mathcal{O}_N(1)(-p_1 - p_2 - p_3) \oplus \mathcal{O}_N(-p_3) \oplus \mathcal{O}_N(-p_1 - p_2), \\ \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \end{aligned}$$

under an isomorphism $M \simeq \mathbb{P}^1$, which has no global sections, as desired.

5 Degeneration with a rational quartic curve

Let X be a nondegenerate curve of degree d and genus g in $\mathbb{P}^4$. Let (p_1, q_1) , (p_2, q_2) , (p_3, q_3) be three pairs of general points on X . Let $\Gamma := p_1 + p_2 + p_3 + q_1 + q_2 + q_3$. Denote by L_i the 2-secant line through p_i and q_i . There is a unique fourth line M meeting each of the L_i (by, for example, intersection theory in the Grassmannian of lines in $\mathbb{P}^4$). Call $L_i \cap M = m_i$. As X is nondegenerate, the union $L_1 \cup L_2 \cup L_3 \cup M$ spans $\mathbb{P}^4$. Call this union

$$Y := X \cup_{\Gamma} ((L_1 + L_2 + L_3) \cup_{m_1+m_2+m_3} M),$$

which is a BN-curve by [7, Theorem 1.6]. We will use this particular degeneration to prove interpolation for $N_Y(-1)$ when Y is a curve of degree 11 and genus 8, and for N_Y when Y is a curve of degree 9 and genus 6 (respectively degree 10 and genus 7).

Lemma 5.1 *With notation as above,*

(i) $N_Y(-1)$ satisfies interpolation if

$$\begin{aligned} N_Y(-1)|_X(-p_1 - q_1 - p_2 - q_2 - p_3 - q_3) \\ \simeq N_X(-1)[p_1 \rightarrow q_1][q_1 \rightarrow p_1][p_2 \rightarrow q_2][q_2 \rightarrow p_2][p_3 \rightarrow q_3][q_3 \rightarrow p_3] \end{aligned}$$

satisfies interpolation.

(ii) N_Y satisfies interpolation if

$$N_Y|_X = N_X(p_1 + q_1 + p_2 + q_2 + p_3 + q_3)[p_1 \rightarrow q_1][q_1 \rightarrow p_1] \\ [p_2 \rightarrow q_2][q_2 \rightarrow p_2][p_3 \rightarrow q_3][q_3 \rightarrow p_3]$$

satisfies interpolation. This vector bundle satisfies interpolation if the modified normal bundle

$$N'_X := N_X[q_1 + q_2 + q_3 \rightarrow p_1]$$

satisfies interpolation.

Proof As $L_1 \cup L_2 \cup L_3 \cup M$ spans $\mathbb{P}^4$, the three directions of the L_i at the points m_i are independent.

By Lemma 2.17 (respectively Lemma 5.1), it suffices to prove interpolation for $N_Y(-1)|_{X \cup \Gamma(L_1 \cup L_2 \cup L_3)}(-m_1 - m_2 - m_3)$ (respectively $N_Y|_{X \cup \Gamma(L_1 \cup L_2 \cup L_3)}$) in order to deduce interpolation for $N_Y(-1)$ (respectively N_Y).

For each of the L_i , the three lines $T_{p_i}X$, $T_{q_i}X$ and M span $\mathbb{P}^4$, and so we similarly have that $N_Y|_{L_i} \simeq \mathcal{O}_{L_i}(2)^{\oplus 3}$. To prove interpolation for $N_Y(-1)$ and N_Y , we are free to pick our general points to include a choice of general point x_i on each of the L_i . Twisting down by this point for each L_i , we have

$$N_Y|_{L_i}(-x_i) \simeq \mathcal{O}_{L_i}(1)^{\oplus 3}, \quad N_Y(-1)|_{L_i}(-m_i - x_i) \simeq \mathcal{O}_{L_i}(-1)^{\oplus 3}.$$

To prove interpolation for $N_Y(-1)$, by Lemma 2.15, we may peel off each line L_i in turn to reduce to proving interpolation for

$$N_Y(-1)|_X(-p_1 - q_1 - p_2 - q_2 - p_3 - q_3).$$

To prove interpolation for N_Y , note that evaluation at the two points $p_i, q_i \in L_i \simeq \mathbb{P}^1$ is an isomorphism

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 3} \xrightarrow{\text{ev}} \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}|_{p_i+q_i}.$$

Hence sections of $N_Y(-x_1 - x_2 - x_3)$ on X extend uniquely to $X \cap L_1 \cap L_2 \cap L_3$ and then to all of Y . Therefore by Corollary 2.13, interpolation for $N_Y(-x_1 - x_2 - x_3)$ (and hence for N_Y) follows from interpolation for $N_Y|_X$. The formula for $N_Y|_X$ follows directly from Lemma 2.7.

Now limit p_2 and p_3 to p_1 ; the corresponding modification is tree-like as q_1, q_2 , and q_3 are independent. By the properties of Proposition 2.5, the resulting bundle is therefore

$$N_X(3p_1 + q_1 + q_2 + q_3)[p_1 \rightarrow q_1][q_1 \rightarrow p_1][p_1 \rightarrow q_2][q_2 \rightarrow p_1][p_1 \rightarrow q_3][q_3 \rightarrow p_1] \\ \simeq N_X(p_1 + q_1 + q_2 + q_3)[p_1 \rightarrow q_1 + q_2 + q_3][q_1 + q_2 + q_3 \rightarrow p_1] \\ \simeq N_X(p_1 + q_1 + q_2 + q_3)[q_1 + q_2 + q_3 \rightarrow p_1].$$

Interpolation for this bundle therefore follows from interpolation for $N_X[q_1 + q_2 + q_3 \rightarrow p_1]$. $\square$

5.1 Degree 11, genus 8

By Lemma 5.1, we may reduce interpolation for the twisted normal bundle of a general BN-curve of degree 11 and genus 8 to interpolation for the modified normal bundle $N_X(-1)[p_1 \rightarrow q_1][q_1 \rightarrow p_1][p_2 \rightarrow q_2][q_2 \rightarrow p_2][p_3 \rightarrow q_3][q_3 \rightarrow p_3]$, where X is a BN-curve of degree 7 and genus 3.

We will further degenerate using the following. As before, N'_X denotes some modification of N_X , which is isomorphic to N_X over some open containing all of L . N'_C is then the bundle on C obtained by gluing $N'_X|_{C \setminus C \cap L}$ to N_C along this open.

Lemma 5.2 *Let C be a BN-curve. Let L be a 2-secant line meeting C at two general points x and y ; write X for the union $X := C \cup_{x+y} L$. Suppose that p_1 and p_2 are general points on L and q_1 and q_2 are general points on C , and D_1 and D_2 are divisors on C . If the bundle*

$$N'_C(-1)(y)[x \rightarrow y][2y \rightarrow x][D_1 \rightarrow x][D_2 \rightarrow x]$$

on C satisfies interpolation, then the bundle

$$N''_X := N'_X(-1)[p_1 \rightarrow q_1][D_1 \rightarrow p_1][p_2 \rightarrow q_2][D_2 \rightarrow p_2] \quad (5)$$

on X satisfies interpolation.

Proof Let $x' \in T_x C$ and $y' \in T_y C$ be choices of points on the tangent lines distinct from x and y respectively. If we restrict the bundle N'_X of (5) to L , we have

$$N''_X|_L \simeq N_L(-1)(x+y)[x \rightarrow x'][y \rightarrow y'][p_1 \rightarrow q_1][p_2 \rightarrow q_2] \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1)^{\oplus 2},$$

which has one global section. We will do a calculation in local coordinates to more precisely understand the behavior of this section at x and y as a meromorphic section of $N_L(-1)$.

The twisted normal bundle $N_L(-1) \simeq V \otimes \mathcal{O}_L$ is trivial of rank 3 on L (with $V \simeq H^0(N_L(-1))$). We may therefore assign, to points of $\mathbb{P}^4 \setminus L$, the corresponding normal direction to L in $\mathbb{P}V$. Choose appropriate coordinates for V so that

$$x' \mapsto [1:0:0], \quad y' \mapsto [0:1:0], \quad q_1 \mapsto [1:0:1], \quad q_2 \mapsto [0:1:1].$$

In addition let t be an affine coordinate on $L \simeq \mathbb{P}^1$ so that x corresponds to $t = 0$, y corresponds to $t = 1$, and p_1 and p_2 correspond to $t = a$ and $t = b$ respectively. Then the unique section of $N'_X|_L$ up to scaling is given by (scalar multiples of) the tuple

$$\sigma_t = \left[\frac{a(t-b)}{t}, \frac{(1-b)(t-a)}{t-1}, a-b \right]$$

as a meromorphic function valued in V . The image of this section under the restriction map to x (i.e., $t = 0$) is

$$\sigma_0 = [-ab, a(1-b), a-b]$$

and under restriction to y (i.e., $t = 1$) is

$$\sigma_1 = [a(1-b), (1-a)(1-b), a-b].$$

By Lemma 2.12, it suffices to prove interpolation for the space of sections of $N_X''|_C$ whose values at x and y are in the 1-dimensional subspace given by (σ_0, σ_1) in $N'_{C \cup_{x+y} L}(-1)|_{x+y}$. As interpolation is an open condition, we may limit p_1 and p_2 to x , in which case σ_0 and σ_1 limit to

$$\sigma'_0 = [0, 0, 0], \quad \sigma'_1 = [0, 1, 0].$$

Therefore the sections of $N_X''|_L$ must vanish at x and point towards y' at y . By [1, Lemma 8.4], the bundle $N_X''|_L[y \rightarrow x]$ glues to $N_X''|_L[y \rightarrow y']$ at the point y . Therefore it suffices to prove interpolation for the vector bundle

$$\begin{aligned} N'_{C \cup_{x+y} L}(-1)[D_1 \rightarrow x][D_2 \rightarrow x]|_C(-x)[y \rightarrow x] \\ \simeq N'_C(-1)(y)[x \rightarrow y][2y \rightarrow x][D_1 \rightarrow x][D_2 \rightarrow x] \end{aligned}$$

on C . □

By [7, Theorem 1.6], we may degenerate our BN-curve of degree 7 and genus 3 to the union of a BN-curve of degree 6 and genus 2 and a 2-secant line L meeting C at points x and y . Do this so the points p_1 and p_2 specialize on to L and the remainder of the marked points specialize to C . Then we apply Lemma 5.2 with $N'_X = N_X[p_3 \rightarrow q_3][q_3 \rightarrow p_3]$ and $D_1 = q_1$ and $D_2 = q_2$. We thus reduce to proving interpolation for the bundle

$$N_C(-1)(y)[x \rightarrow y][2y \rightarrow x][q_1 \rightarrow x][q_2 \rightarrow x][p_3 \rightarrow q_3][q_3 \rightarrow p_3].$$

By [7, Theorem 1.6], we may further degenerate C to the union of a curve D of degree 5 and genus 1 and a 2-secant line M , meeting D at points z and w . Let the points p_3 and q_1 specialize to general points on L , and x , q_3 , q_2 , and y specialize to general points on D . Again we apply Lemma 5.2 with $p_1 = p_3$, $p_2 = q_1$, $q_1 = q_3$, $q_2 = x$, $D_1 = q_2$ and $D_2 = \emptyset$. It therefore suffices to prove interpolation for the bundle

$$N_D(-1)(y+w)[x \rightarrow y][2y \rightarrow x][q_2 \rightarrow x][q_3 \rightarrow z][2w \rightarrow z][z \rightarrow w]$$

on D . We may limit w to x , to obtain the bundle

$$N'_D := N_D(-1)(x+y)[x \rightarrow y][2y \rightarrow x][q_2 \rightarrow x][q_3 \rightarrow z][2x \rightarrow z][z \rightarrow x],$$

which has Euler characteristic 0. This bundle sits in a balanced exact sequence

$$\begin{aligned} 0 \rightarrow N_{D \rightarrow x}(-1)(y - q_3 - 2x) \rightarrow N'_D \\ \rightarrow \pi_x|_D^* N_{\pi_x|_D}(-1)(x - y - q_2 - z)[x \rightarrow y][q_3 \rightarrow z][2x \rightarrow z] \rightarrow 0. \end{aligned}$$

Therefore by [1, Proposition 4.16] and the fact that as a nonspecial line bundle $N_{D \rightarrow p}(-1)(y - q_3 - 2x)$ satisfies interpolation [1, Proposition 4.7], it suffices to prove that

$$\begin{aligned} \pi_x|_D^* N_{\pi_x|_D}(-1)(x - y - q_2 - z)[x \rightarrow y][q_3 \rightarrow z][2x \rightarrow z] \\ \simeq \pi_x|_D^* N_{\pi_x|_D}(-1)(-y - q_2 - z)[q_3 + x \rightarrow z][x \rightarrow y + z] \\ \simeq \pi_x|_D^* N_{\pi_x|_D}(-1)(-y - q_2 - z)[q_3 + x \rightarrow z] \end{aligned}$$

satisfies interpolation as a vector bundle on D . As $\pi_x|_D^* \mathcal{O}(1)(y + q_2 + z)$ is a general line bundle on D , it suffices to show that $\pi_x|_D^* N_{\pi_x|_D}[q_3 + x \rightarrow z]$ satisfies interpolation. This is verified by publicly available code in [1, Appendix B] to prove interpolation for certain modifications of normal bundles of nonspecial curves:

```
>>> good(Curve(4, 1, 3).add(P100, 2))
True
```

5.2 Degree 9, genus 6

By Lemma 5.1, interpolation for the normal bundle of a general curve of degree 9 and genus 6 follows from interpolation for the modified normal bundle

$$N'_C := N_C[q_1 + q_2 + q_3 \rightarrow p]$$

of a curve C of degree 5 and genus 1 in $\mathbb{P}^4$. This is verified by publicly available code in [1, Appendix B]:

```
>>> good(Curve(5, 1, 4).add(P100, 3))
True
```

5.3 Degree 10, genus 7

As above, by Lemma 5.1, it suffices to show that for a BN-curve $X \rightarrow \mathbb{P}^4$ of degree 6 and genus 2, and p, q_1, q_2, q_3 general points of X , the modified normal bundle

$$N'_X = N_X[q_1 + q_2 + q_3 \rightarrow p]$$

satisfies interpolation. This is verified by publicly available code in [1, Appendix B]:

```
>>> good(Curve(6, 2, 4).add(P100, 3))
True
```

6 Proofs of the main theorems

In this section we wrap up the preceding computations to prove the theorems and corollaries quoted in the introduction.

6.1 Proof of Theorem 1.4

As interpolation is an open condition [2, Theorem 5.8], it suffices to find one BN-curve C of every possible degree d and genus g (with $\rho(d, g, 4) \geq 0$) whose twisted normal bundle $N_C(-1)$ satisfies interpolation. By Proposition 3.2, this follows for all BN-pairs $(d, g) \notin \{(6, 2), (8, 5), (9, 6), (10, 7)\}$ from checking interpolation for BN-curves with d and g in the following finite list:

$$\{(9, 5), (11, 8), (12, 10), (13, 10), (13, 11), (14, 12)\}.$$

For $(d, g) \in \{(12, 10), (13, 10), (13, 11), (14, 12)\}$, this was proved in Sects. 4.1–4.4 using a degeneration containing a canonical curve. For $(d, g) = (11, 8)$, this was proved in Sect. 5.1 using a degeneration with a rational quartic curve as well as several 2-secant degenerations. Let us show that the twisted normal bundle $N_C(-1)$ of a general BN-curve C of degree 9 and genus 5 satisfies interpolation. As $\chi(N_C(-1)) = 14 \equiv -1 \pmod{3}$, it suffices by Lemma 2.9 to show that for five general points $p_1, \dots, p_5$, the twist down satisfies

$$h^0(N_C(-1)(-p_1 - \dots - p_5)) = 0.$$

As the genus of C is 5, the line bundle $\mathcal{O}_C(1)(p_1 + \dots + p_5)$ is general of degree 14. Therefore this vanishing follows from interpolation for the untwisted normal bundle N_C , which is a special case of [1, Theorem 1.3].

For the converse, we note that $h^1(N_C(-1)) \neq 0$ if $C \rightarrow \mathbb{P}^4$ is a BN-curve of degree and genus in $\{(8, 5), (9, 6), (10, 7)\}$ by [8, Theorem 1.6]. Moreover, N_C does not satisfy interpolation if $C \rightarrow \mathbb{P}^4$ is a BN-curve of degree 6 and genus 2. $\square$

6.2 Proof of Theorem 1.5

For any BN-curve C , if the twisted normal bundle $N_C(-1)$ satisfies interpolation, then the untwisted normal bundle N_C does as well [1, Proposition 4.11]. Therefore for all BN-pairs $(d, g) \notin \{(6, 2), (8, 5), (9, 6), (10, 7)\}$, this is a corollary of Theorem 1.4. If C is of degree 6 and genus 2, then C is nonspecial and N_C was already observed to fail to satisfy interpolation in [1]. In Sects. 5.2 and 5.3 we proved interpolation for $(d, g) = (9, 6)$ (respectively $(d, g) = (10, 7)$).

A BN-curve C of degree 8 and genus 5 in $\mathbb{P}^4$ is a canonical curve, which is a complete intersection of three quadrics; its normal bundle is therefore

$$N_C \simeq (K_C^{\otimes 2})^{\oplus 3}.$$

Interpolation for N_C thus reduces to interpolation for $K_C^{\otimes 2}$, which holds since $K_C^{\otimes 2}$ is nonspecial [1, Proposition 4.7]. $\square$

6.3 Proof of Corollaries 1.2 and 1.6

When $\rho(d, g, 4) \geq 0$, write $\overline{M}_{g,n}(\mathbb{P}^4, d)^\circ$ for the unique component dominating $\overline{M}_g$ of the Kontsevich space of n -marked genus g curves of degree d in $\mathbb{P}^4$. If $n \leq f(d, g, 4)$, and $p_1, \dots, p_n$ are general points on C , then $\chi(N_C(-p_1 - \dots - p_n)) \geq 0$ and interpolation for N_C implies that $h^1(N_C(-p_1 - \dots - p_n)) = 0$. But the obstruction to smoothness of the map extracting the image of the n marked points

$$\overline{M}_{g,n}(\mathbb{P}^4, d)^\circ \rightarrow (\mathbb{P}^4)^n$$

lies in $H^1(N_C(-p_1 - \dots - p_n))$; thus this map is smooth at the point $(f: C \rightarrow \mathbb{P}^4, p_1, \dots, p_n)$, and hence generically smooth. Therefore the map is dominant. For degree 6 and genus 2, even though the normal bundle does not satisfy interpolation; however it is still true that for $9 = f(6, 2, 4)$ general points $p_1, \dots, p_9$ on C , the twist $h^1(N_C(-p_1 - \dots - p_9)) = 0$. Therefore, the above evaluation map is still generically smooth and a general curve of this degree and genus still passes through nine general points [1, Corollary 1.4].

Similarly, fix a hyperplane $H \subset \mathbb{P}^4$; then interpolation for $N_C(-1)$ implies that the map extracting the section by H (when it is transverse) and the image of the n points

$$\overline{M}_{g,n}(\mathbb{P}^4, d)^\circ \dashrightarrow H^d \times (\mathbb{P}^4)^n,$$

defined by

$$(f: C \rightarrow \mathbb{P}^4, p_1, \dots, p_n) \mapsto (f(C) \cap H, f(p_1), \dots, f(p_n)),$$

is smooth, and hence dominant, for $n \leq f(d, g, 4) - d$. In the case $(d, g) = (6, 2)$, if p_1, p_2, p_3 are general points on C , then as $3 > 2$, we have that $\mathcal{O}_C(1)(p_1 + p_2 + p_3)$ is a general line bundle of degree $9 = f(6, 2, 4)$. Therefore the fact that C passes through nine general points implies that its hyperplane section is general and it passes through three independently general points.

We conclude with the three cases $(d, g) \in \{(8, 5), (9, 6), (10, 7)\}$. In these cases $h^1(N_C(-1)) \neq 0$, and hence the hyperplane section is not a general collection of d points in $\mathbb{P}^3$; however $d - 1$ of these points are general, and all d points are general subject to the condition that they are distinct points on the complete intersection of $11 - d$ quadrics [8, Theorem 1.6].

6.4 Degree 8, genus 5

Let $D = p_1 + \cdots + p_7$ be a collection of $d - 1$ points in a general hyperplane section, then we will show that $N_C(-D)$ satisfies interpolation. We have,

$$\chi(N_C(-D)) = 15 \geq \text{genus}(C) \cdot \text{rk}(N_C).$$

Therefore interpolation follows from interpolation for N_C as in [1, Proposition 4.12]. As above this suffices to show that a general curve of degree 8 and genus 5 passes through $f(8, 5, 4) = 12$ points that are general subject to the constrain that seven of them lie in a hyperplane.

In fact, something a priori stronger follows. The hyperplane section of a curve of degree 8 and genus 5 (which is the complete intersection of three quadrics in $\mathbb{P}^4$) is the complete intersection of three quadrics in $\mathbb{P}^3$; thus, some subset D of seven of them determines the 8th. Therefore any canonical curve in $\mathbb{P}^4$ passing through these seven general points D in a hyperplane passes through the 8th point automatically. In this way, the previous result implies that one may pass a curve of degree 8 and genus 5 through $f(d, g, 4) + 1$ points that are general subject to the constraint that d lie in a hyperplane and are distinct points on the complete intersection of $11 - d$ quadrics.

6.5 Degree 9, genus 6

We first show that for $D = p_1 + \cdots + p_8$ a collection of $d - 1$ general points in a hyperplane, $N_C(-D)$ satisfies the property of interpolation. As above this implies the corollary about passing a general such curve through $f(d, g, 4)$ points subject to the constraint that $d - 1$ lie in a hyperplane. When $(d, g) = (9, 6)$, we have $\chi(N_C(-D)) = 16$. It suffices, therefore, (by Lemma 2.9) to show that for a general divisor Γ of degree 5, the twist down has no higher cohomology:

$$h^1(N_C(-\Gamma - D)) = 0.$$

As $D = \mathcal{O}_C(1)(-p)$, with p general on C , the above twist $N_C(-\Gamma - D) \simeq N_C(-1)(p - D)$. If Γ is a general collection of five points, then $\mathcal{O}_C(p - D)$, and hence $\mathcal{O}_C(-1)(p - D)$, is a general line bundle. The result therefore follows from Theorem 1.5.

In a slightly different direction, we show that a curve of degree 9 and genus 6 in $\mathbb{P}^4$ passes through $f(9, 6, 4) = 13$ points that are general subject to the constrain that $d = 9$ of them lie in a hyperplane and lie on the complete intersection of two quadrics, i.e., an elliptic normal curve. Note that an elliptic normal curve in $\mathbb{P}^3$ passes through eight general points and is determined by eight general points on the curve.

Fix nine general points $p_1, \dots, p_9$ on an elliptic normal curve $E \subset H \simeq \mathbb{P}^3$ and five other general points $q_1, \dots, q_5$ in $\mathbb{P}^4$. We will show that there exists a curve of degree 9 and genus 6 through the first 13 points $p_1 + \cdots + p_9 + q_1 + \cdots + q_4$ but not through all $14 = f(9, 6, 4) + 1$ points.

Let L be a line in $\mathbb{P}^4$ which is general relative to $p_1, \dots, p_7$ and q_1, q_2, q_3 (and so in particular disjoint from $p_1, \dots, p_7, q_1, q_2, q_3$).

Lemma 6.1 *The subscheme*

$$q_1 \cup q_2 \cup q_3 \cup p_1 \cup \dots \cup p_7 \cup L \subset \mathbb{P}^4$$

lies on a smooth del Pezzo surface of degree 4.

Proof We will show that if S is some del Pezzo surface, L is a line on S , and q_1, q_2, q_3 are general points on S and $p_1, \dots, p_7$ are general points in a hyperplane section of S , then

$$H^0(N_S) \rightarrow H^0(N_S|_{q_1 \cup q_2 \cup q_3 \cup p_1 \cup \dots \cup p_7 \cup L})$$

is surjective, and $h^1(N_S) = 0$, so that these points and line are general by deformation theory. As $N_S \simeq \mathcal{O}_S(2)^{\oplus 2}$, we see from Kodaira vanishing that $h^1(N_S) = 0$. As the points q_i are general on S and so impose independent conditions on sections of a line bundle, it suffices to show that

$$H^0(\mathcal{O}_S(2)) \rightarrow H^0(\mathcal{O}(2)|_{p_1 \cup \dots \cup p_7 \cup L})$$

is surjective, for which it suffices to show the composition

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3 \cup L}(2)) \rightarrow H^0(\mathcal{O}(2)|_{p_1 \cup \dots \cup p_7}) \oplus H^0(\mathcal{O}_L(2))$$

is surjective. Consider the second map first; the map evidently surjects onto $H^0(\mathcal{O}_L(2))$, so it suffices to check that the kernel surjects onto $H^0(\mathcal{O}_{p_1 \cup \dots \cup p_7}(2))$. Write $p = L \cap H$. Note that any quadric on H vanishing at p can be extended to a quadric on $\mathbb{P}^4$ vanishing on L (for example by pullback under the projection map from some point on $L \setminus \{p\}$). The restriction of the kernel to H is therefore isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}_p)$, where $\mathcal{I}_p$ is the ideal sheaf of the point p , which surjects onto $H^0(\mathcal{O}_{p_1 \cup \dots \cup p_7}(2))$ as the eight points $p_1, \dots, p_7, p$ are general in $\mathbb{P}^3$. $\square$

To complete the proof, define a point p on E by the relation

$$\mathcal{O}_E(p_1 + \dots + p_9)(-2) = \mathcal{O}_E(p).$$

Note that p is general in H independent from $p_1, \dots, p_7, q_1, q_2, q_3, q_4$. Let L be the line in $\mathbb{P}^4$ joining p and q_4 . Then by Lemma 6.1 above, there exists a smooth del Pezzo surface S through $q_1 \cup q_2 \cup q_3 \cup p_1 \cup \dots \cup p_7 \cup L$. The hyperplane section $S \cap H$ is an elliptic normal curve containing the 8 points $p_1, \dots, p_7$ and p on E , and hence must be equal to E . It therefore contains the 13 points $p_1, \dots, p_9$ and $q_1, \dots, q_4$.

The curve class $2H + L$ is base-point-free of degree 9 and arithmetic genus g satisfying

$$2g - 2 = (H + L) \cdot (2H + L) = 8 + 3 - 1 = 10 \implies g = 6.$$

The general member of this linear system on S is therefore a smooth (hence BN by [6]) curve of degree 9 and genus 6 in $\mathbb{P}^4$. It suffices only to show that such a curve can be passed through 4 general points on S and 8 general points in a hyperplane section. By deformation theory, it suffices to show that $h^1(N_{C/S}(-p_1 - \cdots - p_8 - q_1 - \cdots - q_4)) = 0$. We have

$$\begin{aligned} h^1(N_{C/S}(-p_1 - \cdots - p_8 - q_1 - \cdots - q_4)) \\ &= h^1(\mathcal{O}_C(2H + L)(-p_1 - \cdots - p_8 - q_1 - \cdots - q_4)) \\ &= h^1(\mathcal{O}_C(H + L)(p_9 - q_1 - \cdots - q_4)) \\ &= h^1(K_C(p_9)(-q_1 - \cdots - q_4)). \end{aligned}$$

As $h^1(K_C(p_9)) = 0$, and $q_1, \dots, q_4$ are general points, the twist $K_C(p_9)(-q_1 - \cdots - q_4)$ is still nonspecial as desired.

Finally let us show that there does not exist a curve of degree 9 and genus 6 passing through all 14 points $p_1, \dots, p_9$ and $q_1, \dots, q_5$. The strategy will be similar to the constructive method above, however we will find a smooth del Pezzo containing the first eight (general) points $p_1, \dots, p_8$, and $q_1, \dots, q_5$; using the fact that any curve of degree 9 and genus 6 containing these points must lie on this surface, we show that such a curve cannot pass through p_9 .

Lemma 6.2 *There is a unique smooth del Pezzo surface of degree 4 through the points*

$$p_1, \dots, p_8, q_1, \dots, q_5,$$

that are general subject to the constraint that the p_i lie in a general hyperplane.

Proof First, we show that there is a unique pencil of quadrics passing through $p_1, \dots, p_8, q_1, \dots, q_5$. As there is a 14-dimensional projective space of quadrics in $\mathbb{P}^4$, this amounts to asserting that the 13 points above impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^4}(2)$. Consider the surjective restriction map to $\mathbb{P}^3$

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(2)).$$

As the eight points $p_1, \dots, p_8$ are general in $\mathbb{P}^3$, they impose independent conditions on sections $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ and thus sections of $\mathcal{O}_{\mathbb{P}^4}(2)$. The points $q_1, \dots, q_5$ then impose independent conditions on sections vanishing at $p_1, \dots, p_8$ as they are general points in $\mathbb{P}^4$.

Next, let S be a (smooth) del Pezzo surface and let $p_1, \dots, p_8$ be a general collection of points on $S \cap H = E$ a general hyperplane section. Let $q_1, \dots, q_5$ be five other general points on S . Denote by Γ the 0-dimensional subscheme of the points $p_1 + \cdots + p_8 + q_1 + \cdots + q_5$. We will show that the map from such marked del Pezzo surfaces to $H^8 \times (\mathbb{P}^4)^5$ is dominant, by deformation theory, by observing that $N_{S/\mathbb{P}^4}$ has no higher cohomology and that the restriction map

$$H^0(N_S) \rightarrow H^0(N_S|_{\Gamma})$$

is surjective. Indeed, $N_S \simeq \mathcal{O}_S(2) \oplus \mathcal{O}_S(2)$ is nonspecial as $\mathcal{O}_S(2)$ is ample and so has no higher cohomology by Kodaira vanishing. To show the surjectivity, note that the restriction map $H^0(\mathcal{O}_S(2)) \rightarrow H^0(\mathcal{O}_E(2))$ is surjective, again for example by Kodaira vanishing, as E is in the class $\mathcal{O}_S(1)$. Since the points $p_1, \dots, p_8$ are general on E , they impose independent conditions on $H^0(\mathcal{O}_E(2))$; therefore they also impose independent conditions on sections of $\mathcal{O}_S(2)$. The points $q_1, \dots, q_5$, being general, impose independent conditions on sections of $\mathcal{O}_S(2)$ vanishing at $p_1, \dots, p_8$. $\square$

Now consider any curve C of degree 9 and genus 6 in $\mathbb{P}^4$ passing through Γ (by Corollary 1.2 there exist such curves). By Riemann–Roch, $h^0(\mathcal{O}_C(2)) = 18 + 1 - 6 = 13$, and so C lies on at least a pencil of quadrics in $\mathbb{P}^4$. As the points of Γ lie on a unique, and general, such pencil, C must also lie on a unique such pencil, and so be contained in the base locus S , a del Pezzo surface of degree 4. As the Picard group of S is countable, C is in one of countably many possible curve classes on S . Therefore $C \cap H \subset E$ is in one of only countably many linear equivalence classes of degree 9 divisors on E . This necessarily contains $p_1 + \dots + p_8$, and so the 9th point of intersection is in one of only countably many classes on E . In particular, it cannot be a general point on E .

6.6 Degree 10, genus 7

Let $D = p_1 + \dots + p_9$ be a collection of points on a general hyperplane section of a curve C of degree 10 and genus 7 in $\mathbb{P}^4$. We will show that $N_C(-D)$ satisfies interpolation, therefore deducing that C passes through $f(10, 7, 4) = 14$ points that are general subject to the constraint that nine of them lie in a hyperplane. In this case we have

$$\chi(N_C(-D)) = 17 \equiv -1 \pmod{3}.$$

Therefore it suffices to show that for a general divisor Γ of degree 6,

$$h^0(N_C(-D - \Gamma)) = 0.$$

As before writing $D = \mathcal{O}_C(1)(-p)$, with p general on C , we have that $\mathcal{O}_C(-1)(p - D)$ is general. Therefore this result again follows from Theorem 1.5.

In [8] it was shown that the hyperplane section $H \cap C$ of a curve C of degree 10 and genus 7 is a general collection of 10 points on a quadric in $H \simeq \mathbb{P}^3$. We show here that there exists a curve of degree 10 and genus 7 through a collection of $f(10, 7, 4) + 1 = 15$ points that are general subject to the condition that 10 lie on a quadric in H .

We begin in a similar way to our approach for (9, 6). Fix general points $p_1, \dots, p_{10}$ on a quadric Q in $H \simeq \mathbb{P}^3$, and five more general points $q_1, \dots, q_5$ in $\mathbb{P}^4$.

Lemma 6.3 *There exists a unique smooth quadric through the 14 points*

$$p_1, \dots, p_9, q_1, \dots, q_5,$$

that are general subject to the constraint that the p_i lie in a hyperplane. (Equivalently through the 15 points $p_1, \dots, p_{10}, q_1, \dots, q_5$ that are general subject to the constraint that the p_i lie in a quadric in a hyperplane.)

Proof As there is a 14-dimensional projective space of quadrics in $\mathbb{P}^4$, it suffices to show that these 14 points impose independent conditions on sections of $H^0(\mathcal{O}_{\mathbb{P}^4}(2))$. The points $p_1, \dots, p_9$ impose independent conditions on $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$, being general on $\mathbb{P}^3$. Furthermore, restriction is a surjection

$$H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(2))$$

so the points $p_1, \dots, p_9$ impose at least as many conditions on $H^0(\mathcal{O}_{\mathbb{P}^4}(2))$. The general points $q_1, \dots, q_5$ impose independent conditions on sections of $\mathcal{O}_{\mathbb{P}^4}(2)$ vanishing at $p_1, \dots, p_9$, and so the whole collection imposes independent conditions on quadrics.

Now we show that this quadric is general. As above, let S be a general quadric in $\mathbb{P}^4$ and let $q_1, \dots, q_5$ be general points on S and $p_1, \dots, p_9$ be general points on $H \cap S$. Then it suffices by deformation theory to verify that $N_{S/\mathbb{P}^4}$ has no higher cohomology and that

$$H^0(N_S) \rightarrow H^0(N_S|_{p_1 \cup p_2 \cup \dots \cup p_9 \cup q_1 \cup \dots \cup q_5})$$

is surjective. To show $N_S \simeq \mathcal{O}_S(2)$ has no higher cohomology, we use the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(n-2) \rightarrow \mathcal{O}_{\mathbb{P}^4}(n) \rightarrow N_S(n) \simeq \mathcal{O}_S(n) \rightarrow 0,$$

and the vanishing of the higher cohomology of $\mathcal{O}_{\mathbb{P}^4}(n)$ and $\mathcal{O}_{\mathbb{P}^4}(n-2)$ for $n \geq 1$, which shows that $\mathcal{O}_S(n)$ has no higher cohomology for $n \geq 1$ (in particular for $n = 2$).

To show that these 14 points impose independent conditions on sections of $\mathcal{O}_S(2)$, we first note that the restriction map $H^0(\mathcal{O}_S(2)) \rightarrow H^0(\mathcal{O}_{S \cap H}(2))$ is surjective (since $h^1(\mathcal{O}_S(1)) = 0$ by the above). Since $p_1, p_2, \dots, p_9$ are general on $S \cap H$, and $h^0(\mathcal{O}_{S \cap H}(2)) = h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)) = 9$, they impose independent conditions on sections of $\mathcal{O}_{S \cap H}(2)$ and thus on sections of $\mathcal{O}_S(2)$. Since $q_1, q_2, \dots, q_5$ are general, they impose independent conditions on

$$\ker(H^0(\mathcal{O}_S(2)) \rightarrow H^0(\mathcal{O}_{S \cap H}(2))) \simeq H^0(\mathcal{O}_S(1)).$$

Combining this, our 14 points impose independent conditions on sections of $\mathcal{O}_S(2)$, as desired. $\square$

We showed above that there exists a curve C of degree 10 and genus 7 through $p_1, \dots, p_9, q_1, \dots, q_5$. As $h^0(\mathcal{O}_C(2)) = 20 + 1 - 7 = 14$, the curve C lies on a quadric. As the 14 points lie on a unique quadric S (which is smooth if the points are general), C lies on this quadric.

Finally we claim that the rank 2 bundle $N_{C/S}(-1)$ satisfies interpolation — this is sufficient to prove the desired result as the points $p_1, \dots, p_{10}$ are general points in a

hyperplane section of this quadric and $q_1, \dots, q_5$ are general points on this quadric. From

$$0 \rightarrow N_{C/S}(-1) \rightarrow N_C(-1) \rightarrow \mathcal{O}_C(1) \rightarrow 0,$$

we have $\chi(N_{C/S}(-1)) = 10$. It suffices to show that after twisting down by five general points $q_1, \dots, q_5$ on C , we have $h^0(N_{C/S}(-1)(-q_1 - \dots - q_5)) = 0$. By the above exact sequence, it suffices to show that

$$h^0(N_C(-1)(-q_1 - \dots - q_5)) = 0.$$

To show this, we will degenerate C to the union of a curve X of degree 6 and genus 2 and a degenerate rational normal curve $M \cup_{m_1+m_2+m_3} (L_1 \cup L_2 \cup L_3)$, where $L_i \cap M = m_i$ and $L_i \cap C = z_i + w_i$ is a 2-secant line as in the setup of Sect. 5. Specialize the points q_i so that two lie on M and one lies on each of the L_i . By [7, Theorem 1.6] this is a BN-curve and as in Lemma 5.1 (using Lemma 2.17) we have that on global sections

$$\begin{aligned} h^0(N_C(-1)(-q_1 - \dots - q_5)) \\ = h^0(N_X(-1)[z_1 \rightarrow w_1][w_1 \rightarrow z_1][z_2 \rightarrow w_2][w_2 \rightarrow z_2][z_3 \rightarrow w_3][w_3 \rightarrow z_3]). \end{aligned}$$

Further degenerate X as in Lemma 5.2 to the union of a curve Y of degree 5 and genus 1 and a 2-secant line L meeting Y at points x and y . The points x_1 and x_2 specialize onto L and the remainder specialize onto Y . The union is a BN-curve and the global sections glue to give that

$$\begin{aligned} h^0(N_Y(-1)(y)[x \rightarrow y][2y \rightarrow x][w_1 \rightarrow x][w_2 \rightarrow x][z_3 \rightarrow w_3][w_3 \rightarrow z_3]) \\ = h^0(N_X(-1)[z_1 \rightarrow w_1][w_1 \rightarrow z_1][z_2 \rightarrow w_2][w_2 \rightarrow z_2][z_3 \rightarrow w_3][w_3 \rightarrow z_3]) \\ = h^0(N_C(-1)(-q_1 - \dots - q_5)). \end{aligned}$$

The bundle

$$N'_Y := N_Y(-1)(y)[x \rightarrow y][2y \rightarrow x][w_1 \rightarrow x][w_2 \rightarrow x][z_3 \rightarrow w_3][w_3 \rightarrow z_3]$$

sits in an exact sequence

$$\begin{aligned} 0 \rightarrow N_{Y \rightarrow x}(-1)(y - x - z_3 - w_3) \rightarrow N'_Y \\ \rightarrow N_{\pi_x|Y}(-1)(-y - w_1 - w_2)[x \rightarrow y][z_3 \rightarrow w_3][w_3 \rightarrow z_3] \rightarrow 0 \end{aligned}$$

$N_{Y \rightarrow x}(-1)(y - x - z_3 - w_3) \simeq \mathcal{O}_Y(x + y - z_3 - w_3)$ is a general line bundle of degree 0 on an elliptic curve, and hence has no global sections. It thus remains to show

$$h^0(N_{\pi_x|Y}(-1)(-y - w_1 - w_2)[x \rightarrow y][z_3 \rightarrow w_3][w_3 \rightarrow z_3]) = 0.$$

Since $\chi(N_{\pi_x|Y}(-1)(-y - w_1 - w_2)[x \rightarrow y][z_3 \rightarrow w_3][w_3 \rightarrow z_3]) = -1$ and $\mathcal{O}_C(1)(y + w_1 + w_2)$ is a general line bundle of degree 2 on an elliptic curve, it

suffices to show $N_{\pi_x|Y}[x \rightarrow y][z_3 \rightarrow w_3][w_3 \rightarrow z_3]$ satisfies interpolation. Limiting y to w_3 , it suffices to show interpolation for the bundle

$$N_{\pi_x|Y}[x \rightarrow w_3][z_3 \rightarrow w_3][w_3 \rightarrow z_3].$$

This is verified by publicly available code in [1, Appendix B]:

```
>>> good(Curve(4, 1, 3).add(P101).add(P100))
True
```

Acknowledgements We would like to thank Joe Harris for his guidance, as well as Ravi Vakil, Izzet Coskun, and members of the MIT and Harvard math departments for helpful conversations.

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