Electrostatic energy of interaction between uniformly charged hemispherical surfaces

Orion Ciftja[1]
[1]*Department of Physics, Prairie View A&M University, Prairie View, Texas 77446, USA**
(Dated: December 2, 2021)

The purpose of this study is to gain insights on the nature of the electrostatic energy of interaction between two uniformly charged hemispherical surfaces with constant surface charge density. The present system of these two interacting bodies represents a very difficult scenario since hemispherical surfaces lack the spherical symmetry of a uniformly charged spherical surface or a solid sphere counterpart. For this reason, an exact analytical calculation of the electrostatic interaction energy is not possible for an arbitrary orientation. However, this work shows that analytical results are possible for scenarios where the system of the two interacting hemispherical surfaces manifests some form of inherent symmetry. We identify such situations where combination of suitable mathematical tools appropriate for the axial symmetry together with transformations that apply to a spherical system of coordinates can reduce the difficulty of the problem. We show that there are some special cases where this very difficult integral problem can be reduced to a simpler one of summation of an infinite series. The approach is illustrated by showing explicitly the calculation of the interaction energy between the two uniformly charged hemispherical surfaces when they are brought together so that they are touching each other across the "equator".

Keywords: Modeling, Electrostatic energy, Hemispherical surface, Uniform surface charge density.

I. INTRODUCTION

The calculation of the electrostatic energy of interaction between two charged bodies is impossible to be carried out in analytical form if the two bodies have an arbitrary shape. As a result, the main focus of analytical studies in electrostatics[1–6] are systems consisting of regular bodies possessing some symmetry such as spherical surfaces/shells, solid spheres, disks, rings, etc. Even for these cases, calculations are very difficult if the charge distribution is arbitrary or unknown. In many cases, the equilibrium charge distribution (that makes a body an equipotential) is impossible te determine analytically if the body is slightly more complicated than a spherical suface or a disk[7–9]. Given the situation, a very common approach widely used is to assume a uniform distribution of charge over the volume/surface/length of the bodies resulting, respectively, in a constant volume/surface/length charge density[10,11].

A perfect example of a body that illustrates such a point of view would be a charged hemispherical surface, namely, one of the halves of a spherical surface as divided by the "equator". The equilibrium charge distribution on a spherical surface (that makes the body an equipotential) happens to be a uniform charge distribution. However, this does not mean that the equilibrium charge distribution on a hemispherical surface will remain uniform if the two halves of the spherical surface are divided from the whole. In fact, as far as we know, finding the equilibrium charge distribution on a hemispherical surface remains an unsolved problem. Therefore, in order to shed some light on the electric properties of a charged hemispherical surface one assumes that the charge is uniformly distributed resulting in a constant surface charge density. A uniformly charged hemispherical surface lacks spherical symmetry, but may offer the possibility of an analytical treatment by virtue of the system still retaining axial symmetry.

In view of these considerations, the wishful thinking is that one might be able to calculate the electrostatic energy of interaction between two uniformly charged hemispherical surfaces as long as the two bodies are not arbitrarily positioned in space relative to each other. This means that it is essential to have such a configuration of this system that retains some form of symmetry. With a little bit of thinking one reaches the conclusion that a geometric configuration that has axial symmetry is one represented by two identical uniformly charged hemispherical surfaces that are coaxial, for instance, in such a way that their "equatorial" planes are parallel and separated by an arbitrary finite distance or, at the most, they can be so close that they touch each other on the "equator". In this work, we consider precisely this system and attempt an exact analytical calculation of the resulting electrostatic energy of interaction. It turns out that even this setup is a very challenging. The only instance that led to an exact analytical result is the one where the two hemispherical surfaces are brought together so that they are touching across the "equator".

The paper is organized in the following form: In Section II we introduce the model under consideration and explain some basic theoretical concepts. In Section III we show the key mathematical calculations and the main results that we obtained. In Section IV we summarize the essence of the work and give few concluding remarks.

II. MODEL

Let us consider a system consisting of two identical uniformly charged hemispherical surfaces each with radius, R and each containing the same amount of charge, Q. This means that the charge distribution in each of the

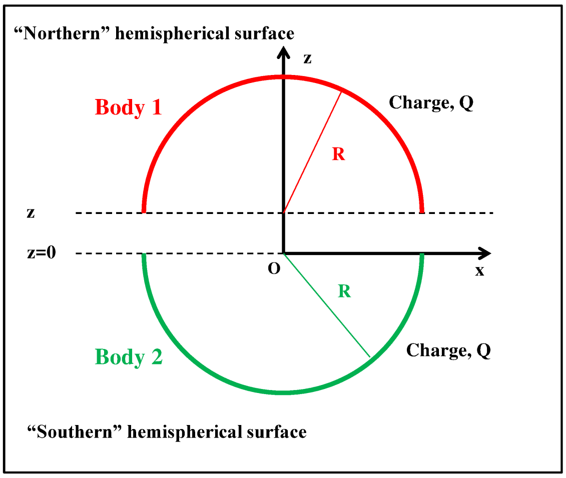

FIG. 1: A system of two identical uniformly charged hemispherical surfaces with radius, R and containing the same amount of charge, Q. The two hemispherical surfaces are considered coaxial. The "equatorial" plane of the "southern" hemispherical surface is on the $z = 0$ plane of the system of coordinates while the "equatorial" plane of the "northern" hemispherical surface is on an arbitrary $z \geq 0$ plane. For simplicity of view, the system is projected on the $y = 0$ plane.

two hemispherical surfaces has a constant surface charge density:

$$\sigma = \frac{Q}{2\,\pi\,R^2}\ . \tag{1}$$

The two hemispherical surfaces are coaxial. The respective "equatorial" planes of the two hemispherical surfaces are parallel to each other and are separated by an arbitrary distance that may become zero when the two planes touch each other across the "equator". The system projected on the $y = 0$ plane is shown in Fig. 1. The system of coordinates is shown in Fig. 1 with origin chosen to correspond to the center of the "southern" hemispherical surface. Based on this setup, the "northern" hemispherical surface (Body 1) has its "equatorial" plane at some arbitraty $z \geq 0$ plane. On the other hand, the "southern" hemispherical surface (Body 2) has its "equatorial" plane on the $z = 0$ plane all the time. The special situation of the two hemispherical surfaces touching each other accross the "equator" occurs when we have $z = 0$ for the "northern" hemispherical surface.

The electrostatic energy of interaction between the two bodies is denoted as $U_{12}(z)$ since we know from the symmetry of the problem that it will depend only on the variable, z and reads:

$$U_{12}(z) = k_e \int_{Body\,1} dQ_1 \int_{Body\,2} dQ_2\, \frac{1}{|\vec{r}_1 - \vec{r}_2|}\ , \tag{2}$$

where k_e is Coulomb's electric constant, dQ_1 is an elementary charge located on the "northern" hemispherical surface (Body 1) at position vector, $\vec{r}_1$, dQ_2 is an elementary charge located on the "southern" hemispherical surface (Body 2) at position vector, $\vec{r}_2$. The domain of integration for Body 2 is easy to write in spherical coordinates. However, the domain of integration for Body 1 is not at all easy to write in spherical coordinates if $z > 0$. Our efforts to obtain an analytical result for $U_{12}(z)$ at an arbitrary $z \geq 0$ separation were not succesful with the exception of the case $z = 0$ that we report in this work. As one will see, even this special case, is quite difficult. The case $z = 0$ corresponds to that scenario where the two hemispherical surfaces touch each other on the "equator" with coinciding equatorial planes both on the $z = 0$ plane. It is simple and we leave it to the reader to check that, indeed, the domains of integration in spherical coordinates simplify considerably when $z = 0$. For such a configuration, one has:

$$U_{12}(z=0) = k_e\,\sigma^2\,R^4 \int_0^{\pi/2} d\theta_1\,\sin\theta_1 \int_0^{2\,\pi} d\phi_1 \int_{\pi/2}^{\pi} d\theta_2\,\sin\theta_2 \int_0^{2\,\pi} d\phi_2\,\frac{1}{|\vec{r}_1 - \vec{r}_2|}\ , \tag{3}$$

where $dQ_1 = \sigma\,R^2\,\sin\theta_1\,d\theta_1\,d\phi_1$ is an elementary surface on Body 1, $dQ_2 = \sigma\,R^2\,\sin\theta_2\,d\theta_2\,d\phi_2$ is an elementary surface on Body 2 and θ_i, ϕ_i $(i = 1, 2)$ are, respectively, the polar and the azimuthal (longitudinal) angles. The calculation of the above quantity in Eq.(3) is not straightforward. Nevertheless and despite the challenges, we succeeded on calculating it exactly. We achieved this succes by using an approach that we recently applied to the calculation of the electrostatic self-energy of a uniformly charged hemispherical surface[12].

A drawing of the system of two uniformly charged hemispherical surfaces for such a case projected on the $y = 0$ plane is shown in Fig. 2. This drawing is useful for the representation of the three-dimensional (3D) vectors $\vec{r}_1$, $\vec{r}_2$ and key angular variables in spherical coordinates that appear in the expression of Eq.(3) as well as subsequent calculations.

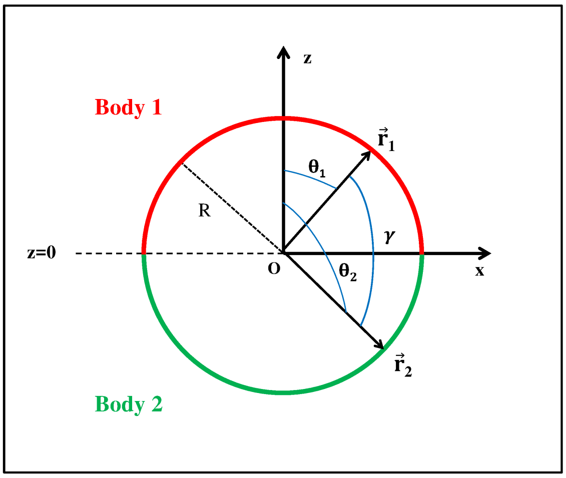

FIG. 2: Two identical uniformly charged hemispherical surfaces with radius, R containing the same amount of charge, Q are brought close together as to touch each other along the "equatorial" ($z = 0$) plane. The polar angles, θ_1 and θ_2 for the respective 3D vectors, $\vec{r}_1$ and $\vec{r}_2$ are explicitly shown. The angle, γ is the angle between these two vectors and, for the present system of spherical coordinates, is calculated from $\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)$. For simplicity of view, the system is projected on the $y = 0$ plane.

III. CALCULATIONS AND RESULTS

The mathematical approach that we mentioned relies on the theory of Legendre polynomials and the following well known expansion of the Coulomb term:

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{\sqrt{r_1^2 - 2\,r_1\,r_2\,\cos\gamma + r_2^2}} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\,P_l(\cos\gamma)\ , \tag{4}$$

where $r_<$ denotes the smaller of $r_1 = |\vec{r}_1|$ and $r_2 = |\vec{r}_2|$, $r_>$ denotes the larger of r_1 and r_2, $P_l(\cos\gamma)$ are Legendre polynomials and γ is the angle between the two vectors $\vec{r}_1$ and $\vec{r}_2$. The above expansion is always valid when $r_1 \neq r_2$.

However, the case in Eq.(3) corresponds to:

$$r_1 = r_2 = R\ . \tag{5}$$

By following the remarks in pg. 740 of Ref.[13] one can prove that:

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{R}\sum_{l=0}^{\infty} P_l(\cos\gamma) \quad ; \quad r_1 = r_2 = R\ , \tag{6}$$

except for angles $\gamma = 0$ and π where the infinite sum diverges. The case $r_1 = r_2 = R$ and $\gamma = 0$ means that $\vec{r}_1 = \vec{r}_2$ and, thus, it is obvious that there is a singularity. The case $r_1 = r_2 = R$ and $\gamma = \pi$ is a peculiarity. However, replacing $1/|\vec{r}_1 - \vec{r}_2|$ by the infinite sum in Eq.(6) and avoiding the special angle $\gamma = \pi$ from the integration process is not going to cause any trouble when the integrals are calculated.

After sheding some light on this technical point, one substitutes the Coulomb term given by the expansion from Eq.(6) into Eq.(3). Keeping an eye on the specific details of the current model but by following the same mathematical steps as in Ref.[12] one obtains:

$$U_{12}(z=0) = \frac{k_e\,Q^2}{R}\sum_{l=0}^{\infty}\left\{\left[\int_0^1 dx\,P_l(x)\right]\left[\int_{-1}^0 dx\,P_l(x)\right]\right\}\ . \tag{7}$$

The quantity in Eq.(7) that we want to calculate involves integrals of Legendre polynomials over half a range and an infinite sum over such resulting integrals. It turns out that all these integrals can be done exactly in analytical form as shown for some values of l in Table. I.

TABLE I: Exactly calculated integrals for few values of l ranging from $l = 0$ to $l_{max} = 10$ where the chosen value of $l_{max} = 10$ has no particular significance except for being sufficiently small as to lead to a quick exact calculation of the integrals.

l	$\int_0^1 dx\,P_l(x)$	$\int_{-1}^0 dx\,P_l(x)$
0	1	1
1	1/2	-1/2
2	0	0
3	-1/8	1/8
4	0	0
5	1/16	-1/16
6	0	0
7	-5/128	5/128
8	0	0
9	7/256	-7/256
10	0	0

A useful mathematical result that one can easily prove is:

$$\int_{-1}^0 dx\,P_l(x) = (-1)^l\int_0^1 dx\,P_l(x) \quad ; \quad l = 0, 1, 2, \ldots\ , \tag{8}$$

This mathematical formula allows us to write the expression for $U_{12}(z=0)$ in Eq.(7) as:

$$U_{12}(z=0) = C\,\frac{k_e\,Q^2}{R}\ , \tag{9}$$

where

$$C = \sum_{l=0}^{\infty}\left\{(-1)^l\left[\int_0^1 dx\,P_l(x)\right]^2\right\}\ . \tag{10}$$

By knowing that $\int_0^1 dx\, P_0(x) = 1$ and $\int_0^1 dx\, P_{2\,k}(x) = 0 \quad ; \quad k = 1, 2, \ldots$ one writes C in Eq.(10) as:

$$C = 1 - \sum_{k=0}^{\infty} \left[\int_0^1 dx\, P_{2\,k+1}(x) \right]^2 . \tag{11}$$

With help from Eq.(13) of Ref.[12] one can express the constant C in Eq.(11) as:

$$C = 1 - \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{2^{2\,k+1}\,(k+1)} \frac{(2\,k)!}{(k!)^2} \right]^2 . \tag{12}$$

We used symbolic computation software[14] to calculate the above finite sum with the final result:

$$C = 2 - \frac{4}{\pi} . \tag{13}$$

Hence, the electrostatic energy of interaction between two uniformly charged hemispherical surfaces that are brought together so that they are touching across the "equator" has the exact value:

$$U_{12}(z=0) = \Big(2 - \frac{4}{\pi}\Big) \frac{k_e\, Q^2}{R} , \tag{14}$$

where each of two hemispherical surfaces has radius, R and charge, Q. At this juncture, we also became aware that the correctness of the above result can be checked by a simpler line of analysis. We mentioned earlier that, in a previous work[12] we were able to calculate the electrostatic potential energy stored in a hemispherical surface with uniform surface charge distribution. In the work in question, we showed that the self-energy of a uniformly charged hemispherical surface with total charge Q equals $(2/\pi)\, k_e\, Q^2/R$. Since the "northern" hemispherical surface is identical to the "southern" one, the self-energies of each of them are equal, $U_{11} = U_{22} = (2/\pi)\, k_e\, Q^2/R$. The full spherical surface (with total charge, $2\,Q$) formed after the two hemispherical surfaces touch each other along the "equator" will have a self-energy of $(1/2)\, k_e (2\,Q)^2/R$. The self-energy of the uniformly charged spherical surface (with total charge $2\,Q$) must equal the sum of the self-energies of its two hemispherical surfaces (each with charge Q), plus the energy of interaction, $U_{12}(z=0)$ between them. It is reassuring to see that the expression in Eq.(14) follows from here confirming again the exactness of our result derived via a whole different approach.

As stated earlier, an analytical result for $U_{12}(z > 0)$ does not seem possible to attain by using the current method. It is quite likely that it is impossible to calculate exactly $U_{12}(z > 0)$ in analytic form with any method. Therefore, numerical methods are the only tools left that could be used in future works to calculate $U_{12}(z > 0)$ for the configuration in Fig. 1 at an arbitrary separation $z > 0$. As in any numerical implementation, one must carefully control and scrutinize the numerical accuracy of such calculations.

Combining numerical and analytical results would substantially increase the utility of the work while retaining the interesting mathematical derivation. In this sense, the current exact result for $U_{12}(z=0)$ can serve as a gauge to measure the accuracy any numerical calculation that, in principle, can be applied at arbitrary values of $z \geq 0$. From a more practical perspective, the system depicted in Fig. 1 can also be viewed as a capacitor with electrodes having a hemispherical shape, but, for such a case, the respective hemispherical surfaces should contain $\pm Q$ charges. Capacitance can be derived from the total energy stored for such a system and, thus, the present calculations help in this direction.

IV. CONCLUSIONS

The only instances where we have seen the model of two coaxial uniformly charged hemispherical surfaces being considered is from the perspective of the net force exerted by one hemispherical surface on the other under the assumption that they are touching each other on the "equator"[15–17]. For such an occurrence, the problem is solved by using symmetry arguments and special methods that do not apply to the calculation of the energy of interaction between them. This is a hint that the counterpart calculation of the electrostatic interaction energy between two coaxial uniformly charged hemispherical surfaces is a much more complicated problem as highlighted by this work.

The objective of this study is to address this issue and gain some insight on the nature of the electrostatic energy of interaction between these two bodies. Despite the fact that the two interacting hemispherical surfaces are coaxial, it turns out that this a very challenging problem to tackle. As a result, we found out that an exact analytical calculation of the electrostatic interaction energy is not possible for an arbitrary separation except for the special case when the two hemispherical surfaces are touching accross the "equator". For such a situation, a combination of suitable mathematical tools and expansions appropriate for the axial symmetry of the system reduces the difficulty of the problem. As shown in this work, the starting very difficult integral problem can be ultimately simplified to the calculation of an infinite sum which can be done exactly.

As final remarks, we point out that this work can be of interest to a wide audience of researchers working on the field of electrostatics[18] or opto-electronic materials[19]. Furthermore, the exact analytical result obtained can also be helpful to researchers dealing with numerical calculations. For example, it can be utilized to test and enhance the accuracy and the stability of computarional routines used in numerical calculation of integrals[20].

Acknowledgments

This research was supported in part by National Science Foundation (NSF) grant no. DMR-2001980.

* Electronic address: ogciftja@pvamu.edu

[1] T. LaFave Jr., J. Electrostat. **72**, 39 (2014).
[2] T. LaFave Jr., J. Electrostat. **69**, 414 (2011).
[3] F. R. Zypman, Am. J. Phys. **74** (4), 295 (2006).
[4] U-R. Kim , W. Han, D-W. Jung , J. Lee, and C. Yu, Eur. J. Phys. **42**, 045205 (2021).
[5] A. Charyyev and M. S. Shikakhwa, Eur. J. Phys. **39**, 065204 (2018).
[6] V. Bochko and Z. K. Silagadze, Eur. J. Phys. **41**, 045201 (2020).
[7] D. J. Griffiths and Y. Li, Am. J. Phys. **64**, 706 (1996).
[8] R. H. Good, Am. J. Phys. **65**, 155 (1997).
[9] G. L. Pollack and D. R. Stump, Am. J. Phys. **75**, 485 (2007).
[10] O. Ciftja, Results Phys. **15**, 102684 (2019).
[11] O. Ciftja, Eur. J. Phys. **42**, 025204 (2021).
[12] O. Ciftja, J. Electrostat. **111**, 103579 (2021).
[13] George B. Arfken and Hans J. Weber, Mathematical methods for physicists, Fifth Edition, Harcourt/Academic Press, Burlington, Massachusetts, USA (2001).
[14] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, IL, USA (2012).
[15] D. J. Griffiths, Introduction to Electrodynamics, Third Edition, Prentice Hall, Upper Saddle River, New Jersey, USA (1999).
[16] X. Xie and X. Huang, Am. J. Phys. **62**, 952 (1994).
[17] A. DasGupta, Eur. J. Phys. **28**, 705 (2007).
[18] T. M. Minteer, Eur. J. Phys. **35**, 035022 (2014).
[19] J. P. McClymer, Rev. Sci. Instrum. **90**, 046101 (2019).
[20] D. Finocchiaro, M. Pellegrini, and P. Bientinesi, J. Comput. Phys. **146**, 707 (1998).