On the Cartan decomposition for classical random matrix ensembles

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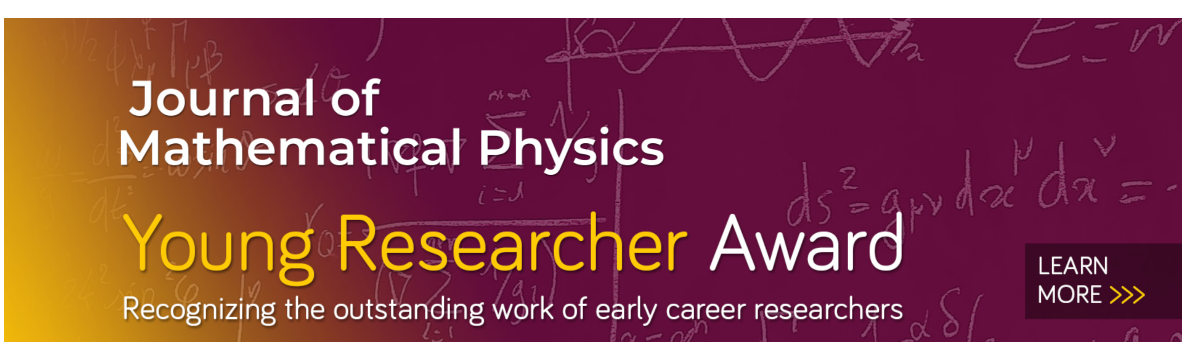

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ABSTRACT

We complete Dyson's dream by cementing the links between symmetric spaces and classical random matrix ensembles. Previous work has focused on a one-to-one correspondence between symmetric spaces and many but not all of the classical random matrix ensembles. This work shows that we can completely capture all of the classical random matrix ensembles from Cartan's symmetric spaces through the use of alternative coordinate systems. In the end, we have to let go of the notion of a one-to-one correspondence. We emphasize that the KAK decomposition traditionally favored by mathematicians is merely one coordinate system on the symmetric space, albeit a beautiful one. However, other matrix factorizations, especially the generalized singular value decomposition from numerical linear algebra, reveal themselves to be perfectly valid coordinate systems that one symmetric space can lead to many classical random matrix theories. We establish the connection between this numerical linear algebra viewpoint and the theory of generalized Cartan decompositions. This, in turn, allows us to produce yet more random matrix theories from a single symmetric space. Yet, again, these random matrix theories arise from matrix factorizations, though ones that we are not aware have appeared in the literature.

I. INTRODUCTION

Random matrix theory (RMT) is a big subject touching so many fields of mathematics, science, and engineering. For such a subject, it is helpful to have a means of cataloging the objects to be studied and a theory that covers the objects in the catalog. In 1962, Dyson[1–4] was the first to propose a systematic approach to RMT. In the beginning of Ref. 4, he states his noble intent:

> *To bring together and unify three trends of thought which have grown up independently during the last thirty years.*

which he enumerates as (i) group representations including time-inversion, (ii) Weyl's theory of matrix algebras, and (iii) RMT.

Around a decade later, Dyson hit upon the idea that symmetric spaces should play a key role (Ref. 5, Sec. V). Dyson's suggestion was taken up in famous papers by Zirnbauer *et al.*[6,7] and others.[8,9] These papers mainly focus on the noncompact cases. On the mathematical side, inspired by Katz and Sarnak,[10,11] Dueñez detailed connections to RMT for the compact symmetric spaces.[12,13]

Nonetheless, we felt there was a gap. When one juxtaposes (i) the well-established theory of classical random matrix ensembles with (ii) the RMTs associated with symmetric spaces, ensembles are missing. In particular, only very special Jacobi ensembles (the left side of Fig. 2) seem to be making the symmetric space list. More precisely, if one starts with a symmetric space, one has to make what we call a coordinate system choice, what others might call a matrix factorization choice. This choice has been the map $\Phi : K \times A \to G/K;\ (k,a) \mapsto kaK$ of Cartan, which we could call the KAK decomposition. (Although it is often called Cartan's KAK decomposition, Cartan was not aware of $G = KAK$.) See Fig. 1.

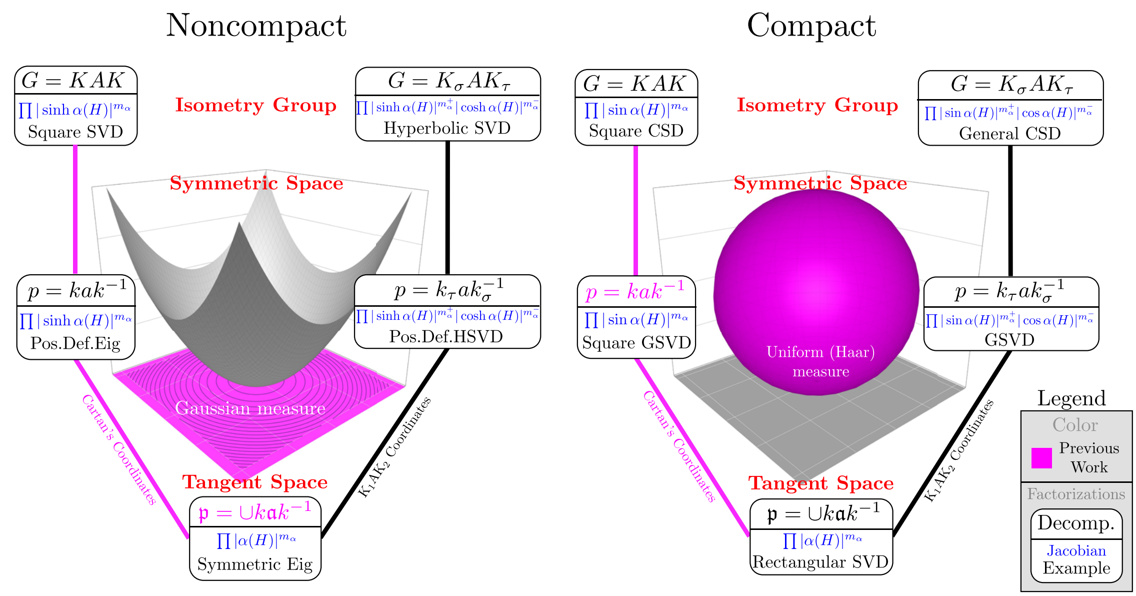

FIG. 1. Families of matrix factorizations associated with a symmetric space, its tangent space, and its isometry group: Shown above are the skeleton of five factorizations associated with noncompact (left) and compact (right) symmetric spaces. Each serves as coordinate systems on the respective manifolds. Previous approaches (manifold, coordinate system, and measure) are shown in magenta. Examples of the linked factorizations/coordinate systems are shown.

We show that coordinate systems from the generalized Cartan (K_1AK_2) decomposition associate a single symmetric space to multiple RMTs. Letting go of the historical bias of the KAK decomposition, the full set of Jacobi ensembles (the right side of Fig. 2) emerges, thereby leading to the complete list of classical random matrix ensembles. Of course, there is much mathematical precedent in differential geometry to letting go of any one special coordinate system.

A. Classical random matrix ensembles

The objects that we are interested in are the classical random matrix ensembles. Well-established conventions in random matrix theory agree that the ensembles in this class consist of the Hermite, Laguerre, Jacobi, and circular ensembles built from matrices of integer sizes and involve entries that are real, complex, or quaternion. (Dyson denoted $\beta = 1, 2, 4$, and other authors in mathematics denote $\alpha = 2/\beta = 2, 1, 1/2$.)

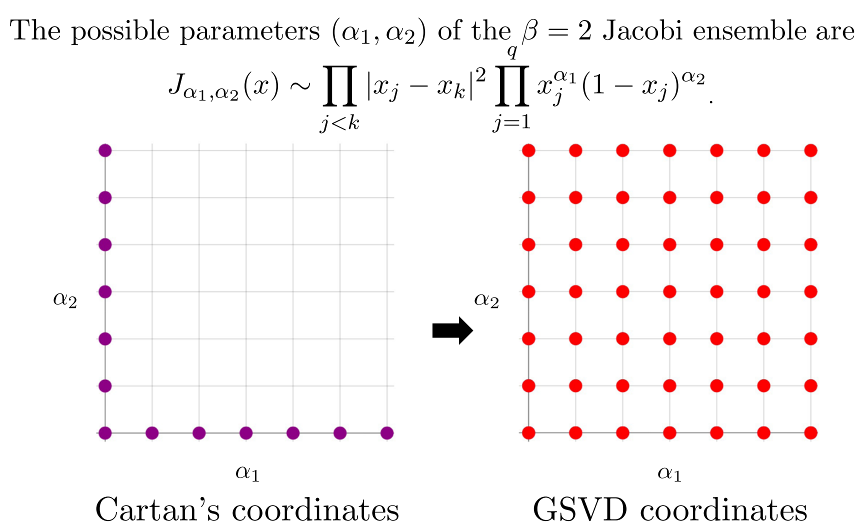

FIG. 2. The parameter space $(\alpha_1, \alpha_2) \in (-1, \infty)^2$ of the $\beta = 2$ Jacobi ensemble obtained from Cartan's coordinates (KAK) (left) and the generalized singular value decomposition coordinates (K_1AK_2) (right).

The term "classical random matrix ensembles" may be found in the following well-known references:

- Chapter 1 of Forrester's paper[14] has the title "Classical Random Matrix Ensembles," and the even sections (1.2, 1.4, 1.6, and 1.8) are explicitly Hermite, circular, Laguerre, and Jacobi in that order. (Odd sections have discussions related to these ensembles.) Forrester's comprehensive book[15] deals exclusively Hermite, Laguerre, Jacobi and circular ensembles in Chaps. 1–3 where the preface states: "eigenvalue p.d.f. of the various classical β-ensembles given in Chaps. 1–3." Then, later in Chap. 5.4, he further justifies the terminology by pointing out the four weights from classical orthogonal polynomial theory.
- In Ref. 16, Chap. 4.1 is entitled "Joint distribution of eigenvalues in the classical matrix ensembles" and specifically covers exactly the Hermite, Laguerre, Jacobi, and circular ensembles.
- The first author's 2005 *Acta Numerica* article (Ref. 17, Sec. 4).

If one starts with the list of ten infinite families of Cartan's symmetric spaces (we will not discuss finite families of the exceptional types) and asks to characterize which classical random matrix ensembles are covered, answers could be found in Ref. 8 (Table 1), Ref. 9 (Table 1) (noncompact cases), and Ref. 13 (Table 1) (compact cases). However, turning the question around, if one starts with the classical random matrix ensembles and asks whether symmetric spaces are adequate to explain all of them, we find that the answer is a big "almost," as the Jacobi ensembles are not adequately covered. To be precise, the Jacobi densities associated with compact symmetric spaces BDI, AIII, and CII from the previous attempts by the KAK decomposition are the following joint probability densities with $\beta = 1, 2, 4$ (up to constant) and integers $p \geq q$,

$$\text{KAK decomposition}: \prod_{j<k} |x_j - x_k|^{\beta} \prod_{j=1}^{q} x_j^{\frac{\beta}{2}-1} (1-x_j)^{\frac{\beta(p-q+1)}{2}-1}, \tag{1.1}$$

where we observe the powers of x_j's restricted to $\frac{\beta}{2} - 1$. The possible parameters of (1.1) are described in the left side of Fig. 2. Additional four compact symmetric spaces DIII, BD, C, and CI add four more Jacobi ensembles,[13] but they are not sufficient to cover the two dimensional parameter set of the Jacobi ensembles.

B. Coordinate systems on the Grassmannian manifold

It is always interesting when a branch of applied mathematics reverses direction and provides guidance to pure mathematics. In this work, we focus on the role of the generalized singular value decomposition (GSVD) from numerical linear algebra.[18,19]

From an applied viewpoint, the Jacobi ensembles are elegantly generated in software with commands such as `svdvals (randn(p,s),randn(q,s))` in languages such as Julia, which is computed by taking the GSVD of two i.i.d. normal matrices with the same number of columns.[20,21] From a pure viewpoint, this is a pushforward of the uniform measure on the Grassmannian manifold onto a maximal Abelian subgroup A (with a fixed Weyl chamber) along the generalized Cartan (K_1AK_2) decomposition (Fig. 3).[22,23]

For example, take a Grassmannian point with any $\beta = 1, 2, 4$ from $O(n)/(O(n-s) \times O(s))$ (respectively, with complex or quaternionic unitary groups) and represent it by the $n \times s$ orthogonal (respectively, complex or quaternionic unitary) matrix X. [More precisely, we treat the Grassmannian manifold as the quotient $V_s(\mathbb{R}^n)/O(s)$ where $V_s(\mathbb{R}^n)$ is the Stiefel manifold. We are allowed to multiply any $O \in O(s)$ on the right side of X.] For any $p, q \geq s$ satisfying $p + q = n$, we have the following coordinate system of X arising from the GSVD[24] of the first p rows and the last q rows of X (for an alternative viewpoint, see Ref. 25):

$$X = \begin{bmatrix} U & \\ & V \end{bmatrix} \begin{bmatrix} C \\ S \end{bmatrix} = \begin{bmatrix} UC \\ VS \end{bmatrix}, \tag{1.2}$$

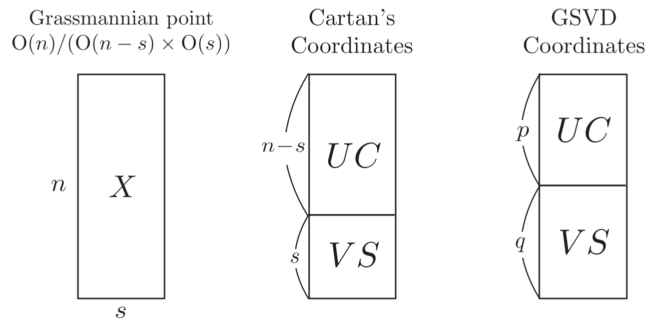

FIG. 3. Cartan's coordinate system (KAK) and GSVD coordinate systems (K_1AK_2) on the Grassmannian manifold $O(n)/(O(n-s) \times O(s))$.

where U, V are $p \times s, q \times s$ orthogonal (respectively, complex or quaternionic unitary) matrices and C, S are $s \times s$ diagonal matrices with cosine and sine values. Deduced joint probability densities[21] $(p, q \geq s)$ are the following (up to constant):

$$\mathrm{K_1AK_2 \ decomposition\ (GSVD)}: \quad \prod_{j<k} |x_j - x_k|^{\beta} \prod_{j=1}^{s} x_j^{\frac{\beta(q-s+1)}{2}-1}(1-x_j)^{\frac{\beta(p-s+1)}{2}-1},$$

where the case $q = s$ represents the usual KAK decomposition case (1.1).

As can be seen, the classical Jacobi parameters are quantized as they are integer multiples of $\beta/2$. Random matrix models that remove this quantization, thereby going beyond the classical, appear in Refs. 20, 26, and 27. In Sec. VII, we also illustrate that some Jacobi ensembles can arise from symmetric spaces that are outside the traditional quantization (Fig. 6).

C. Contributions of this paper

This work shows that a symmetric space can be associated with multiple random matrix theories (Fig. 4). Letting go of the arbitrariness of the choice of the KAK decomposition coordinate system allows us to choose other coordinate systems on symmetric spaces, thereby leading us to the complete list of classical random matrix ensembles (Secs. V, VI, VIII, and IX). Many of these coordinate systems are sometimes better known as matrix factorizations, used widely in matrix models of the classical ensembles.[15,17,20,26,27] However, in Sec. VII, we compute new families of the Jacobi ensemble parameters from coordinate systems that have not been known before.

This work also endeavors to make the Lie theory more widely accessible by simplifying and modernizing key ideas and proofs in Ref. 28. Cartan's theory[29–32] as developed by Helgason[28,33] is a crowning mathematical achievement, and it is our hope to open up this theory for the benefit of all. Indeed, in Ref. 34 (p. 428), Helgason writes about the difficulty of understanding Cartan's writings:

> *[Cartan] was one of the great mathematicians of the period, but his papers were quite a challenge. Hermann Weyl, in reviewing a book by Cartan from 1937 writes: "Cartan is undoubtedly the greatest living master in differential geometry. . . I must admit that I found the book like most of Cartan's papers, hard reading."*

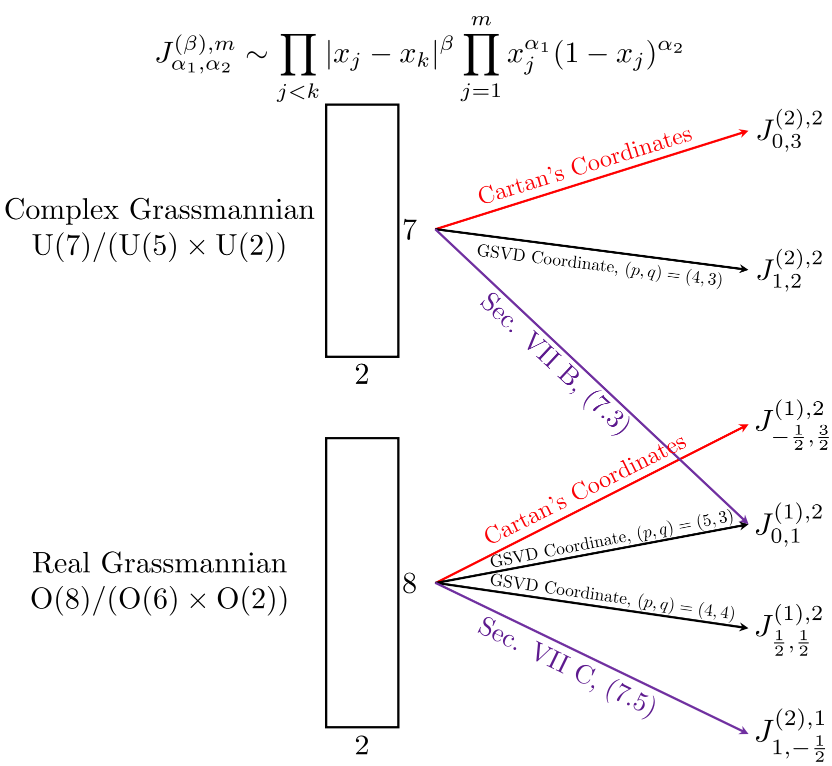

FIG. 4. Examples illustrating the lack of a one-to-one relationship between symmetric spaces and classical random matrix theories: A complex Grassmannian (top) obtains three Jacobi ensembles. A real Grassmannian (bottom) obtains four Jacobi ensembles. In particular, the $\beta = 1$ Jacobi ensemble $J^{(1),2}_{0,1}$ can be obtained from both symmetric spaces. Interestingly, a complex Grassmannian can lead to (top purple) a real RMT in the sense that $\beta = 1$. Similarly, a real Grassmannian obtains $\beta = 2$ RMT (bottom purple).

In the same vein, while we are admirers of Helgason's extensive work, we authors must admit that we, in turn, found Refs. 28 and 33 hard reading as well, and this paper attempts to introduce the theory by couching the ideas in terms of what we call ping pong operators.

Summarizing our work, we have the following:

- We use the coordinate systems of the K_1AK_2 decomposition that connects a single symmetric space to multiple random matrices (Fig. 4), completing the list of associated classical random matrix ensembles.
- We translate some of the key concepts in Cartan's theory of symmetric spaces into easier to follow linear algebra (Sec. III).
- We provide coordinate systems (matrix factorizations) of symmetric spaces that have not been discussed in random matrix context, obtaining new parameter families of the Jacobi ensemble (Sec. VII).

II. BACKGROUND

A. Joint densities of classical random matrix ensembles

Dyson introduced the $\beta = 1, 2, 4$ circular ensembles[1,4] in 1962. Earlier expositions on circular ensembles could be found on Hurwitz[35] and Weyl.[36] Hermite ensembles were introduced by Wigner.[37–39] Laguerre and Jacobi ensembles could be found as early as 1939 in the statistics literature by Fisher,[40] Roy,[41] or Hsu.[42] The physics literature first touches upon the idea of Laguerre and Jacobi with the 1963 thesis of Leff.[43] The following list is the joint probability densities (without normalization constants) of classical random matrix ensembles ($\beta = 1, 2, 4$):

- Circular: $\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta$, $(\theta_1, \ldots, \theta_n) \in [0, 2\pi)^n$;
- Hermite: $\prod_{j<k} |\lambda_j - \lambda_k|^\beta e^{-\sum \frac{\lambda_j^2}{2}}$ $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$;
- Laguerre: $\prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^m \lambda_j^\alpha e^{-\sum \frac{\lambda_j}{2}}$ $(\lambda_1, \ldots, \lambda_m) \in [0, \infty)^m$;
- Jacobi: $\prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^m x_j^{\alpha_1} (1 - x_j)^{\alpha_2}$ $(x_1, \ldots, x_m) \in [0, 1]^m$.

In particular, the parameters $\alpha, \alpha_1, \alpha_2 > -1$ are quantized as integer multiples of $\frac{\beta}{2}$, i.e., $\frac{\beta}{2}(N + 1) - 1$ for some non-negative integer N.

B. Symmetric space and the generalized Cartan decomposition

In this section, we introduce the theory related to the generalized Cartan decomposition. For readers without preliminary knowledge in Lie theory, we recommend skipping to Sec. III, which follows a more modern linear algebra approach.

Let G/K_σ be a Riemannian symmetric space with a real reductive noncompact Lie group G and its maximal compact subgroup K_σ. Let σ be the Cartan involution on $\mathfrak{g} := \mathrm{Lie}(G)$. Then, $\mathfrak{g} = \mathfrak{k}_\sigma + \mathfrak{p}_\sigma$ is the Cartan decomposition. Let τ be another involution on $\mathfrak{g}$ such that $\tau\sigma = \sigma\tau$, and let $\mathfrak{g} = \mathfrak{k}_\tau + \mathfrak{p}_\tau$ be the ± 1 eigenspace decomposition by τ. Denote by K_τ the analytic subgroup of G with tangent space $\mathfrak{k}_\tau$. Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}_\tau \cap \mathfrak{p}_\sigma$ and define $A := \exp(\mathfrak{a})$. We introduce the (noncompact) *generalized Cartan decomposition* (Ref. 22, Theorem 4.1).

Theorem 2.1 *(generalized Cartan decomposition,* K_1AK_2 *decomposition). With the above setting, we have the following decomposition of* G*:*

$$G = K_\tau A K_\sigma. \tag{2.1}$$

That is, for any $g \in G$*, we have* $k_1 \in K_\tau, k_2 \in K_\sigma$ *and* $a \in A$ *such that* $g = k_1 a k_2$*.*

We often use the equivalent name "K_1AK_2 decomposition" for simplicity. Note that if $\tau = \sigma$ (i.e., $K = K_\sigma = K_\tau$), we recover the usual KAK decomposition, $G = KAK$. The generalized Cartan decomposition in the work of Flensted-Jensen[22] is originally intended for the case where G is noncompact. The compact analog is developed by Hoogenboom[23] [Theorem 3.6].

Theorem 2.2 *(generalized Cartan decomposition; compact case). Let* G/K_σ *and* G/K_τ *be two compact Riemannian symmetric spaces. Let* $\mathfrak{g} = \mathfrak{k}_\sigma + \mathfrak{p}_\sigma$ *and* $\mathfrak{g} = \mathfrak{k}_\tau + \mathfrak{p}_\tau$ *be the corresponding eigenspace decompositions of* $\mathfrak{g} = \mathrm{Lie}(G)$*. Then, for a maximal Abelian subalgebra* $\mathfrak{a}$ *of* $\mathfrak{p}_\sigma \cap \mathfrak{p}_\tau$ *and* $A = \exp(\mathfrak{a})$*, we have*

$$G = K_\tau A K_\sigma.$$

From the space of linear functionals $\mathfrak{a}^*$, we collect eigenvalues of an adjoint representation (the commutator) of $\mathfrak{a}$ on $\mathfrak{g}$ and call these eigenvalues the roots of the K_1AK_2 decomposition. By fixing the Weyl chamber, we obtain a set of positive roots Σ^+. Details of the theory of the K_1AK_2 decomposition and its root system can be found in Flensted-Jensen,[22,44] Hoogenboom,[23] Matsuki,[45–47] and Kobayashi.[48] The

K_1AK_2 decomposition is also studied in the context of spherical harmonics and intertwining functions.[49,50] Refine the root space $\mathfrak{g}_\alpha$ of a root α by ±1 eigenspaces of $\sigma\tau$. Let the two dimensions be $m_\alpha^\pm$.

Let dk_σ, dk_τ be the Haar measures of K_σ, K_τ, respectively. Let dH be the Euclidean measure on $\mathfrak{a}$. The Jacobian of the K_1AK_2 decomposition is the following:

Theorem 2.3 *(Jacobian of the K_1AK_2 decomposition[23,44]). Let dg be the Haar measure on G, and let $H \in \mathfrak{a}$. We have the Jacobian and the integral formula corresponding to the change of variables associated with the K_1AK_2 decomposition,*

$$\int_G f(g)dg = \int_{K_\tau}\int_{K_\sigma}\int_{\mathfrak{a}^+} f(k_\sigma \exp(H)k_\tau)d\mu(H)dk_\sigma dk_\tau,$$

where for noncompact G,

$$d\mu(H) \propto \prod_{\alpha\in\Sigma^+}(\sinh \alpha(H))^{m_\alpha^+}(\cosh \alpha(H))^{m_\alpha^-}dH, \tag{2.2}$$

and for compact G,

$$d\mu(H) \propto \prod_{\alpha\in\Sigma^+}(\sin \alpha(H))^{m_\alpha^+}(\cos \alpha(H))^{m_\alpha^-}dH. \tag{2.3}$$

Similar results on the KAK decomposition and the restricted roots of symmetric spaces can be found in standard Lie group textbooks.[28,33,51–53] In the KAK case, the Jacobian (2.2) reduces down to $\prod(\sinh \alpha(H))^{m_\alpha}$ as we do not have -1 eigenspace of $\sigma\tau$ so that $m_\alpha = m_\alpha^+$.[33,54,55]

Theorems 2.1–2.3 are decompositions of the group G. These decompositions can also be applied to the symmetric space G/K_σ. The following map Φ is the K_1AK_2 decomposition of the Riemannian symmetric space G/K_σ. The map Φ is also called the Hermann action,[56,57] nonstandard polar coordinates,[58] and non-Cartan parameterization.[59] In the KAK case ($K = K_\sigma = K_\tau$), Helgason called this the polar coordinate decomposition[33] and credits Cartan[30] for this map. Since the G-invariant measure of G/K inherits the Haar measure of G, the identical Jacobian is obtained for the decomposition of a symmetric space.[60]

Theorem 2.4 *(K_1AK_2 decomposition of G/K_σ). Given a K_1AK_2 decomposition $G = K_\sigma AK_\tau$ with the Riemannian symmetric space G/K_σ, we have the map Φ,*

$$\Phi : K_\tau \times A \to G/K; \quad (k_\tau, a) \mapsto k_\tau aK. \tag{2.4}$$

Suppose $H \in \mathfrak{a}$, $a = \exp(H)$. For the G-invariant measure dx of G/K_σ, $dk_\tau = \mathrm{Haar}(K_\tau)$, and the Euclidean measure dH on $\mathfrak{a}$, $dx = dk_\tau d\mu(H)$ holds where the Jacobian $d\mu(H)$ is given in (2.2) if G is noncompact and (2.3) if G is compact.

Remark 2.5 [representing $G/K \cong P$: gK (coset) or $p \in P$?]. In the standard KAK decomposition, the Jacobian (2.2) [respectively, (2.3)] only has sinh (respectively, sin) terms as we discussed above. This result could be found in many literature, where some authors[28,44,55,61] use $\prod \sinh \alpha(H)$ as the Jacobian, whereas other authors[13,54,62] use $\prod \sinh(\alpha(H)/2)$. This gap is due to the difference in the realization of a symmetric space G/K as a subset $P \subset G$. The former uses the right coset representative, i.e., $G/K \to P$ as $gK \mapsto p$, where $g = pk$ is its group level Cartan decomposition. Then, the action of G on G/K is given as $(g_1, g_2K) \mapsto g_1g_2K$. The latter authors use the map $G/K \to P$ such that $gK \mapsto g(\sigma g)^{-1}$, where σ is the group level involution. The G-action is $(g_1, g_2) \mapsto g_1g_2(\sigma g_1)^{-1}$, $g_1 \in G, g_2 \in P$. In terms of Theorem 2.4, the latter gives the map Φ such that $(k, a) \mapsto ka^2k^{-1}$ since

$$g(\sigma g)^{-1} = pk\sigma(pk)^{-1} = pk(p^{-1}k)^{-1} = pkk^{-1}p = p^2 = kak^{-1}kak^{-1} = ka^2k^{-1},$$

which explains the extra factor $\frac{1}{2}$ applied to H where $a = \exp(H)$. Moreover, these two identifications define the map $\Phi : K \times A \to P$ with the same k, a as

$$\Phi : (k, a) \mapsto kaK \qquad \text{or} \qquad \Phi : (k, a) \mapsto ka^2k^{-1}, \tag{2.5}$$

depending on the author's notational choice explained above. This coordinate system Φ is sometimes called the polar coordinate decomposition, e.g., see Ref. 33 (p. 402).

Example 2.6 (G/K vs P: a symmetric positive definite matrix). Let us take a look at the two realizations in Remark 2.5 for $G/K = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, where P is the set of all symmetric positive definite matrices. Let S be a fixed positive definite symmetric matrix, with its eigendecomposition $S = Q\Lambda Q^T$, with $Q \in \mathrm{O}(n)$. The coset representation of S is $Q\Lambda \cdot \mathrm{O}(n) \in G/K$ as $Q\Lambda = (Q\Lambda Q^T)Q$ is the polar decomposition. With the realization of $P \cong G/K$, the point in G/K is represented by the matrix $S = Q\Lambda Q^T$.

Finally, we have the Lie algebra counterpart of Theorem 2.4 when $K = K_\sigma = K_\tau$.

Theorem 2.7. *For a noncompact Riemannian symmetric space G/K with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. We have*

$$\Psi : K \times \mathfrak{a} \to \mathfrak{p}; \quad (k, H) \mapsto kHk^{-1}, \tag{2.6}$$

equivalently the decomposition $\mathfrak{p} = \cup_{k\in K} k\mathfrak{a}k^{-1}$ with the Jacobian $d\mu$ given as

$$d\mu(H) \propto \prod_{\alpha\in\Sigma^+} |\alpha(H)|^{m_\alpha}, \tag{2.7}$$

where $H \in \mathfrak{a}$ and Σ is the restricted root system with dimensions m_α. The measure on $\mathfrak{p}$ is the Euclidean measure.

C. A symmetric space: one RMT or many RMTs?

The answer to the title question of this section is that both one and many can be construed as correct. To explain how this is possible requires teasing apart the assumptions behind the words "associated with." Certainly,[6,8,9,13] associate one random matrix with one symmetric space. However, the example of the GSVD coordinate systems discussed in Sec. I B associates multiple Jacobi densities with one symmetric space, the Grassmannian manifold. In Ref. 59, another example is illustrated as the "non-Cartan parameterization" for the special case of $(G, K_\sigma, K_\tau) = (\mathrm{U}(n), \mathrm{O}(n), \mathrm{U}(p) \times \mathrm{U}(q))$. (A similar approach may be found in Ref. 63.) This is discussed in Sec. VII B.

The reconciliation is that indeed it is true that the required maps (2.4) when $K = K_\sigma = K_\tau$, i.e., $\Phi(k, a) = kaK = kak^{-1}$ (compact) or the map (2.6) $\psi(k, H) = kHk^{-1}$ (noncompact) lead to a unique random matrix theory associated with a given symmetric space G/K. This is unique in a sense that any geodesic on the symmetric space G/K could be transformed to the geodesic on A with the above maps.

However, if we relax the condition so that we are allowed to choose K_τ under the generalized Cartan decomposition framework, we can associate multiple random matrix theories to one symmetric space. The GSVD coordinate systems in Sec. I B illustrate this viewpoint. The real Grassmannian manifold $G/K = \mathrm{O}(n)/(\mathrm{O}(n-s) \times \mathrm{O}(s))$ has the map $\Phi : (k, a) \mapsto kaK$ for $K = K_\sigma = K_\tau$ explicitly written as $X = \begin{bmatrix} U & \\ & V \end{bmatrix}\begin{bmatrix} C \\ S \end{bmatrix} \cdot \mathrm{O}(s)$, where U, V are $(r-s) \times s, s \times s$ orthogonal matrices. On the other hand, if we let $K_\tau = \mathrm{O}(p) \times \mathrm{O}(q)$, we have multiple maps $\Phi : (k_\tau, a) \mapsto k_\tau aK$ written as $X = \begin{bmatrix} U & \\ & V \end{bmatrix}\begin{bmatrix} C \\ S \end{bmatrix} \cdot \mathrm{O}(s)$, where U, V are $p \times s, q \times s$ orthogonal matrices.

Starting from Sec. V, we discuss (i) random matrices arising from the $\mathrm{K_1AK_2}$ decompositions of compact symmetric spaces (Theorem 2.4 or 2.2) and (ii) random matrices arising from the Lie algebra decomposition of noncompact symmetric spaces (Theorem 2.7). The associated decompositions are well explained by matrix factorizations in numerical linear algebra. As we pointed out, the resulting Jacobi ensembles cover the full parameter set of the classical Jacobi densities, thereby completing the classification from the classical RMT point of view.

III. CARTAN'S IDEA: A MODERNIZED APPROACH

The Jacobian of the KAK ($\mathrm{K_1AK_2}$) decomposition, equivalently the determinant of the differential of the map $\Phi : K \times A \to P$ (in Theorem 2.4 and Remark 2.5), is computed in several references.[28,54,55] The proof of (2.2) is can also be found in Refs. 23 and 44. However, the proof can be inaccessible to some audiences. Meanwhile, individual cases of the KAK decomposition, recognized as matrix factorizations, show up in many areas of mathematics, and some were discovered in various formats by specialists in numerical linear algebra. Motivated by random matrix theory (and sometimes perturbation theory in numerical analysis), Jacobians of these factorizations were often computed case-by-case using the matrix differentials and wedging of independent elements.[15,21,26,64,73]

In this section, we provide a generalization of such individual Jacobian computations and compare it to the general technique Helgason proposed. With appropriate translation of terminologies and maps in Lie theory into linear algebra, we observe the both methods are indeed the same process but have been illustrated in different languages for a long time. We start out by introducing some important concepts in Lie theory accessible to an audience with a good background in linear algebra and perhaps some basic geometry. Then, in Table II, we present a line-by-line correspondence between Helgason's derivation and the proof by matrix differentials.

A. The ping pong operator, ping pong vectors, and ping pong subspaces

We will start with a concrete 2×2 linear operator so as to establish the notions of the *ping pong operator*, *ping pong vectors*, *ping pong subspaces*, and the relationship to eigenvectors. Then, we will define a "bigger" linear operator ad_H that acts on 2×2 spaces exactly in the manner we are about to describe.

We introduce the 2×2 matrix

$$M := \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which we will call a *2 × 2 ping pong operator*, and we will call $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ the *ping pong vectors* of M, in that M bounces these two vectors into α times the other,

$$M\begin{bmatrix}1\\0\end{bmatrix} = \alpha\begin{bmatrix}0\\1\end{bmatrix}, \qquad M\begin{bmatrix}0\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix}.$$

Furthermore, M has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$, with eigenvalues $\alpha, -\alpha$. We will call the eigenvalue a *root* of M.

Also worth pointing out are the matrix exponential and matrix sinh of M,

$$e^M = \begin{bmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{bmatrix} \qquad \text{and} \qquad \sinh M = \frac{1}{2}(e^M + e^{-M}) = \sinh\alpha\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

and thus, we see that $\sinh M$ is another ping pong operator with scaling $\sinh\alpha$. Figure 5 plots the action of a ping pong matrix and its exponential, with notations that we will use in Secs. III D and III E, i.e., the ping pong operator is denoted ad_H, p_j and k_j are the ping pong vectors, and x_j and θx_j are the eigenvectors. The right side of Fig. 5 shows the action of e^M and portrays $\sinh(M)$ as a projection of e^M on the p_j direction.

We now go beyond 2×2 matrices and suggest the more general $2N \times 2N$ ping pong matrix M_N, with N roots, $\alpha_1, \ldots, \alpha_N$, N pairs of ping pong vectors $(k_1, p_1), \ldots, (k_N, p_N)$ along with eigenvectors $(x_1, y_1), \ldots, (x_N, y_N)$,

$$M_N = \begin{bmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \alpha_N \\ & & & \alpha_N & 0 \end{bmatrix},$$

$$k_j, p_j, x_j, y_j = \begin{bmatrix}\vdots\\1\\0\\\vdots\end{bmatrix}, \begin{bmatrix}\vdots\\0\\1\\\vdots\end{bmatrix}, \begin{bmatrix}\vdots\\1\\1\\\vdots\end{bmatrix}, \begin{bmatrix}\vdots\\1\\-1\\\vdots\end{bmatrix}, \qquad j = 1, 2, \ldots, N,$$

where the $2j - 1$ and $2j$ positions are 0 or ± 1 and all other entries of these vectors are 0. The matrices $\exp(M_N)$ and $\sinh M_N$ are block versions of the 2×2 case.

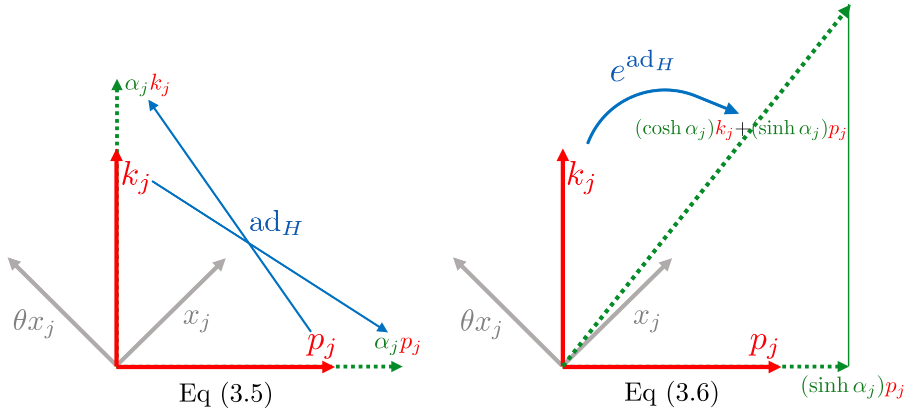

FIG. 5. The eigenmatrices x_j, θx_j and ping pong matrices k_j, p_j (3.4) in the tangent space $\mathfrak{g}$. The operators are illustrated in blue lines. The operator ad_H and ping pong relationship (left) and the operator e^{ad_H} on k_j to p_j (right). The left map shows the factor of α_j, which is a building block of the Jacobian $\prod_j |\alpha_j(H)|$ (2.7). The factor of $\sinh\alpha_j$ in the right map builds the Jacobian $\prod_j |\sinh\alpha_j(H)|$ (2.2).

We may define the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ (using the "mathfrak" Fraktur letters "k" and "p") to be the span of the k_j and p_j, respectively. Note that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal complements as subspaces. A key "ping pong" relationship between these subspaces is that

$$M_N k \in \mathfrak{p} \text{ if } k \in \mathfrak{k},$$
$$M_N p \in \mathfrak{k} \text{ if } p \in \mathfrak{p}.$$

Thus, if we consider $M_N|_{\mathfrak{k}}$, the restriction of M_N to $\mathfrak{k}$, we have an operator from $\mathfrak{k}$ to $\mathfrak{p}$. Evidently, $M_N|_{\mathfrak{k}}$ as a matrix may be obtained by taking the even rows and odd columns of M_N. The result is a diagonal matrix with the α_j on the diagonal. Similarly, $\sinh(M_N)|_{\mathfrak{k}}$ is a diagonal matrix with $\sinh(\alpha_j)$ on the diagonal. We then get the important result that

$$\det(\sinh(M_N)|_{\mathfrak{k}}) = \prod_{j=1}^{N} \sinh \alpha_j,$$

the product of the hyperbolic sines of the roots.

Given a linear operator $\mathcal{L}$ on a vector space with nonzero eigenvalues $\pm\lambda$, the following lemma constructs a pair of ping pong vectors from $\mathcal{L}$:

Lemma 3.1. For a linear operator $\mathcal{L}$ defined on any vector space, assume that $\pm\lambda$ are both nonzero eigenvalues of $\mathcal{L}$. Let x and y be the corresponding eigenvectors, i.e., $\mathcal{L}x = \lambda x$ and $\mathcal{L}y = -\lambda y$. Define two vectors $k := x + y, p := x - y$. Then, k, p are ping pong vectors. Furthermore, we have for the operator $\exp(\mathcal{L})$,

$$e^{\mathcal{L}} k = \cosh \lambda k + \sinh \lambda p, \qquad e^{\mathcal{L}} p = \sinh \lambda k + \cosh \lambda p.$$

The proof is a straightforward extension of the discussion in previous paragraphs.

Remark 3.2. For the reader who wants to know the upcoming significance of this fact for Jacobians of matrix factorizations, it turns out (or maybe as the reader already observed in Sec. II) that the Jacobian will be the product of $\sinh \alpha$'s. Just as the matrix $\sinh\left(\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}\right)$ takes one of the ping pong vectors to $\sinh \alpha$ times the other, the key piece of the differential map will consist of multiple ping pong relationships, each one sending one ping pong vector to another.

B. The Kronecker product, linear operator ad_X, and its exponential

Lie theory picks out operators $\mathcal{L}$ that exactly have the properties in Sec. III A. Our vector spaces are now matrix spaces, and our operators are linear operators on a matrix space. We introduce the Lie bracket, denoted by $[X, Y]$, defined as $[X, Y] = XY - YX$ (the commutator). The Kronecker product notation is very helpful in this context. We define the Kronecker product notation as a linear operator on a matrix space. [Many authors would write $\mathrm{vec}(BXA^T) = (A \otimes B)\mathrm{vec}(X)$, but we omit the "vec" as we believe it is always clear from context. In a computer language such as Julia, one would write `kron(A,B) * vec(X) = vec(B*X*A′)`],

$$(A \otimes B)X = BXA^T. \tag{3.1}$$

With this, we can express the Lie bracket with Kronecker products,

$$(I \otimes X - X^T \otimes I)Y = XY - YX.$$

Consider the Lie bracket as a linear operator (determined by X) applied to Y, and call this operator ad_X (abbreviation for "adjoint"),

$$\mathrm{ad}_X = I \otimes X - X^T \otimes I,$$
$$\mathrm{ad}_X(Y) = [X, Y].$$

This will be the important ping pong operator $\mathcal{L}$. The operator exponential of ad_X (equivalently, the matrix exponential of $I \otimes X - X^T \otimes I$) is given in the following lemma:

Lemma 3.3. For the linear operator ad_X, *the following holds for* $e^{\mathrm{ad}_X} := \sum_{j=0}^{\infty} \frac{(\mathrm{ad}_X)^n}{n!}$ *and* $\sinh \mathrm{ad}_x = (e^{\mathrm{ad}_X} + e^{-\mathrm{ad}_X})/2$*:*

$$e^{\mathrm{ad}_X} = \exp(I \otimes X - X^T \otimes I) = (e^{-X})^T \otimes e^X, \tag{3.2}$$

$$e^{\mathrm{ad}_X} Y = e^X Y e^{-X} \qquad \text{and} \qquad (\sinh \mathrm{ad}_x)Y = (e^X Y e^{-X} - e^{-X} Y e^X)/2. \tag{3.3}$$

Proof. The proof is straightforward by identity (3.1). $e^X Y e^{-X} = \big((e^{-X})^T \otimes e^X\big)Y$ and $e^{\mathrm{ad}_X} Y = \exp(I \otimes X - X^T \otimes I)Y$. It is left to prove $(e^{-X})^T \otimes e^X = \exp(I \otimes X - X^T \otimes I)$. Since $I \otimes X$ commutes with $X^T \otimes I$, we have

$$\exp(I \otimes X - X^T \otimes I) = e^{I\otimes X} e^{-X^T \otimes I} = (I \otimes e^X)((e^{-X})^T \otimes I) = (e^{-X})^T \otimes e^X,$$

proving the result. The sinh result follows trivially. □

C. Antisymmetric and symmetric matrices: An important first example of symmetric space as ping pong spaces

In our first example, our vector space is $n \times n$ real matrices. Consider

$$\mathfrak{k} = \{\text{Antisymmetric matrices}\},$$
$$\mathfrak{p} = \{\text{Symmetric matrices}\}.$$

The ping pong operator that will bounce $\mathfrak{k}$ and $\mathfrak{p}$ around will be $\mathrm{ad}_H = I \otimes H - H^T \otimes I$, where H is the diagonal matrix

$$H = \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix}.$$

Note that the operator ad_H sends an antisymmetric matrix to a symmetric matrix and a symmetric matrix to an antisymmetric matrix.

What does this have to do with Jacobians of matrix factorizations, such as the symmetric positive definite eigenvalue factorization? Consider a perturbation of Q when forming $S = Q\Lambda Q^T$. An infinitesimal antisymmetric perturbation $Q^T dQ$ is mapped into a dS, an infinitesimal symmetric perturbation. This is the very linear map from the tangent space of Q to that of S that we wish to understand, so perhaps it is not surprising we would want to restrict our ping pong operator from $\mathfrak{k}$ to $\mathfrak{p}$. We invite the reader to check that the corresponding eigenmatrices and ping pong matrices of ad_H may be found in the first column of Table I.

D. General $\mathfrak{k}$ and $\mathfrak{p}$ arise from an involution θ

We proceed to construct more important general operators $\mathcal{L}$ that have the property in the assumption of Lemma 3.1. This is where the theory of Lie groups and symmetric spaces need to be brought in. Upon doing so, we will obtain two linear spaces of matrices $\mathfrak{k}$, $\mathfrak{p}$, and also a space $\mathfrak{a}$.

TABLE I. Examples of eigenmatrices x_l, θx_l and ping pong matrices k_l, p_l. $k_l = x_l + \theta x_l$ and $p_l = x_l - \theta x_l$ as defined in (3.4). k_l, p_l are normalized to have ± 1 entries. A block structure on row/columns j, k and $j' := p + j$ and $k' := p + k$ are filled up with 0 and ± 1.

$\frac{G}{K}$	$\frac{\mathrm{GL}(n,\mathbb{R})}{\mathrm{O}(n)}$	$\frac{\mathrm{U}(n)}{\mathrm{O}(n)}$	$\frac{\mathrm{O}(p,q)}{\mathrm{O}(p)\times\mathrm{O}(q)}$	$\frac{\mathrm{O}(n)}{\mathrm{O}(p)\times\mathrm{O}(q)}$
x_l	$\begin{array}{ccc} & j & k \\ j & 0 & 1 \\ k & 0 & 0 \end{array}$	...	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & 1 & & 1 \\ k & -1 & & 1 & \\ \hline j' & & 1 & & 1 \\ k' & 1 & & -1 & \end{array}$	...
θx_l	$\begin{array}{ccc} & j & k \\ j & 0 & 0 \\ k & -1 & 0 \end{array}$	...	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & 1 & & -1 \\ k & -1 & & -1 & \\ \hline j' & & -1 & & 1 \\ k' & -1 & & -1 & \end{array}$	...
k_l	$\begin{array}{ccc} & j & k \\ j & 0 & 1 \\ k & -1 & 0 \end{array}$	$\begin{array}{ccc} & j & k \\ j & 0 & 1 \\ k & -1 & 0 \end{array}$	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & 1 & & \\ k & -1 & & & \\ \hline j' & & & & 1 \\ k' & & & -1 & \end{array}$	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & 1 & & \\ k & -1 & & & \\ \hline j' & & & & 1 \\ k' & & & -1 & \end{array}$
p_l	$\begin{array}{ccc} & j & k \\ j & 0 & 1 \\ k & 1 & 0 \end{array}$	$\begin{array}{ccc} & j & k \\ j & 0 & i \\ k & i & 0 \end{array}$	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & & & 1 \\ k & & & 1 & \\ \hline j' & & 1 & & \\ k' & 1 & & & \end{array}$	$\begin{array}{ccccc} & j & k & j' & k' \\ j & & & & 1 \\ k & & & 1 & \\ \hline j' & & -1 & & \\ k' & -1 & & & \end{array}$

For the reader not familiar with Lie groups, one need only imagine a continuous set of matrices that are a subgroup of real, complex, or quaternion matrices. The tangent space $\mathfrak{g}$ is just a vector space of matrix differentials at the identity. One key example is the compact Lie group $\mathrm{O}(n)$ (the group of square orthogonal matrices) and its tangent space at the identity $\mathfrak{g}_{\mathrm{O}(n)}$: the set of antisymmetric matrices. Another key example is all $n \times n$ invertible matrices $\mathrm{GL}(n,\mathbb{R})$ (a noncompact Lie group) and its tangent space $\mathfrak{g}_{\mathrm{GL}(n,\mathbb{R})}$, consisting of all $n \times n$ matrices.

Cartan noticed that important matrix factorizations start with two ingredients: the **tangent space** $\mathfrak{g}$ (at the identity) of a Lie group G and an **involution θ** on $\mathfrak{g}$ (i.e., $\theta^2 = \mathrm{Id}$ and $\theta[X,Y] = [\theta X, \theta Y]$) An example of θ is $\theta(X) = -X^T$ on $\mathfrak{g}$ for $G = \mathrm{GL}(n,\mathbb{R})$. Among matrices in $\mathfrak{g}$, we select two kinds of matrices. The ones fixed by the involution θ, and the ones negated by θ. Denote each set by $\mathfrak{k}$ and $\mathfrak{p}$,

$$\mathfrak{k} := \{g \in \mathfrak{g} : \theta(g) = g\}, \qquad \mathfrak{p} := \{g \in \mathfrak{g} : \theta(g) = -g\}.$$

[For $\mathrm{GL}(n,\mathbb{R})$, these are the antisymmetric and symmetric matrices respectively.]

The next important player is $\mathfrak{a} \subset \mathfrak{p}$. Readers familiar with the singular value decomposition know the special role of diagonal matrices in the SVD as they list the very important "singular values." Diagonal matrices have the nice property that linear combinations are still diagonal, they commute (the Lie bracket of any two are zero), and they are symmetric (the $\mathfrak{p}$ of our first example). The generalization of this is to take a $\mathfrak{p}$ and find a maximal subalgebra where every matrix commutes. This is the maximal subspace $\mathfrak{a} \subset \mathfrak{p}$ such that for all $a_1, a_2 \in \mathfrak{a}$, $[a_1, a_2] = 0$.

If $H \in \mathfrak{a}$, then $S = Q\Lambda Q^T$ is a symmetric positive definite eigendecomposition, with $\Lambda = e^H$. In the rest of the section, we will be focusing on factorizations of the form $Q\Lambda Q^{-1}$, where Λ is a matrix exponential of $H \in \mathfrak{a}$. (These will be more general than eigendecompositions, as Q may not be orthogonal, and Λ may not be diagonal.) In particular, we will compute the Jacobian of perturbations with respect to Q, holding H constant, and thus, necessarily the Jacobian will be defined in terms of H.

From here, we assume that the Lie group G is noncompact. The compact case will be discussed after completing the noncompact case. Pick $H \in \mathfrak{a}$, and recall that ad_H is a linear operator on $\mathfrak{g}$. The operator ad_H will play the role of $\mathcal{L}$, the ping pong operator. We decompose $\mathfrak{g}$ into the eigenspaces of ad_H. For any eigenpair (α_j, x_j) of ad_H, i.e., $\mathrm{ad}_H(x_j) = [H, x_j] = \alpha_j x_j$, we observe (for $\alpha_j \neq 0$)

$$\mathrm{ad}_H(\theta x_j) = [H, \theta x_j] = -[-H, \theta x_j] = -[\theta H, \theta x_j] = -\theta([H, x_j]) = -\alpha_j \theta x_j,$$

which implies that the eigenvalues $\pm\alpha_j$ always exist in pairs, with the corresponding eigenmatrices x_j and θx_j. This satisfies the assumption of Lemma 3.1, from which we can now construct our ping pong matrices,

$$k_j := x_j + \theta x_j, \qquad p_j := x_j - \theta x_j, \tag{3.4}$$

with the ping pong relationship by the operator ad_H,

$$\mathrm{ad}_H\, k_j = \alpha_j p_j, \qquad \mathrm{ad}_H\, p_j = \alpha_j k_j. \tag{3.5}$$

In addition, the relationship by the operator e^{ad_H} follows:

$$e^{\mathrm{ad}_H} k_j = \cosh \alpha_j k_j + \sinh \alpha_j p_j, \tag{3.6}$$

$$e^{\mathrm{ad}_H} p_j = \sinh \alpha_j k_j + \cosh \alpha_j p_j. \tag{3.7}$$

The ping pong matrices k_j, p_j, eigenmatrices $x_j, \theta x_j$ and the relationships (3.5), (3.6) are illustrated in Fig. 5.

As we mentioned in Remark 3.2 and Sec. III C, the role of ping pong matrices k_j, p_j is crucial. **The map e^{ad_H} (particularly, $\sinh \mathrm{ad}_H$) is the main ingredient constructing the differential map $d\Phi$** of the factorization $\Phi : (Q, \Lambda) \mapsto Q\Lambda Q^{-1}$. The operator e^{ad_H} is applied to k_j and then projected to the span of p_j as in Fig. 5 (right), leaving only the $\sinh \alpha_j$ factor.

We now compute the full basis of $\mathfrak{k}$ and $\mathfrak{p}$. The collection $\cup_j\{x_j, \theta x_j\}$ is a full basis for the union of eigenspaces with nonzero eigenvalues. Since $\mathrm{span}(\{x_j, \theta x_j\}) = \mathrm{span}(\{k_j, p_j\})$ for any j, $\cup_j\{k_j, p_j\}$ is another full basis for the eigenspaces with nonzero eigenvalues. Interestingly, we observe $\theta k_j = k_j$ and $\theta p_j = -p_j$, which identifies $\cup_j\{k_j\}$ and $\cup_j\{p_j\}$ as subsets of the basis of $\mathfrak{k}$ and $\mathfrak{p}$, respectively. The remaining case is the zero eigenspace. When $\alpha_j = 0$, there are two possibilities. First, if x_j and θx_j are independent of each other, we can still obtain k_j and p_j as before and add them to $\cup_j\{k_j\}$ and $\cup_j\{p_j\}$. Second, if x_j and θx_j are colinear, θx_j is either x_j or $-x_j$. If $\theta x_j = x_j$, we collect such x_j and name the set K_z. Similarly, if $\theta x_j = -x_j$, then we put them in P_z. Since we analyzed both nonzero and zero eigenspaces, we have obtained a full basis of $\mathfrak{g}$, which is $(\cup_j\{k_j, p_j\}) \cup K_z \cup P_z$. Refining once more, $\mathrm{span}((\cup_j\{k_j\}) \cup K_z) = \mathfrak{k}$ and $\mathrm{span}((\cup_j\{p_j\}) \cup P_z) = \mathfrak{p}$.

E. The operators $\mathrm{ad}_H, e^{\mathrm{ad}_H}$, and the subspaces $\mathfrak{k}, \mathfrak{p}$

In Sec. III D, we obtained the basis of $\mathfrak{k}$ and $\mathfrak{p}$, in terms of ping pong matrices, by linearly combining eigenmatrices of the operator ad_H. We now illustrate the relationship of the basis of $\mathfrak{k}$ and $\mathfrak{p}$ under e^{ad_H}, just like we illustrated the operator M_N in Sec. III A. In the $k_1, \ldots, k_N$ and

$p_1, \ldots, p_N$ basis, we have the following:

$$e^{\mathrm{ad}_H}\begin{bmatrix} k_1 \\ p_1 \\ \vdots \\ k_N \\ p_N \end{bmatrix} = \begin{bmatrix} \cosh\alpha_1 & \sinh\alpha_1 & & & \\ \sinh\alpha_1 & \cosh\alpha_1 & & & \\ & & \ddots & & \\ & & & \cosh\alpha_N & \sinh\alpha_N \\ & & & \sinh\alpha_N & \cosh\alpha_N \end{bmatrix}\begin{bmatrix} k_1 \\ p_1 \\ \vdots \\ k_N \\ p_N \end{bmatrix}. \tag{3.8}$$

We are now ready to carefully investigate the map $d\Phi$ using (3.8).

Remark 3.4. Results in Lie theory imply that the eigenmatrices x_j and θx_j of ad_H are independent of the choice of $H \in \mathfrak{a}$. In other words, the complete basis of $\mathfrak{g}$ and $\mathfrak{k}$, $\mathfrak{p}$ obtained above does not care about a specific choice of H. Furthermore, the eigenvalues $\pm\alpha_j$ are functions of H, and these eigenvalue assigning functions $\tilde{\alpha}_j : H \mapsto \alpha_j \in \mathbb{R}$ are more properly called the *restricted roots*. It can be inferred from the separation of the basis that $\mathfrak{k}$, $\mathfrak{p}$ together form the whole tangent space $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}. \tag{3.9}$$

F. Symmetric spaces

The reader may have noticed that our discussions have focused on the Lie algebras rather than the Lie groups themselves. It is a point of fact that Lie groups are mostly useful to define the factorizations of our interest, but Lie algebras are where the Jacobian "lives," and hence, this is the most important place to concentrate. For the interested reader, the subgroup K of G is picked such that its tangent space is exactly $\mathfrak{k}$ [one easy way to imagine such a subgroup is to define $K := \exp(\mathfrak{k})$], and we now obtain a *symmetric space* G/K.

It can be proven that for the noncompact Lie group, there exists a unique involution θ such that the subgroup K is the maximal compact subgroup of G. We call θ the *Cartan involution*, and (3.9) is called the *Cartan decomposition*. Furthermore, the subset $P := \exp(\mathfrak{p})$ plays an important role as its elements serve as representatives of the cosets in G/K. Regarding the identification of G/K as elements in P, refer to Remark 2.5, where we point out as an example, taking $G/K = \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ that an element of G/K has the form of a coset gK, then gg^T may be a representative of the coset in $\mathfrak{p}$. While some authors use $(gg^T)^{1/2}$, the key point being each choice is well defined independent of choice of representative.

G. When G is a compact Lie group

Upon considering the compact cases, it is helpful to make use of a certain duality between compact and noncompact symmetric spaces. We again start with a noncompact Lie group G and the Cartan involution θ. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Then, define a new space,

$$\mathfrak{g}_C := \mathfrak{k} + i\mathfrak{p}, \tag{3.10}$$

where i is the imaginary unit. The result in Lie theory implies that the new vector space $\mathfrak{g}_C$ is the tangent space of a compact Lie group, say, G_C. In Table I, the first and third columns labeled $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ and $\mathrm{O}(p,q)/(\mathrm{O}(p)\times\mathrm{O}(q))$ are noncompact tangent spaces. Their compact duals are, respectively, the second and fourth columns labeled $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{O}(n)/(\mathrm{O}(p)\times\mathrm{O}(q))$.

Matrixwise, the ping pong matrices $k_j \in \mathfrak{k}, p_j \in \mathfrak{p}$ of $\mathfrak{g}$ are brought back to a new set of ping pong matrices $k_j \in \mathfrak{k}_C, ip_j \in \mathfrak{p}_C$ in $\mathfrak{g}_C$. Let us denote them by $\tilde{k}_j := k_j$ and $\tilde{p}_j := ip_j$. The role of the subspace $\mathfrak{a}$ is now played by $i\mathfrak{a}$ replacing ad_H by ad_{iH}. We deduce a set of similar relationships for $\tilde{k}_j, \tilde{p}_j$ under ad_{iH},

$$\mathrm{ad}_{iH}(\tilde{k}_j) = \alpha_j\tilde{p}_j, \qquad \mathrm{ad}_{iH}(\tilde{p}_j) = -\alpha_j\tilde{k}_j.$$

In matrix form,

$$\mathrm{ad}_{iH}\begin{bmatrix} \tilde{k}_j \\ \tilde{p}_j \end{bmatrix} = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}\begin{bmatrix} \tilde{k}_j \\ \tilde{p}_j \end{bmatrix}, \tag{3.11}$$

which leads to the compact version of (3.6) and (3.7),

$$\exp(\mathrm{ad}_{iH})\begin{bmatrix} \tilde{k}_j \\ \tilde{p}_j \end{bmatrix} = \begin{bmatrix} \cos\alpha_j & \sin\alpha_j \\ -\sin\alpha_j & \cos\alpha_j \end{bmatrix}\begin{bmatrix} \tilde{k}_j \\ \tilde{p}_j \end{bmatrix}. \tag{3.12}$$

At the group level, the symmetric spaces G/K and G_C/K are called the duals of each other, and they appear in the same row of standard symmetric space charts. An example of eigenmatrices x_j, θx_j and ping pong matrices for some symmetric spaces and their duals are presented in Table I.

H. Jacobian of the map Φ

We provide a generalized algorithm for finding a Jacobian of the decomposition $\Phi(Q,\Lambda) = Q\Lambda Q^{-1}$ [as we defined in (2.5)], where $\Lambda \in A := \exp(\mathfrak{a}), Q \in K$. $\mathfrak{k}$ and $\mathfrak{p}$ from Sec. III D are the tangent spaces of K and P, respectively. As mentioned, we follow Helgason's derivation (Ref. 28, Theorem 5.8 of Chap. I) and start by directly translating his proof into simple linear algebra terms. In Table II, we have Helgason's derivation (left) compared in the same row with linear algebra (Right). Table II is using the noncompact symmetric space G/K but the compact case is identical with replacing $\sin\alpha_j$ by $\sinh\alpha_j$.

From the last line of Table II, we can finish the story with two different directions, depending on the choice of the volume measure. First, if we use a ***G*-invariant measure** (the "canonical measure") of P, the measure is invariant under the map $d\tau$ or $d\tilde{\tau}$ (by definition of the invariant measure). Thus, we can disregard $d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})$ [or $d\tau(ka)$] so that the Jacobian of $d\tilde{\Phi}$ (or $d\Phi$) only depends on the differential map $k_j \mapsto (\sinh\alpha_j)p_j$. Since $\cup_j\{k_j\}$ and $\cup_j\{p_j\}$ are both orthonormal bases, we obtain the Jacobian (2.2),

$$\prod_{\alpha\in\Sigma^+} \sinh\, \alpha(H).$$

Note that eigenvalues $\pm\alpha_j$ belong to x_j and θx_j have the same corresponding k_j. [see (3.4) and above.] Thus, we only take the positive roots Σ^+ above.

The second choice of measure is the **Euclidean measure**, which is a wedge product of independent entrywise differentials. In this case, the procedure is identical up to the factor $\sinh\alpha_j$, but the map $d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})$ [equivalently $d\tau(ka)$] cannot be ignored. One needs to carefully compute the differential map $d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})p_j = Q\Lambda^{\frac{1}{2}}p_j\Lambda^{\frac{1}{2}}Q^{-1}$ under the Euclidean measure. We can further use the fact that conjugation by the matrix Q always preserves the Euclidean measure, since the subgroup K is always a set of matrices with an orthogonal/unitary type of property. Thus, one needs to compute the map $p_j \mapsto \Lambda^{\frac{1}{2}}p_j\Lambda^{\frac{1}{2}}$ and multiply its Jacobian by $\prod_{\alpha\in\Sigma^+}\sinh\,\alpha(H)$.

TABLE II. Line-by-line translation of the classical proof to linear algebra proof.

Classical notation (Ref. 28, p. 187, Proof of Theorem 5.8, Chap. I)	Linear algebra notation (matrix factorizations)
Definitions	
$\Phi: K\times A \to G/K$	$\tilde{\Phi}: K\times A \to P$
$\Phi: (k,a)\mapsto kaK$	$\tilde{\Phi}: (Q,\Lambda)\mapsto Q\Lambda Q^{-1}$ $(\Lambda^{\frac{1}{2}} = a, Q = k)$
$d\tau(g_0): (G/K)_o \to (G/K)_{g_0\cdot o}$	$d\tilde{\tau}(g_0): X\mapsto g_0X(\theta g_0)^{-1}$
$d\pi: \mathfrak{g}\to (G/K)_o$	$(\theta k = k, k\in K, \theta p = p^{-1}, p\in P)$
At $k\in K$, fix a tangent vector $d\tau(k)T_i^\alpha$	At $Q\in K$, fix a tangent vector dQ
At Id, basis element $T_i^\alpha \in \mathfrak{k}$	At Id, basis element $Q^{-1}dQ = k_j\in\mathfrak{k}$
Derivations	
$2d\Phi(d\tau(k)T_i^\alpha, 0)$[a]	$d\tilde{\Phi}(dQ,0) = d(Q\Lambda Q^{-1})$ (with $d\Lambda = 0$)
$= d\pi(2kT_i^\alpha a)$	$= dQ\Lambda Q^{-1} + Q\Lambda dQ^{-1}$
$= d\tau(ka)d\pi(2\mathrm{Ad}(a^{-1})T_i^\alpha)$[b]	$= d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})\left[\Lambda^{-\frac{1}{2}}(Q^{-1}dQ\Lambda + \Lambda dQ^{-1}Q)\Lambda^{-\frac{1}{2}}\right]$[c]
$= d\tau(ka)d\pi(\mathrm{Ad}(a^{-1})T_i^\alpha - \mathrm{Ad}(a)T_i^\alpha)$	$= d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})\left[\Lambda^{-\frac{1}{2}}k_j\Lambda^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}k_j\Lambda^{-\frac{1}{2}}\right]$
[Let H be such that $\exp(H) = a = \Lambda^{\frac{1}{2}}$]	[Note that $d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})X = Q\Lambda^{\frac{1}{2}}X\Lambda^{\frac{1}{2}}Q^{-1}$]
$= d\tau(ka)d\pi(e^{-\mathrm{ad}H}T_i^\alpha - e^{\mathrm{ad}H}T_i^\alpha)$	$= d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})\big[\exp(H^T\otimes I - I\otimes H)k_j$ $- \exp(I\otimes H - H^T\otimes I)k_j\big]$ [by (3.3)]
$= d\tau(ka)d\pi(-\alpha(H)^{-1}[H,T_i^\alpha]2\sinh\,\alpha(H))$	$= d\tilde{\tau}(Q\Lambda^{\frac{1}{2}})[(-2\sinh\alpha_j)p_j]$ [by (3.8)]

[a] Since $\Lambda^{\frac{1}{2}} = a$, we have $2d\Phi = d\tilde{\Phi}$.
[b] This is $(d\tau(ka)\circ d\pi)(\mathrm{Ad}(a^{-1})T_i^\alpha)$.
[c] Both $dQ\Lambda Q^{-1}$ and $Q\Lambda dQ^{-1}$ are at $Q\Lambda Q^{-1}$ and should be brought back to identity (inside bracket).

Remark 3.5. For the compact Lie group G, we have $\sinh \alpha_j$ replaced by $\sin \alpha_j$ everywhere. Moreover, the last Jacobian computation step $p_j \mapsto \Lambda^{\frac{1}{2}} p_j \Lambda^{\frac{1}{2}}$ can be omitted for the compact cases, since $\Lambda^{\frac{1}{2}}$ is an orthogonal/unitary matrix for the compact cases. The map $d\tilde{\tau}(\Lambda^{\frac{1}{2}})$ preserves the Euclidean measure as $d\tilde{\tau}(Q)$.

I. Extension to the generalized Cartan decomposition

In the previous paragraphs, we studied the Jacobian of the usual Cartan decomposition. We now proceed to consider the generalized Cartan decomposition (Theorems 2.1 and 2.2), its Jacobian (2.2), (2.3), and the extension of Table II. The derivations are analogous, analyzing subspaces of $\mathfrak{g}$, but one should now proceed with four tangent subspaces, $\mathfrak{k}_\tau \cap \mathfrak{k}_\sigma$, $\mathfrak{k}_\tau \cap \mathfrak{p}_\sigma$, $\mathfrak{p}_\tau \cap \mathfrak{k}_\sigma$, and $\mathfrak{p}_\tau \cap \mathfrak{p}_\sigma$. Earlier work on these Jacobian related derivations may be found in Refs. 23 and 44. The maximal subspace $\mathfrak{a}$ is now defined inside $\mathfrak{p}_\tau \cap \mathfrak{p}_\sigma$. We start with the same strategy: the tangent space $\mathfrak{g}$ is decomposed into the eigenspaces of the linear operator ad_H with $H \in \mathfrak{a}$. The eigenvalues $\pm\alpha_j$ still come in pairs, but we have two eigenmatrices $x_j, \tau\sigma x_j$ for eigenvalue α_j and two eigenmatrices $\tau x_j, \sigma x_j$ for eigenvalue $-\alpha_j$. We define four vectors v_1, v_2, w_1, w_2 with the same roles as k_j and p_j played before,

$$v_1 := x_j + \tau x_j + \sigma x_j + \tau\sigma x_j \in \mathfrak{k}_\tau \cap \mathfrak{k}_\sigma, \qquad v_2 := x_j - \tau x_j - \sigma x_j + \tau\sigma x_j \in \mathfrak{p}_\tau \cap \mathfrak{p}_\sigma,$$
$$w_1 := x_j - \tau x_j + \sigma x_j - \tau\sigma x_j \in \mathfrak{p}_\tau \cap \mathfrak{k}_\sigma, \qquad w_2 := x_j + \tau x_j - \sigma x_j - \tau\sigma x_j \in \mathfrak{k}_\tau \cap \mathfrak{p}_\sigma,$$

and these have similar ping pong relationships by ad_H like k_j and p_j,

$$\mathrm{ad}_H(v_1) = \alpha_j v_2, \qquad \mathrm{ad}_H(v_2) = \alpha_j v_1,$$
$$\mathrm{ad}_H(w_1) = \alpha_j w_2, \qquad \mathrm{ad}_H(w_2) = \alpha_j w_1.$$

We can similarly extend (3.8) and other relationships and proceed as in Table II to obtain (2.2) and (2.3).

IV. RANDOM MATRIX ENSEMBLES: COMPACT AND NONCOMPACT

A. Compact symmetric spaces

In compact cases, the random matrices could be simply determined from the Haar measure of the compact Lie group G,[12,13] since the compactness of G turns the Haar measure into a probability measure. In Secs. V, VI, VII, we discuss random matrix ensembles based on ten types of Riemannian symmetric space classification by Cartan. For the triple (G, K_σ, K_τ), we start with the cases where G/K_σ and G/K_τ are of the same types in Secs. V and VI. Then, in Sec. VII, we will discuss the "mixed types" where G/K_σ and G/K_τ are different types under Cartan's classification.

B. Noncompact symmetric spaces

Sections VIII and IX discuss classical random matrix ensembles associated with noncompact symmetric spaces. Hermite and Laguerre eigenvalue joint densities arise as result of (2.2) using Theorem 2.4 on noncompact symmetric spaces. As opposed to compact Lie groups and symmetric spaces where the Haar measure or G-invariant measure can be normalized by a constant to a probability measure, invariant measures on noncompact manifolds cannot be normalized to one by constants. A normalizing factor $\mathcal{S}$ should be introduced to complete the construction of a probability measure. Therefore, random matrices on a noncompact manifold face an innate problem if we proceed analogous to Secs. V and VI:

- The choice of the probability measure on noncompact G/K is not unique.

In Ref. 13, Dueñez also addressed this problem along the noncompact duals.

As we push the measure forward to the subgroup A, the resulting measure should be a symmetric function of independent generators of A. Hence, the probability measure $\mathcal{I}(g)$ of the random matrix ensemble is the Haar or G-invariant measure on G or G/K, multiplied by some symmetric function $\mathcal{S}$ on A,

$$\mathcal{I}(g) = \mathcal{S}(a)\mu(g),$$

where $g = k_1 a k_2$ or $g = kak^{-1}$ and $\mu(g)$ is an invariant measure. Using (2.2), the measure on A is induced,

$$\mathcal{I}(g) = dk \cdot \mathcal{S}(a)\left(\prod_{\alpha\in\Sigma^+} \sinh\, \alpha(H)\right) dH_1 \ldots dH_{\dim(\mathrm{A})},$$

which means that even though the measure $\mathcal{I}$ changes, the measure on A still differs only by a normalization function. The traditional choice of $\mathcal{S}$ has been made such that $\mathcal{I}(g)$ can be constructed from independent Gaussian distributions endowed on matrix entries. In fact, one could also endow a Gaussian distribution on the Riemannian manifold (symmetric space) itself.[65]

An alternative approach that appears in Ref. 6 is to put a probability measure on the tangent space of the symmetric space, $\mathfrak{p}$. In particular, independent Gaussian distribution endowed on the elements of $\mathfrak{p}$ give rise to Hermite and Laguerre ensembles by Theorem 2.7. We will follow this alternative approach.

C. Non-probability measure of noncompact groups

As discussed in Sec. IV B, the Haar measure of a noncompact group G or a noncompact symmetric space G/K is not a probability measure. However, we can force an analog of a random matrix theory. Imagine, for example, a noncompact $\mathrm{K}_1\mathrm{AK}_2$ decomposition $G = K_\sigma A K_\tau$ with $(G, K_\sigma, K_\tau) = (\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(n), \mathrm{O}(p,q))$. This is called the hyperbolic SVD[66] where any real invertible matrix M is factored into the product of an orthogonal matrix O, a positive diagonal matrix Λ, and an indefinite orthogonal matrix V. From the Haar measure and (2.2) of $\mathrm{GL}(n,\mathbb{R})$, one obtains the Jacobian,

$$\prod_{\substack{1\le j<k\le p\\ p<j<k\le n}} |\lambda_j - \lambda_k| \prod_{\substack{1\le j\le p\\ p<j\le n}} |\lambda_j + \lambda_k| \prod_{j=1}^{n} |\lambda_j|^{-\frac{2n+1}{2}}\, d\lambda_1 \ldots d\lambda_n,$$

where λ_j is the squared diagonal entries of Λ for all j's.

One can impose a Gaussian-like density function (although not a probability density) on the group $\mathrm{GL}(n,\mathbb{R})$, such as $\exp(-\mathrm{tr}(gI_{p,q}g^T)/2)\prod dg_{jk}$, where $I_{p,q} = \mathrm{diag}(I_p, -I_q)$. In terms of independent entries of g, this is

$$\prod_{\text{first } p \text{ columns}} e^{-g_{jk}^2/2} \prod_{\text{last } q \text{ columns}} e^{g_{jk}^2/2} \prod dg_{jk}. \tag{4.1}$$

Since the Haar measure of $\mathrm{GL}(n,\mathbb{R})$ is $|\det(g)|^{-n}\prod dg_{jk}$, (4.1) becomes [after integrating out $\mathrm{O}(n)$ and $\mathrm{O}(p,q)$]

$$\prod_{j<k}|\lambda_j - \lambda_k| \prod_{j=1}^{n} |\lambda|^{-\frac{n+1}{2}} e^{-\Sigma \lambda_j/2} d\lambda_1 \ldots d\lambda_n,$$

where $\lambda_1, \ldots, \lambda_p \ge 0$ are the first p squared diagonal values of Λ and $\lambda_{p+1}, \ldots, \lambda_n \le 0$ are the last q squared diagonal values of Λ, multiplied by -1. Extending this approach to find a proper random matrix probability measure on noncompact Lie groups and symmetric spaces with joint probability densities on the subgroup A is still an open problem.

V. COMPACT AI, A, and AII: CIRCULAR ENSEMBLES

The joint probability density of the circular ensemble is $(\beta = 1, 2, 4)$

$$E_n^{(\beta)}(\theta) \propto \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

Circular ensembles $\beta = 1, 2,$ and 4 (COE, CUE, and CSE) arise as the eigenvalues of special unitary matrices. As we discuss in the Introduction, circular ensembles are completely classified by (compact) symmetric spaces of the types AI, A, and AII, respectively.[5,13] The $\mathrm{K}_1\mathrm{AK}_2$ decomposition associated with each symmetric space recovers the KAK decomposition. The restricted root system (and dimensions) of AI, A, and AII are given as the following $(1 \le j < k \le n)$:

$\alpha(H)$	$\pm(h_j - h_k)$
m_α	β

(5.1)

Since we have compact symmetric spaces, we use (2.3) from either Theorem 2.2 or 2.4 with these root systems.

A. Compact AI, $\beta = 1$ COE

The compact symmetric space AI is $G/K = \mathrm{U}(n)/\mathrm{O}(n)$. The involution on $\mathrm{U}(n)$ has no free parameter and the $\mathrm{K}_1\mathrm{AK}_2$ decomposition is equivalent to the KAK decomposition of $\mathrm{U}(n)/\mathrm{O}(n)$. (In other words, we only have Cartan's coordinate system.) The maximal Abelian torus A is

$$A = \{\text{Diagonal matrices with entries } e^{ih_j}, \text{ where } h_j \in \mathbb{R}\}.$$

From the KAK decomposition, we obtain $U = O_1 D O_2$, a factorization of a unitary matrix U into the product of two orthogonal matrices $O_1, O_2 \in \mathrm{O}(n)$ and a unit complex diagonal matrix $D \in A$. This decomposition first appears in Ref. 67, and we will call this the *ODO decomposition*. The corresponding Jacobian (up to constant) from (2.3) using (5.1), $\beta = 1$ is (with the change of variables $\theta_j = 2h_j$)

$$\left(\prod_{j<k} \sin(h_j - h_k)\right) dh_1 \ldots dh_n \;\propto\; \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}| d\theta_1 \ldots d\theta_n.$$

This is the joint density of the COE. In other words, doubled angles in the diagonal of D from the ODO decomposition of a Haar distributed unitary matrix is the COE distribution. Moreover, if we identify G/K as the set of unitary symmetric matrices P, the map (2.5) is the factorization $S = O\Lambda O^T$, the eigendecomposition of a unitary symmetric matrix S with real eigenvectors O. In terms of Remark 2.5, $U = O_1 D O_2$ becomes $S = UU^T = O_1 D^2 O_1^T$, where $\Lambda = D^2$. To obtain the COE, we can utilize both factorizations:

- Two times the angles of the unit diagonal values of D from the ODO decomposition of $U \in \mathrm{Haar}(\mathrm{U}(n))$.
- The angles of the (unit) eigenvalues of a unitary symmetric matrix obtained from UU^T, $U \in \mathrm{Haar}(\mathrm{U}(n))$.

Remark 5.1. The second algorithm above would be obvious since the days of Dyson,[1,4] while we are not aware of the first algorithm appearing in the literature.

B. Compact A, $\beta = 2$ CUE

The symmetric space of compact type A is $G/K = \mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n)$. The restricted root system returns to the usual root system A_n of the classical semisimple Lie algebra. A maximal torus of $\mathrm{U}(n)$ is a Cartan subalgebra of $\mathrm{U}(n)$. Weyl's integration formula agrees with (2.3) obtaining the CUE, which is the eigenvalues of a Haar distributed unitary matrix. The derivation of the CUE can be found in many random matrix textbooks.[15,64,68]

C. Compact AII, $\beta = 4$ CSE

The involution $X \mapsto -J_n^T X^T J_n$, where $J_n := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ on the tangent space of $\mathrm{U}(2n)$, results in the symmetric space $\mathrm{U}(2n)/\mathrm{Sp}(n)$, where $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$. A choice of maximal Abelian torus A is

$$A = \{\mathrm{diag}(\tilde{D}, \tilde{D}) : \tilde{D} = \mathrm{diag}(e^{ih_1}, \ldots, e^{ih_n}),\ h_j \in \mathbb{R}\}.$$

Again from the KAK decomposition, we obtain $U = Q_1 D Q_2$, a factorization of a $2n \times 2n$ unitary matrix U into the product of two unitary symplectic matrices $Q_1, Q_2 \in \mathrm{Sp}(n)$ and a unit complex diagonal matrix $D \in A$. We call this the *QDQ decomposition*. The corresponding Jacobian from (2.3) using (5.1) ($\beta = 4$) is

$$\left(\prod_{j<k} \sin^4(h_j - h_k)\right) dh_1 \ldots dh_n \quad \propto \quad \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^4 \, d\theta_1 \ldots d\theta_n,$$

with the change of variables $\theta_j = 2h_j$. This is the CSE distribution. Similarly, as in Sec. V A, the eigendecomposition of the unitary skew-Hamiltonian matrix obtained by $UJ_nU^TJ_n^T$, $U \in \mathrm{Haar}(2n)$ is equivalent to the map (2.5). Two numerical algorithms for sampling the CSE are as follows:

- Two times the angles of the first n unit diagonal values of D from the QDQ decomposition of $U \in \mathrm{Haar}(\mathrm{U}(2n))$.
- The angles of the first n (unit) eigenvalues of a unitary skew-Hamiltonian matrix obtained by $UJ_nU^TJ_n^T$ with $U \in \mathrm{Haar}(\mathrm{U}(2n))$.

VI. COMPACT BDI, AIII, and CII: JACOBI ENSEMBLES

The joint probability density of the Jacobi ensemble is ($\beta = 1, 2, 4$),

$$J_{\alpha_1,\alpha_2}^{(\beta),m}(x) \propto \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^m x_j^{\alpha_1} (1 - x_j)^{\alpha_2}.$$

In Refs. 12 and 13, Jacobi ensembles $\beta = 1, 2, 4$ arise from the KAK decompositions of seven compact symmetric spaces, BDI, AIII, CII, DIII, BD, C, and CI. In particular, types BDI, AIII, and CII give multiple Jacobi densities as follows (for integers $p \geq q$):

$$\prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^q x_j^{\frac{\beta}{2}-1} (1 - x_j)^{\frac{\beta(p-q+1)}{2}-1},$$

and the powers of x_j's are fixed to $\frac{\beta}{2} - 1$. The remaining four cases add four more parameter points, which could be found in Refs. 12 and 13. In this paper, we omit these four cases as these do not have any further results, as they only have Cartan's coordinates (no free parameter for the Cartan involution).

The K_1AK_2 decomposition $G = K_\tau A K_\sigma$ of the compact types BDI-I, AIII-III, CII-II are exactly the *CS decomposition* (CSD)[69,70] of orthogonal, unitary, and unitary symplectic matrices, respectively. The decomposition Φ of the symmetric space (Theorem 2.4) is the GSVD coordinate systems we discussed in Secs. I B and II C. Assume $r \geq p \geq q \geq s$ and $n = p + q = r + s$ throughout this section. We note that with the KAK decomposition, only the cases $p = r, q = s$ are obtained for the CSD. The root system associated with the K_1AK_2 decomposition is the following ($1 \leq j < k \leq s$):

$$\begin{array}{c|c|c|c|} \alpha(H) & \pm(\theta_j \pm \theta_k) & \pm\theta_j & \pm 2\theta_j \\ \hline m_\alpha^+ & \beta & \beta(p-s) & \beta - 1 \\ \hline m_\alpha^- & 0 & \beta(q-s) & 0 \\ \hline \end{array}. \tag{6.1}$$

For all three β, we have the identical maximal Abelian subgroup A,

$$A = \{n \times n \text{ matrices with the block structure} \begin{bmatrix} C & & S \\ & I_{p-q} & \\ -S & & C \end{bmatrix}\},$$

where $C, S \in \mathbb{R}^{s\times s}$ are diagonal matrices with cosine, sine values of $\theta_1, \ldots, \theta_s$ on diagonal entries, respectively.

A. Compact BDI-I, $\beta = 1$ Jacobi

With the involution $X \mapsto I_{p,q} X I_{p,q}$ on the tangent space of $\mathrm{O}(n)$, we obtain the symmetric space BDI, $G/K = \mathrm{O}(n)/(\mathrm{O}(p) \times \mathrm{O}(q))$, where $I_{p,q} := \mathrm{diag}(I_p - I_q)$. With two symmetric pairs $[\mathrm{O}(n), \mathrm{O}(p) \times \mathrm{O}(q)]$ and $[\mathrm{O}(n), \mathrm{O}(r) \times \mathrm{O}(s)]$, we obtain the K_1AK_2 decomposition BDI-I,

$$\begin{bmatrix} n - \text{by} - n \\ \text{Orthogonal} \end{bmatrix} = \begin{bmatrix} O_p & \\ & O_q \end{bmatrix} \begin{bmatrix} C & & S \\ & I_{n-2s} & \\ -S & & C \end{bmatrix} \begin{bmatrix} O_r & \\ & O_s \end{bmatrix}.$$

This is the real CSD. [Equivalently, one can imagine the GSVD of (1.2).] From (2.3) using (6.1) $\beta = 1$, we obtain the Jacobian

$$d\mu(H) \propto \prod_{j<k} (\sin(\theta_j - \theta_k)\sin(\theta_j + \theta_k)) \prod_j \Big((\sin\theta_j)^{(p-s)} (\cos\theta_j)^{(q-s)}\Big) d\theta_1 \ldots d\theta_s.$$

Using trigonometric identities with change of variables $x_j = \cos^2\theta_j = \frac{1+\cos(2\theta_j)}{2}$,

$$d\mu(H) \propto \prod_{j<k} |x_j - x_k| \prod_{j=1}^{s} x_j^{\frac{1}{2}(q-s+1)-1} (1 - x_j)^{\frac{1}{2}(p-s+1)-1}\, dx_1 \ldots dx_s,$$

which is the joint density of the $\beta = 1$ Jacobi ensemble $J^{(1),s}_{\alpha_1,\alpha_2}$ if we let $\alpha_1 = \frac{1}{2}(q - s + 1) - 1$, $\alpha_2 = \frac{1}{2}(p - s + 1) - 1$. This result agrees with Ref. 20, Theorem 1.5, where the squared CSD cosine values of a Haar distributed orthogonal matrix are distributed as $\beta = 1$ Jacobi ensemble. Moreover, recall the fact that the QL decomposition $G = QL$ (a lower triangular analog of the QR decomposition) of an $n \times n$ independent Gaussian matrix G obtains a Haar distributed orthogonal matrix Q. Since the GSVD[18,19] is equivalent to the combination of the QL decomposition and the CSD, one can take the GSVD of a real independent Gaussian matrix to obtain the same $\beta = 1$ Jacobi ensemble. Two associated numerical algorithms are as follows ($a = q - s, b = p - s$):

- The squared CSD cosine values of a Haar distributed $m \times m$ orthogonal matrix ($m = 2s + a + b$) with row/column partitions $(s + a, s + b)$ and $(s, s + a + b)$.
- The squared cosine values, where the tangent values are the generalized singular values of real $(s + a) \times s$ and $(s + b) \times s$ Gaussian matrices.

B. Compact AIII-III, β = 2 Jacobi

Two symmetric pairs of compact AIII type are $[\mathrm{U}(n), \mathrm{U}(p) \times \mathrm{U}(q)]$ and $[\mathrm{U}(n), \mathrm{U}(r) \times \mathrm{U}(s)]$. The K_1AK_2 decomposition of the group G is the CSD of unitary matrices and the decomposition of $G/K_\sigma = \mathrm{U}(n)/(\mathrm{U}(r) \times \mathrm{U}(s))$ are the complex GSVD described in Sec. I B and Eq. (1.2). Using (2.3) with the root system (6.1), $\beta = 2$, and change of variables $x_j = \cos^2\theta_j$ as above, we obtain the Jacobian

$$\prod_{j<k}(\sin(\theta_j-\theta_k)\sin(\theta_j+\theta_k))^2\prod_j\Big((\sin\theta_j)^{2(p-s)}(\cos\theta_j)^{2(q-s)}\ \sin(2\theta_j)\Big)d\theta_1\dots d\theta_s$$
$$\propto\prod_{j<k}|x_j-x_k|^2\prod_j x_j^{q-s}(1-x_j)^{p-s}dx_1\dots dx_s,$$

which is the $\beta=2$ Jacobi density $J^{(2),s}_{\alpha_1,\alpha_2}$ with $\alpha_1=q-s,\alpha_2=p-s$. Numerically, the following could be utilized to obtain $\beta=2$ Jacobi densities $(a=q-s,b=p-s)$:

- The squared CSD cosine values of a Haar distributed $m\times m$ unitary matrix $(m=2s+a+b)$ with row/column partitions $(s+a,s+b)$ and $(s,s+a+b)$.
- The squared cosine values, where the tangent values are the generalized singular values of complex $(s+a)\times s$ and $(s+b)\times s$ Gaussian matrices.

C. Compact CII-II, β = 4 Jacobi

Jacobi densities with $\beta=4$ are similarly obtained from two symmetric spaces $\mathrm{Sp}(n)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$ and $\mathrm{Sp}(n)/(\mathrm{Sp}(r)\times\mathrm{Sp}(s))$, where both are compact type CII. We identify $\mathrm{Sp}(n)$ as the quaternionic unitary group, $\mathrm{U}(n,\mathbb{H}):=\{g\in\mathrm{GL}(n,\mathbb{H})|g^Dg=I_n\}$. The $\mathrm{K}_1\mathrm{AK}_2$ decomposition is the CSD of a quaternionic unitary matrix. Using (2.3) with the root system (6.1) $\beta=4$, we obtain the following Jacobian with the change of variables $x_j=\cos^2\theta_j$:

$$\prod_{j<k}(\sin(\theta_j-\theta_k)\sin(\theta_j+\theta_k))^4\prod_j\Big((\sin\theta_j)^{4(p-s)}(\cos\theta_j)^{4(q-s)}\ \sin^3(2\theta_j)\Big)d\theta_1\dots d\theta_s$$
$$\propto\prod_{j<k}|x_j-x_k|^4\prod_j x_j^{2(q-s)+1}(1-x_j)^{2(p-s)+1}\,dx_1\dots dx_s,$$

which is the $\beta=4$ Jacobi density $J^{(4),s}_{\alpha_1,\alpha_2}$ with $\alpha_1=2(q-s)+1,\alpha_2=2(p-s)+1$. The associated numerical algorithm is the following $(a=q-s,b=p-s)$:

- The squared cosine CS values of a Haar distributed $m\times m$ quaternionic unitary matrix $(m=2s+a+b)$ with row/column partitions $(s+a,s+b)$ and $(s,s+a+b)$.

Remark 6.1. Again, one can use the GSVD on quaternionic Gaussian matrices to obtain the classical $\beta=4$ Jacobi ensemble.

VII. COMPACT MIXED TYPES: MORE CIRCULAR AND JACOBI

In this section, we show even more cases such that a single symmetric space leading to multiple random matrix theories. We introduce $\mathrm{K}_1\mathrm{AK}_2$ decompositions with two compact symmetric spaces, each from different Cartan types. The classification of such $\mathrm{K}_1\mathrm{AK}_2$ decompositions is studied in Ref. 47, with the computation of corresponding root systems. As always the names of these decompositions are combinations of two Cartan types, i.e., AI-II represents $(G,K_\sigma,K_\tau)=(\mathrm{U}(2n),\mathrm{O}(2n),\mathrm{Sp}(2n))$.

A. Compact AI-II

The two compact symmetric spaces are types AI and AII, $\mathrm{U}(2n)/\mathrm{O}(2n)$ and $\mathrm{U}(2n)/\mathrm{Usp}(2n)$. A maximal Abelian subalgebra $\mathfrak{a}\subset\mathfrak{p}_\sigma\cap\mathfrak{p}_\tau$ is the set of all matrices $\mathrm{diag}(i\theta_1,\dots,i\theta_n,i\theta_1,\dots,i\theta_n)$ for $(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$. The subgroup A is the following:

$$A=\{\mathrm{diag}(\tilde D,\tilde D):\tilde D=\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_n})\}.$$

The root system is given as

$$\begin{array}{c|c|}\alpha(H) & \pm(\theta_j-\theta_k)\\ \hline m_\alpha^+ & 2\\ \hline m_\alpha^- & 2\end{array}\ . \tag{7.1}$$

Using (2.3), we obtain the Jacobian $(\xi_j=4\theta_j)$,

$$|e^{i\xi_j}-e^{i\xi_k}|^2d\xi_1\dots d\xi_n,$$

which is the joint probability density of the CUE. Hence, we obtain another sampling method for the CUE.

B. Compact AI-III, CI-II

The two symmetric spaces in each case are the following:

$$G/K_\tau, G/K_\sigma = \mathrm{U}(n)/\mathrm{O}(n), \mathrm{U}(n)/(\mathrm{U}(p)\times\mathrm{U}(q)),$$
$$G/K_\tau, G/K_\sigma = \mathrm{U}(n,\mathbb{H})/\mathrm{U}(n), \mathrm{U}(n,\mathbb{H})/(\mathrm{U}(p,\mathbb{H})\times\mathrm{U}(q,\mathbb{H})).$$

The subgroup A is computed as follows:

$$A = \{n\times n \text{ matrices with the block structure} \begin{bmatrix} C & & \eta S \\ & I_{p-q} & \\ \eta S & & C \end{bmatrix}\},$$

where C, S are $q \times q$ diagonal matrices with cosine and sine values of q angles $\theta_1, \ldots, \theta_q$ on their diagonals. The imaginary unit η is i for AI-III ($\beta = 1$) and $\eta = j, k$ for CI-II ($\beta = 2$). [If we select the subgroup K of $\mathrm{U}(n,\mathbb{H})/\mathrm{U}(n)$ to be the unitary group with the imaginary unit j, we could also obtain $\eta = i$.] The root system is the following ($\beta = 1, 2$):

$\alpha(H)$	$\pm(\theta_j \pm \theta_k)$	$\pm\theta_j$	$\pm 2\theta_j$
m_α^+	β	$\beta(p-q)$	$\beta - 1$
m_α^-	β	$\beta(p-q)$	β

(7.2)

Using (2.3) with the above root system above, we obtain the following Jacobian:

$$\prod_{j<k}|x_j - x_k|^\beta \prod_{j=1}^{q} x_j^{\frac{\beta(p-q+1)}{2}-1}(1-x_j)^{\frac{\beta-1}{2}}, \tag{7.3}$$

where $x_j = \sin^2 2\theta_j$ for all j. The $\beta = 1$ case of (7.3) can be obtained from the CS decomposition approach too, with $(n+1)\times(n+1)$ orthogonal matrix and partitions $(p, q+1)$ and $(p+1, q)$ [see Fig. 4]. The parameters of $\beta = 2$ (7.3) cannot be obtained by the complex CSD and, thus, fall outside of the classical parameters.

C. Compact DI-III, AII-III

Another family of the $\mathrm{K_1AK_2}$ decomposition arise from the following pairs of compact symmetric spaces ($\beta = 2, 4$):

$$G/K_\tau, G/K_\sigma = \mathrm{O}(2n)/\mathrm{U}(n), \mathrm{O}(2n)/(\mathrm{O}(2p)\times\mathrm{O}(2q)),$$
$$G/K_\tau, G/K_\sigma = \mathrm{U}(2n)/\mathrm{U}(n,\mathbb{H}), \mathrm{U}(2n)/(\mathrm{U}(2p)\times\mathrm{U}(2q)).$$

Under Cartan's classification, they are types DI-III and AII-III, respectively. The subgroup A can be computed as

$$A = \{2n\times 2n \text{ matrices with the block structure} \begin{bmatrix} I_{p-q} & & & \\ & C\otimes I_2 & & S\otimes J_1 \\ & & I_{p-q} & \\ & S\otimes J_1 & & C\otimes I_2 \end{bmatrix}\},$$

where I_2 is the 2×2 identity matrix, $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and C, S are $q\times q$ diagonal matrices with cosines and sines of $\theta_1, \ldots, \theta_q$ on their diagonals. The root system is given as follows ($\beta = 2, 4$):

$\alpha(H)$	$\pm(\theta_j \pm \theta_k)$	$\pm\theta_j$	$\pm 2\theta_j$
m_α^+	β	$\frac{\beta}{2}(p-q)$	$\beta - 1$
m_α^-	β	$\frac{\beta}{2}(p-q)$	$\frac{\beta}{2} - 1$

(7.4)

Again, using (2.3) with the root system above, we obtain the following Jacobian, with the change of variables $x_j = \sin^2\theta_j$ for all j:

$$\prod_{j=1}^{q} x_j^{\frac{\beta(p-q+2)}{4}-1}(1-x_j)^{\frac{\beta-4}{4}}\prod_{j<k}|x_j - x_k|^\beta. \tag{7.5}$$

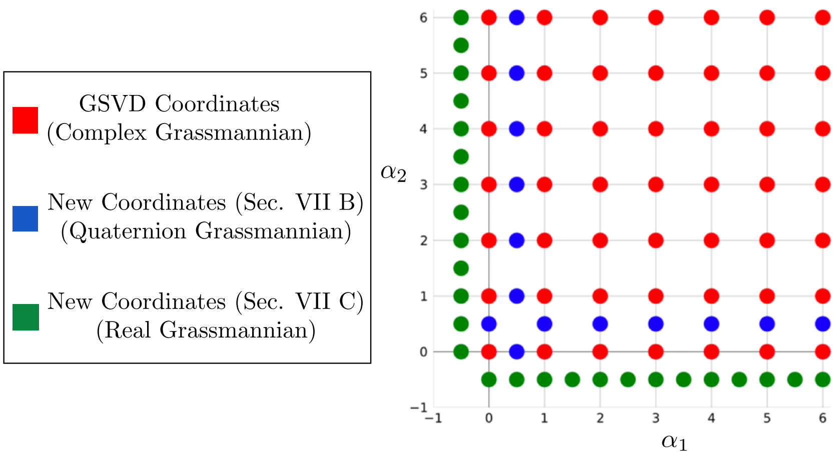

FIG. 6. The parameter space $(\alpha_1, \alpha_2) \in (-1, \infty)^2$ of the $\beta = 2$ Jacobi ensemble covered by symmetric spaces. The GSVD coordinate systems on the complex Grassmannian manifold (AIII-III) discussed in Sec. VI covers red dots. A new coordinate system on the quaternionic (respectively, real) Grassmannian manifold discussed in Sec. VII B (respectively, Sec. VII C) of type CI-II (respectively, DI-III) represent blue (respectively, green) dots.

They are $\beta = 2, 4$ Jacobi ensembles. Both cases could not be obtained from the classical CSD approach, so they are all non-classical parameters of the Jacobi ensemble. To see this at once, we compare three $\beta = 2$ Jacobi densities each from Secs. VI B, VII B and VII C. Figure 6 shows the possible parameters α_1, α_2 of the $\beta = 2$ Jacobi ensemble obtained from each approach.

VIII. NONCOMPACT AI, A, AND AII: HERMITE ENSEMBLES

While Sec. VII contains essentially new random matrix theories, Secs. VIII and IX review the Hermite and Laguerre ensembles for completeness.[6–9,59]

The joint probability density of the Hermite ensemble is $(\beta = 1, 2, 4)$,

$$H_n^{(\beta)}(\lambda) \propto \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^{n} e^{-\lambda_j^2/2}.$$

Hermite ensembles $\beta = 1, 2,$ and 4 (GOE, GUE, and GSE) arise as the eigenvalues of symmetric, Hermitian, and self-dual Gaussian matrices. Hermite ensembles can be thought as the Gaussian measure endowed on the tangent space of noncompact symmetric spaces of the types AI, A, and AII. The connection between these symmetric spaces and Hermite ensembles are made by Theorem 2.7. The decomposition Ψ (2.6) in Theorem 2.7 is the eigendecomposition of symmetric, Hermitian, and self-dual matrices. The maximal Abelian subalgebra $\mathfrak{a}$ is the collection of all real diagonal matrices, $\mathrm{diag}(h_1, \ldots, h_n)$. The restricted root system is the following $(1 \le j < k \le n)$:

$$\begin{array}{r|c|} \alpha(H) & \pm(h_j - h_k) \\ m_\alpha & \beta \end{array}. \tag{8.1}$$

A. Noncompact AI, β = 1 GOE

The dual of the compact symmetric space type AI, the noncompact symmetric space type AI, is $G/K = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, represented by the set $\mathcal{S}_n$ of all symmetric positive definite matrices. The tangent space at the identity of $\mathcal{S}_n$, $\mathfrak{p}$, is the set of all real symmetric matrices. The Gaussian measure on $\mathfrak{p}$ is, for $p \in \mathfrak{p}$, $\exp(-\mathrm{tr}(p^T p)/2)dp$,where dp is the Euclidean measure on $\mathfrak{p}$. From (2.7) using (8.1) $\beta = 1$, we obtain (integrate out dk)

$$\exp(-\mathrm{tr}(p^T p)/2)dp \propto \prod_{j<k} |\lambda_j - \lambda_k| \prod_{j=1}^{n} e^{-\lambda_j^2/2} d\lambda_1 \ldots d\lambda_n$$

for the eigenvalues of p, $\lambda_j = h_j$. This is the joint density of the GOE.

B. Noncompact A, β = 2 GUE

The noncompact symmetric space type A is $G/K = \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$, represented by $\mathcal{H}_n$, the set of all Hermitian positive definite matrices. The tangent space at the identity of $\mathcal{H}_n$, $\mathfrak{p}$, is the set of all complex Hermitian matrices. The Gaussian measure on $\mathfrak{p}$ is, for $p \in \mathfrak{p}$, $\exp(-\mathrm{tr}(p^H p)/2)dp$, where dp is the (real) Euclidean measure on $\mathfrak{p}$. From (2.7) using (8.1) $\beta = 2$, we obtain

$$\exp(-\mathrm{tr}(p^H p)/2)dp \propto \prod_{j<k}|\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} e^{-\lambda_j^2/2} d\lambda_1 \dots d\lambda_n$$

for the eigenvalues of p, $\lambda_j = h_j$. This is the joint density of the GUE.

C. Noncompact AII, β = 4 GSE

The noncompact symmetric space type AII is $G/K = \mathrm{GL}(n,\mathbb{H})/\mathrm{U}(n,\mathbb{H})$. We use $\mathrm{U}(n,\mathbb{H})$ instead of $\mathrm{Sp}(n)$ to clearly indicate the quaternionic realization. G/K can be represented by the set of all quaternionic self-dual positive definite matrices, $\mathcal{QH}_n$. Again, the tangent space at the identity $\mathfrak{p}$ is the set of all quaternionic self-dual matrices. The Gaussian measure on $\mathfrak{p}$ is, for $p \in \mathfrak{p}$, $\exp(-\mathrm{tr}(p^D p)/2)dp$, where dp is the (real) Euclidean measure on $\mathfrak{p}$. From (2.7) using (8.1) $\beta = 4$, we obtain

$$\exp(-\mathrm{tr}(p^D p)/2)dp \propto \prod_{j<k}|\lambda_j - \lambda_k|^4 \prod_{j=1}^{n} e^{-\lambda_j^2/2} d\lambda_1 \dots d\lambda_n$$

for the eigenvalues of p, $\lambda_j = h_j$. This is the joint density of the GSE.

IX. NONCOMPACT BDI, AIII, and CII: LAGUERRE ENSEMBLES

The joint probability density of the Laguerre ensemble is ($\beta = 1, 2, 4$)

$$L_{\alpha,m}^{(\beta)}(\lambda) \propto \prod_{j<k}|\lambda_j - \lambda_k|^\beta \prod_{j=1}^{m} \lambda_j^\alpha e^{-\lambda_j/2}.$$

Laguerre ensembles $\beta = 1, 2, 4$ arise from Theorem 2.7 applied to noncompact symmetric spaces BDI, AIII, CII, DIII, BD, C, and CI. The last four cases of types DIII, BD, C, and CI are well-studied in Ref. 6, and we again omit these cases as discussed in Sec. VI. In particular, the first three symmetric spaces give the following Laguerre densities ($\beta = 1, 2, 4$ and $p \geq q$):

$$\prod_{j<k}|\lambda_j - \lambda_k|^\beta \prod_{j=1}^{q} \lambda_j^{\frac{\beta(p-q+1)}{2}-1} e^{-\lambda_j/2},$$

as these λ_j values are the squared singular values of $p \times q$ i.i.d. Gaussian matrices. Equivalently, the eigenvalues of the matrix $A^\dagger A \in \mathbb{F}^{q\times q}$ are frequently used for sampling purpose, where $\dagger$ is the conjugate transposition. The tangent spaces of noncompact symmetric spaces of the types BDI, AIII, and CII are

$$\left\{ \begin{bmatrix} 0 & X \\ X^\dagger & 0 \end{bmatrix} : X \text{ is } p \times q \text{ matrix} \right\}, \tag{9.1}$$

and a choice of maximal Abelian subalgebra $\mathfrak{a}$ is the set with X being (nonsquare) diagonal matrix with diagonal elements $h_1, \dots, h_q$. The KAK decomposition $G = KAK$ of the noncompact symmetric spaces BDI, AIII, and CII is the *hyperbolic CS decomposition* (HCSD).[71,72] The decomposition $\mathfrak{p} = \cup_{k\in K} k\mathfrak{a}k^{-1}$ is the $p \times q$ SVD on upper right $p \times q$ corner. The restricted roots are the following ($\beta = 1, 2, 4$):

$\alpha(H)$	$\pm(h_j \pm h_k)$	$\pm h_j$	$\pm 2h_j$
m_α	β	$\beta(p-q)$	$\beta - 1$

(9.2)

A. Noncompact BDI, β = 1 Laguerre

The noncompact symmetric space type BDI is $G/K = \mathrm{O}(p,q)/(\mathrm{O}(p) \times \mathrm{O}(q))$. The tangent space $\mathfrak{p}$ (9.1) has the Gaussian measure as i.i.d. Gaussian distribution endowed on the elements of X. For $M \in \mathfrak{p}$, it is $\exp(-\mathrm{tr}(M^T M))d\mathfrak{p}$. From (2.7) using (9.2) $\beta = 1$, we obtain

$$\exp(-\mathrm{tr}(M^T M))d\mathfrak{p} \propto \prod_{j<k}|\lambda_j - \lambda_k| \prod_{j=1}^{q} e^{-\lambda_j/2} \lambda_j^{\frac{p-q-1}{2}} d\lambda_1 \dots d\lambda_q,$$

with the change of variables $\lambda_j = h_j^2$. Thus, the values $\lambda_1, \ldots, \lambda_q$ are the squared singular values of the upper right corner of M. The obtained measure is the joint density of the $\beta = 1$ Laguerre ensemble.

B. Noncompact AIII, β = 2 Laguerre

The noncompact symmetric space type AIII is $G/K = \mathrm{U}(p,q)/(\mathrm{U}(p)\times\mathrm{U}(q))$. The tangent space (9.1) has the Gaussian measure as i.i.d. complex Gaussian distribution endowed on the elements of X. For $M \in \mathfrak{p}$, that is $\exp(-\mathrm{tr}(M^H M))d\mathfrak{p}$. From (2.7) and using (9.2) $\beta = 2$, we obtain

$$\exp(-\mathrm{tr}(M^H M))d\mathfrak{p} \propto \prod_{j<k}|\lambda_j - \lambda_k|^2 \prod_{j=1}^{q} e^{-\lambda_j/2}\lambda_j^{p-q} d\lambda_1 \ldots d\lambda_q,$$

with the change of variables $\lambda_j = h_j^2$. Again, the values $\lambda_1, \ldots, \lambda_q$ are the squared singular values of the upper right corner of M. The obtained measure is the joint density of the $\beta = 2$ Laguerre ensemble.

C. Noncompact CII, β = 4 Laguerre

The noncompact symmetric space CII is $G/K = \mathrm{U}(p,q,\mathbb{H})/(\mathrm{U}(p,\mathbb{H})\times\mathrm{U}(q,\mathbb{H}))$. The tangent space (9.1) has the Gaussian measure as i.i.d. quaternionic Gaussian distribution endowed on the elements of X. For $M \in \mathfrak{p}$, that is $\exp(-\mathrm{tr}(M^D M))d\mathfrak{p}$. From (2.7) and using (9.2) $\beta = 4$, we obtain

$$\exp(-\mathrm{tr}(M^D M))d\mathfrak{p} \propto \prod_{j<k}|\lambda_j - \lambda_k|^4 \prod_{j=1}^{q} e^{-\lambda_j/2}\lambda_j^{2(p-q)+1} d\lambda_1 \ldots d\lambda_q,$$

with the change of variables $\lambda_j = h_j^2$. The values $\lambda_1, \ldots, \lambda_q$ are the squared singular values of the upper right corner of M. The obtained measure is the joint density of the $\beta = 4$ Laguerre ensemble.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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