Calculation of the Coulomb self-energy of a spherical surface with uniform surface charge density using the Fourier transform method

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Undergraduate students in upper levels of physics or engineering programs learn the theory of Fourier series and integral transform method from mathematics courses. Nevertheless, they rarely see the application of such a method to solving problems in calculus-based physics courses that deal with topics such as electrostatics or magnetism. In this work, we illustrate the utility of the Fourier transform method by considering and solving via such a technique a representative problem that arises in electrostatics. The chosen case study is that of a spherical surface with uniform surface charge density and the calculation of its electrostatic Coulomb self-energy. By solving this problem by using the Fourier transform technique we also draw attention to the pedagogical aspects of the treatment. In particular, we stress the point that the Fourier transform method should be treated at more depth in calculus-based physics courses for undergraduate students.

Keywords: Coulomb self-energy, Uniform surface charge density, Fourier transform method.

I. INTRODUCTION

Fourier series and integral transforms are important examples of transformations that have been very useful in mathematical analysis and physical applications[1,2]. Applications of Fourier transform method may be found in many diverse theoretical and applied sciences areas[3]. Therefore, it is desirable for undergraduate students in physics and engineering to first encounter the Fourier method when dealing with physical systems and not when they learn solving differential equations in mathematics courses. Beside its practical use in mathematical physics, the Fourier transform is also of fundamental importance in many other subjects[4–7]. However, despite its relevance, the Fourier transform method is not covered in typical calculus-based physics courses that deal with electricity and magnetism[8–13]. We believe that these courses, in particular when dealing with the subject of electrostatics, offer several good examples to illustrate the use of the Fourier transform method to solve problems of relevance. For example, one can consider the problem of the electrostatic properties of a spherical surface with uniform surface charge density and show a step by step implementation of the Fourier integral transform method to obtain the Coulomb self-energy stored in the body.

In this work, we will explain the application of the Fourier integral transform method in the context of one particular example that arises in electrostatics. This is the problem of a spherical surface with uniform surface charge density, namely, a charged spherical counductor. The idea is to calculate the Coulomb self-energy of the selected object by means of such a method in such a way that all the key mathematical steps are explained in a clear pedagogical manner. This approach has the added benefit of introducing the method as a powerful tool that can be used to tackle even more challenging problems that involve the calculation of multi-dimensional integrals. While the case study chosen is meant to be simple, the important message that we would like to transmit is that of a very useful tool that can be applied to solve much more complicated situations. In other words, we hint that the Fourier transform method is the tool to go in many problems of electrostatics where other approaches are not applicable[14,15]. For instance, problems of such nature would be the calculation of the electrostatic potential or Coulomb self-energy of more complicated charged bodies[16–19] that are not uniformly charged spherical surfaces or solid spheres.

The paper is organized as follows: In Section II we show some basic theory and formulas of electrostatics. This way the proper audience is quickly introduced to the topic. In Section III we list some known results that apply to the electrostatic properties of a spherical surface with uniform surface charge density. In Section IV we introduce the Fourier integral transform method formalism and show its use to solve the problem under consideration. In Section V we provide some concluding remarks and highlight the useful features of the method in various aspects.

II. BASIC THEORY OF ELECTROSTATICS

Let's consider a discrete system of N point charges, q_i at positions, $\vec{r}_i$ in space. The total electrostatic energy of the system (the Coulomb self-energy) can be written as:

$$U = \frac{k_e}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{q_i \, q_j}{|\vec{r}_i - \vec{r}_j|} \ , \tag{1}$$

where k_e is Coulomb's electric constant and the factor "2" is placed to avoid double-counting. The quantity in Eq.(1) may be rewritten as:

$$U = \frac{1}{2} \sum_{i=1}^{N} q_i \, V(\vec{r}_i) \ , \tag{2}$$

where

$$V(\vec{r}_i) = k_e \sum_{j \neq i}^{N} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \ , \qquad (3)$$

represents the electrostatic potential created by all charges $j \neq i$ at the location, $\vec{r}_i$ of charge, q_i.

Let us now consider a body with continuous charge distribution over a given length, surface or volume domain, D. Let's assume that the total charge contained in that region is Q. For the sake of full generality, the charge distribution may be arbitrary. This means that, line, surface or volume charge density (depending on the nature of the charged body) may vary from point to point. The continuum counterpart to the expression in Eq.(1) becomes:

$$U = \frac{k_e}{2} \int_D dQ \int_D dQ' \ \frac{1}{|\vec{r} - \vec{r}'|} \ , \qquad (4)$$

where dQ and dQ' represent elementary charges located around points $\vec{r}$ and $\vec{r}'$, respectively. The continuum counterpart to the expression in Eq.(2) becomes:

$$U = \frac{1}{2} \int_D dQ \, V(\vec{r}) \ , \qquad (5)$$

where

$$V(\vec{r}) = k_e \int_D dQ' \ \frac{1}{|\vec{r} - \vec{r}'|} \ . \qquad (6)$$

At this point, let's also avoid subtler discussions on why the $j = i$ terms that are tacitly implied when writing the integrals in Eq.(4) and Eq.(5) do not cause trouble since we want to keep the level of this presentation rather basic.

One also needs to remember the relationship between electrostatic potential, $V(\vec{r})$ and electrostatic field, $\vec{E}(\vec{r})$:

$$\vec{E}(\vec{r}) = -\vec{\nabla} \, V(\vec{r}) \ . \qquad (7)$$

where $\vec{\nabla}$ is the gradient operator. If we regard the electrostatic field as a real physical entity, possessing energy, we may want to rewrite the equation for U entirely in terms of the electrostatic field produced by the charge density. We skip the details and list below the resulting expression:

$$U = \frac{\epsilon_0}{2} \int_{All \ Space} d^3r \, |\vec{E}(\vec{r})|^2 \ , \qquad (8)$$

where now the integral is over all space (not confined to the domain region, D where the charge is located). In most cases the electrostatic potential is easier to calculate than the electrostatic field. As a result, one typically finds it easier to calculate the Coulomb self-energy from Eq.(5) once it has obtained the potential especially if the charged body is not simple[20,21].

III. SPHERICAL SURFACE WITH UNIFORM SURFACE CHARGE DENSITY

Let us consider a charged spherical surface, namely, a spherical conductor with radius, R containing a total charge, Q. The charge is spread uniformly on its surface. Thus, the constant uniform surface charge density of the body may be written as:

$$\sigma = \frac{Q}{4 \, \pi \, R^2} \ . \qquad (9)$$

For convenience, one chooses a spherical system of coordinates with origin at the center of the spherical surface. As a vector quantity, the electrostatic field created by a uniformly charged spherical surface has only a radial component, $\vec{E}(r)$. Its magnitude, $E(r) = |\vec{E}(r)|$ is written as:

$$E(r) = \begin{cases} 0 \quad ; \quad 0 \leq r < R \\ \frac{k_e \, Q}{r^2} \quad ; \quad R \leq r < \infty \ , \end{cases} \qquad (10)$$

where $r = |\vec{r}| \geq 0$ is the radial distance. The electrostatic field is not continous at $r = R$. The electrostatic potential, $V(r)$ is calculated from the electrostatic field resulting in:

$$E(r) = -\frac{dV(r)}{dr} \quad ; \quad dV(r) = -E(r) \, dr \ . \qquad (11)$$

One has to integrate over $E(r)$ to obtain $V(r)$ while being careful to enforce the continuity of the potential all over space:

$$V(r) = \begin{cases} \frac{k_e \, Q}{R} \quad ; \quad 0 \leq r < R \\ \frac{k_e \, Q}{r} \quad ; \quad R \leq r < \infty \ . \end{cases} \qquad (12)$$

One can calculate the electrostatic Coulomb self-energy of the uniformly charged spherical surface either from the potential, or from the field. For example, if one wants to calculate U from the electrostatic potential, then Eq.(5) for a constant uniform surface charge density leads to:

$$U = \frac{\sigma}{2} \iint_D dS \ V(r) = \frac{Q}{2} \ V(r = R) = \frac{1}{2} \, \frac{k_e \, Q^2}{R} \ . \qquad (13)$$

The same result as in Eq.(13) will be obtained if one calculates U from Eq.(8) with help from the expression for the electrostatic field in Eq.(10).

$$U = \frac{\epsilon_0}{2} \int_R^{\infty} dr \, 4 \, \pi \, r^2 \, E(r)^2 = \frac{1}{2} \, \frac{k_e \, Q^2}{R} \ . \qquad (14)$$

Note that the integration in Eq.(14) starts from $r = R$ since the electrostatic field is zero for $0 \leq r < R$.

IV. CALCULATION USING THE FOURIER TRANSFORM METHOD

Let's consider the elementary charges,

$$dQ = \sigma\, dS \quad ; \quad dQ' = \sigma\, dS' \;, \tag{15}$$

located on the spherical surface at position $\vec{r}$ and $\vec{r}'$, respectively, where the elementary surface areas on the spherical surface are:

$$dS = R^2\, d\theta\, \sin\theta\, d\varphi \quad ; \quad dS' = R^2\, d\theta'\, \sin\theta'\, d\varphi' \;. \tag{16}$$

The Coulom self-energy the uniformly charged spherical surface can be written as:

$$U = \frac{k_e}{2}\, \sigma^2 \iint\limits_D dS \iint\limits_D dS' \frac{1}{|\vec{r} - \vec{r}'|} \;, \tag{17}$$

where D is spherical surface (integration) domain containg the charge. In spherical coordinates, such a domain is:

$$D \;:\; \left\{ r = r' = R \;;\; 0 \le (\theta, \theta') \le \pi \;;\; 0 \le (\varphi, \varphi') < 2\,\pi \right\} , \tag{18}$$

where θ, θ' are the polar angles, φ, φ' are the azimuthal (longitudinal) angles and $r{=}r' = R$ are the radial variables of respective vectors, $\vec{r}$ and $\vec{r}'$. On can write the expression in Eq.(17) more explicitly as:

$$U = \frac{k_e}{2}\, \left(\sigma\, R^2\right)^2 \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\,\pi} d\varphi \int_0^{\pi} d\theta'\, \sin\theta' \int_0^{2\,\pi} d\varphi' \frac{1}{|\vec{r} - \vec{r}'|} \;. \tag{19}$$

Let us now illustrate how the Fourier integral transform method can be used to calculate the integral above. To this effect, let us define the pair of three-dimensional (3D) Fourier integral transform functions, $F(\vec{k})$ and $f(\vec{r})$ as follows:

$$\begin{cases} F(\vec{k}) = \int d^3r\, \exp\left(i\, \vec{k}\, \vec{r}\right)\, f(\vec{r}) \;, \\[2ex] f(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, \exp\left(-i\, \vec{k}\, \vec{r}\right)\, F(\vec{k}) \;, \end{cases} \tag{20}$$

where $\vec{k}$ and $\vec{r}$ are 3D vectors, $i = \sqrt{-1}$ is the imaginary unit and the integration extends over all space. Based on the result that:

$$\int d^3r\, \exp\left(i\, \vec{k}\, \vec{r}\right)\, \frac{1}{r} = \frac{4\,\pi}{k^2} \;, \tag{21}$$

we can write

$$\frac{1}{|\vec{r} - \vec{r}'|} = \int \frac{d^3k}{(2\pi)^3}\, \exp\left[-i\, \vec{k}\, (\vec{r} - \vec{r}')\right]\, \frac{4\,\pi}{k^2} \;, \tag{22}$$

where $r = |\vec{r}| \ge 0$ and $k = |\vec{k}| \ge 0$. By substituting Eq.(22) into Eq.(19), one obtains:

$$U = \frac{k_e}{2}\, \left(\sigma\, R^2\right)^2 \int \frac{d^3k}{(2\pi)^3}\, \frac{4\,\pi}{k^2} \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\,\pi} d\varphi\, \exp\left(-i\, \vec{k}\, \vec{r}\right) \int_0^{\pi} d\theta'\, \sin\theta' \int_0^{2\,\pi} d\varphi'\, \exp\left(+i\, \vec{k}\, \vec{r}'\right) \;. \tag{23}$$

Sometimes, simple-looking integrals over angular variables are very challinging[22], but in the present case the calculation is straightforward:

$$\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\,\pi} d\varphi\, \exp\left(\pm i\, \vec{k}\, \vec{r}\right) = \int_0^{\pi} d\theta'\, \sin\theta' \int_0^{2\,\pi} d\varphi'\, \exp\left(\pm i\, \vec{k}\, \vec{r}\,'\right) = (4\,\pi)\, \frac{\sin(k\, R)}{(k\, R)} \;, \tag{24}$$

where one must remember that $|\vec{r}| = |\vec{r}\,'| = R$.

One substitutes the result from Eq.(24) into Eq.(23)

and after a little of algebra obtains:

$$U = \frac{k_e}{2}\,\left(\sigma\, 4\,\pi\, R^2\right)^2\,\frac{2}{\pi}\int_0^{\infty} dk\,\left[\frac{\sin(k\,R)}{(k\,R)}\right]^2 \,. \qquad (25)$$

One knows from Eq.(9) that, $\left(\sigma\, 4\,\pi\, R^2\right)^2 = Q^2$. Therefore, after simple mathematical arrangements, one has:

$$U = \frac{1}{2}\frac{k_e\, Q^2}{R}\,\frac{2}{\pi}\int_0^{\infty} dx\,\left(\frac{\sin x}{x}\right)^2 \,. \qquad (26)$$

At this juncture, one uses the formula:

$$\int_0^{\infty} dx\left(\frac{\sin x}{x}\right)^2 = \frac{\pi}{2}\,. \qquad (27)$$

This leads to the final result for the Coulomb self-energy:

$$U = \frac{1}{2}\frac{k_e\, Q^2}{R}\,, \qquad (28)$$

in agreement with Eq.(13).

V. CONCLUSIONS

In this work, we calculated the Coulomb self-energy of a charged spherical surface with uniform surface charge density by using the Fourier integral transform method. The Fourier series and integral transforms are known to undergraduate students in various science, engineering and/or mathematics disciplines. However, the use of the method to solve physics problems in physics undergraduate courses is not widespread. For thir reason, we chose a popular model found in virtually all physics textbooks and showed how to apply the Fourier transform method to calculate the resulting electrostatic Coulomb self-energy for such a model. The main message to transmit is not that we solved a specific problem, but that the implementation of the method is general. For this reason, we explained all the steps involved in a pedagogical way and provided all the necessary details in abundance.

The Fourier transform method, in particular when used for the calculation of the Coulomb self-energy of any given arbitrary charged body, enables one to simplify the multi-dimentional integral of a two-particle Coulomb term, $1/|\vec{r}-\vec{r}\,'|$ into a series of simpler integrals of one-particle functions. This reduces the difficulty of the problem and may lead to exact analytical results even in complicated cases of a regular charged body such as a uniformly charged elliptical plate[23]. Overall, the implementation of the Fourier transform method will result in a mathematical simplication of the problem which is a welcomed help for those cases where numerical calculations are necessary[24,25]. Furthermore, this work illustrates the implementation of a sophisticated mathematical tool to a receptive audience of students and teachers that should find it useful. The solution of this example as well as few similar cases can help one to enrich teaching and learning of calculus-based physics at undergraduate level. This means that the results reported here may be of interest to a rather broad audience of students and teachers working in the physical sciences.

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