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Brief paper

On the robustness of networked cooperative tracking systems☆

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ABSTRACT

This paper provides an analytical framework for the robustness of networked multi-agent systems (MAS). It is well-known that a single-agent linear quadratic regulator (LQR) system can guarantee 60° phase margin and infinite gain margin. However, for networked MAS, there exist no theoretical results on guaranteed stability margins, due to the complexity caused by the interplay of communication structure and agents' dynamics. In this paper, we analyze the effect of communication graph topology on the robustness properties of networked cooperative tracking systems with local LQR designs. For such systems, we provide closed-form expressions of phase and gain margins modulated by their graph topology, following a Lyapunov type of analysis. We further derive upper bounds of stability margins for MAS with general graph topology, through a structural analysis based on the algebraic graph theory. We prove that the directed tree communication topology is among the most robust graph topology that promises the best stability margins, which are as good as the ones in a single-agent LQR system

1. Introduction

Stability margins, i.e., gain margin and phase margin, describe the ability of a control system to maintain stability in the presence of perturbations, and have been adopted as the measures for robustness for decades (Franklin, Powell, & Emami-Naeini, 1994). Studies of stability margins largely focus on single-agent systems, including both single-input single-output (SISO) systems and multi-input multi-output (MIMO) systems. For SISO systems, the scalar Nyquist approach and Bode analysis have been developed to find phase and gain margins (Bode, 1945; Horowitz, 2013). Since the 1970s, a number of attempts have been directed to extend the robustness analysis from SISO systems to MIMO systems (Doyle, 1979; MacFarlane, 1972; MacFarlane & Belletrutti, 1973; McMorran, 1970; Rosenbrock, 1969; Safonov, 1982; Safonov & Athans, 1977; Tsao, Lee, & Augenstein, 1998). In a very first effort of this direction, paper (Safonov & Athans, 1977) introduced the concept of multi-loop robustness subject to simultaneous phase and gain perturbations in multiple loops, and showed that the linear quadratic regulator (LQR) possesses $\pm 60°$ phase margin, 50% gain reduction, and infinite gain margin, following a Lyapunov type of analysis. From the viewpoint of system transfer function matrix, a generalization of the classical scalar Nyquist approach and Bode analysis to MIMO systems was investigated in Doyle (1979), by exploiting the characteristics of singular values, single vectors, and the spectral norm of the closed-loop system transfer matrix. Based on the singular value analysis, the μ-analysis framework was then established, with the purpose of bounding the stability margins of diagonally perturbed MIMO systems (Balakrishnan, 2002; Halton, Iordanov, & Mooney, 2015; Lawrence, Tits, & Van Dooren, 2000; Safonov, 1982; Tsao et al., 1998; Zhou, Doyle, & Glover, 1996). However, all of the aforementioned studies assume a single-agent system, which is limited in scope considering the many networked real-world system applications. The stability margin analysis for networked multi-agent systems (MAS) is challenging considering the complexity caused by the interplay of communication structure and agents' dynamics. In this paper, we develop a framework to analyze the phase and gain margins of networked MAS, which is a first attempt in the literature per the knowledge of the authors.

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Networked MAS have attracted extensive attention due to their wide applications in mobile robots, unmanned air vehicles (UAVs), sensor networks, and satellite formation (Beard, McLain, & Goodrich, 2002; Bender, 1991; Ren, Beard, & Atkins, 2005; Russell Carpenter, 2002). In general, networked MAS can be classified into two categories: leaderless consensus systems and leader follower tracking systems, depending on whether a leader exists or not (Movric & Lewis, 2013; Zhang, Feng, Yang, & Liang, 2014). For the leaderless consensus problem, or commonly referred as the *cooperative regulator problem*, distributed controllers have been designed for agents to achieve consensus by utilizing the information received from their immediate neighbors in the communication network (Fax & Murray, 2004; Jadbabaie, Lin, & Morse, 2003; Ren, Beard, & Atkins, 2007; Zhang, Lewis, & Qu, 2011). Consensus value is usually a function of agents' initial states dependent on network topology and agents' dynamics. For the leader follower consensus problem, or called *cooperative tracking problem*, a leader communicates to at least one agent, and all agents are controlled to synchronize their states to the state trajectory generated by the leader (Lewis, Zhang, Hengster-Movric, & Das, 2013; Movric & Lewis, 2013; Zhang et al., 2014; Zhang, Lewis, & Das, 2011). Optimal controller design for cooperative tracking systems has been studied in Zhang et al. (2014), Zhang, Lewis, and Das (2011), Zhang, Lewis, and Qu (2011). In particular, Zhang, Lewis, and Das (2011) developed a local LQR design for agents with identical linear time-invariant dynamics, and showed that the local LQR design guarantees unbounded synchronization regions on arbitrary digraphs containing a spanning tree. Zhang et al. (2014) developed an optimality criterion that promises the existence of a global optimal controller under certain conditions by the inverse optimality method. Although some properties of the cooperative tracking systems, e.g., optimality and stability, have been studied in the aforementioned works, the analysis of robustness in the presence of perturbations is still missing. In addition, the effect of communication graph topology on the robustness properties also remains to be investigated.

This paper studies the robustness properties of networked cooperative tracking systems using the Lyapunov analysis and the algebraic graph theory. The contributions of this paper are fourfold. First, phase and gain margins of networked cooperative tracking systems are derived in closed-form, by analyzing the stability conditions of perturbed systems. Second, graph topology characteristics relating to stability margins are developed, through an eigen-analysis. Third, the upper bounds of phase and gain margins for MAS of general communication graph topology are obtained, by integrating the robustness analysis with the graph topology analysis. Fourth, we prove that the directed tree topology is among the most robust communication graph topology that promises the best stability margin performances, which can be as good as the ones in a single-agent LQR system.

This paper is organized as follows. Section 2 introduces preliminaries and definitions. Section 3 formulates the perturbed cooperative tracking systems. Section 4 investigates stability margins of networked MAS. Section 5 analyzes the graph topology characteristics relating to stability margins, and Section 6 concludes the paper.

2. Preliminaries

We introduce notations and definitions in Section 2.1 and preliminaries on communication graph in Section 2.2 to facilitate the analysis in this paper.

2.1. Notations and definitions

(1) The space $\mathcal{L}_2^n$ is defined as the set of all piecewise continuous functions $x : [0, \infty) \to \mathbb{R}^n$ such that

$$\|x\|_{\mathcal{L}_2} = \left(\int_0^\infty x^T(t)x(t)dt \right)^{\frac{1}{2}} < \infty,$$

i.e., the space $\mathcal{L}_2^n$ defines the set of all square-integrable functions $x(t)$ (Khalil, 2002).

(2) The extension $\mathcal{L}_{2e}^n$ of $\mathcal{L}_2^n$ is defined as

$$\mathcal{L}_{2e}^n = \{x | x_\tau \in \mathcal{L}_2^n, \forall \tau \geqslant 0\},$$

where $x_\tau(t)$ is a truncation of $x(t)$ defined as

$$x_\tau(t) = \begin{cases} x(t) & 0 \leqslant t \leqslant \tau, \\ 0 & t > \tau. \end{cases}$$

(3) Define the inner-product $\langle x, y\rangle$ for piecewise continuous functions $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ as (Khalil, 2002; Safonov & Athans, 1977)

$$\langle x, y\rangle = \int_0^\infty x^T(t)y(t)dt.$$

(4) An *operator* $\mathscr{P}$ is defined as a function that acts on vector spaces and maps a vector of functions into another vector of functions. For example, a dynamical system can be viewed as an operator which maps input time-functions into output time-functions, i.e., $y(t) = \mathscr{P}u(t)$, where $y(t)$ and $u(t)$ are the output and input time-functions, respectively, and $\mathscr{P}u(t)$ means that the operator $\mathscr{P}$ acts on the input $u(t)$ (Safonov, 1982; Safonov & Athans, 1977).

(5) An operator $\mathscr{P}$ with $\mathscr{P}\mathbf{0} = \mathbf{0}$, where $\mathbf{0}$ is a zero vector and $\mathscr{P}\mathbf{0}$ represents that the operator $\mathscr{P}$ acts on a zero vector, is said to have finite gain if there exists a constant $k < \infty$ such that $\|\mathscr{P}x\| < k\|x\|$ for all square-integrable x (Safonov & Athans, 1977).

2.2. Communication graph

Consider a group of N agents connected by a weighted communication graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. Here $\mathcal{N}$ is the set of agents, $\mathcal{N} = \{1, 2, \ldots, N\}$, and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of edges. An edge starting from agent j to agent i is denoted as (j, i), which means that information flows from j to i. The graph adjacency matrix is denoted as $\mathcal{A} = [a_{ij}]$, where a_{ij} is the weight of edge (j, i) and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. It is assumed that the graph is simple, i.e., there is no repeated edge or self-loop. Denote the set of neighbors of agent i as $\mathcal{N}_i$, i.e., $\mathcal{N}_i = \{j | (j, i) \in \mathcal{E}\}$. Denote the in-degree matrix as $\mathcal{D}$, i.e., $\mathcal{D} = \text{diag}(d_1, d_2, \ldots, d_N)$, where diag$(\cdot)$ means placing the elements in the parenthesis as diagonal entries, and d_i is the ith row sum of $\mathcal{A}$: $d_i = \sum_j a_{ij}$. Define the graph Laplacian matrix as L, $L = \mathcal{D} - \mathcal{A}$, which has all row sums equaling zero.

3. Problem formulation

Consider a group of N agents distributed on a communication graph $\mathcal{G}$ with identical perturbed linear dynamics

$$\dot{x}_i = Ax_i + B\mathscr{P}u_i, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the perturbed state vector, $u_i \in \mathbb{R}^m$ is the control input vector, and $i = 1, 2, \ldots, N$. A and B are the drift and input matrices, respectively. The perturbation $\mathscr{P}$ is a finite-gain operator with $\mathscr{P}\mathbf{0} = \mathbf{0}$. The MAS with homogeneous perturbations have wide applications including, e.g., mobile robots tracking and UAV formation. Consider, for example, a group of robots cooperatively fulfill a task in a confined space. With the friction formulated as perturbation, the effect of such perturbation on each robot is approximately the same, i.e., homogeneous perturbations.

Assumption 1. The pair (A, B) is stabilizable.

Assumption 2. The graph $\mathcal{G}$ contains at least one spanning tree whose root node can observe the leader's state information.

The dynamics of the leader, indexed with 0, is given by

$$\dot{x}_0 = Ax_0, \tag{2}$$

where $x_0 \in \mathbb{R}^n$ is the state vector of the leader. The communication between the leader and agent i is captured by the pinning gain $g_i \geqslant 0$. $g_i > 0$ means that the leader's state information can be observed by agent i. Denote the pinning matrix as G, then $G = \mathrm{diag}(g_1, g_2, \dots, g_N) \in \mathbb{R}^{N\times N}$. Note that the leader dynamics is not required to be stable. Denote λ_i $(i = 1, 2, \dots, N)$ as the eigenvalues of $L + G$ and $Re\{\lambda_i\}$ as the real part of λ_i.

In cooperative tracking systems, all agents aim to synchronize their states to the state trajectory of the leader, i.e., $\lim_{t\to\infty}(x_i(t) - x_0(t)) = \mathbf{0}$ for all $i = 1, 2, \dots, N$, (Lewis et al., 2013; Zhang, Lewis, & Das, 2011). Define the neighborhood synchronization error for agent i as

$$\varepsilon_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + g_i(x_i - x_0). \tag{3}$$

We consider a state feedback control protocol for each agent i to be

$$u_i = cK\varepsilon_i, \tag{4}$$

where $c > 0$ is a scalar coupling gain and $K \in \mathbb{R}^{m\times n}$ is the feedback control gain matrix. These controllers are distributed in the sense that each agent only uses the local tracking error ε_i.

Define the global synchronization error of the perturbed systems as $\delta = x - \underline{x_0}$, where x is the global state, $x = [x_1^T, x_2^T, \dots, x_N^T]^T \in \mathbb{R}^{nN}$, and $\underline{x_0} = \mathbf{1}_N \otimes x_0 \in \mathbb{R}^{nN}$. $\mathbf{1}_N$ is an N-vector of ones, and $\otimes$ is the Kronecker product. The dynamics of the global synchronization error is

$$\begin{aligned}\dot{\delta} &= \dot{x} - \dot{\underline{x_0}} = (I_N \otimes A - c(L+G) \otimes B\mathscr{P}K)\delta \\ &= A_c\delta,\end{aligned} \tag{5}$$

where $A_c = I_N \otimes A - c(L+G) \otimes B\mathscr{P}K$.

We consider the following local LQR feedback gain for each agent

$$K = R^{-1}B^TP. \tag{6}$$

Here, P is the positive definite solution of the control algebraic Riccati equation (ARE),

$$A^TP + PA + Q - PBR^{-1}B^TP = \mathbf{0}, \tag{7}$$

where $Q = Q^T \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix and $R = R^T \in \mathbb{R}^{m\times m}$ is a positive definite matrix. As proved in Zhang, Lewis, and Das (2011), such a local LQR design makes the cooperative tracking systems asymptotically stable if there is no perturbation and $c \geq \frac{1}{2}Re\{\lambda_i\}$, $\forall i = 1, 2, \dots, N$.

In this paper, we are interested in the robustness performances of the cooperative tracking systems of local LQR design. It is known that for a single-agent system, the local LQR design guarantees a $\pm 60^\circ$ phase margin, a 50% gain reduction, and an infinite gain margin (Lewis, Vrabie, & Syrmos, 2012; Safonov & Athans, 1977). However, the stability margin analysis for networked MAS is still an open question. This paper aims to answer this question by deriving closed-form phase and gain margin expressions and characterizing the effect of communication graph topology on the robustness performances of networked MAS. An example of the perturbed four-agent systems with local LQR design is shown in Fig. 1, where $H(s)$ is the transfer matrix of the perturbation $\mathscr{P}$ and $i = 1, 2, 3, 4$.

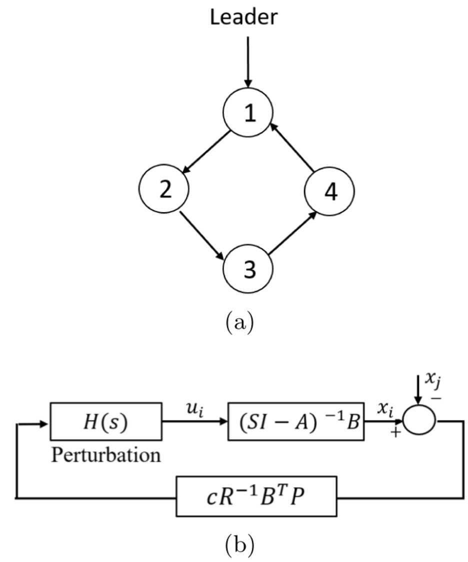

Fig. 1. An example of perturbed networked MAS with communication topology (a) and local perturbed system (b).

4. Robustness analysis

This section studies the robustness of cooperative tracking systems through investigating stability conditions on the perturbation $\mathscr{P}$ in the perturbed systems.

Lemma 1 (*Lewis et al., 2013, Lemma 3.3*)**.** *Under Assumption 2, the matrix $L + G$ is nonsingular. Moreover, the eigenvalues λ_i satisfy $Re\{\lambda_i\} > 0$, for all $i = 1, 2, \dots, N$.*

The next theorem provides a necessary and sufficient condition for the stability of the perturbed cooperative tracking systems. The proof follows a similar development as in Zhang, Lewis, and Das (2011), but is developed for perturbed systems. The proof is omitted here due to the page limit.

Theorem 1. *The global synchronization error of the perturbed system (5) is asymptotically stable if and only if the following systems*

$$\dot{\xi}_i = (A - c\lambda_i B\mathscr{P}K)\xi_i, \tag{8}$$

are asymptotically stable for all $i = 1, 2, \dots, N$.

Theorem 1 shows that the stability of the global system δ depends on the stability of the local systems ξ_i in (8). The next theorem shows that the stability of systems ξ_i depends only on the real part of the dynamics, i.e., $A - cRe\{\lambda_i\}B\mathscr{P}K$. The proof follows a similar development as in Zhang, Lewis, and Das (2011) and is omitted here.

Theorem 2. *The global synchronization error of the perturbed system (5) is asymptotically stable if and only if the following systems,*

$$\dot{\zeta}_i = (A - cRe\{\lambda_i\}B\mathscr{P}K)\,\zeta_i, \tag{9}$$

are asymptotically stable for all $i = 1, 2, \dots, N$.

4.1. Phase and gain margins

In this subsection, we first find conditions on the perturbation $\mathscr{P}$ that guarantee the stability of ζ_i. The phase and gain margins of the cooperative tracking systems then follow. The proof of Theorem 3 is in Appendix A.

Theorem 3. *Consider the cooperative tracking systems in (1)–(7). If the perturbation $\mathscr{P}$ satisfies the following inequality*

$$\langle \bar{u}_i, (2cRe\{\lambda_i\}\mathscr{P} - I)R^{-1}\bar{u}_i \rangle \geqslant 0 \tag{10}$$

for all $\bar{u}_i \in \mathbb{R}^m$ and $i = 1, 2, \ldots, N$, then

(1) the following inequality holds,

$$\zeta_i^T(0)P\zeta_i(0) \geqslant \langle \zeta_i, Q\zeta_i \rangle; \tag{11}$$

(2) if additionally, $[Q^{\frac{1}{2}}, A]$ is detectable, then the systems ζ_i in (9) are asymptotically stable.

The following theorem derives the condition on the perturbation $\mathscr{P}$ in the frequency domain, for the case when $\mathscr{P}$ is a linear operator. The proofs of Theorem 4 and Corollary 5 are in Appendices B and C, respectively.

Theorem 4. *Let the perturbation $\mathscr{P}$ be a linear time-invariant operator $\mathscr{H}$ with a finite-gain and a proper transfer function $H(j\omega)$. If*

$$2cRe\{\lambda_i\}H(j\omega)R^{-1} + 2cRe\{\lambda_i\}R^{-1}H^*(j\omega) - R^{-1} \geqslant 0 \tag{12}$$

holds for all ω and $i = 1, 2, \ldots, N$, and $[Q^{\frac{1}{2}}, A]$ is detectable, then the systems ζ_i in (9) are asymptotically stable.

Corollary 5. *Let the matrix R in (6) be diagonal, i.e., $R = diag(r_1, r_2, \ldots, r_m)$, where r_l $(l = 1, 2, \ldots, m)$ are diagonal elements of the matrix R. Let the perturbation $\mathscr{P}$ be diagonal such that $\mathscr{P}u_i = [(\mathscr{P}_1 u_{i,1})^T, (\mathscr{P}_2 u_{i,2})^T, \cdots, (\mathscr{P}_m u_{i,m})^T]^T$. If each element of the perturbation, $\mathscr{P}_l$, is linear time-invariant with proper transfer function $H_l(j\omega)$, and*

$$Re\{H_l(j\omega)\} \geqslant \frac{1}{2cRe\{\lambda_i\}}$$

holds for all $i = 1, 2, \ldots, N$, then the systems ζ_i in (9) are asymptotically stable.

Denote $\underline{\lambda_R}$ as the minimum value of $Re\{\lambda_i\}$ for all $i = 1, 2, \ldots, N$, i.e., $\underline{\lambda_R} = min_{i\in\mathcal{N}} Re\{\lambda_i\}$. Guaranteed phase and gain margins of cooperative tracking systems are expressed in closed-form in Theorem 6.

Theorem 6. *Let the matrix R and the perturbation $\mathscr{P}$ be diagonal. The cooperative tracking systems (1)–(7) have a guaranteed phase margin $\pm arccos\frac{1}{2c\underline{\lambda_R}}$, a guaranteed gain reduction tolerance $\frac{1}{2c\underline{\lambda_R}}$, and an infinite gain margin.*

Proof. This can be derived naturally from Corollary 5, by expressing $H_l(j\omega)$ in its polar form.

Remark 1. Compared to the single-agent LQR system, which has a $\pm 60°$ phase margin, a 50% gain reduction, and an infinite gain margin, the stability margins of the multi-agent cooperative tracking systems depend on characteristics of the communication graph topology, i.e., $\underline{\lambda_R}$, and the coupling gain c. In particular, given a fixed c, a larger $\underline{\lambda_R}$ leads to better phase and gain margins, and given a fixed communication graph, a larger c results in better robustness performance.

In the next section, we study properties of $\underline{\lambda_R}$ to further explore the effect of communication graph topology on the stability margins of networked MAS.

5. Graphical results on phase and gain margins

In this section, we first study the range of $\underline{\lambda_R}$ following an algebraic graph theory analysis. We show that $0 < \underline{\lambda_R} \leqslant 1$ holds for general communication graph topology, and then prove that the directed tree graph permits the maximum $\underline{\lambda_R}$, i.e., $\underline{\lambda_R} = 1$. Finally, we provide graphical results on the guaranteed phase and gain margins.

5.1. $\underline{\lambda_R}$ In communication graph topology

We denote $\mathbf{Z}$ as the set of all real square matrices whose off-diagonal elements are all non-positive.

Assumption 3. The communication graph $\mathcal{G}$ is an unweighted graph, i.e., $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $g_i = 1$ if agent i can observe the leader.

The next theorem investigates the maximum and minimum values of $\underline{\lambda_R}$ for general communication graph topology. The proof is in Appendix D.

Theorem 7. *For any communication graph topology satisfying Assumptions 2 and 3, the following inequality holds,*

$$0 < \underline{\lambda_R} \leqslant 1. \tag{13}$$

Theorem 7 provides the maximum and minimum values of $\underline{\lambda_R}$ for cooperative tracking systems of general graph topology. Theorem 8 finds a class of special graph topology that leads to the maximum $\underline{\lambda_R}$ among all possible communication graphs satisfying Assumptions 2 and 3. The proof of Theorem 8 is in Appendix E.

Theorem 8. *For the cooperative tracking systems of a directed tree communication graph $\mathcal{G}$, $\underline{\lambda_R} = 1$ under Assumptions 2 and 3.*

Comparing with general graph topology where $\underline{\lambda_R} \leqslant 1$ according to Theorem 7, it is straightforward to conclude that the directed tree graph promises the maximum $\underline{\lambda_R}$ among all possible communication graphs.

5.2. Graphical results on phase and gain margins

Theorem 9 finds the upper bounds of stability margins for networked cooperative tracking systems of general graph topology. As in a directed tree graph, $\underline{\lambda_R} = 1$ and $c \geq \frac{1}{2}\underline{\lambda_R}$ always holds if $c = 1$ is selected, Theorem 9 can be derived naturally from Theorems 6–8.

Theorem 9. *For the cooperative tracking systems (1)–(7), the directed tree communication graph is among the most robust graph topology that promises the best stability margin performances: $\pm arccos\frac{1}{2c}$ phase margin, $\frac{1}{2c}$ gain reduction tolerance, and infinite gain margin. The performances are as good as the single-agent LQR system when $c = 1$, i.e., $\pm 60°$ phase margin, 50% gain reduction, and infinite gain margin.*

Remark 2. Theorem 9 shows that among all possible communication graphs, directed tree is one of the special graphs that promise the best stability margins, which are as good as the ones in a single-agent LQR system when $c = 1$. This result can be understood intuitively as follows. In the directed tree graph, the control of each agent is uniquely decided by its root agent, but not any other agents. Each agent synchronizes to its root node based on the state information received from the root node. This architecture is equivalent to that of a single-agent LQR system. As each agent behaves the same as the single-agent LQR system, the robustness of the whole cooperative system in terms of guaranteed phase and gain margins is also equivalent to the single-agent LQR system.

6. Concluding remarks

This paper studies the phase and gain margins of networked cooperative tracking systems. We find that the robustness of the cooperative tracking systems is dependent upon the communication graph topology. In particular, both phase and gain margins are functions of $\underline{\lambda_R}$, which is the minimum real part of the eigenvalues of $L+G$. Motivated by this connection, the ranges of $\underline{\lambda_R}$ for general communication graphs are further studied. We find that $0 < \underline{\lambda_R} \leqslant 1$ holds for any possible communication graphs, and $\underline{\lambda_R} = 1$ if the communication graph is a directed tree. Graphical results on the bounds of phase and gain margins are then analyzed through connecting the robustness and graph analysis. In particular, we show that the directed tree is among the most robust graph topology that promises the best stability margins, which are as good as the ones in a single-agent LQR system when $c = 1$.

Appendix A. Proof of Theorem 3

Denote $\zeta_{i\tau}$ as a truncation of ζ_i, i.e.,

$$\zeta_{i\tau}(t) = \begin{cases} \zeta_i(t) & 0 \leqslant t \leqslant \tau, \\ 0 & t > \tau. \end{cases}$$

Combining (9) and the feedback gain K in (6), one has

$$\begin{aligned} &\zeta_i^T(0)P\zeta_i(0) \\ &= \zeta_i^T(\tau)P\zeta_i(\tau) - \int_0^\tau \frac{d}{dt}\left(\zeta_i^T(t)P\zeta_i(t)\right)dt \\ &= \zeta_i^T(\tau)P\zeta_i(\tau) - \int_0^\tau 2\zeta_i^T(t)P\dot{\zeta}_i(t)dt \\ &\geqslant -\int_0^\tau 2\zeta_i^T(t)P\dot{\zeta}_i(t)dt \\ &= -\int_0^\tau 2\zeta_i^T(t)P\left(A - cRe\{\lambda_i\}B\mathscr{P}K\right)\zeta_i(t))dt \\ &= -2\langle \zeta_{i\tau}, P(A - cRe\{\lambda_i\}B\mathscr{P}R^{-1}B^TP)\zeta_{i\tau}\rangle \\ &= \langle \zeta_{i\tau}, (Q - PBR^{-1}B^TP + 2cRe\{\lambda_i\}PB\mathscr{P}R^{-1}B^TP)\zeta_{i\tau}\rangle \\ &= \langle \zeta_{i\tau}, Q\zeta_{i\tau}\rangle + \langle \zeta_{i\tau}, \left(PB(2cRe\{\lambda_i\}\mathscr{P} - I)R^{-1}B^TP\right)\zeta_{i\tau}\rangle. \end{aligned}$$

Let $\Pi_i = (2cRe\{\lambda_i\}\mathscr{P} - I)\,R^{-1}$ and $\bar{u}_i = B^TP\zeta_{i\tau}$. The following inequality holds,

$$\begin{aligned} &\zeta_i^T(0)P\zeta_i(0) - \langle \zeta_{i\tau}, Q\zeta_{i\tau}\rangle \geqslant \langle \zeta_{i\tau}, PB\Pi_iB^TP\zeta_{i\tau}\rangle \\ &= \langle B^TP\zeta_{i\tau}, \Pi_iB^TP\zeta_{i\tau}\rangle = \langle \bar{u}_i, \Pi_i\bar{u}_i\rangle. \end{aligned} \tag{14}$$

If $\mathscr{P}$ satisfies (10), then the following inequality holds according to (14),

$$\zeta_i^T(0)P\zeta_i(0) \geqslant \langle \zeta_{i\tau}, Q\zeta_{i\tau}\rangle.$$

Taking the limit $\tau \to \infty$, then the first statement in (11) follows.

Note that $\zeta_i^T(0)P\zeta_i(0) \geqslant \langle \zeta_i, Q\zeta_i\rangle$ implies that $\langle \zeta_i, Q\zeta_i\rangle$ is bounded. If additionally, $[Q^{\frac{1}{2}}, A]$ is detectable, then ζ_i is square-integrable (Safonov & Athans, 1977). Because $\mathscr{P}$ has a finite gain and ζ_i is square-integrable, $\dot{\zeta}_i$ is also square-integrable. Since both ζ_i and $\dot{\zeta}_i$ are square-integrable, ζ_i is asymptotically stable (Safonov & Athans, 1977), which proves the second statement.

Appendix B. Proof of Theorem 4

From (12) and the Parseval's theorem (Desoer & Vidyasagar, 1975), we have

$$\begin{aligned} &\langle \bar{u}_i, (2cRe\{\lambda_i\}\mathscr{P} - I)\,R^{-1}\bar{u}_i\rangle \\ &= \frac{1}{2}\Big(\langle \bar{u}_i, (2cRe\{\lambda_i\}\mathscr{P} - I)\,R^{-1}\bar{u}_i\rangle \\ &\quad + \langle (2cRe\{\lambda_i\}\mathscr{P} - I)\,R^{-1}\bar{u}_i, \bar{u}_i\rangle\Big) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{U}_i^*(j\omega)\Big(cRe\{\lambda_i\}\big(H(j\omega)R^{-1} \\ &\quad + R^{-1}H^*(j\omega)\big) - R^{-1}\Big)\bar{U}_i(j\omega)d\omega \\ &\geqslant 0, \end{aligned}$$

where $\bar{U}_i(j\omega)$ is the Fourier transform of $\bar{u}_i$, $\bar{U}_i^*(j\omega)$ is the Hermitian of $\bar{U}_i(j\omega)$, and $H^*(j\omega)$ is the Hermitian of $H(j\omega)$.

As $\langle \bar{u}_i, (2cRe\{\lambda_i\}\mathscr{P} - I)\,R^{-1}\bar{u}_i\rangle \geqslant 0$, the systems in (9) are asymptotically stable from Theorem 3.

Appendix C. Proof of Corollary 5

Taking $H(j\omega) = \text{diag}(H_1(j\omega), H_2(j\omega), \ldots, H_m(j\omega))$, where $l = 1, 2, \ldots, m$, then one has

$$\begin{aligned} &2cRe\{\lambda_i\}\left(r_l^{-1}\left(H_l(j\omega) + H_l^*(j\omega)\right)\right) - r_l^{-1} \\ &= r_l^{-1}\left(2cRe\{\lambda_i\}Re\{H_l(j\omega)\} - 1\right) \geqslant 0 \end{aligned}$$

for all $l = 1, 2, \ldots, m$.

As such, the condition (12) is satisfied. According to Theorem 4, the systems in (9) are asymptotically stable.

Appendix D. Proof of Theorem 7

The lower limit $\underline{\lambda_R} > 0$ is straightforward from Lemma 1. We now show that $\underline{\lambda_R} \leqslant 1$ holds by using a contradiction method.

Assume $\underline{\lambda_R} > 1$ under contradiction. Then $Re\{\lambda_i\} > 1$ holds for all $i = 1, 2, \ldots, N$. With this assumption, there exists a real number $\beta > 1$, such that $Re\{\lambda_i\} - \beta > 0$ holds for all $i = 1, 2, \ldots, N$. Denote $\alpha_i = \lambda_i - \beta$. α_i is then an eigenvalue of the matrix $L + G - \beta I_N$, i.e.,

$$(L + G - \beta I_N)\omega_i = (\lambda_i - \beta)\omega_i = \alpha_i\omega_i,$$

where ω_i is the ith eigenvector of $L + G$, i.e., $(L + G)\omega_i = \lambda_i\omega_i$. Because $\underline{\lambda_R} > 1$, $Re\{\alpha_i\} > 0$ holds for all $i = 1, 2, \ldots, N$. Next we show that there exists at least one α_i such that $Re\{\alpha_i\} \leqslant 0$, which contradicts the assumption that $\underline{\lambda_R} > 1$.

Under Assumption 3, $\beta I_N - G$ has all positive diagonal elements. Denote the minimum diagonal element of $\beta I_N - G$ as γ, then $\gamma > 0$, and $\beta I_N - G$ can be rewritten as $\beta I_N - G = \gamma I_N + E$, where E is an $N \times N$ diagonal matrix with non-negative diagonal elements. As such, the matrix $L + G - \beta I_N$ can be rewritten as

$$L + G - \beta I_N = L - (\beta I_N - G) = L - \gamma I_N - E.$$

Note that the minimum eigenvalue of the Laplacian matrix L is 0. As such, the minimum eigenvalue of $L - \gamma I$ is negative. As a result, there exists at least one principal minor of the matrix $L - \gamma I$ that is negative (Fiedler & Ptak, 1962).

Denote $|M|$ as the negative principal minor of $L - \gamma I$ with the minimum order, i.e., all principal minors of $L - \gamma I$ that have lower orders are positive. Denote the order of $|M|$ as $k (k \leqslant N)$. Assume M has the following form

$$M = \begin{bmatrix} m & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where m is a scalar, the row vector $M_{12} \in \mathbb{R}^{1\times(k-1)}$, the column vector $M_{21} \in \mathbb{R}^{(k-1)\times 1}$, and the square matrix $M_{22} \in \mathbb{R}^{(k-1)\times(k-1)}$. Since $|M| < 0$, one has

$$\begin{vmatrix} m & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = (m - M_{12}M_{22}^{-1}M_{21})|M_{22}| < 0.$$

Since the matrix M_{22} is of $k-1$ order, we have $|M_{22}| > 0$. As such,

$$m - M_{12}M_{22}^{-1}M_{21} < 0. \tag{15}$$

Then we consider the principal minor of $L - \gamma I - E$, $|M - \bar{E}|$, where $\bar{E}$ is a submatrix of E. $M - \bar{E}$ has the following form,

$$M - \bar{E} = \begin{bmatrix} m - e & M_{12} \\ M_{21} & M_{22} - E_{22} \end{bmatrix}, \tag{16}$$

where $e \geqslant 0$ is a scalar, and E_{22} is a $k-1$ by $k-1$ square diagonal matrix with non-negative elements. The determinant of $M - \bar{E}$ is

$$\begin{aligned} |M - \bar{E}| &= \begin{vmatrix} m - e & M_{12} \\ M_{21} & M_{22} - E_{22} \end{vmatrix} \\ &= \left((m - e) - M_{12}(M_{22} - E_{22})^{-1}M_{21}\right)|M_{22} - E_{22}|. \end{aligned} \tag{17}$$

Since $|M - \bar{E}|$ is a principal minor of $L - \gamma I - E$, we can determine the sign of the eigenvalues of $L - \gamma I - E$, i.e., α_i, by checking the sign of $|M - \bar{E}|$. To do so, we consider two cases: (1) $|M_{22} - E_{22}| \leqslant 0$, and (2) $|M_{22} - E_{22}| > 0$. For the first case, it is straightforward that there exists at least one $\alpha_i \leqslant 0$, which contradicts the assumption that $\underline{\lambda_R} > 1$. For the second case, let us prove that

$$(m - e) - M_{12}(M_{22} - E_{22})^{-1}M_{21} < 0, \tag{18}$$

which leads to the result that $|M - \bar{E}| < 0$ according to (17). $|M - \bar{E}| < 0$ indicates that there exists at least one $\alpha_i < 0$, which contradicts the assumption $\underline{\lambda_R} > 1$. Noticing that e is a non-negative number, it is clear that if the following equation holds,

$$m - M_{12}(M_{22} - E_{22})^{-1}M_{21} < 0, \tag{19}$$

then (18) holds.

Compare (15) and (19). Because (15) holds, to show (19), we only need to show that

$$M_{22}^{-1} \leqslant (M_{22} - E_{22})^{-1}. \tag{20}$$

Here "$\leqslant$" is element by element comparison. Note that $M_{22} \in \mathbf{Z}$ and $(M_{22} - E_{22}) \in \mathbf{Z}$. As such, $M_{22}^{-1} \geqslant 0$ and $(M_{22} - E_{22})^{-1} \geqslant 0$ hold. Because $(M_{22} - E_{22})^{-1} = M_{22}^{-1} + M_{22}^{-1}E_{22}(M_{22} - E_{22})^{-1}$, Eq. (20) holds.

Appendix E. Proof of Theorem 8

For a directed tree, the Laplacian matrix is a lower triangular matrix, i.e.,

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -a_{21} & 1 & 0 & \cdots & 0 \\ -a_{31} & -a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ -a_{N1} & -a_{N2} & -a_{N3} & \cdots & 1 \end{bmatrix},$$

where $a_{ij} = 1$ if and only if $a_{ik} = 0$ $\forall k \neq j$. As such, we have $\underline{\lambda_R} = 1$.

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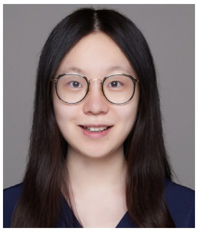

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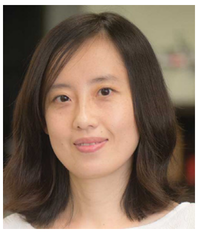

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