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Maximal Gain and Phase Margins Attainable by PID Control

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***Abstract*—In this paper, we study the gain and phase margins achievable by PID controllers in stabilizing linear time-invariant (LTI) systems. The problem under consideration amounts to determining the largest ranges of gain and phase variations such that there exists a single PID controller capable of stabilizing all the plants within the variation ranges. We consider low-order systems, notably the first- and second-order systems. For each class of these systems, we derive explicit expressions of the maximal gain and phase margins achievable. The results demonstrate analytically how a plant's unstable poles and nonminimum phase zeros may confine the maximal gain and phase margins attainable by PID control, which lead to a number of useful observations. First, for minimum phase systems, we show that the maximal gain and phase margins achievable by PID controllers coincide with those by general LTI controllers. Second, for nonminimum phase systems, we show that LTI controllers perform no better than twice than PID controllers, in the sense that the maximal gain and phase margins achievable by general LTI controllers are within a factor of two of those by PID controllers, whereas the former is measured on a logarithmic scale and latter on a linear scale. Finally, we show that PID and PD controllers achieve the same maximal margins, indicating that integral control is immaterial in improving a system's robustness in feedback stabilization. These results thus provide useful insights into PID control, and from a system robustness perspective, offer an interpretation on the effectiveness of PID controllers.**

I. Introduction

The primary goal of feedback control system design is to maintain stability and performance robustness in the presence of uncertainties and disturbances. Various criteria can be used to measure robustness, of which the gain and phase margin are two conspicuous measures of robustness employed in the classical control design theory and practice. These measures define the relative stability degree, or the distance from instability of a system when subject to gain or phase perturbations in the system's frequency response. In spite of a variety of available robustness measures thanks to the developments of robust optimal control theory [1], including, e.g., gap metric and structured singular value, for their simplicity and effectiveness the gain and phase margins continue to be the essential attributes in the contemporary robustness analysis (see, e.g., [2], [3], [4], [5], [6], [7] and the references therein). Accordingly, the maximal gain and phase margins provide fundamental limits of robust feedback stabilization, serving to determine the largest ranges of gain and phase variations so that a system can be robustly stabilized. In turn, they also serve as indicators on how difficult a feedback system is to control.

A vast volume of work is in existence on gain margin and phase margin problems, and various time- and frequency-domain analysis techniques have been explored for linear time-invariant (LTI) systems (see, e.g., [8], [9], [10], [11] and the references therein). Of these, it has been long well-known that linear quadratic (LQ) state-feedback regulators are capable of achieving satisfactory stability margins for continuous-time feedback systems. In particular, when reaching optimality in the limit, optimal LQ controllers can produce an infinite gain margin and a $\pm 60^\circ$ phase margin [12], a property later found to hold in general for multi-input, multi-output systems [2]. It is also known, however, that when extended to LQG optimal control, these desirable margins may vanish [13]. Furthermore, as shown in [14], there exists no counterpart of this property to discrete-time systems, but generally a limit to gain and phase margins achievable by LQ control.

Our focus in this paper dwells on the gain and phase margins attainable by PID control. Earlier results on the gain and phase margin enhancement by PID controllers include, e.g., [15], [16], where simple formulas and schemes were devised to tune PID controller parameters such that the required gain and phase margins can be fulfilled. As a step further beyond, in the present paper we seek to quantify analytically the maximal gain and phase margins achievable by PID controllers. The problem amounts to determining the variation ranges of the system's gain and phase so that a single PID controller can robustly stabilize the entire family of the plants for all possible variations within the ranges, thus furnishing fundamental limits of robustness as measured by the gain and phase margins. For this purpose, we consider low-order systems. Specifically, we consider first- and second-order unstable plants, with one or two unstable poles and possibly nonminimum phase zeros. It is useful to note that the gain and phase margin

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maximization problems are meaningful only for unstable plants; sheer by their nature, the problems become moot for stable plants: the maximal gain margin is infinitely large and the maximal phase margin is equal to 180^o. It is also worth pointing out that with its limited degree of freedom, PID control is effective essentially for first- and second-order plants [17], [18]; in fact, in industrial control systems, where PID control is most prevalent, one typically employs first- and second-order plant models. This simplicity in the plant model, albeit somewhat restrictive, renders the problems analytically tractable. On the other hand, high-order plants will generally pose a challenging parametric, nonlinear optimization problem, to which such analytical results become unavailable.

Our contribution can be summarized as follows. In Section II, we introduce the gain and phase margin maximization problems achievable by PID control, including the general PID controller structure and the subclasses of proportional, PI, and PD controllers. The gain and phase margin maximization problems will then be formulated and tackled as parametric nonlinear programming problems, for first-order systems in Section III and second-order systems in Section IV. Section V shows the maximal gain and phase margins achieved by PI controllers and provides a special case for second-order systems. In each case, we derive explicit expressions of the maximal achievable gain and phase margins, which demonstrate explicitly how plant unstable poles and nonminimum phase zeros may limit the margins achievable. In an intuitively plausible finding, our results throughout show that the integral control tends to reduce the gain and phase margins. As such, PID and PD controllers achieve the same maximal margins. The paper concludes in Section VI.

Due to space constraint, throughout this paper we omit all the technical proofs.

II. Preliminaries and Problem Formulation

We consider the feedback system depicted in Fig.1, in which $P(s)$ represents the plant model, and $K(s)$ a finite-dimensional stabilizing LTI controller. Suppose that $P(s)$ is stabilized by $K(s)$. Then if $P(s)$ is perturbed by uncertainties in the form of gain or phase within a small range, by continuity, the controller $K(s)$ can still stabilize the plant despite the variations. But how large may the gain or phase variation be tolerated, before the closed-loop system loses its stability?

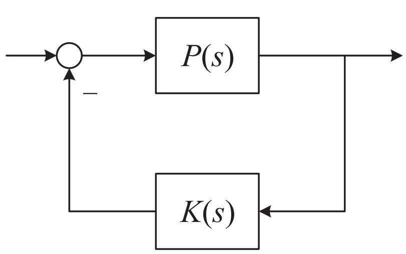

Fig. 1: Standard feedback control structure

The *gain/phase margin maximization problems* address the question above. Consider the family of plants

$$\mathscr{P}_\alpha = \left\{ \alpha P(s) \ : \ 1 \le \alpha < \mu \ \right\}. \tag{1}$$

For a given plant model $P(s)$, its maximal gain margin is defined as

$$k_M = \sup\{\mu : \text{There exists some } K(s) \text{ stabilizing } \alpha P(s), \ \forall \alpha \in [1, \mu)\}.$$

Similarly, consider the family of plants

$$\mathcal{Q}_\theta = \left\{ e^{-j\theta} P(s) \ : \ \theta \in [-\nu, \nu] \ \right\}. \tag{2}$$

The maximal phase margin is defined by

$$\theta_M = \sup\{\nu : \text{There exists some } K(s) \text{ stabilizing } e^{-j\theta} P(s), \ \forall \theta \in [-\nu, \nu]\}.$$

Stated in words, the maximal gain margin k_M and the maximal phase margin θ_M determine, respectively, the maximal ranges of gain and phase variations within which the families of $\mathscr{P}_\alpha$ and $\mathcal{Q}_\theta$ can be robustly stabilized by a LTI controller.

Remark 2.1: More generally, we may consider $\underline{\mu} \le \alpha \le \bar{\mu}$. Suppose that $\mu_0 P(s)$ can be stabilized for some $\mu_0 \in [\underline{\mu}, \bar{\mu}]$. We may define analogously

$$\bar{k}_M = \sup\{\bar{\mu} : \text{There exists some } K(s) \text{ stabilizing } \alpha P(s), \ \forall \alpha \in [\mu_0, \bar{\mu})\},$$

and

$$\underline{k}_M = \inf\{\underline{\mu} : \text{There exists some } K(s) \text{ stabilizing } \alpha P(s), \ \forall \alpha \in (\underline{\mu}, \mu_0]\}.$$

A more general notion of the maximal gain margin can then be defined as $\bar{k}_M / \underline{k}_M$.

Define the system's complementary sensitivity function by

$$T(s) = P(s) K(s) [1 + P(s) K(s)]^{-1}.$$

The following result, quoted from [19], [20], provides analytical expressions of k_M and θ_M, and shows that they can be determined by solving a standard $\mathcal{H}_\infty$ optimal control problem.

Proposition 2.1:

(i). If P is stable or minimum-phase, then the maximal gain margin $k_M = \infty$. Otherwise,

$$k_M = \left(\frac{\gamma_{\text{opt}} + 1}{\gamma_{\text{opt}} - 1} \right)^2, \tag{3}$$

(ii). If P is stable or minimum-phase, then the maximal phase margin $\theta_M = \pi$. Otherwise,

$$\theta_M = 2 \sin^{-1} \frac{1}{\gamma_{\text{opt}}}, \tag{4}$$

where

$$\gamma_{\text{opt}} = \inf \left\{ \|T(s)\|_\infty : \ K(s) \text{ stabilizes } P(s) \right\},$$

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with $\|T(s)\|_\infty$ being the $\mathcal{H}_\infty$ norm of the complementary sensitivity function $T(s)$.

Of particular interest in this paper are maximal robustness margins achievable by PID controllers; that is, $K(s) = K_{PID}(s)$, where

$$K_{PID}(s) = k_p + \frac{k_i}{s} + k_d s. \tag{5}$$

Correspondingly, the maximal gain and phase margins achievable by PID controllers are defined by

$$k_M^{PID} = \sup\{\mu : \text{There exists } K_{PID}(s) \text{ stabilizing } \alpha P(s)\ \forall \alpha \in [1,\mu)\},$$

$$\theta_M^{PID} = \sup\{\nu : \text{There exists } K_{PID}(s) \text{ stabilizing } e^{-j\theta} P(s)\ \forall \theta \in [-\nu,\nu]\}.$$

Likewise, it is of interest to consider subclasses of PID controllers, such as proportional, PI, and PD controllers, given by

$$K_P(s) = k_p, \quad K_{PI}(s) = k_p + \tfrac{k_i}{s}, \quad K_{PD}(s) = k_p + k_d s, \tag{6}$$

respectively. We shall denote the corresponding maximal gain and phase margins with the corresponding superscripts P, PI, and PD. In the sequel, at times it will also be more convenient to quantify the gain margins on the logarithmic scale, i.e., by $\log k_M$ and $\log k_M^{PID}$, respectively.

Throughout this paper we consider first- and second-order plants. This consideration stems partly from the fact that industrial processes are often modeled by low-order, and in fact, mostly first-order systems, and partly due to the limitation of PID controllers in controlling high-order dynamics. Indeed, in the latter vein, it is worth noting that PID control is known to be essentially limited to first- and second-order systems [18], [21], and that in general PID controllers may not be able to stabilize certain third- and higher-order unstable systems.

III. First-Order Unstable Systems

In this section, we provide explicit expressions of maximal gain and phase margins of first-order systems achievable by PID controllers. The results show explicitly the dependence of these measures on the system's unstable pole and non-minimum phase zero.

A. PI Control

We first consider the first-order unstable plants,

$$P(s) = \frac{\beta_0 s + \beta_1}{s-p}, \quad p > 0. \tag{7}$$

Without loss of generality, it is assumed that $\beta_0 \geq 0$ and $\beta_1 \neq 0$. Note that in the case $\beta_0 \neq 0$, derivative control will result in an improper system. For this reason, in this section, we shall focus on PI controllers.

Theorem 3.1: Let $P(s)$ be given by (7). Then the following statements hold:

(i). For $\beta_0 > 0$,

$$k_M^{PI} = k_M^P = \begin{cases} \max\left\{\frac{|\beta_1|/\beta_0}{p}, \frac{p}{|\beta_1|/\beta_0}\right\}, & \beta_1 < 0 \\ \infty, & \beta_1 > 0 \end{cases},$$
$$\theta_M^{PI} = \theta_M^P = \begin{cases} \cos^{-1}\frac{2\sqrt{\beta_0|\beta_1|/p}}{\beta_0 + (|\beta_1|/p)}, & \beta_1 < 0 \\ \pi, & \beta_1 > 0 \end{cases}. \tag{8}$$

(ii). For $\beta_0 = 0$,

$$\begin{aligned} k_M^{PID} &= k_M^{PD} = k_M^P = \infty, \\ \theta_M^{PID} &= \theta_M^{PD} = 2\theta_M^P = \pi. \end{aligned} \tag{9}$$

The explicit expressions given in Theorem 3.1 lead us to a number of useful observations. First, it is clear that integral control has no effect in maximizing either the gain or phase margin. This is consistent with one's intuition; integral control has its essential utility in tracking reference signals, which is seen as a conflict with a system's stability robustness and henceforth with increasing the system's gain and phase margins. Secondly, we note that for minimum phase plants ($\beta_0 > 0$ and $\beta_1 > 0$) of relative degree zero, proportional control suffices to achieve the maximum possible infinite gain margin and a phase margin of $\pm 180°$. For nonminimum phase plants ($\beta_0 > 0$ and $\beta_1 < 0$), it is instructive to consider

$$P(s) = \frac{s-z}{s-p}, \quad p > 0, \ z > 0. \tag{10}$$

In this case, it follows from Theorem 3.1 that

$$\begin{aligned} k_M^{PI} &= k_M^P = \max\left\{\frac{z}{p}, \frac{p}{z}\right\}. \\ \theta_M^{PI} &= \theta_M^P = \cos^{-1}\frac{2\sqrt{z/p}}{1+(z/p)}. \end{aligned} \tag{11}$$

It is interesting to see that

$$\theta_M^{PI} = \theta_M^P = \cos^{-1}\frac{2\sqrt{k_M^P}}{1+k_M^P} = \cos^{-1}\frac{2\sqrt{k_M^{PI}}}{1+k_M^{PI}}. \tag{12}$$

A subsequent comparison of (11) with Proposition 2.1 (cf. Corollary 3.1) shows that for a first-order nonminimum phase plant, the maximal phase margin achievable by proportional control is half that achievable by general LTI controllers, and when measured on the logarithmic scale, the maximal gain margin is also half that by LTI controllers.

We conclude this section with a comparison of the maximal gain and phase margins herein with those achievable by the general LTI controllers. The following corollary states the comparison for both the PI controller $K_{PI}(s)$. The results show that for a first-order nonminimum phase plant, the maximal gain and phase margins achievable by a proportional controller are half as good as those by general LTI controllers.

Corollary 3.1: Let $P(s)$ be given by (10). Then

$$\begin{aligned} k_M &= \left(k_M^{PI}\right)^2 = \left(k_M^P\right)^2. \\ \theta_M &= 2\theta_M^{PI} = 2\theta_M^P. \end{aligned} \tag{13}$$

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Example 3.1: In this example, we give a companion between the maximal margins attainable by PI controllers and those by general LTI controllers. In this case we take $p = 2$ and let z vary in the interval $[0.5, 8]$. Fig.2 plots the corresponding gain margins $20\log_{10} k_M^{PI}$ and $20\log_{10} k_M$ (both in dB) as a function of z, while Fig.3 shows the corresponding phase margins θ_M^{PI} and θ_M. We find from the figures that when $z = p$, the gain and phase margins both vanish.

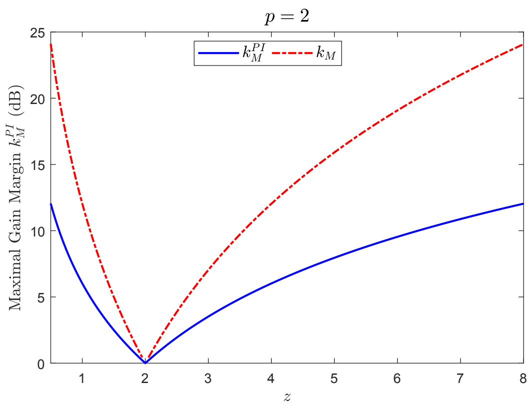

Fig. 2: Maximal gain margin k_M^{PI} of system (10) with comparison to k_M in [19]

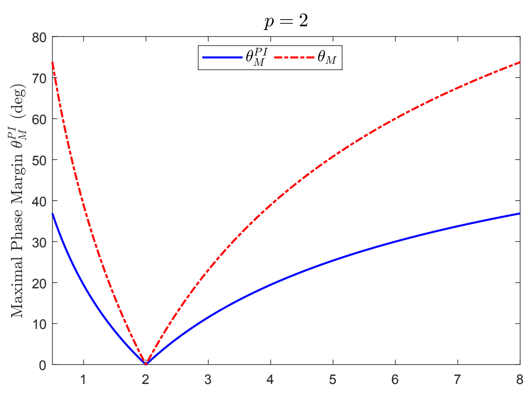

Fig. 3: Maximal phase margin θ_M^{PI} of system (10) with comparison to θ_M in [19]

IV. Second-Order Unstable Systems

This section presents the main results for the second-order unstable plants, which include minimum phase systems and nonminimum phase systems. In general, the computation of the gain and phase margins for second-order plants poses a more difficult problem. Throughout this section, our development seeks to recast the gain and phase maximization problems as one of nonlinear programming. Since the derivations are lengthy, we omit the proofs of the subsequent results. For their essential flavor, we employ the KKT condition to obtain values of the proportional and integral gains as the necessary solution to the nonlinear programming problems under consideration, which show that unequivocally in all cases, the optimal integral gain is $k_i = 0$, and the optimal proportional gain k_p is a certain boundary value. The maximal gain and phase margins are then obtained by solving a univariate optimization problem, defined in terms of a function of the derivative gain k_d alone.

A. Minimum Phase Systems

We begin with minimum phase plants that contain a pair of unstable poles p_1, p_2, described by

$$P(s) = \frac{\beta_0 s + \beta_1}{(s - p_1)(s - p_2)}, \quad \mathrm{Re}(p_1) > 0, \quad \mathrm{Re}(p_2) > 0, \tag{14}$$

where $\beta_0 \geq 0, \beta_1 > 0$. To ensure that $P(s)$ is a real rational plant, we assume that p_1 and p_2 are either real poles or a complex conjugate pair.

Theorem 4.1: Let $P(s)$ be given by (14). Then the following statements hold:

(i).

$$\begin{aligned} k_M^P &= \begin{cases} \text{none}, & \text{when } \beta_0 = 0 \\ \infty, & \text{when } \beta_0 \neq 0 \end{cases}, \\ k_M^{PID} &= k_M^{PD} = \infty, \end{aligned} \tag{15}$$

(ii).

$$\begin{aligned} \theta_M^P &= \begin{cases} \text{none}, & \text{when } \beta_0 = 0 \\ \frac{\pi}{2}, & \text{when } \beta_0 \neq 0 \end{cases}, \\ \theta_M^{PID} &= \theta_M^{PD} = \begin{cases} \frac{\pi}{2}, & \text{when } \beta_0 = 0 \\ \pi, & \text{when } \beta_0 \neq 0 \end{cases}. \end{aligned} \tag{16}$$

Remark 4.1: It is clear from Theorem 4.1 that for second-order systems the maximal gain and phase margins achievable by PID control are the same as those by general LTI controllers, provided that the plant is minimum phase and has a relative degree no greater than one. On the other hand, if the plant does have a relative degree greater than one, then the maximal phase margin is reduced to $\pi/2$, despite that the maximal gain margin remains unchanged.

B. Nonminimum Phase Systems

Now we extend our results to second-order nonminimum phase plants. We consider first the plant described by

$$P(s) = \frac{s - z}{(s - p_1)(s - p_2)}, \quad \mathrm{Re}(p_1) > 0, \quad \mathrm{Re}(p_2) > 0, \tag{17}$$

where likewise, we assume that p_1 and p_2 are either real or complex conjugate.

Theorem 4.2: Let $P(s)$ be given by (17). For $K_{PD}(s)$ and $K_{PID}(s)$ to stabilize $P(s)$, it is necessary that $p_1 + p_2 \neq (p_1 p_2/z) + z$. Then under this condition, the following statements hold:

(i). For $p_1 + p_2 < (p_1 p_2/z) + z$, then

$$k_M^{PID} = k_M^{PD} = \begin{cases} \dfrac{z^2}{z(p_1+p_2) - p_1 p_2}, & \text{if } p_1+p_2 > p_1 p_2/z, \\ \dfrac{p_1 p_2}{z(p_1+p_2) - z^2}, & \text{if } p_1+p_2 \leq p_1 p_2/z. \end{cases} \tag{18}$$

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Otherwise, for $p_1+p_2>(p_1p_2/z)+z$,

$$k_M^{PID}=k_M^{PD}=\frac{z\,(p_1+p_2)}{z^2+p_1p_2}. \tag{19}$$

(ii). Define

$$\hat{\theta}\,(k_p)=\sum_{i=1}^{2}\tan^{-1}\frac{\omega(k_p)}{p_i}-\tan^{-1}\frac{\omega(k_p)}{z}-\tan^{-1}\frac{\omega(k_p)}{k_p}, \tag{20}$$

where

$$\omega(k_p)=\sqrt{\frac{p_1^2p_2^2-z^2k_p^2}{k_p^2+z^2-p_1^2-p_2^2}}. \tag{21}$$

Then for $p_1+p_2<(p_1p_2/z)+z$,

$$\theta_M^{PID}=\theta_M^{PD}=\begin{cases}\max\left\{|\hat{\theta}\,(p_1+p_2-z)\,|,\ |\hat{\theta}\,(\bar{k}_p)\,|\right\}, & \text{if } p_1+p_2>z,\\ \max\left\{|\hat{\theta}\,(0^+)\,|,\ |\hat{\theta}\,(\bar{k}_p)\,|\right\}, & \text{if } p_1+p_2\le z,\end{cases} \tag{22}$$

where $\bar{k}_p\in(p_1+p_2-z,\ p_1p_2/z)$ is a positive solution to the polynomial equation

$$\begin{aligned}zk_p^5-z^2k_p^4&+z\left((p_1+p_2)(z-p_1-p_2-p_1p_2/z)+2p_1p_2\right)k_p^3\\&+2p_1^2p_2^2k_p^2+\left(p_1p_2(p_1+p_2)(p_1^2+p_2^2-p_1p_2)+zp_1p_2\cdot\right.\\&\left.(p_1p_2-z(p_1+p_2))\right)k_p+p_1^2p_2^2\left(z^2-p_1^2-p_2^2\right)=0.\end{aligned} \tag{23}$$

Otherwise, for $p_1+p_2>(p_1p_2/z)+z$,

$$\theta_M^{PID}=\theta_M^{PD}=\max\left\{\hat{\theta}\,(p_1+p_2-z)\,,\ \hat{\theta}\,(\bar{k}_p)\right\}, \tag{24}$$

where $\bar{k}_p\in(p_1p_2/z,\ p_1+p_2-z)$ is a solution to the equation (23).

Remark 4.2: Again, similar to the core of first-order plants, Theorem 4.2 reveals that the integral control has no effect to improve the gain and phase margins; in fact, from the proof of Theorem 4.2, one can see that a non-zero integral control gain k_i will make the gain and phase margins smaller. On the other hand, the theorem also shows that the derivative control improves the margins. Consider, for example,the case $p_1+p_2\le p_1p_2/z$ in (18). In this case, it follows from theorem 4.2 and (26) that

$$k_M^{PD}=\frac{p_1+p_2}{p_1+p_2-z}k_M^P.$$

Example 4.1: In this example, we draw a companion between the maximal margins achievable by PID controllers and those by general LTI controllers. Towards this end, it was found in [22] that

$$\gamma_{\text{opt}}=\prod_{i=1}^{2}\left|\frac{p_i+z}{p_i-z}\right|.$$

We consider the second-order system (17) with two real poles and a pair of unstable complex conjugate poles, respectively.

Real unstable poles $p_1>0,p_2>0$: In this case we take $p_1=2$, $p_2=6$ and let z vary in the interval $[0.5,8]$. Fig.4 plots the corresponding gain margins $20\log_{10}k_M^{PID}$ and $20\log_{10}k_M$ (both in dB) as a function of z. Likewise, when $z=p_1,p_2$, the gain margin vanishes.

Complex conjugate unstable poles $p_1=\sigma+j\nu,p_2=\sigma-j\nu$, $\sigma>0$: We take $\sigma=4$, $\nu=1$ and let z vary in the interval $[0.5,8]$ in a similar manner. Fig.5 shows the corresponding gain margins $20\log_{10}k_M^{PID}$ and $20\log_{10}k_M$ (both in dB) as a function of z.

In both cases, we observe from the figure that k_M is no greater than $\left(k_M^{PID}\right)^2$, and when measured on the logarithmic scale, k_M is no greater than twice k_M^{PID}. A deeper investigation reveals that this is true in general, as evidenced by the following corollary.

Corollary 4.1: Let $P(s)$ be given by (17). Then,

$$k_M^{PID}\le k_M\le\left(k_M^{PID}\right)^2. \tag{25}$$

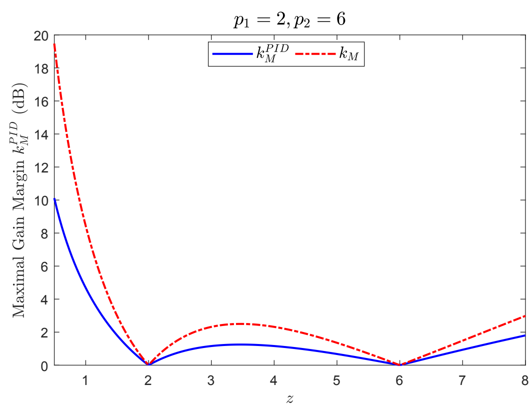

Fig. 4: Maximal gain margin k_M^{PID} of system (17) with comparison to k_M in [19]

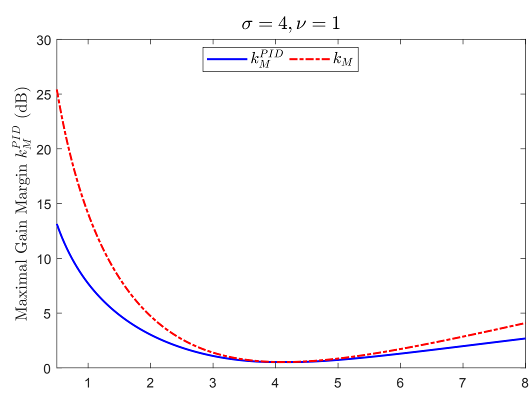

Fig. 5: Maximal gain margin k_M^{PID} of system (17) with comparison to k_M in [19]

V. PI Control and Special Cases

In the preceding section, we have shown that the maximal phase margin can be determined by solving a polynomial equation. In this section we consider PI controllers and other related problems. We show that in these cases, explicit expressions of the maximal phase margins can be obtained,

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which solutions of polynomial equations are no longer needed. The results too reinforce the proceeding conclusion that the integral gain has no effect in improving the gain and phase margins.

Theorem 5.1: Let $P(s)$ be given by (17). For $K_{PI}(s)$ to stabilize $P(s)$, it is necessary that $p_1 + p_2 < p_1 p_2/z$. Then under this condition, the following statements hold:

(i).

$$k_M^{PI} = k_M^P = \frac{p_1 p_2}{z\,(p_1 + p_2)}. \tag{26}$$

(ii).

$$\theta_M^{PI} = \theta_M^P = \tan^{-1}\frac{\omega_0}{z} - \tan^{-1}\frac{\omega_0}{p_1} - \tan^{-1}\frac{\omega_0}{p_2}, \tag{27}$$

where

$$\omega_0 = \left\{ \frac{z\,(p_1+p_2)}{2\,(p_1+p_2-z)} \left(\frac{p_1^2+p_2^2}{p_1+p_2} - z - p_1p_2/z + \sqrt{(z - p_1p_2/z)^2 + (p_1-p_2)^2 \left(1 - 2\frac{z + p_1p_2/z}{p_1+p_2}\right)} \right) \right\}^{1/2}.$$

The optimal PI coefficients are given by $k_i^* = 0$, and

$$k_p^* = \sqrt{\frac{p_1+p_2}{z}(\omega_0^2 + p_1p_2)}. \tag{28}$$

VI. Conclusion

In this paper we have studied the gain and phase margins of LTI systems attainable using PID controllers. The problem is to seek the largest margins of gain and phase, which are the intrinsic limits under which a PID controller may exist to stabilize the family of plants with gain and phase values varying within the margins. We derived analytical expressions for the maximal gain and phase margins for first- and second-order plants. Of these, we found that for minimum phase systems up to the second order, the maximal gain and phase margins achievable by PID controllers coincide with those by general LTI controllers, and that for nonminimum phase systems, the maximal gain and phase margins achievable by LTI controllers are at most twice those by PID controllers; here the gain margin is measured on the logarithmic scale and the phase margin is measured on the linear scale. It is worth noting that in all cases, the maximal gain and phase margins are attained with no integral control; in other words, the maximal gain and phase margins achievable by PID controllers coincide with those by PD controllers. This is consistent with one's intuition, since integral control is generally intended for achieving such performance objective as reference tracking, which by nature is in conflict with stability robustness as measured by gain and phase margins. The results consequently shed useful lights into PID control, and reaffirm analytically, from a system robustness perspective, long-held heuristics on the effectiveness of PID controllers.

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