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Brief paper

Functional observers with linear error dynamics for discrete-time nonlinear systems ☆

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ABSTRACT

This work deals with the problem of designing observers for the estimation of a single function of the states for discrete-time nonlinear systems. Necessary and sufficient conditions for the existence of lower order functional observers with linear dynamics and linear output map are derived. The results provide a direct generalization to Luenberger's linear theory of functional observers.

1. Introduction

In control theory, a functional observer is an auxiliary system that is driven by the available system outputs and mirrors the dynamics of a physical process in order to estimate one or more functions of the system states (Luenberger, 1966, 1971). Besides being of theoretical importance, the use of functional observers arises in many applications. For example, functional estimates are useful in feedback control system design because the control signal is often a linear combination of the states, and it is possible to utilize a functional observer to directly estimate the feedback control signal (Kravaris, 2016; Luenberger, 1966, 1971). From a practical point of view, the most common class of applications is related to condition monitoring of dynamic systems.

Over the past fifty years, considerable research has been carried out on estimating functions of the state vector for linear systems ever since Luenberger introduced the concept of functional observers in 1966 (Luenberger, 1966) and proved that it is feasible to construct a functional observer with number of states equal to observability index minus one. Subsequent research has focused on lower order functional observers where necessary and sufficient conditions for their existence and stability have been derived (Darouach, 2000; Fairman & Gupta, 1980; Moore & Ledwich, 1975; Tsui, 1986), and parametric approaches to the design of lower order functional observers (Trinh & Fernando, 2011; Trinh, Tran, & Nahavandi, 2006) and algorithms for solving the functional observer design conditions have also been developed (Fairman & Gupta, 1980; Moore & Ledwich, 1975; Trinh, Nahavandi, & Tran, 2008; Tsui, 1986). In a parallel direction, the problem of designing unknown input/ disturbance decoupled functional observers (Trinh, Fernando, & Nahavandi, 2004; Trinh, Teh, & Ha, 2008; Xiong & Saif, 2003) and functional observers for systems with time delays (Trinh, 1999; Trinh, Teh, & Fernando, 2010) have also been tackled. In fact, strong connections between the design of functional observers for linear systems with unknown inputs and the design of delay-free functional observers for time-delay systems have been established (Trinh & Fernando, 2011). This implies that the design of linear functional observers for these systems can be done under the general framework of linear functional unknown input observers (Trinh & Fernando, 2011).

For nonlinear systems, there has been significant literature in the theory of full-state observers, with a variety of methods and approaches (Andrieu & Praly, 2006; Bernard & Andrieu, 2018; Califano, Monaco, & Normand-Cyrot, 2003; Ciccarella, Dalla Mora, & Germani, 1993, 1995; Kazantzis & Kravaris, 1998, 2001; Lee & Nam, 1991; Xiao et al., 2003). In particular, in the context of exact linearization methods, Luenberger theory has been extended to nonlinear systems in a direct and analogous manner for both continuous (Andrieu & Praly, 2006; Bernard & Andrieu, 2018; Kazantzis & Kravaris, 1998) and discrete-time systems (Kazantzis & Kravaris, 2001; Xiao et al., 2003). Recently, the authors developed a direct generalization of Luenberger's functional observers to continuous time nonlinear systems (Kravaris & Venkateswaran, 2021). The main goal of this study is to develop a generalization of Luenberger's functional observers to discrete-time nonlinear systems.

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The outline of this present study is as follows. In the next couple of sections, the notion of functional observer for discrete time nonlinear systems will be defined in a manner completely analogous to Luenberger's definition (Luenberger, 1966, 1971) for linear systems and different approaches to solve the functional observer design problem will be outlined. Following this, notions of observer error linearization will be defined, and then necessary and sufficient conditions will be derived for the solution of the linearization problem, as well as a simple formula for the resulting functional observer. Finally, the methodology will be tested on a mathematical example.

2. Functional observers for discrete-time nonlinear systems

Consider a discrete-time nonlinear system described by:

$$\begin{aligned} x(k+1) &= F(x(k)) \\ y(k) &= H(x(k)) \\ z(k) &= q(x(k)) \end{aligned} \tag{2.1}$$

where:
$x \in \mathbb{R}^n$ is the system state
$y \in \mathbb{R}^p$ is the vector of measured outputs
$z \in \mathbb{R}$ is the (scalar) output to be estimated
and $F: \mathbb{R}^n \to \mathbb{R}^n, H: \mathbb{R}^n \to \mathbb{R}^p, q: \mathbb{R}^n \to \mathbb{R}$ are smooth nonlinear functions. The objective is to construct a functional observer of order $\nu < n$, which generates an estimate of the output z, driven by the output measurement y.

In complete analogy to Luenberger's construction for the linear case, we seek a mapping

$$\xi = T(x) = \begin{bmatrix} T_1(x) \\ \vdots \\ T_\nu(x) \end{bmatrix}$$

from $\mathbb{R}^n$ to $\mathbb{R}^\nu$, to immerse system (2.1) to a νth order system ($\nu < n$), with input y and output z:

$$\begin{aligned} \xi(k+1) &= \varphi(\xi(k), y(k)) \\ z(k+1) &= \omega(\xi(k), y(k)) \end{aligned} \tag{2.2}$$

But in order for system (2.1) to be mapped to system (2.2) under the mapping T(x), the following relations have to hold

$$\varphi(T(x), H(x)) = T(F(x)) \tag{2.3}$$

$$\omega(T(x), H(x)) = q(x) \tag{2.4}$$

The foregoing considerations lead to the following definition of a functional observer:

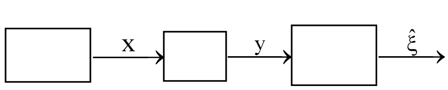
Definition 1. Given a dynamic system

$$\begin{aligned} x(k+1) &= F(x(k)) \\ y(k) &= H(x(k)) \\ z(k) &= q(x(k)) \end{aligned} \tag{2.1}$$

where $F: \mathbb{R}^n \to \mathbb{R}^n, H: \mathbb{R}^n \to \mathbb{R}^p, q: \mathbb{R}^n \to \mathbb{R}$ are smooth nonlinear functions, *y* is the vector of measured outputs and z is the scalar output to be estimated, the system

$$\begin{aligned} \hat{\xi}(k+1) &= \varphi(\hat{\xi}(k), y(k)) \\ \hat{z}(k+1) &= \omega(\hat{\xi}(k), y(k)) \end{aligned} \tag{2.5}$$

is called a functional observer for (2.1) if in the series connection



the overall dynamics

$$\begin{aligned} x(k+1) &= F(x(k)) \\ \hat{\xi}(k+1) &= \varphi(\hat{\xi}(k), H(x(k))) \end{aligned}$$

possesses an invariant manifold $\hat{\xi} = T(x)$ with the property that $q(x) = \omega(T(x), H(x))$.

If the functional observer (2.5) is initialized consistently with the system (2.1) i.e. if
$\hat{\xi}(0) = T(x(0))$, then $\hat{\xi}(k) = T(x(k))$, $\forall\, k \in \mathbb{N}$ and therefore
$\hat{z}(k) = \omega\left(\hat{\xi}(k), y(k)\right) = \omega(T(x(k)), H(x(k))) = q(x(k))$
$\forall\, k \in \mathbb{N}$, which means that the functional observer will be able to exactly reproduce z(k).

In the presence of initialization errors, additional stability requirements will need to be imposed on the $\hat{\xi}$-dynamics, for the estimate $\hat{z}(k)$ to asymptotically converge to z(k).

At this point, it is important to examine the special case of a linear system, where $F(x) = Fx$, $H(x) = Hx$, $q(x) = qx$ with F, H, q being $n \times n$, $p \times n$, $1 \times n$ matrices respectively, and a linear mapping $T(x) = Tx$ is considered. Definition 1 tells us that for a linear time-invariant system

$$\begin{aligned} x(k+1) &= Fx(k) \\ y(k) &= Hx(k) \\ z(k) &= qx(k) \end{aligned} \tag{2.6}$$

the system

$$\begin{aligned} \hat{\xi}(k+1) &= A\hat{\xi}(k) + By(k) \\ \hat{z}(k) &= C\hat{\xi}(k) + Dy(k) \end{aligned} \tag{2.7}$$

will be a functional observer if the following conditions are met:

$$TF = AT + BH$$

$$q = CT + DH$$

for some $\nu \times n$ matrix T. These are exactly the discrete-time version of Luenberger's conditions for a functional observer for linear continuous time-invariant systems (Luenberger, 1966, 1971).

3. Designing lower order functional observers

For the design of a functional observer, one must be able to find a continuous map $T(x) = \begin{bmatrix} T_1(x) \\ \vdots \\ T_\nu(x) \end{bmatrix}$ to satisfy conditions (2.3) and (2.4) i.e. such that $T_j(F(x)), j = 1, \ldots, \nu$ is a function of $T_1(x), \ldots, T_\nu(x)$, H(x) and q(x) is a function of $T_1(x), \ldots, T_\nu(x)$, H(x)

However, such scalar functions $T_1(x), \ldots, T_\nu(x)$ may not exist, if ν is too small. Moreover, even when they do exist, there is an additional very important requirement:
Since $T(F(x)) = \varphi(T(x), H(x))$ will define the right-hand side of the functional observer's dynamics, it must be such that the functional observer's dynamics is stable and the decay of the error is sufficiently rapid. This paper will address the functional observer design problem, focusing on *finding conditions under which low-order functional observers are feasible*

4. Exact linearization of a functional observer

The concept of exact observer linearization has been formulated in the discrete-time literature for full-state observers. We will start this section with a brief necessary review, following Kazantzis and Kravaris (2001), Xiao et al. (2003) and Brivadis, Andrieu, and Serres (2019). Subsequently we will propose an extension of the concept of exact linearization to discrete-time functional observers.

Consider a discrete-time nonlinear system

$$x(k+1) = F(x(k)) \tag{4.1}$$
$$y(k) = H(x(k))$$

If a mapping ξ $=T(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^\nu$ can be found to map system (4.1) to a linear system

$$\xi(k+1) = A\xi(k) + By(k) \tag{4.2}$$

where A and B are ν x ν and ν x p matrices respectively, the idea is to use system (4.2) as the basis for a state observer. The mapping T(x) must satisfy

$$T(F(x)) = AT(x) + BH(x) \tag{4.3}$$

Assuming for the moment that the functional equation (4.3) can be solved, *it will be possible to reconstruct the state if the mapping $x \to \xi$ is injective or if $x \to (\xi, y)$ is injective.* The observer will then consist of a replica of (4.2)

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + By(k) \tag{4.4}$$

along with an algebraic equation to calculate the state estimate. In particular,

– for $\nu = n$ (full-order observer), the state estimate will be calculated as $\hat{x} = T^{-1}(\hat{\xi})$,

– for $\nu = n - p$ (reduced-order observer), the state estimate will be the solution of $\begin{Bmatrix} T(\hat{x}) = \hat{\xi} \\ H(\hat{x}) = y \end{Bmatrix}$.

In both cases, the observer's error dynamics will follow

$$\hat{\xi}(k+1) - T(x(k+1)) = A\left(\hat{\xi}(k) - T(x(k))\right) \tag{4.5}$$

which is linear, and it will converge exponentially to 0 when A has spectral radius of less than 1.

In summary, existence of a full-state observer with linear error dynamics reduces to two main questions:

i. Solvability of the functional equation (4.3)
ii. Injectivity of the mapping $x \to T(x)$ or $x \to (T(x), H(x))$.

i. has affirmative answer under mild assumptions (see specific results later in this section), whereas for **ii.** to hold, appropriate observability or backward distinguishability conditions on (4.1) must be imposed (see Brivadis et al., 2019; Kazantzis & Kravaris, 2001; Xiao et al., 2003).

Let us now consider the functional observer, as defined by Definition 1 in Section 2. In the spirit of full-state observer linearization, we seek for a functional observer of the form (2.5) whose dynamics is linear, i.e.

$$\varphi(\xi, y) = A\xi + By \tag{4.6}$$

Then, condition (2.3) will become the functional equation (4.3), but we also need to satisfy condition (2.4), which states that q(x) must be expressible as a function of T(x) and H(x). This is a functional dependence condition, whose satisfaction depends on the order of the functional observer:

– If $\nu = n - p$ and the reduced-order full-state observer linearization problem can be solved, this will automatically solve the functional observer linearization problem, *for any* functional q(x).

– If $\nu < n - p$, the functional dependence requirement from (2.4) may or may not be feasible, depending on q(x) and the observer's dynamics.

In summary, the possibility of extension of exact linearization to functional observers reduces to two main questions:

i. Solvability of the functional equation (4.3)
ii. Compatibility of T(x) with the functional q(x), in the sense of the functional dependence specified through (2.4).

Thus, we see that the solvability problem for the functional equation (4.3) is common both for the functional and the full-state observer linearization problems. This problem has been resolved, and there are alternative existence results, depending on the regularity assumptions on the functions F and H (Lipschitz continuity or local analyticity).

The following Proposition is an immediate consequence of a Theorem by Brivadis, et al. (see Theorem 2 in Brivadis et al. (2019)).

Proposition 1. *Assume that* F *is invertible, and* F^{-1}*and* H *are globally Lipschitz. Also, let* A *and* B *be* $\nu\times\nu$ *and* $\nu\times p$ *matrices respectively, with the spectral radius* $\rho(A) < \min\left\{1, 1/K_{F^{-1}}\right\}$, *where* $K_{F^{-1}}$ *is the Lipschitz constant of* F^{-1}. *Then, there exists a continuous function* T(x) *from* $\mathbb{R}^n$ *to* $\mathbb{R}^\nu$ *that satisfies* (4.3).

Moreover, under additional conditions, it is possible to prove that the solution is unique (see Theorem 4 in Brivadis et al. (2019)).

The following Proposition is an immediate consequence of Smajdor's Theorem (Smajdor, 1968) (see also Kazantzis & Kravaris, 2001).

Proposition 1'. *Let* $F: \mathbb{R}^n \to \mathbb{R}^n$, $H: \mathbb{R}^n \to \mathbb{R}^p$ *be real analytic functions with* $F(0) = 0, H(0) = 0$. *Also, let* A *and* B *be* $\nu \times \nu$ *and* $\nu \times p$ *and matrices respectively. Suppose that the eigenvalues* k_i *of* $\frac{\partial F}{\partial x}(0)$ *all lie either entirely inside or outside the unit disc, and are not related to the eigenvalues* $\lambda_j (j = 1, 2, \ldots, \nu)$ *of* A *through any equation of the form* $\prod_{i=1}^{n} k_i^{m_i} = \lambda_j$ *with* m_i *nonnegative integers, not all zero. Then the system of functional equations* (4.3) *with initial condition* $T(0) = 0$, *admits a unique analytic solution* T(x) *in a neighborhood of* $x = 0$.

Because the subproblem of solvability of the functional equation (4.3) has been resolved, the focus of the present paper will be on the second subproblem: *under what conditions could the solution T(x) be compatible with q(x), in the sense specified through* (*2.4*). The goal will be to find conditions to check feasibility of lowering the order of the functional observer, below (n – p).

In the next section we will study a special form of the functional observer linearization problem, where in addition to requiring linear observer dynamics, we will also require linearity of the observer's output map. In particular, we will consider the following:

Functional Observer Linearization Problem

Given a system of the form (2.1), *find a functional observer of the form*

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + By(k) \tag{4.7}$$
$$\hat{z}(k) = C\hat{\xi}(k) + Dy(k)$$

where A, B, C, D *are* $\nu \times \nu$, $\nu \times p$, $1 \times \nu$, $1 \times p$ *matrices respectively, with* A *having stable eigenvalues. Equivalently, find a continuously differentiable mapping* $T: \mathbb{R}^n \to \mathbb{R}^\nu$ *such that*

$$T(F(x)) = AT(x) + BH(x) \tag{4.3}$$

and

$$q(x) = CT(x) + DH(x) \tag{4.8}$$

Assuming that the above problem can be solved, the resulting error dynamics will be linear:

$$\hat{\xi}(k+1) - T(x(k+1)) = A\left(\hat{\xi}(k) - T(x(k))\right)$$

$$\hat{z}(k) - z(k) = C\left(\hat{\xi}(k) - T(x(k))\right) \tag{4.9}$$

from which $\hat{z}(k) - z(k) = CA^k\left(\hat{\xi}(0) - T(x(0))\right)$.

With the matrix A having eigenvalues in the interior of the unit disc, the effect of the initialization error $\hat{\xi}(0) - T(x(0))$ will die out, and $\hat{z}(k)$ will approach $z(k)$ asymptotically.

Remark. It is possible to formulate a linearization problem in a slightly more general manner by including additive nonlinear output injection terms in the functional observer and a possibly nonlinear output map

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + \mathcal{B}(y(k)) \tag{4.10}$$
$$\hat{z}(k) = \omega(\hat{\xi}(k), y(k))$$

where $\mathcal{B}\colon \mathbb{R}^p \to \mathbb{R}^\nu$ is the nonlinear output injection term. This generalization will also be considered in the next section.

5. Necessary and sufficient conditions for solvability of the functional observer linearization problem

To be able to develop a practical approach for designing functional observers, it would be helpful to develop criteria to check if for a given set of ν eigenvalues, there exists a functional observer whose error dynamics is governed by these eigenvalues. This will be done in the present Section for the Functional Observer Linearization Problem

The main result is as follows:

Proposition 2. *For a nonlinear system of the form (2.1), there exists a functional observer of the form*

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + By(k) \tag{4.7}$$
$$\hat{z}(k) = C\hat{\xi}(k) + Dy(k)$$

with the eigenvalues of A *being the roots of a given polynomial* $\lambda^\nu + \alpha_1\lambda^{\nu-1} + \cdots + \alpha_{\nu-1}\lambda + \alpha_\nu$, *if and only if* $qF^\nu(x) + \alpha_1 qF^{\nu-1}(x) + \cdots + \alpha_{\nu-1}qF(x) + \alpha_\nu q(x)$ *is* $\mathbb{R}$*-linear combination of* $H_j(x), H_jF(x), \ldots, H_jF^\nu(x), j = 1, \ldots, p$, *where in the above we have used the notation* $F^j(x) = \underbrace{F \circ F \ldots F \circ F(x)}_{j\ times}$ *and* $H_jF(x) = (H_j \circ F)(x)$.

Proof. (i) *Necessity*: Suppose that there exists $T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_\nu(x) \end{bmatrix}$ such that (4.3) is satisfied, i.e

$$\begin{bmatrix} T_1F(x) \\ T_2F(x) \\ \vdots \\ T_\nu F(x) \end{bmatrix} = A\begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_\nu(x) \end{bmatrix} + \begin{bmatrix} B_1H(x) \\ B_2H(x) \\ \vdots \\ B_\nu H(x) \end{bmatrix}$$

where $B_1, \ldots, B_\nu$ denote the rows of the matrix B. Now, we find that for k = 1,2,3…

$$\begin{bmatrix} T_1F^k(x) \\ T_2F^k(x) \\ \vdots \\ T_\nu F^k(x) \end{bmatrix} = A^k\begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_\nu(x) \end{bmatrix} + \begin{bmatrix} (A^{k-1}B)_1H(x) + (A^{k-2}B)_1HF(x) + \cdots + (B_1HF^{k-1}(x)) \\ (A^{k-1}B)_2H(x) + (A^{k-2}B)_2HF(x) + \cdots + (B_2HF^{k-1}(x)) \\ \vdots \\ (A^{k-1}B)_\nu H(x) + (A^{k-2}B)_\nu HF(x) + \cdots + (B_\nu HF^{k-1}(x)) \end{bmatrix}$$

and we can calculate

$$\begin{aligned} &T_iF^\nu(x) + \alpha_1T_iF^{\nu-1}(x) + \cdots + \alpha_\nu T_i(x) \\ &= \left(A^\nu + \alpha_1A^{\nu-1} + \cdots + \alpha_\nu I\right)_i \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_\nu(x) \end{bmatrix} \\ &+ ((A^{\nu-1}B)_i + \alpha_1(A^{\nu-2}B)_i + \cdots \\ &+ \alpha_{\nu-1}B_i)H(x) + ((A^{\nu-2}B)_i + \cdots \\ &+ \alpha_{\nu-2}B_i)HF(x) + \cdots + (B_iHF^{\nu-1}(x)) \end{aligned}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ are the coefficients of the characteristic polynomial of the matrix A. By the Cayley–Hamilton theorem, $A^\nu + \alpha_1A^{\nu-1} + \cdots + \alpha_\nu I = 0$ and we have,

$$\begin{aligned} &T_iF^\nu(x) + \alpha_1T_iF^{\nu-1}(x) + \cdots + \alpha_\nu T_i(x) \\ &= ((A^{\nu-1}B)_i + \alpha_1(A^{\nu-2}B)_i + \cdots \\ &+ \alpha_{\nu-1}B_i)H(x) + ((A^{\nu-2}B)_i + \cdots \\ &+ \alpha_{\nu-2}B_i)HF(x) + \cdots + (B_iHF^{\nu-1}(x)) \end{aligned}$$

At the same time the mapping T(x) must satisfy (4.8) and we can conclude

$$\begin{aligned} &qF^\nu(x) + \alpha_1qF^{\nu-1}(x) + \cdots + \alpha_\nu q(x) \\ &= \left(CA^{\nu-1}B + \alpha_1CA^{\nu-2}B + \cdots + \alpha_{\nu-1}CB \right. \\ &\left. + \alpha_\nu D\right) H(x) \\ &+ \left(CA^{\nu-2}B + \cdots + \alpha_{\nu-2}CB + \alpha_{\nu-1}D\right) HF(x) + \cdots \\ &+ (CB + \alpha_1D)HF^{\nu-1}(x) + DHF^\nu(x) \end{aligned}$$

i.e.,

$$\begin{aligned} &qF^\nu(x) + \alpha_1qF^{\nu-1}(x) + \cdots + \alpha_\nu q(x) \\ &= \beta_0HF^\nu(x) + \beta_1HF^{\nu-1}(x) + \cdots + \beta_{\nu-1}HF(x) + \beta_\nu H(x) \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \beta_0 &= D \\ \beta_1 &= CB + \alpha_1D \\ \beta_2 &= CAB + \alpha_1CB + \alpha_2D \\ &\vdots \\ \beta_{\nu-1} &= CA^{\nu-2}B + \cdots + \alpha_{\nu-2}CB + \alpha_{\nu-1}D \\ \beta_\nu &= CA^{\nu-1}B + \alpha_1CA^{\nu-2}B + \cdots + \alpha_{\nu-1}CB + \alpha_\nu D \end{aligned} \tag{5.2}$$

which proves that $qF^{\nu-1}(x) + \alpha_1qF^{\nu-1}(x) + \cdots + \alpha_{\nu-1}qF(x) + \alpha_\nu q(x)$ is $\mathbb{R}$-linear combination of $H_j(x), H_jF(x), \ldots, H_jF^\nu(x), j = 1, \ldots, p$,

(ii) *Sufficiency*: Suppose that $qF^\nu(x) + \alpha_1qF^{\nu-1}(x) + \cdots + \alpha_{\nu-1}qF(x) + \alpha_\nu q(x)$ is $\mathbb{R}$-linear combination of $H_j(x), H_jF(x), \ldots, H_jF^\nu(x), j = 1, \ldots, p$, i.e. there exist constant row vectors β_0, β_1, …, $\beta_{\nu-1}$, β_ν that satisfy (5.1).

Consider the functional equation:

$$T(F(x)) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_\nu \\ 1 & 0 & \cdots & 0 & -\alpha_{\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -\alpha_2 \\ 0 & \cdots & 0 & 1 & -\alpha_1 \end{bmatrix} T(x) + \begin{bmatrix} \beta_\nu - \alpha_\nu\beta_0 \\ \beta_{\nu-1} - \alpha_{\nu-1}\beta_0 \\ \beta_{\nu-2} - \alpha_{\nu-2}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1\beta_0 \end{bmatrix} H(x) \tag{5.3}$$

It is straightforward to verify that

$$T(x) = \begin{bmatrix} \begin{pmatrix} -\beta_0 HF^{\nu-1}(x) - \cdots - \beta_{\nu-2}HF(x) - \beta_{\nu-1}H(x) \\ +qF^{\nu-1}(x) + \alpha_1 qF^{\nu-2}(x) + \cdots + \alpha_{\nu-1}q(x) \end{pmatrix} \\ \vdots \\ -\beta_0 HF(x) - \beta_1 H(x) + qF(x) + \alpha_1 q(x) \\ -\beta_0 H(x) + q(x) \end{bmatrix} \tag{5.4}$$

satisfies the functional equation (5.3) and we see that its νth component is $T_\nu(x) = -\beta_0 H(x) + q(x)$, therefore,

$$q(x) = [0\ 0 \ldots 0 1]\, T(x) + \beta_0 H(x) \tag{5.5}$$

Hence T(x) given by (5.4) satisfies conditions (4.3) and (4.8) for the solution of the Functional Observer Linearization Problem and system (4.7) with

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_\nu \\ 1 & 0 & \cdots & 0 & -\alpha_{\nu-1} \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & -\alpha_2 \\ 0 & \cdots & 0 & 1 & -\alpha_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_\nu - \alpha_\nu\beta_0 \\ \beta_{\nu-1} - \alpha_{\nu-1}\beta_0 \\ \beta_{\nu-2} - \alpha_{\nu-2}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1\beta_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad D = \beta_0 \tag{5.6}$$

is a functional observer.

It is important to observe that the sufficiency part of the proof is constructive, and it immediately leads to a design method for the functional observer.

Once a set of constant row vectors $\beta_0, \beta_1, \ldots, \beta_{\nu-1}, \beta_\nu \in \mathbb{R}^p$ have been found to satisfy (5.1) for a specific characteristic polynomial $\lambda^\nu + \alpha_1\lambda^{\nu-1} + \cdots + \alpha_{\nu-1}\lambda + \alpha_\nu$, formula (5.6) immediately gives the A, B, C and D matrices of the linear functional observer.

Also, it should be noted that there may be multiple sets of $\beta_0, \beta_1, \ldots, \beta_\nu \in \mathbb{R}^p$ that satisfy (5.1), leading to multiple solutions for the functional observer linearization problem.

Remark 5.1. The linear functional observer is amenable to a slight generalization involving additive nonlinear output injection terms:

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + \mathcal{B}(y(k)) \tag{4.10}$$

$$\hat{z}(k+1) = \omega(C\hat{\xi}(k), y(k))$$

and in particular, with an implicitly defined output map:

$$\hat{\xi}(k+1) = A\hat{\xi}(k) + \mathcal{B}(y(k))$$

$$\hat{z}(k) \text{ is the solution of } \wp\left(\hat{z}(k), y(k)\right) = C\hat{\xi}(k) \tag{5.7}$$

where $\wp : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ is a function such that the equation $\wp(z, y) = \zeta$ is uniquely solvable with respect to z.

A system of the above form will be a functional observer for system (2.1) if there exists a mapping T(x) that satisfies

$$T(F(x)) = AT(x) + \mathcal{B}(H(x)) \tag{5.8}$$

and

$$\wp(q(x), H(x)) = CT(x) \tag{5.9}$$

Following the same steps as in the proof of Proposition 2, we can prove the following:

Proposition 2'. *For a nonlinear system of the form (2.1), there exists a functional observer of the form (5.7) with the eigenvalues of A being the roots of a given polynomial $\lambda^\nu + \alpha_1\lambda^{\nu-1} + \cdots + \alpha_{\nu-1}\lambda + \alpha_\nu$, if and only if there exist functions $\beta_0 : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ invertible with respect to its first argument, and $\beta_1, \ldots, \beta_\nu : \mathbb{R}^p \to \mathbb{R}$ such that*

$$\begin{aligned} &\beta_0\left(qF^\nu(x), HF^\nu(x)\right) + \alpha_1\beta_0\left(qF^{\nu-1}(x), HF^{\nu-1}(x)\right) + \cdots \\ &+ \alpha_{\nu-1}\beta_0(qF(x), HF(x)) + \alpha_\nu\beta_0(q(x), H(x)) \\ &= \beta_1(H(x)) + \beta_2(HF(x)) + \cdots + \beta_\nu\left(HF^{\nu-1}(x)\right) \end{aligned} \tag{5.10}$$

Proof. (i) *Necessity*: Following the same steps as in the proof of Proposition 2, we have

$$\begin{bmatrix} T_1F^\nu(x) + \alpha_1 T_1 F^{\nu-1}(x) + \cdots + \alpha_{\nu-1}T_1F(x) + \alpha_\nu T_1(x) \\ \vdots \\ T_\nu F^\nu(x) + \alpha_1 T_\nu F^{\nu-1}(x) + \cdots + \alpha_{\nu-1}T_\nu F(x) + \alpha_\nu T_\nu(x) \end{bmatrix}$$
$$= \left(A^{\nu-1} + \alpha_1 A^{\nu-2} + \cdots + \alpha_{\nu-1}I\right)\begin{bmatrix} \mathcal{B}_1(H(x)) \\ \vdots \\ \mathcal{B}_\nu(H(x)) \end{bmatrix} + \left(A^{\nu-2} + \alpha_1 A^{\nu-3} + \cdots + \alpha_{\nu-2}I\right)\begin{bmatrix} \mathcal{B}_1(HF(x)) \\ \vdots \\ \mathcal{B}_\nu(HF(x)) \end{bmatrix} + \cdots$$
$$+ (A + \alpha_1 I)\begin{bmatrix} \mathcal{B}_1\left(HF^{\nu-2}(x)\right) \\ \vdots \\ \mathcal{B}_\nu\left(HF^{\nu-2}(x)\right) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1\left(HF^{\nu-1}(x)\right) \\ \vdots \\ \mathcal{B}_\nu(HF^{\nu-1}(x)) \end{bmatrix}$$

Using (5.9) and the previous expression, we can also conclude that

$$\begin{aligned} &\wp(qF^\nu(x), HF^\nu(x)) + \alpha_1\wp(qF^{\nu-1}(x), HF^{\nu-1}(x)) + \cdots \\ &+ \alpha_{\nu-1}\wp(qF(x), HF(x)) + \alpha_\nu\wp(q(x), H(x)) \\ &= \left(CA^{\nu-1} + \alpha_1 CA^{\nu-2} + \cdots + \alpha_{\nu-1}C\right)\begin{bmatrix} \mathcal{B}_1(H(x)) \\ \vdots \\ \mathcal{B}_\nu(H(x)) \end{bmatrix} \\ &+ \left(CA^{\nu-2} + \cdots + \alpha_{\nu-2}C\right)\begin{bmatrix} \mathcal{B}_1(HF(x)) \\ \vdots \\ \mathcal{B}_\nu(HF(x)) \end{bmatrix} + \cdots \end{aligned}$$

$$+ (CA + \alpha_1 C) \begin{bmatrix} \mathcal{B}_1 \left(HF^{\nu-2}(x)\right) \\ \vdots \\ \mathcal{B}_\nu \left(HF^{\nu-2}(x)\right) \end{bmatrix} + C \begin{bmatrix} \mathcal{B}_1 \left(HF^{\nu-1}(x)\right) \\ \vdots \\ \mathcal{B}_\nu (HF^{\nu-1}(x)) \end{bmatrix}$$

or

$$\beta_0 \left(qF^\nu(x), HF^\nu(x)\right) + \alpha_1 \beta_0 \left(qF^{\nu-1}(x), HF^{\nu-1}(x)\right) + \cdots + \alpha_{\nu-1}\beta_0(qF(x), HF(x)) + \alpha_\nu \beta_0(q(x), H(x)) = \beta_1(H(x)) + \beta_2(HF(x)) + \cdots + \beta_\nu \left(HF^{\nu-1}(x)\right) \tag{5.11}$$

where

$$\begin{aligned} \beta_0(z, y) &= \wp(z, y) \\ \beta_1(y) &= \left(CA^{\nu-1} + \alpha_1 CA^{\nu-2} + \cdots + \alpha_{\nu-1}C\right)\mathcal{B}(y) \\ \beta_2(y) &= \left(CA^{\nu-2} + \cdots + \alpha_{\nu-2}C\right)\mathcal{B}(y) \\ &\vdots \\ \beta_{\nu-1}(y) &= (CA + \alpha_1 C)\mathcal{B}(y) \\ \beta_\nu(y) &= C\mathcal{B}(y) \end{aligned} \tag{5.12}$$

(ii) *Sufficiency*: Assuming that (5.10) holds, we can follow the same steps as in the proof of Proposition 2 and prove that

$$\begin{aligned} T_1(x) = {} & \beta_0 \left(qF^{\nu-1}(x), HF^{\nu-1}(x)\right) + \alpha_1 \beta_0 \left(qF^{\nu-2}(x), HF^{\nu-2}(x)\right) \\ & + \cdots + \alpha_{\nu-2}\beta_0(qF(x), HF(x)) \\ & + \alpha_{\nu-1}\beta_0(q(x), H(x)) - \beta_2(H(x)) - \cdots \\ & - \beta_\nu \left(HF^{\nu-2}(x)\right) \\ & \vdots \\ T_{\nu-2}(x) = {} & \beta_0 \left(qF^2(x), HF^2(x)\right) + \alpha_1\beta_0(qF(x), HF(x)) + \\ & \alpha_2\beta_0(q(x), H(x)) - \beta_{\nu-1}(H(x)) - \beta_\nu(HF(x)) \\ T_{\nu-1}(x) = {} & \beta_0(qF(x), HF(x)) + \alpha_1\beta_0(q(x), H(x)) - \beta_\nu(H(x)) \\ T_\nu(x) = {} & \beta_0(q(x), H(x)) \end{aligned} \tag{5.13}$$

satisfies (5.8) and (5.9) with

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_\nu \\ 1 & 0 & \cdots & 0 & -\alpha_{\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -\alpha_2 \\ 0 & \cdots & 0 & 1 & -\alpha_1 \end{bmatrix}, \quad \mathcal{B}(y) = \begin{bmatrix} \beta_1(y) \\ \beta_2(y) \\ \beta_3(y) \\ \vdots \\ \beta_\nu(y) \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \wp(q(x), H(x)) = \beta_0(z, y) \tag{5.14}$$

Therefore, system (5.7) with $A, \mathcal{B}(.), C, \wp(., .)$ given by (5.14) is a functional observer.

6. Lower order functional observers for linear systems

The results of the previous section can now be specialized to linear time-invariant systems. The following is an immediate consequence to Proposition 2.

Proposition 3. *For a linear time-invariant system of the form*

$$\begin{aligned} x(k+1) &= Fx(k) \\ y(k) &= Hx(k) \\ z(k) &= qx(k) \end{aligned} \tag{2.6}$$

there exists a functional observer of the form

$$\begin{aligned} \hat{\xi}(k+1) &= A\hat{\xi}(k) + By(k) \\ \hat{z}(k) &= C\hat{\xi}(k) + Dy(k) \end{aligned} \tag{2.7}$$

with the eigenvalues of A being the roots of a given polynomial $\lambda^\nu + \alpha_1\lambda^{\nu-1} + \cdots + \alpha_{\nu-1}\lambda + \alpha_\nu$*, if and only if*

$$\left(qF^\nu + \alpha_1 qF^{\nu-1} + \cdots + \alpha_{\nu-1}qF + \alpha_\nu q\right) \in \textit{span}\left\{H_j, H_jF, \ldots, H_jF^\nu, j = 1, \ldots, p\right\} \tag{6.1}$$

The above Proposition provides a simple and easy-to-check feasibility criterion for a lower-order functional observer with a pre-specified set of eigenvalues governing the error dynamics. Moreover, an immediate consequence of Proposition 3 is the following:

Corollary. Consider a linear time-invariant system of the form (2.6) with observability index ν_o (Luenberger, 1966) i.e. the least positive integer such that rank of $\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{\nu_0-1} \end{bmatrix} = n$. Then, there exists a functional observer of the form (2.7) of order $\nu = \nu_o - 1$ and arbitrarily assigned eigenvalues.

The result of the Corollary, derived through a different approach, is exactly the discrete-time version of Luenberger's result for functional observers for continuous linear time-invariant systems (Luenberger, 1966, 1971).

7. Case study

Consider the following discrete-time dynamical system with states $x = [x_1, x_2, x_3, x_4]$.

$$\begin{aligned} x_1(k+1) &= 0.9967x_1(k) - e^{-x_3(k)}x_2(k)x_1(k) \\ x_2(k+1) &= 0.9967x_2(k) - 3e^{-x_3(k)}x_2(k)x_1(k) \\ x_3(k+1) &= 0.9867x_3(k) + 40e^{-x_3(k)}x_2(k)x_1(k) \\ &\quad + 0.01x_4(k) \\ x_4(k+1) &= 0.83x_4(k) + 0.15x_3(k) \\ y_1(k) &= x_3(k) \\ y_2(k) &= x_4(k) \end{aligned} \tag{7.1}$$

The initial condition is $x_1(0) = 2, x_2(0) = -1, x_3(0) = 23, x_4(0) = 14$. The objective is to use the measurements $y_1(k)$ and $y_2(k)$ to estimate the sum $x_1(k) + x_2(k)$.

To this end, a scalar functional observer is built ($\nu = 1$) and the necessary and sufficient condition (5.1) is satisfied for the following choice of $\beta_0, \beta_1 \in \mathbb{R}^2$ and $\alpha_1 \in \mathbb{R}$

$$\begin{aligned} \beta_0 &= [-0.1, 1] \\ \beta_1 &= [-0.05133, -0.829] \\ \alpha_1 &= -0.9967 \end{aligned} \tag{7.2}$$

The resulting functional observer is

$$\begin{aligned} \hat{\xi}(k+1) &= -0.9967\hat{\xi}(k) - 0.151y_1(k) + 0.1677y_2(k) \\ \hat{z}(k) &= \hat{\xi}(k) - 0.1y_1(k) + y_2(k) \end{aligned} \tag{7.3}$$

The estimate generated by the functional observer and the estimation error are plotted in Fig. 1.

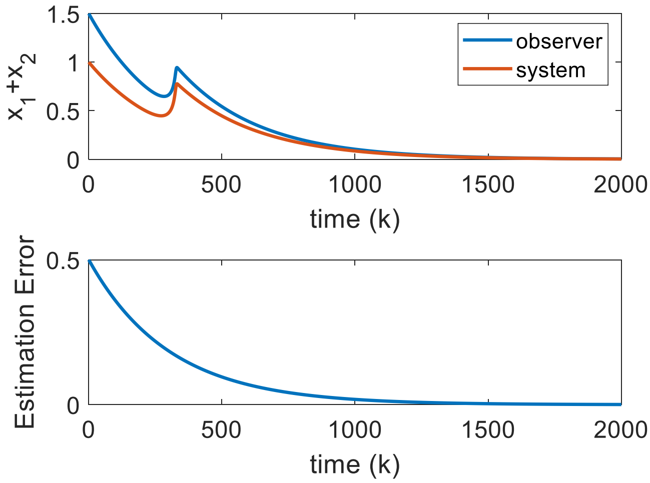


Fig. 1. Top—Estimates and true profiles in the presence of initialization error ($\hat{\xi}(0) - T(x(0)) = 0.5$) where T(x) is given by (5.4). Bottom—Estimation error ($\hat{z}(k) - z(k)$).

8. Conclusions

A generalization of Luenberger's functional observer to discrete-time nonlinear systems is presented in this work. The problem of exact linearization of the functional observer dynamics has been studied and conditions for the linearization to be feasible have been derived including a simple formula for the design of the resulting functional observer.

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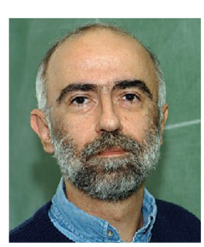

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