Data-Driven Optimal Control of Bilinear Systems

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***Abstract*— This paper develops a method to learn optimal controls from data for bilinear systems without a priori knowledge of the dynamics. Given an unknown bilinear system, we characterize when the available data is sufficiently informative to solve the optimal control problem. This characterization leads us to propose an online control experiment design procedure that guarantees that any input/state trajectory can be represented as a linear combination of collected input/state data matrices. Leveraging this representation, we transform the original optimal control problem into an equivalent data-based optimization problem with bilinear constraints. We solve the latter by iteratively employing a convex-concave procedure to find a locally optimal control sequence. Simulations show that the performance of the proposed data-based approach is comparable with model-based methods.**

***Index Terms*— Data-driven control, biliner systems**

I. Introduction

THE widespread availability of data, together with increasing computational capabilities to store, process, and manipulate it, has boosted the research activity in learning, modeling, and control of dynamical phenomena across science and engineering. Data-driven control has emerged as an appealing way of leveraging this data surge by employing solid theoretical principles to design controllers that do not require explicit a priori knowledge of the plant to be controlled. This paper contributes to this body of work by studying the data-driven synthesis of optimal control laws for bilinear systems.

Literature Review. Data-driven control approaches include indirect and direct methods [1]. Indirect methods identify system models from data prior to proceeding to the synthesis of model-based controllers, while direct approaches bypass the intermediate modeling step and construct controllers directly from data. A diverse range of factors, including the complexity of the plant, the cost and practicality of performing system identification, and the amount and quality of the available data, play a key role in the suitability and performance of each of these approaches, see e.g., [2], [3]. The direct data-driven approach has been particularly fruitful for linear systems, where tools from behavioral theory [4] have allowed to express the system trajectories in terms of sufficiently-rich data. This has resulted in the synthesis of feedback stabilizing controllers [5], [6], optimal control laws [7]–[9], predictive controllers [10], [11], network controllers [11], [12], control experiment design [13], optimization-based controllers [14], and extensions to various types of nonlinear systems [15]–[17], including flat [18], second-order [19], and linear time-varying systems [20]. Here we focus on direct data-driven control of bilinear systems as a building block for future work on more complex nonlinear systems. These systems are often viewed as the bridge between linear and nonlinear systems due to their special properties [21]. Moreover, [22] shows that control-affine nonlinear systems can be exactly bilinearized. The recent work [15] proposes a local stabilizing data-driven controller design for bilinear systems. Here, we focus on the synthesis of optimal controllers. Model-based approaches to optimal control of bilinear systems include [23]–[25], which treat them as time-varying linear systems and solve the optimization problem by applying iteratively the Pontryagin's maximum principle, and [26], which gives a lower bound on the minimum control energy required to steer the bilinear system using the reachability Gramian.

Statement of Contributions. We consider discrete-time bilinear control systems and study the point-to-point optimal control problem over a finite time horizon. We assume the system matrices are unknown and seek to learn the optimal control from input/state data. We introduce the notion of T-persistently exciting data to characterize when it is sufficiently informative for reconstructing the optimal control over the time horizon T. Under this hypothesis, we show that any input/state trajectory can be represented as a linear combination of the collected input/state data. Owing to the nonlinear nature of bilinear systems, the problem of ensuring that data is T-persistently exciting requires us to introduce an online control experiment design. We show our design is guaranteed to yield T-persistently exciting data in a finite number of steps. Building on this, we pose the optimal control synthesis problem as a data-based optimization with bilinear constraints. We show that a local solution to this nonconvex problem can be found by iteratively solving the convexified problems that result from applying a convex-concave approximation procedure. Simulations show similar performance between the proposed data-based approach and model-based methods.

II. Problem Formulation

Consider[1] the discrete-time bilinear control system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \Big[\sum_{j=1}^{n} \mathbf{x}_j(t)\mathbf{N}_j\Big]\mathbf{u}(t), \tag{1}$$

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[1]We denote by $\mathbb{R}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ the sets of real, non-negative integer, and positive integer numbers, resp. Let I, and $\mathbf{0}$ and $\mathbf{1}$ denote the identity matrix, and zero and all-ones vector/matrix, resp. Given $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}^d$ and $i, j \in \mathbb{Z}_{\geq 0}$, $i \leq j$, $f_{[i,j]}$ is the restriction of f to $[i, j]$ in vector form, i.e., $f_{[i,j]} = [f(i)^\top \; f(i+1)^\top \; \cdots \; f(j)^\top]^\top$, and $f_{\{i,j\}}$ the sequence $\{f(i), \ldots, f(j)\}$. For $\mathbf{X} = [\mathbf{x}_1^\top \; \mathbf{x}_2^\top \; \cdots \; \mathbf{x}_j^\top]^\top \in \mathbb{R}^{ij}$ with $\mathbf{x}_1, \cdots, \mathbf{x}_j \in \mathbb{R}^i$, $\mathcal{H}_k(\mathbf{X})$ denotes the Hankel matrix of depth $k \in \mathbb{Z}_{>0}$, with $k \leq j$,

$$\mathcal{H}_k(\mathbf{X}) := \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{j-k+1} \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{j-k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_j \end{bmatrix} \in \mathbb{R}^{ik \times (j-k+1)}.$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{R}^m$ are the system state and input, respectively, and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{N}_j \in \mathbb{R}^{n\times m}, j = 1,\dots,n$ are system matrices. Denoting $\mathbf{N} = [\mathbf{N}_1\ \mathbf{N}_2\ \cdots\ \mathbf{N}_n] \in \mathbb{R}^{n\times mn}$, the dynamics (1) is

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}(\mathbf{x}(t)\otimes\mathbf{u}(t)). \tag{2}$$

We make the following assumption.

Assumption II.1. *The pair* $(\mathbf{A}, [\mathbf{B}\ \mathbf{N}])$ *is controllable.*

Note that Assumption II.1 is weaker than asking for the bilinear system (2) to be controllable. Given initial $\mathbf{x}_0$ and target $\mathbf{x_f}$ states, we consider the following (point-to-point) optimal control problem over the time horizon T,

$$\begin{aligned} \min_{\mathbf{u}_{[0,T-1]}} \quad & \sum_{t=0}^{T-1} \mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t) \\ \text{s.t.} \quad & \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \Big[\sum_{j=1}^{n} \mathbf{x}_j(t)\mathbf{N}_j\Big]\mathbf{u}(t), \\ & \mathbf{x}(0) = \mathbf{x_0},\ \mathbf{x}(T) = \mathbf{x_f}. \end{aligned} \tag{P1}$$

Here, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{R} \in \mathbb{R}^{m\times m}$ are positive semi-definite. The minimum-energy control problem corresponds to $\mathbf{Q} = \mathbf{0}$ and $\mathbf{R} = I$. This optimization is nonconvex and its closed-form solution is not known in general. The optimality conditions of (P1) lead to a nonlinear two-point boundary-value problem, for which there is no analytical solution available [26].

We address the following problem: assume the system matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{N}_j, j = 1,\dots,n$ are unknown and, instead, we have access to input/state data of a control experiment of (2), that is, a control input sequence $\mathbf{u}_{\{0,L-1\}}$ along with the corresponding state sequence $\mathbf{x}_{\{0,L\}}$ of (2). Our objective is to develop an algorithmic procedure to learn from the data the optimal control sequence $\mathbf{u}^\star_{\{0,T-1\}}$ that solves (P1).

III. T-Persistently Exciting Data for Optimal Control of Bilinear Systems

In this section, we characterize when the available data is sufficient to solve the optimal control problem and discuss a procedure to design the control experiment. To motivate our discussion, we start by considering the linear system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3}$$

(corresponding to $\mathbf{N} = \mathbf{0}$ in (2)). Let $\mathbf{x}_{\{0,L\}}$ be a state sequence generated by (3) with input sequence $\mathbf{u}_{\{0,L-1\}}$. According to Willems' fundamental lemma [4], [27], and assuming the pair $(\mathbf{A},\mathbf{B})$ is controllable, if $\mathbf{u}_{\{0,L-1\}}$ is persistently exciting[2] of order $n+T$, then $\tilde{\mathcal{G}}_T(L) := [\mathbf{x}_{[0,L-T]}; \mathcal{H}_T(\mathbf{u}_{[0,L-1]})] \in \mathbb{R}^{(n+mT)\times(L-T+1)}$ is full-row rank. This ensures that for any input/state trajectory $(\bar{\mathbf{u}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T-1]})$ of length T of the linear system (3), there exist some $\tilde{\alpha} \in \mathbb{R}^{L-T+1}$ such that

$$\begin{bmatrix} \bar{\mathbf{x}}_{[0,T-1]} \\ \hline \bar{\mathbf{u}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_T(\mathbf{x}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \end{bmatrix} \tilde{\alpha}.$$

Given matrices $\mathbf{Y}$ and $\mathbf{Z}$, $[\mathbf{Y}\ \mathbf{Z}]$ and $[\mathbf{Y}; \mathbf{Z}]$ denote their row- and column-concatenations, resp. We use $\mathbf{Z}^\dagger$ and $\operatorname{Im}\mathbf{Z}$ to represent the pseudo-inverse and image space of $\mathbf{Z}$, resp. Finally, $\otimes$ denotes the Kronecker product, while $\|\cdot\|$ represents the Euclidean norm.

[2] $f_{\{0,L-1\}}$ is persist. exciting of order k if $\mathcal{H}_k(f_{[0,L-1]})$ is full-row rank.

Now consider the original bilinear system (2). If we regard $\mathbf{x}(t)\otimes\mathbf{u}(t)$ as an independent input, then the dynamics corresponds to a linear system with input matrix $[\mathbf{B}\ \mathbf{N}]$ and control input $\mathbf{v}(t) = [\mathbf{u}(t); \mathbf{x}(t)\otimes\mathbf{u}(t)]$. Willems' fundamental lemma applied to this linear system implies that, under Assumption II.1, if $\mathbf{v}_{\{0,L-1\}}$ is persistently exciting of order $n+T$, then $[\mathbf{x}_{[0,L-T]}; \mathcal{H}_T(\mathbf{v}_{[0,L-1]})]$ is full-row rank, which then ensures that any input/state trajectory $(\bar{\mathbf{v}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T-1]})$ of length T of system (2) can be represented by

$$\begin{bmatrix} \bar{\mathbf{x}}_{[0,T-1]} \\ \hline \bar{\mathbf{v}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_T(\mathbf{x}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{v}_{[0,L-1]}) \end{bmatrix} \alpha, \tag{4}$$

for some $\alpha \in \mathbb{R}^{L-T+1}$. However, as we know, the input $\mathbf{v}$ is not independent, and ensuring it is persistently exciting is not guaranteed by simply asking for $\mathbf{u}$ to be so. These observations motivate our ensuing definitions and technical treatment.

Remark III.1. *(Fundamental lemma for nonlinear systems):* Our exposition above uses the fundamental lemma in the context of bilinear systems by interpreting the bilinear term as an input. Recent literature on data-driven control has pursued similar ideas for different classes of nonlinear systems by relying on linear expressions in lifted coordinates, e.g., Hammerstein and Wiener [17], linear parameter-varying [20], second-order Volterra [19] and flat [18] systems. ▷

A. Parametrization of state trajectories

We next introduce the notion of T-persistently exciting data.

Definition III.2. *(T-persistently exciting data for optimal control of bilinear systems): Let* $\mathbf{x}_{\{0,L\}}$ *be a state sequence generated by* (2) *with input sequence* $\mathbf{u}_{\{0,L-1\}}$*. The data* $\mathbf{x}_{\{0,L\}}$, $\mathbf{u}_{\{0,L-1\}}$ *is* T-persistently exciting *if*

$$\mathcal{G}_T(L) := \begin{bmatrix} \mathcal{H}_1(\mathbf{x}_{[0,L-T]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{x}\otimes\mathbf{u}_{[0,L-1]}) \end{bmatrix} \in \mathbb{R}^{(n+mT+mnT)\times(L-T+1)}.$$

is full-row rank.

This definition requires as a necessary condition that $L \geq (mn+m+1)T+n-1$. We point out that $\mathcal{G}_T(L)$ being full-row rank is equivalent to $[\mathcal{H}_1(\mathbf{x}_{[0,L-T]}); \mathcal{H}_T(\mathbf{v}_{[0,L-1]})]$ being full-row rank, as both matrices can be obtained from each other by row permutation. Using (2), one can obtain the relation (5), where $\mathcal{O}_T \in \mathbb{R}^{nT\times n}$, $\mathcal{P}_T \in \mathbb{R}^{nT\times mT}$, $\mathcal{Q}_T \in \mathbb{R}^{nT\times mnT}$. If $\mathcal{G}_T(L)$ is full-row rank, it immediately follows that

$$[\mathcal{O}_T\ \mathcal{P}_T\ \mathcal{Q}_T] = \mathcal{H}_T(\mathbf{x}_{[1,L]})\mathcal{G}_T(L)^\dagger.$$

Remark III.3. *(T-persistently exciting data for optimal control versus for identification and stabilization):* When $T = 1$, we have $[\mathcal{O}_1\ \mathcal{P}_1\ \mathcal{Q}_1] = [\mathbf{A}\ \mathbf{B}\ \mathbf{N}]$. Therefore, 1-persistently exciting data corresponds to the standard notion of persistently exciting data, describing data needed for system identification [5], [6]. Moreover, if $\mathcal{H}_1(\mathbf{x}_{[0,L-1]})$ is full-row rank, then under knowledge of an upper bound on $\|\mathbf{N}\|$, one can construct locally stabilizing controllers directly from data, cf. [15]. Notice $\mathcal{G}_T(L)$ is full-row rank $\Rightarrow$ $\mathcal{G}_1(L)$ is full-row rank $\Rightarrow$ $\mathcal{H}_1(\mathbf{x}_{[0,L-1]})$ is full-row rank. We deduce that T-persistently exciting data comprises data needed for system identification and local stabilization. Note that, although $\mathcal{O}_T, \mathcal{P}_T$ and $\mathcal{Q}_T$

$$\mathcal{H}_T(\mathbf{x}_{[1,L]}) = \Bigg[\underbrace{\begin{matrix}\mathbf{A}\\ \mathbf{A}^2\\ \vdots\\ \mathbf{A}^T\end{matrix}}_{\mathcal{O}_T}\;\Bigg|\;\underbrace{\begin{matrix}\mathbf{B} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{AB} & \mathbf{B} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}^{T-1}\mathbf{B} & \mathbf{A}^{T-2}\mathbf{B} & \cdots & \mathbf{B}\end{matrix}}_{\mathcal{P}_T}\;\Bigg|\;\underbrace{\begin{matrix}\mathbf{N} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{AN} & \mathbf{N} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}^{T-1}\mathbf{N} & \mathbf{A}^{T-2}\mathbf{N} & \cdots & \mathbf{N}\end{matrix}}_{\mathcal{Q}_T}\Bigg]\mathcal{G}_T(L) \tag{5}$$

can be constructed only using $\mathbf{A}, \mathbf{B}$ and $\mathbf{N}$ when $\mathcal{G}_1(L)$ is full-row rank, this is not enough to express any input/state trajectory of length T as a linear combination of the collected input/state data, and thus $\mathcal{G}_1(L)$ being full-row rank is not sufficient to recover optimal controls. Another observation is that, in the linear case ($\mathbf{N} = \mathbf{0}$), according to [9], optimal controls over the time horizon T can be learned if $\tilde{\mathcal{G}}_T(L)$ is full-row rank. One can identify and globally stabilize linear systems directly using data if $\tilde{\mathcal{G}}_1(L)$ is full-row rank, cf. [5]. Since $\tilde{\mathcal{G}}_T(L)$ is full-row rank $\Rightarrow \tilde{\mathcal{G}}_1(L)$ is full-row rank, this reinforces the parallelism between the bilinear and linear cases regarding data conditions for different control problems. $\triangleright$

We have established that any input/state trajectory of (2) admits a data-based representation of the form (4). The next result establishes when the converse is also true, i.e., when a trajectory of the form (4) corresponds to a trajectory of (2).

Lemma III.4. *(Data-based representation of input/state trajectory in terms of T-persistently exciting data): Let $\mathbf{x}_{\{0,L\}}$ and $\mathbf{u}_{\{0,L-1\}}$ be a T-persistently exciting data set. Then*

(i) Any input/state trajectory $(\bar{\mathbf{u}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T]})$ of system (2) can be represented as

$$\left[\frac{\bar{\mathbf{x}}_{[0,T]}}{\bar{\mathbf{u}}_{[0,T-1]}}\right] = \left[\frac{\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]})}{\mathcal{H}_T(\mathbf{u}_{[0,L-1]})}\right]\alpha$$

for some $\alpha \in \mathbb{R}^{L-T+1}$;

(ii) Conversely, let $\alpha \in \mathbb{R}^{L-T+1}$ such that

$$\bar{\mathbf{x}} \otimes \bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{x} \otimes \mathbf{u}_{[0,L-1]})\alpha, \tag{6}$$

where $\bar{\mathbf{x}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{x}_{[0,L-1]})\alpha$ and $\bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\alpha$. Then, $\left[\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}); \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\right]\alpha$ is an input/state trajectory of (2) over the time horizon T.

Proof. For *(i)*, note that any input/state trajectory $(\bar{\mathbf{x}}_{[0,T]}, \bar{\mathbf{u}}_{[0,T-1]})$ of (2) is uniquely determined by $\bar{\mathbf{x}}(0) \in \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,L-T]})$ and $\bar{\mathbf{u}}_{[0,T-1]} \in \operatorname{Im}\mathcal{H}_T(\mathbf{u}_{[0,L-1]})$. Recalling that $\mathcal{G}_T(L)$ is full-row rank, *(i)* follows. For *(ii)*, let α satisfy (6) and consider the initial state $\bar{\mathbf{x}}(0) = \mathcal{H}_1(\mathbf{x}_{[0,L-T]})\alpha$ and input sequence $\bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\alpha$. Then,

$$\begin{aligned}\bar{\mathbf{x}}_{[1,T]} &= [\mathcal{O}_T\ \mathcal{P}_T\ \mathcal{Q}_T]\left[\frac{\bar{\mathbf{x}}(0)}{\frac{\mathcal{H}_T(\bar{\mathbf{x}} \otimes \bar{\mathbf{u}}_{[0,T-1]})}{\bar{\mathbf{u}}_{[0,T-1]}}}\right]\\ &= [\mathcal{O}_T\ \mathcal{P}_T\ \mathcal{Q}_T]\,\mathcal{G}_T(L)\alpha = \mathcal{H}_T(\mathbf{x}_{[1,L]})\alpha,\end{aligned}$$

where we have employed (5). The conclusion follows by noting $\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}) = [\mathcal{H}_1(\mathbf{x}_{[0,L-T]}); \mathcal{H}_T(\mathbf{x}_{[1,L]})]$. ☐

B. Online experiment for T*-persistence of excitation*

We discuss next how to ensure that the available data is T-persistently exciting. Based on our discussion above, $\mathbf{v}_{\{0,L-1\}}$ being persistently exciting of order $n+T$ is enough to ensure the T-persistence of excitation of the data for bilinear systems. In contrast to the linear case, where $\mathbf{u}_{\{0,L-1\}}$ can be designed to be persistently exciting of any order by selecting control inputs offline, the persistence of excitation of $\mathbf{v}_{\{0,L-1\}}$ depends on both the control input $\mathbf{u}(t)$ and the system state $\mathbf{x}(t)$. Due to the unknown nonlinear dynamics, there is no available closed-form expression of $\mathbf{x}(t)$ in terms of $\mathbf{u}(t)$. Hence, selecting control inputs offline may not guarantee $\mathbf{v}_{\{0,L-1\}}$ to be persistently exciting of order $n+T$, which motivates an online approach to design $\mathbf{u}$. To tackle this, we draw inspiration from [13], [16] to propose an experiment design approach for bilinear systems that yields T-persistently exciting data. We start with some useful facts.

Proposition III.5. *(Scaled persistently exciting input returns a full-row rank Hankel matrix of state data): Consider system (2) and further assume that the pair $(\mathbf{A}, \mathbf{B})$ is controllable. Then, for any input sequence $\mathbf{u}_{\{0,L-1\}}$ that is persistently exciting of order $n+k$, there exists $\bar{\varepsilon}$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, the input sequence $\varepsilon\mathbf{u}_{\{0,L-1\}}$ with initial state $\mathbf{x}(0) = \mathbf{0}$ ensures $\mathcal{H}_k(\mathbf{x}_{[1,L]})$ is full-row rank.*

We omit the proof for space reasons, but note that the result follows by using for the higher-order case of $k \geq 1$ the same arguments employed in [16] for the case of $k = 1$ (note that the initial state $\mathbf{x}(0) = \mathbf{0}$ remains the same after scaling by ε). The next result generalizes [13, Thm. 2] to bilinear systems.

Proposition III.6. *(Property on the left kernel of $\mathcal{G}_T(t)$ when it is not full-row rank): Suppose $\mathcal{G}_T(t)$ is not full-row rank for some $t \geq T$. If*

$$\begin{bmatrix}\mathbf{x}(t-T+1)\\ \hline \mathbf{u}_{[t-T+1,t-1]}\\ \hline \mathbf{x} \otimes \mathbf{u}_{[t-T+1,t-1]}\end{bmatrix} \in \operatorname{Im}\begin{bmatrix}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-2]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-2]})\end{bmatrix}, \tag{7}$$

then there must exist $\xi \in \mathbb{R}^n$, $\eta_1, \ldots, \eta_T \in \mathbb{R}^m$, and $\chi_1, \ldots, \chi_T \in \mathbb{R}^{mn}$ such that the following holds

$$[\xi^\top\ \eta_1^\top\ \cdots\ \eta_T^\top\ \chi_1^\top\ \cdots\ \chi_T^\top]\mathcal{G}_T(t) = \mathbf{0}, \tag{8}$$

with at least one in $\{\eta_T, \chi_T\}$ not equal to $\mathbf{0}$.

Proof. We reason by contradiction. Suppose all vectors of the form $[\xi^\top\ \eta_1^\top\ \cdots\ \eta_T^\top\ \chi_1^\top\ \cdots\ \chi_T^\top]$ in the left kernel of $\mathcal{G}_T(t)$ satisfy that both η_T and χ_T are equal to $\mathbf{0}$. Then,

$$[\xi^\top\ \eta_1^\top \cdots \eta_{T-1}^\top\ \chi_1^\top \cdots \chi_{T-1}^\top]\begin{bmatrix}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-2]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-2]})\end{bmatrix} = \mathbf{0}.$$

Combining this with (7), we deduce that

$$[\xi^\top\ \eta_1^\top \cdots \eta_{T-1}^\top\ \chi_1^\top \cdots \chi_{T-1}^\top]\begin{bmatrix}\mathcal{H}_1(\mathbf{x}_{[0,t-T+1]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-1]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-1]})\end{bmatrix} = \mathbf{0}.$$

Combining this with the fact that

$$\begin{bmatrix} \mathcal{H}_1(\mathbf{x}_{[1,t-T+1]}) \\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[1,t-1]}) \\ \hline \mathcal{H}_{T-1}(\mathbf{x}\otimes\mathbf{u}_{[1,t-1]}) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \mathcal{G}_T(t),$$

we obtain

$$\begin{bmatrix} \xi^\top\mathbf{A} & \xi^\top\mathbf{B} & \eta_1^\top & \cdots & \eta_{T-1}^\top & \xi^\top\mathbf{N} & \chi_1^\top & \cdots & \chi_{T-1}^\top \end{bmatrix} \mathcal{G}_T(t) = \mathbf{0}.$$

Consequently, given our hypothesis of contradiction, η_{T-1} and χ_{T-1} must both be equal to $\mathbf{0}$. Following a similar procedure iteratively, we conclude that $\eta_{T-1} = \cdots = \eta_1 = \mathbf{0}$ and $\chi_{T-1} = \cdots = \chi_1 = \mathbf{0}$. This implies that $\operatorname{Im}\mathcal{G}_T(t) = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) \times \mathbb{R}^{(m+mn)T}$. Left multiplying by $[\mathbf{A}\ \mathbf{B}\ \mathbf{0}\ \mathbf{N}\ \mathbf{0}]$ on both sides, we obtain $\mathbf{A}\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) + \operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{N} = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[1,t-T+1]})$. Since $\mathbf{x}(t-T+1) \in \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$, then $\mathbf{A}\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) + \operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{N} = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$. This implies $\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$ is an $\mathbf{A}$-invariant subspace containing $\operatorname{Im}[\mathbf{B}\ \mathbf{N}]$. Since the reachable subspace of the pair $(\mathbf{A}, [\mathbf{B}\ \mathbf{N}])$ is $\mathbb{R}^n$ by Assumption II.1, and the fact that it is also the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Im}[\mathbf{B}\ \mathbf{N}]$, we deduce that $\mathbb{R}^n \subseteq \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$. Therefore $\mathbb{R}^{(m+mn)T+n} \subseteq \operatorname{Im}\mathbf{x}_{[0,t-T]} \times \mathbb{R}^{(m+mn)T} = \operatorname{Im}\mathcal{G}_T(t)$, which contradicts the fact that $\mathcal{G}_T(t)$ is not full-row rank. □

Based on Propositions III.5 and III.6, now we introduce the online control experiment procedure in Algorithm 1 to ensure the data is T-persistently exciting. The underlying idea of the strategy is to increase the row rank of $\mathcal{G}_T(t)$ at each step.

Algorithm 1 Online control experiment design

Input: $\mathbf{x}(0) = \mathbf{0}$, $\|\mathbf{u}(i)\| < \epsilon$ for $i = 0, \dots, T-1$ s.t. $\mathcal{G}_T(T) \neq \mathbf{0}$, ϵ sufficient close to 0, $t := T$, $k := 1$
repeat
  while $\mathcal{H}_{n+k}(\mathbf{u}_{[0,t-1]})$ is full-row rank **do**
    $k \leftarrow k+1$ ▷ Increase order
  end while
  if (7) holds **then**
    select $\xi \in \mathbb{R}^n$, $\eta = \begin{bmatrix}\eta_1^\top \dots \eta_T^\top\end{bmatrix}^\top \in \mathbb{R}^{mT}$, and $\chi = \begin{bmatrix}\chi_1^\top \dots \chi_T^\top\end{bmatrix}^\top \in \mathbb{R}^{mnT}$ s.t. (8) holds, with $\begin{bmatrix}\eta_T^\top & \chi_T^\top\end{bmatrix} \neq \mathbf{0}$
    if $\eta_T^\top + \chi_T^\top(\mathbf{x}(t)\otimes I) \neq \mathbf{0}$ **then**
      choose $\|\mathbf{u}(t)\| < \epsilon$ s.t. $\xi^\top\mathbf{x}(t-T+1) + \eta^\top\mathbf{u}_{[t-T+1,t]} + \chi^\top\mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]} \neq \mathbf{0}$ holds
    else
      choose $\|\mathbf{u}(t)\| < \epsilon$ s.t. rowrk $(\mathcal{H}_{n+k}(\mathbf{u}_{[0,t]}))$ increases
    end if
  else
    choose $\|\mathbf{u}(t)\| < \epsilon$ arbitrarily
  end if
  $t \leftarrow t+1$ ▷ Update iteration
until $\mathcal{G}_T(t)$ is full-row rank
$L \leftarrow t$
Output: Full-row rank $\mathcal{G}_T(L)$

Theorem III.7. *(Online control experiment design for T-persistently exciting data): Let $(\mathbf{A},\mathbf{B})$ be controllable and design the control experiment for system* (2) *according to Algorithm 1. Then the output $\mathcal{G}_T(L)$ is full-row rank.*

Proof. Given $t \geq T$, assume $\mathcal{G}_T(t)$ is not full-row rank. If (7) does not hold, it is easy to see that any choice of $\mathbf{u}(t)$ leads to rowrk $(\mathcal{G}_T(t+1)) >$ rowrk $(\mathcal{G}_T(t))$. Hence, we concentrate on the case when (7) holds. In this case, from Proposition III.6, we know there exist $\xi \in \mathbb{R}^n$, $\eta_1, \dots, \eta_T \in \mathbb{R}^m$, and $\chi_1, \dots, \chi_T \in \mathbb{R}^{mn}$, with at least one in $\{\eta_T, \chi_T\}$ not equal to $\mathbf{0}$ making (8) hold. We aim to design $\mathbf{u}(t)$ to satisfy $\xi^\top\mathbf{x}(t-T+1) + \eta^\top\mathbf{u}_{[t-T+1,t]} + \chi^\top\mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]} \neq \mathbf{0}$ so that $[\mathbf{x}(t-T+1); \mathbf{u}_{[t-T+1,t]}; \mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]}]$ does not belong to $\operatorname{Im}\mathcal{G}_T(t)$, which ensures rowrk $(\mathcal{G}_T(t+1)) >$ rowrk $(\mathcal{G}_T(t))$. Such $u(t)$ can be found as long as $\eta_T^\top + \chi_T^\top(\mathbf{x}(t)\otimes I) \neq \mathbf{0}$. If this is not the case, any selection of $\mathbf{u}(t)$ will not affect whether the row rank of $\mathcal{G}_T(t)$ will increase or not at this time step. We prove by contradiction that this situation will not occur indefinitely under Algorithm 1. Suppose $\eta_T^\top + \chi_T^\top(\mathbf{x}(\ell)\otimes I) = \mathbf{0}$ holds for all $\ell \geq t$, it then follows that $\eta_T^\top + \chi_T^\top(\mathcal{H}_1(\mathbf{x}_{[t,\ell]})\alpha\otimes I) = \mathbf{0}$ for any $\alpha \in \mathbb{R}^{\ell-t+1}$ with $\mathbf{1}^\top\alpha = 1$. According to Algorithm 1, the order k is increased, followed by an input selection that makes $\mathcal{H}_{n+k}(\mathbf{u}_{[0,\ell]})$ full-row rank. Let ℓ sufficiently large so that $k > t$. In this case, $\mathcal{H}_{n+t}(\mathbf{u}_{[0,\ell]})$ is full-row rank and, using Proposition III.5, $\mathcal{H}_t(\mathbf{x}_{[1,\ell]})$ is full-row rank too. The latter implies that $\mathcal{H}_1(\mathbf{x}_{[t,\ell]})$ is full-row rank. Since at least one in $\{\eta_T, \chi_T\}$ is not equal to $\mathbf{0}$, there must exist $\alpha \in \mathbb{R}^{\ell-t+1}$ with $\mathbf{1}^\top\alpha = 1$ such that $\eta_T^\top + \chi_T^\top(\mathcal{H}_1(\mathbf{x}_{[t,\ell]})\alpha\otimes I) \neq \mathbf{0}$ holds, which is a contradiction. This shows that Algorithm 1 increases the row rank of $\mathcal{G}_T(t)$ by one after finitely many steps, and hence, it eventually terminates with a full-row rank $\mathcal{G}_T(L)$. □

Note that the controllability assumption on the pair $(\mathbf{A},\mathbf{B})$ is only necessary to ensure Algorithm 1 is successful, cf. Theorem III.7. Our design methodology below is still valid as long as a full row-rank matrix $\mathcal{G}_T(L)$ can be obtained.

IV. Data-driven control design

Here, we describe an algorithmic procedure to find a local solution of the optimal control problem (P1) using T-persistently exciting data. Our first step is to provide an equivalent data-based representation of the optimization. We then iteratively apply a convex-concave procedure to solve it efficiently. The next result provides a data-based formulation of (P1), provided the available data is T-persistently exciting.

Theorem IV.1. *(Data-based reformulation of optimal control problem): Given a T-persistently exciting data set $\mathbf{x}_{\{0,L\}}$ and $\mathbf{u}_{\{0,L-1\}}$,* (P1) *is equivalent to the data-based optimization:*

$$\begin{aligned} \min_{\alpha} \quad & \sum_{t=0}^{T-1} \bar{\mathbf{x}}^\top(t)\mathbf{Q}\bar{\mathbf{x}}(t) + \bar{\mathbf{u}}^\top(t)\mathbf{R}\bar{\mathbf{u}}(t) \\ \text{s.t.} \quad & \begin{bmatrix} \bar{\mathbf{x}}_{[0,T]} \\ \hline \bar{\mathbf{u}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \end{bmatrix}\alpha, \\ & \bar{\mathbf{x}}(0) = \mathbf{x_0}, \bar{\mathbf{x}}(T) = \mathbf{x_f}, \text{(6) holds.} \end{aligned} \tag{P2}$$

The proof of this result readily follows from Lemma III.4. Notice that the optimization problem (P2) is nonconvex because of the presence of bilinear terms $\alpha_i\alpha_j$, $i,j \in \{1,\dots,L-T+1\}$, in the constraints. Here, we describe a convex–concave procedure from [28] that can be iteratively employed to solve it. We first describe the bilinear terms with new variables $r_{i,j} = \alpha_i \alpha_j$, which we employ in the constraints in (P2) to make them all become affine. We represent this set of constraints by $\mathcal{A}_1(\alpha, r) = \mathbf{0}$. Additionally, we write each equality $r_{i,j} = \alpha_i \alpha_j$ with the following equivalent representation

$$(\alpha_i + \alpha_j)^2 - (\alpha_i^2 + \alpha_j^2) - 2r_{i,j} \le 0,$$
$$(\alpha_i^2 + \alpha_j^2) - (\alpha_i + \alpha_j)^2 + 2r_{i,j} \le 0.$$

We gather all these new nonconvex constraints in the expression $\mathcal{C}_1(\alpha) - \mathcal{C}_2(\alpha) + \mathcal{A}_2(r) \le 0$, where $\{\mathcal{C}_i\}_{i=1}^2$, and $\mathcal{A}_2$ are vector-valued convex and affine functions, resp. Using $\mathcal{C}_0$ to denote the convex cost function, (P2) reads

$$\begin{aligned} \min_{\alpha, r} \quad & \mathcal{C}_0(\alpha) \\ \text{s.t.} \quad & \mathcal{C}_1(\alpha) - \mathcal{C}_2(\alpha) + \mathcal{A}_2(r) \le \mathbf{0}, \\ & \mathcal{A}_1(\alpha, r) = \mathbf{0}. \end{aligned} \tag{P3}$$

The inequality in (P3) can be convexified linearizing the concave function $-\mathcal{C}_2$. We perform such convexification iteratively to yield Algorithm 2, which has non-polynomial time complexity [28]. At each iteration, the algorithm solves a convex quadratically constrained quadratic program, with complexity [29] $O(\sqrt{M}(M + N)N^2)$ (M constraints and N variables). Here, $M = (L - T)^2 + L + T + 3$ and $N = \frac{(L-T+1)(L-T+2)}{2}$. The next result follows from [28, Sec. 1.3].

Lemma IV.2. *(Convergence to critical point of* (P2)*): Given a feasible initial point α^0, all iterates of Algorithm 2 are feasible, $\{\mathcal{C}_0(\alpha^k)\}_{k=1}^{\infty}$ decreases monotonically, and $\{\alpha^k\}_{k=1}^{\infty}$ converges to a critical point $\alpha^\star$ of* (P2)*.*

Algorithm 2 Convex-concave procedure to solve ((P3))

1: **Given** Initial feasible point α^0, $k := 0$.
2: **repeat**
3: Let $\bar{\mathcal{C}}_2(\alpha, \alpha^k) \triangleq \mathcal{C}_2(\alpha^k) + \nabla_\alpha \mathcal{C}_2(\alpha^k)^\top (\alpha - \alpha^k)$ ▷ Convexifying the constraint
4: Set α^{k+1} to be the solution of the convex problem

$$\begin{aligned} \min_{\alpha, r} \quad & \mathcal{C}_0(\alpha) \\ \text{s.t.} \quad & \mathcal{C}_1(\alpha) - \bar{\mathcal{C}}_2(\alpha, \alpha^k) + \mathcal{A}_2(r) \le 0 \\ & \mathcal{A}_1(\alpha, r) = 0 \end{aligned}$$

5: $k \leftarrow k + 1$ ▷ Update iteration
6: **until** convergence

V. Simulation examples

Here we illustrate the effectiveness of the proposed data-based approach in solving (P1) and compare it against the model-based approaches in [24], [26] for bilinear systems.

Example V.1. *(Population control):* We consider a population control problem introduced in [24, Example 1] evolving in continuous time. For the horizon $T = 20$, we use a first-order Euler discretization with stepsize 0.1. The resulting discrete-time bilinear system is $\mathbf{x}(t+1) = \mathbf{x}(t) + 0.1\mathbf{x}(t)\mathbf{u}(t)$. We take $\mathbf{Q} = \mathbf{R} = 1$ and consider $\mathbf{x_0} = 1$, $\mathbf{x_f} = \frac{1}{3}$. We perform a control experiment with $L = 60$ using randomly generated inputs, and verify that the resulting $\mathcal{G}_{20}(60)$ is full-row rank. Algorithm 2 obtains a local optimum $\alpha^\star$ of (P2). Fig. 1(a) shows the trajectories, both displaying similar performance, obtained from the data-based solution in Theorem IV.1 with that of the model-based iterative method [24]. ▷

Example V.2. *(Minimum-energy control problem):* Consider the bilinear system from [26, Example 4.5], $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t)$, where $\mathbf{N} = \text{diag}(0.1, 0.2, 0.3, 0.4, 0.5)$,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.024 & 0 & 0 \\ 1 & 0 & -0.26 & 0 & 0 \\ 0 & 1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & -0.06 \\ 0 & 0 & 0.15 & 1 & 0.5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.5 \end{bmatrix}.$$

We consider the minimum-energy control problem ($\mathbf{Q} = \mathbf{0}$, $\mathbf{R} = I$) with $T = 10$. Let $\mathbf{x_0} = \mathbf{0}$ and $\mathbf{x_f} = [0.0004 \; -0.00038 \; 0.00318 \; 0.00062 \; 0.00219]^\top$. We perform a control experiment with $L = 74$ using Algorithm 1. The execution of Algorithm 1 here increases the row rank of $\mathcal{G}_T(t)$ monotonically for every $t \ge T$ until it becomes full-row rank (i.e., the algorithm never falls into Step 11). We solve (P2) using Algorithm 2. For comparison, we use the Gramian-based lower bound of the optimal cost value obtained in [26],

$$\sum_{t=0}^{T-1} \mathbf{u}^{\star\top}(t)\mathbf{u}^\star(t) \ge \mathbf{x}^\top(T)\mathcal{W}^{-1}\mathbf{x}(T),$$

where $\mathcal{W}$ is the reachability Gramian of the bilinear system. Fig. 1(b) compares this lower bound with the values obtained with the trajectories from the data-based solution in Theorem IV.1, showing a close agreement between the two. ▷

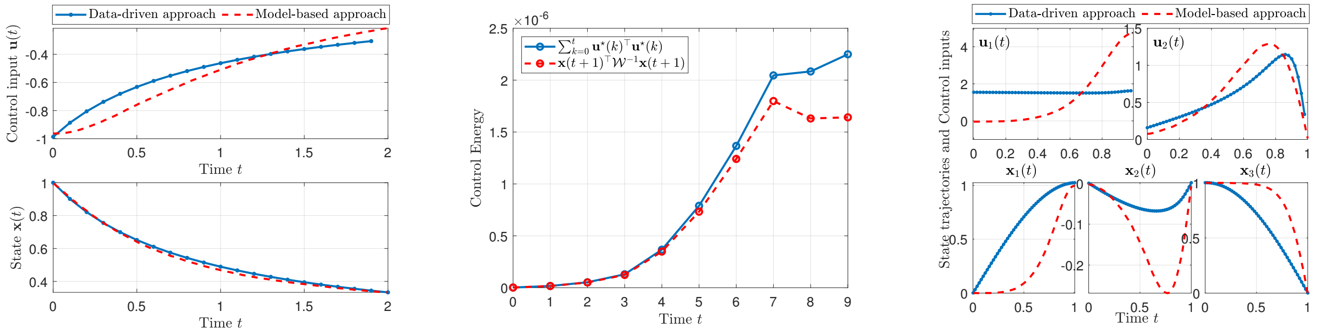
Example V.3. *(Minimum-energy control problem):* We consider a minimum-energy control example from [24, Example 2], for which we use a first-order Euler discretization with stepsize 0.02. The discrete-time bilinear system is $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \left[\sum_{j=1}^{3} \mathbf{x}_j(t)\mathbf{N}_j\right]\mathbf{u}(t)$, with

$$\mathbf{A} = \begin{bmatrix} 1 & -0.01 & 0 \\ 0.01 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{N}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.02 & 0 \end{bmatrix},$$
$$\mathbf{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.02 \end{bmatrix}, \mathbf{N}_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.02 \\ 0 & 0 \end{bmatrix}.$$

We consider $T = 50$ and perform a control experiment with $L = 452$ randomly generated inputs, and verify $\mathcal{G}_{50}(452)$ is full-row rank. We let $\mathbf{x_0} = [0 \; 0 \; 1]^\top$, $\mathbf{x_f} = [1 \; 0 \; 0]^\top$. We solve (P2) using Algorithm 2 to obtain $\alpha^\star$ and compare, cf. Fig. 1(c), the trajectories obtained from the data-based solution in Theorem IV.1 with that of the model-based iterative method [24], showing a better local optimum by the former. ▷

VI. Conclusions

We have presented a data-driven method to learn optimal controls of bilinear systems directly from input/state data without a priori knowledge of the matrices. We have provided an online control experiment design method to obtain



(a) Example V.1

(b) Example V.2

(c) Example V.3

Fig. 1: Performance of the proposed data-driven approach (solid blue lines) versus model-based approaches (dashed red lines). The total cost values are (a) $0.1 \times \sum_{t=0}^{19} \mathbf{x}^2(t) + \mathbf{u}^2(t) = 1.3346$ for the data-based approach and $\int_0^2 \mathbf{x}^2(\tau) + \mathbf{u}^2(\tau) d\tau = 1.3506$ for the model-based iterative method in [24]; (b) $\sum_{t=0}^{9} \mathbf{u}(t)^\top \mathbf{u}(t) = 2.25 \times 10^{-6}$ for the data-based approach and $\mathbf{x}(10)^\top \mathcal{W}^{-1} \mathbf{x}(10) = 1.64 \times 10^{-6}$ for the Gramian-based lower bound in [26]; and (c) $0.02 \times \sum_{t=0}^{49} \mathbf{u}^\top(t)\mathbf{u}(t) = 2.7999$ for the data-based approach and $\int_0^1 \mathbf{u}^\top(\tau)\mathbf{u}(\tau) d\tau = 4.7976$ for the model-based iterative method in [24].

sufficiently informative data and introduced an equivalent data-based reformulation of the original nonconvex optimal control problem and employed an iterative convex-concave algorithmic procedure to solve it. Simulations show data-based optimal control trajectories have comparable performance to those obtained by model-based ones. Future work will explore extensions to noisy data and robustness analysis, weaker notions under which data is sufficient to reconstruct optimal controls, online implementations of the convex-concave procedure as data becomes increasingly available, and distributed implementations for large-scale bilinear networks.

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