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# Multi-rate Sampled-data Observer Design for Nonlinear Systems with Asynchronous and Delayed Measurements

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*Abstract*— **In our previous work, we developed a multi-rate sampled-data observer design method in nonlinear systems with asynchronous sampling. In this article, possible measurement delays are accounted for in the multi-rate observer design. The proposed observer adopts an available multi-rate design in the time interval between two consecutive delayed measurements. A dead time compensation approach is developed to compensate for the effect of delay and update past estimates when a delayed measurement arrives. It is shown that stability of the multi-rate observer is preserved under nonconstant, arbitrarily large delays, in the absence of measurement errors. The proposed multi-rate multi-delay observer is applied to a gas-phase polyethylene reactor example and provides reliable estimates in the presence of nonuniform sampling and nonconstant delays.**

## I. INTRODUCTION

Motivated by many engineering applications, state estimation of a continuous-time dynamical system in the presence of sampled and delayed measurements has received lots of attention. In chemical processes, for example, product quality measurements are usually sampled infrequently and require off-line lab analysis, which inevitably introduces delay as a consequence of sample preparation, analysis and calculation. Process data is usually collected from multiple heterogeneous sensors of different sampling rates and different measurement delays and thus, continuous-time and/or single-rate sampled-data observer design methods from the literature are not directly applicable any more. Thus, the objective of this work is to develop a general methodology of multi-rate multi-delay observer for process monitoring in nonlinear systems.

Most of the observer design methods using delayed output are based on a chain of state observation algorithms, where various types of output delay (e.g., constant, piecewise, time-varying) have been considered. A chain structure algorithm was proposed for globally drift-observable systems with constant measurement delay in [1]. The chain observer consisted of a number of cascaded subsystems, where each subsystem reconstructed the system states at different delayed times. A similar methodology was applied to single-output systems with constant delay in [2], where enhanced design flexibility was achieved. To reduce the number of subsystems and avoid a long oscillatory transient behavior, an alternative cascade structure was developed in [3]. The assumption on constant delay has been relaxed to piecewise constant delay and time-varying delay in recent studies, e.g., [4], [5].

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The delayed output in all the above contributions was assumed to be continuous. As most of the product quality measurements in chemical processes are sampled infrequently and are available with not necessarily small delay, sampling and delay effects need to be simultaneously considered and compensated for in the observer design. A chain observer was designed in a class of triangular nonlinear systems with sampled, delayed measurements in [6]. A robust global exponential observer was proposed for certain classes of nonlinear systems under sampled measurement with a constant delay in [7]. In addition to these single-rate observer design methods considering delay, multi-rate multi-delay observer design was proposed and implemented in a polymerization reactor in [8], [9], where process data from multiple heterogeneous sensors were used in the observer design. Despite the fact that fairly good results have been achieved, stability analysis of a multi-rate multi-delay observer remained open. Other multi-rate estimation approaches based on extended Kalman filter [10], [11] and moving horizon estimation [12], [13] that consider measurement delays have also been studied.

In previous work of the authors [14], the problem of multi-rate observer design was first addressed in linear systems in the absence of measurement delay. Motivated by the single-rate observer design in [15], the multi-rate observer design in [14] was based on an available continuous-time design coupled with multiple, asynchronous inter-sample predictors for the sampled measurements. To handle measurement delays, the authors proposed a multi-rate multi-delay observer design method in linear systems in [16], based on an available multi-rate design combined with dead time compensation. Stability and robustness of the delay-free multi-rate observer were shown to be preserved under nonconstant delays.

In this work, we consider the problem of observer design in nonlinear systems where measurements become available with different sampling rates and different delays. Motivated by the nonlinear multi-rate observer design in [17] and the dead time compensation method in [16], the multi-rate multi-delay observer design is carried out in a two-step manner. First, a delay-free multi-rate observer design [17], outlined in Section II, is adopted as a starting point and estimates of the current state are obtained in the time interval between two consecutive delayed measurements. Second, we propose a dead time compensation approach in Section III, in the same spirit as [16] for linear systems, to handle output delays in nonlinear observer design. It is shown that stability of the underlying multi-rate observer will be preserved under nonconstant and arbitrarily large delays. The proposed observer is tested through a simulation example in Section IV.

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## II. PRELIMINARIES

### A. Notations

- By $\mathcal{K}^+$, we denote the class of positive, continuous functions defined on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geqslant 0\}$. We say that a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. We denote by $\mathcal{K}$ the set of positive definite, increasing and continuous functions. We say that a positive definite, increasing and continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if $\lim_{s\to+\infty} \rho(s) = +\infty$. We denote by $\mathcal{KL}$ the set of all continuous functions $\sigma = \sigma(s,t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the two properties: (i) the mapping $\sigma(\cdot,t)$ is of class $\mathcal{K}$ for each $t \geqslant 0$; (ii) the mapping $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty} \sigma(s,t) = 0$ for each $s \geqslant 0$.
- The set of nonnegative integers is denoted by $\mathbb{Z}_+$.
- $\mathbb{R}^n_+ := \{[x_1,\ldots,x_n]' \in \mathbb{R}^n : x_1 \geqslant 0,\ldots,x_n \geqslant 0\}$. Let $x,y \in \mathbb{R}^n$. We say that $x \leqslant y$ if and only if $(y-x) \in \mathbb{R}^n_+$. We say that a function $\rho : \mathbb{R}^n_+ \to \mathbb{R}_+$ is of class $\mathcal{N}_n$, if $\rho$ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leqslant \rho(y)$ for all $x,y \in \mathbb{R}^n_+$ with $x \leqslant y$.
- For every positive integer $l$ and an open, non-empty set $A \subseteq \mathbb{R}^n$, $C^l(A;\Omega)$ denotes the class of continuous functions on $A$ with continuous derivatives of order $l$, which take values in $\Omega \subseteq \mathbb{R}^m$. $C^0(A;\Omega)$ denotes the class of continuous functions on $A$, which take values in $\Omega$.
- We denote by $\|\cdot\|_{\mathcal{X}}$ the norm of the normed linear space $\mathcal{X}$. By $|\cdot|$, we denote the $\ell_1$-norm of $\mathbb{R}^n$. Let $I \subseteq \mathbb{R}_+$ be an interval and $D \subseteq \mathbb{R}^l$ be a non-empty set. By $\mathcal{L}^\infty_{loc}(I;D)$, we denote the class of all Lebesgue measurable and locally bounded functions $u : I \to D$. For $u \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^n)$, we define the norm $\|u(t)\|_{\mathcal{U}} := \sum_{i=1}^n \sup_{\tau\in[0,t]} |u_i(\tau)|$. Notice that $\sup_{\tau\in[0,t]} |u_i(\tau)|$ denotes the actual supremum of $|u_i(t)|$ on $[0,t]$.

### B. Delay-free Multi-rate Observer Design

This section outlines the main results in [17] on multi-rate observer design for nonlinear systems under asynchronous sampling, in the absence of measurement delays. It is based on a continuous-time design coupled with inter-sample output predictors. The stability and robustness properties of the observer will be reviewed. The delay-free multi-rate observer design will serve as a point of departure when measurement delays are considered.

A reduced-order observer formulation is adopted for multi-output systems, as lower dimensionality can ease implementation of the observer. Continuous estimates of the sampled outputs in each sampling interval can be generated from the inter-sample predictors. Therefore, a reduced-order observer formulation will be the focus of this work.

Consider a nonlinear forward complete system with continuous outputs, where without loss of generality, the outputs are assumed to be a part of the states

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ y(t) &= x_M(t) + v(t) \end{aligned} \tag{1}$$

where $x_R \in \mathbb{R}^{n-m}$ is the unmeasured state, $x_M \in \mathbb{R}^m$ is the remaining state that is directly measured, $y$ is the continuous outputs subject to measurement errors $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^m)$, and $f_R \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m;\mathbb{R}^{n-m})$, $f_M \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m;\mathbb{R}^m)$ with $f_R(0,0) = 0$, $f_M(0,0) = 0$.

Suppose that there exists a robust observer for system (1) with respect to measurement errors, in the sense of Definition 1 in [17]

$$\begin{aligned} \dot{z}(t) &= F(z(t), y(t)) \\ \hat{x}_R(t) &= \Psi(z(t), y(t)) \end{aligned} \tag{2}$$

with $z \in \mathbb{R}^k$ being the observer states, $\hat{x}_R \in \mathbb{R}^{n-m}$ being the state estimates, and $F \in C^1(\mathbb{R}^k \times \mathbb{R}^m;\mathbb{R}^k)$, $\Psi \in C^1(\mathbb{R}^k \times \mathbb{R}^m;\mathbb{R}^{n-m})$ with $F(0,0) = 0$, $\Psi(0,0) = 0$. Hence, there exist functions $\sigma \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu \in \mathcal{K}^+$ and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of systems (1) and (2) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^m)$ exists for all $t \geqslant 0$ and satisfies the following estimates

$$\begin{aligned} |\hat{x}_R(t) - x_R(t)| \leqslant \sigma(|(x_{R,0}, x_{M,0}, z_0)|, t) \\ + \gamma(\|v(t)\|_{\mathcal{U}}), \forall t \geqslant 0 \end{aligned} \tag{3a}$$

$$\begin{aligned} |z(t)| \leqslant \mu(t)[a(|(x_{R,0}, x_{M,0}, z_0)|) \\ + p(\|v(t)\|_{\mathcal{U}})], \forall t \geqslant 0 \end{aligned} \tag{3b}$$

Next we present a robust multi-rate sampled-data observer with respect to measurement errors for multi-rate systems, in the sense of Definition 2 in [17]. It is based on a robust observer (2) coupled with inter-sample predictors. Consider system (1) with asynchronous, sampled outputs

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ y^i(t^i_j) &= x^i_M(t^i_j) + v^i(t^i_j), \quad j \in \mathbb{Z}_+, i = 1,2,\ldots,m \end{aligned} \tag{4}$$

where $t^i_j$ denotes the $j$-th sampling time for the state $x^i_M$, at some sequence of time instants $S = \{t_k\}^\infty_{k=0}$ (a partition of $\mathbb{R}_+$). The sampling times of each sensor are not necessarily uniformly spaced, but satisfying $0 < t^i_{j+1} - t^i_j \leqslant r$ for all $j \in \mathbb{Z}_+$, where $r$ is the maximum sampling period among all the sensors. There is a one-to-one mapping from $\{t_k\}^\infty_{k=0}$ to $\{t^i_j : j \in \mathbb{Z}_+, i = 1,2,\ldots,m\}$.

Consider a multi-rate sampled-data observer design of the following form for all $t \in [t_k, t_{k+1})$

$$\begin{aligned} \dot{z}(t) &= F(z(t), w(t)) \\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)) \\ w^i(t_{k+1}) &= y^i(t_{k+1}) \\ t_{k+1} &= t_k + rd(t_k) \\ \hat{x}_R(t) &= \Psi(z(t), w(t)) \end{aligned} \tag{5}$$

where $w \in \mathbb{R}^m$ is the predicted outputs, $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+;[0,1])$ generates the actual sampling schedule allowed to be time-varying. The multi-rate design (5) consists of a continuous-time observer coupled with $m$ inter-sample predictors. Therefore, the existence of a robust continuous-time observer (2) is a prerequisite for the observer design in a multi-rate system. The inter-sample predictors continuously generate estimates of the sampled outputs in each sampling interval. $w^i(t)$ will get reinitialized once a new measurement $y^i(t_{k+1})$ becomes available, while the rest of the predictor states do not change until their measurements are obtained.

This design offers two attractive features: (i) a continuous-time observer design from the literature can be reused in the context of a multi-rate design by coupling with predictors, (ii) the unmeasured state is reconstructed from the observer, while continuous estimates of the sampled measurements are obtained from the inter-sample predictors. It was seen in [14] that the model-based prediction can better estimate the inter-sample behavior as opposed to a sample-and-hold strategy, especially under large sampling period.

From the main results in [17], suppose that there exists a robust observer (2) for system (1) with respect to measurement errors. Suppose that there exist constants $C^i \geqslant 0$ and functions $\bar{\sigma}^i \in \mathcal{KL}$ for all $i = 1, 2, \ldots, m$, such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of the overall system (i.e., the continuous-time system (1) and the robust observer (2)) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geqslant 0$ and satisfies the following estimate

$$\begin{aligned} &|f^i_M(\Psi(z(t), x_M(t) + v(t)), x_M(t) + v(t)) \\ &- f^i_M(x_R(t), x_M(t))| \\ &\leqslant \bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + C^i \|v(t)\|_{\mathcal{U}}, \forall t \geqslant 0 \end{aligned} \tag{6}$$

In addition, suppose that (i) $3rC^i m < 1$ for $i = 1, 2, \ldots, m$; (ii) $3\gamma(ms) < s$ for all $s > 0$, where $\gamma \in \mathcal{N}_1$ is the gain function in the estimate (3a) of the robust observer.

If the above conditions are satisfied in a continuous-time observer design, then it was proved in [17] that (5) is a robust multi-rate sampled-data observer for system (4) with respect to measurement errors. In other words, there exist functions $\tilde{\sigma}_R, \tilde{\sigma}_M \in \mathcal{KL}$, $\tilde{\gamma}_R, \tilde{\gamma}_M, \tilde{p} \in \mathcal{N}_1$, $\tilde{\mu} \in \mathcal{K}^+$ and $\tilde{a} \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, w_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1]) \times \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of the overall system of (4) and (5) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; [0,1])$ and $v \in \mathcal{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^m)$ satisfies the following estimates

$$\begin{aligned} |\hat{x}_R(t) - x_R(t)| \leqslant \tilde{\sigma}_R(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) \\ + \tilde{\gamma}_R(\|v(t)\|_{\mathcal{U}}), \forall t \geqslant 0 \end{aligned} \tag{7a}$$

$$\begin{aligned} |w(t) - x_M(t)| \leqslant \tilde{\sigma}_M(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) \\ + \tilde{\gamma}_M(\|v(t)\|_{\mathcal{U}}), \forall t \geqslant 0 \end{aligned} \tag{7b}$$

$$\begin{aligned} |(z(t), w(t))| \leqslant \tilde{\mu}(t)[\tilde{a}(|(x_{R,0}, x_{M,0}, z_0, w_0)|) \\ + \tilde{p}(\|v(t)\|_{\mathcal{U}})], \forall t \geqslant 0 \end{aligned} \tag{7c}$$

The input-to-output stability property was established for the observer error and predictor error with respect to measurement noises. The proof was based on the Karafyllis-Jiang vector small-gain theorem (see [18]). Furthermore, the multi-rate design provides robustness with respect to perturbations in the sampling schedule.

## III. MAIN RESULTS

In this section, we adopt an available multi-rate observer design (5) and propose a dead time compensation algorithm to handle possible measurement delays, in the same spirit as [16] for linear systems. Measurement error is not considered (i.e., $v \equiv 0$). The multi-rate multi-delay observer is shown to be asymptotically stable in the presence of nonconstant and arbitrarily large delays, as long as the underlying delay-free multi-rate observer is stable.

### A. Proposed Multi-rate Multi-delay Observer Design

Now consider a multi-rate system (4) with possible delays in the sampled outputs $y^i(t^i_j)$ for all $j \in \mathbb{Z}_+, i = 1, 2, \ldots, m$, in the absence of measurement errors

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)), \quad t \geqslant -\Delta \\ y^i(t^i_j) &= x^i_M(t^i_j - \delta^i_j) \end{aligned} \tag{8}$$

The $j$-th measurement of $x^i_M$ becomes available at $t^i_j$ after some possible delay $\delta^i_j \in [0, \Delta]$. In other words, the output $y^i(t^i_j)$ is a function of the state $x^i_M$ sampled at time $t^i_j - \delta^i_j$. The measurement delay $\delta^i_j$ is not constant but is assumed bounded by a positive real number $\Delta$. The sampling times of each measurement are not necessarily uniformly spaced, but satisfying $0 < \left|(t^i_{j'} - \delta^i_{j'}) - (t^i_j - \delta^i_j)\right| \leqslant r$ for any two consecutive sampling instants.

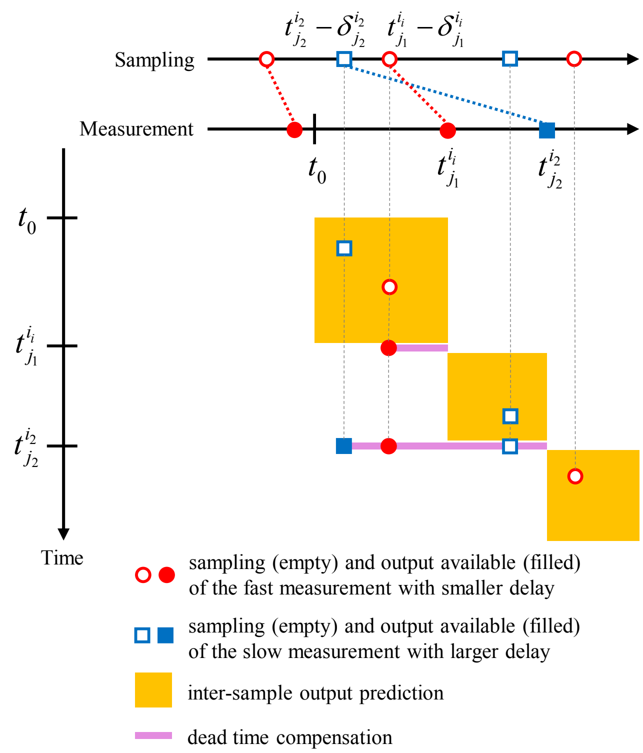


Fig. 1. An illustration of the proposed two-step estimation process of a multi-rate multi-delay observer in the presence of two sampled and delayed measurements starting from $t_0$.

The proposed observer for the multi-rate system (8) with multiple measurement delays is based on a multi-rate design

(5) combined with dead time compensation. As depicted in Fig. 1, the estimation process contains two steps. First, dead time compensation will be triggered once a sampled, delayed measurement is obtained at $t^i_j$. Past estimates are recalculated by integrating the observer and compensator equations from $t^i_j - \delta^i_j$ to $t^i_j$, where any available measurement can be used as an undelayed output and reinitializes the corresponding compensator at its sampling time. The state estimates at $t^i_j$ are consequently updated at the end of the compensation. This step ensures that these available measurements are used in the observer without delay, in the same order as they are sampled. Second, the updated estimates at $t^i_j$ are used as the initial condition of the observer and the inter-sample predictors. The multi-rate multi-delay observer operates as a delay-free multi-rate observer between consecutive sampled measurements.

When a sampled, delayed measurement becomes available at $t^i_j$, dead time compensation is executed to update the past estimates. For all $t \in [t^i_j - \delta^i_j, t^i_j)$ where $\delta^i_j \neq 0$, we propose the following design of a multi-rate observer with dead time compensation

$$\dot{z}(t) = F(z(t), w(t)) \tag{9a}$$

$$\dot{w}(t) = f_M(\Psi(z(t), w(t)), w(t)) \tag{9b}$$

$$w^i(t^i_j - \delta^i_j) = y^i(t^i_j) \tag{9c}$$

$$w^{i'}(t^{i'}_{j'} - \delta^{i'}_{j'}) = y^{i'}(t^{i'}_{j'}), \quad \forall t^{i'}_{j'}, (t^{i'}_{j'} - \delta^{i'}_{j'}) \in [t^i_j - \delta^i_j, t^i_j) \tag{9d}$$

$$\hat{x}_R(t) = \Psi(z(t), w(t)) \tag{9e}$$

where $w \in \mathbb{R}^m$ is the compensator state representing the past estimates for $x_M(t), t \in [t^i_j - \delta^i_j, t^i_j)$. Equation (9c) shows the reinitialization step of the $i$-th dead time compensator by using the delayed measurement $y^i(t^i_j)$ at its sampling time $t^i_j - \delta^i_j$. The available outputs that are sampled and measured between $t^i_j - \delta^i_j$ and $t^i_j$ can be used to reset the compensators at their respective sampling times, as seen in (9d).

*Remark 1:* The observer state $z$, compensator state $w$, state estimates $\hat{x}_R$ and sampled outputs $y^{i'}$ in (9) all represent the past information in the system throughout the dead time compensation, which should be stored in a buffer. The past estimates are recalculated for the purpose of correcting the state estimates at current time $t^i_j$ and consequently, improving the estimation accuracy afterwards. The buffer memory will be finite as long as the upper bound of the delay $\Delta$ is finite, as will be discussed later.

Once the estimates at $t^i_j$ are obtained after the dead time compensation, inter-sample prediction comes into play in the interval between two consecutive measurements at $t^i_j$ and $t^{i'}_{j'}$. For all $t \in [t^i_j, t^{i'}_{j'})$, the multi-rate multi-delay observer is of the following form

$$\begin{aligned} \dot{z}(t) &= F(z(t), w(t)) \\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)) \\ \hat{x}_R(t) &= \Psi(z(t), w(t)) \end{aligned} \tag{10}$$

where $w \in \mathbb{R}^m$ denotes the predicted outputs. The predictors estimate the evolution of the sampled outputs, in the same spirit as in a delay-free multi-rate observer. If an undelayed measurement becomes available at $t^i_j$, inter-sample prediction will run immediately after reinitialization, and no dead time compensation will be needed. Algorithm 1 summarizes the estimation process of the proposed observer.

**Algorithm 1** Algorithm for Multi-rate Multi-delay Observer

**STEP 0**: Initialize $z(t_0)$, $w(t_0)$, and solve (10) for $[t_0, t^i_j)$
**STEP 1**: Calculate $z(t)$ and $w(t)$ when a sampled measurement becomes available at $t^i_j$
**if** $\delta^i_j > 0$ **then** ▷ Dead time compensation
  Solve (9) for $[t^i_j - \delta^i_j, t^i_j)$ and update $z(t^i_j)$, $w(t^i_j)$
**end if**
Reinitialize (10) with $z(t^i_j)$, $w(t^i_j)$, and solve it for $[t^i_j, t^{i'}_{j'})$
**STEP 2**: Set $t^i_j = t^{i'}_{j'}$ and go to Step 1

*Remark 2:* Unlike the aforementioned chain observers where a high dimensionality may be required to reconstruct the state in the case of large measurement delays [1]–[3], the proposed multi-rate multi-delay observer does not require a chain-like structure and the dimension of the observer (9) and (10) is greatly reduced to $n$. Furthermore, it can handle multiple nonconstant measurement delays.

### B. Stability Analysis

Past estimates for all $t \in [t^i_j - \delta^i_j, t^i_j)$ are recalculated in the dead time compensation, once a delayed measurement becomes available at $t^i_j$. Estimates at certain times may be calculated more than once, if the measurement order differs from the sampling order, e.g., the estimates from $(t^{i_1}_{j_1} - \delta^{i_1}_{j_1})$ to $t^{i_1}_{j_1}$ are calculated three times as seen in Fig. 1 (once from inter-sample prediction and twice from dead time compensation). We name the last updated estimates obtained from the multi-rate multi-delay observer "final estimates". We denote $\tilde{t}$ the most-recent sampling time where the measurements of all the samples taken before $\tilde{t}$ (including $\tilde{t}$) are available. It indicates that the final estimates are obtained for all $t \leqslant \tilde{t}$. As the measurements are used in the same order as the way they are sampled in the calculation of the final estimates, the final estimates $z(t)$, $w(t)$ and $\hat{x}_R(t)$ for all $t \leqslant \tilde{t}$ in the multi-rate multi-delay observer are identical to those in a delay-free multi-rate observer, under the same design parameters. Once the final estimates at $\tilde{t}$ are obtained, the stored measurements that are sampled before $\tilde{t}$ can be cleared from the buffer.

Despite the fact that the estimation process in Fig. 1 has two steps, stability of the multi-rate multi-delay observer will be presented in a unified manner, because the essence of both compensation and prediction is to predict the dynamic model forward.

It is straightforward to show that the estimates and predicted outputs are bounded for all $t \geqslant 0$. The fact that system (1) is forward complete implies the existence of functions $\mu \in \mathcal{K}^+$ and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, the solution $(x_R(t), x_M(t))$ of (1) with initial condition $(x_R(0), x_M(0)) = (x_{R,0}, x_{M,0})$ exists and satisfies the following condition for all $t \geqslant 0$

$$|(x_R(t), x_M(t))| \leqslant \mu(t)a(|(x_{R,0}, x_{M,0})|) \tag{11}$$

Obviously, $|(\hat{x}_R(t), w(t))|$ will be bounded before the first measurement becomes available, because the initial condition of the observer is finite. After the first measurement, the estimates in the compensation (or prediction) will be generated by forward predicting the model from $\tilde{t}$ with reinitialization at some sampling instants. The estimates at $\tilde{t}$ are identical to those in a delay-free multi-rate observer, which are bounded from (7a) and (7b). Therefore, $|(\hat{x}_R(t), w(t))|$ is bounded for all $t \geqslant \tilde{t}$.

Because of the previous assumption that the measurement delay in system (8) has a finite upper bound, $\tilde{t}$ will approach infinity as $t$ goes to infinity. From (7a) and (7b), we derive

$$\lim_{\tilde{t} \to +\infty} \hat{x}_R(\tilde{t}) = x_R(\tilde{t}) \tag{12a}$$

$$\lim_{\tilde{t} \to +\infty} w(\tilde{t}) = x_M(\tilde{t}) \tag{12b}$$

Thus, the observer can accurately estimate the actual state in the compensation and prediction as $t$ approaches infinity, in the absence of measurement errors. The reinitialization in the compensation does not affect the convergence property. An attractive feature of the approach is that it can handle the situation where the delayed measurement sequence is not in the same order as the sampling sequence, as seen in Fig. 1.

## IV. NUMERICAL EXAMPLE

In this section, the application of a multi-rate multi-delay observer is explored in an industrial gas-phase polyethylene reactor, where nonuniform sampling and measurement delay of on-line gas chromatography (GC) and off-line lab analysis will be considered. Nonlinear observer design methods will be adopted as the basis of the multi-rate multi-delay observer to deal with process nonlinearities, whereas a linear multi-rate multi-delay observer, derived from a linearized process model, was developed in [16].

A nonlinear reactor model has the form [16]

$$\begin{aligned}
\frac{dY}{dt} &= F_c a_c \quad k_d Y \quad \frac{(R_{M_1} M_{W_1} + R_{M_2} M_{W_2})Y}{B_w} \\
\frac{dT}{dt} &= \frac{H_f + H_g \quad H_{top} \quad H_r \quad H_{pol}}{M_r C_{pr} + B_w C_{ppol}} \\
\frac{d[In]}{dt} &= \frac{F_{In} \quad x_{In} b_t}{V_g} \\
\frac{d[M_1]}{dt} &= \frac{F_{M_1} \quad x_{M_1} b_t \quad R_{M_1}}{V_g} \\
\frac{d[M_2]}{dt} &= \frac{F_{M_2} \quad x_{M_2} b_t \quad R_{M_2}}{V_g} \\
\frac{d[H]}{dt} &= \frac{F_H \quad x_H b_t}{V_g} \\
\frac{dMI_c{}^{\frac{1}{3.5}}}{dt} &= \frac{1}{\tau_r} MI_i{}^{\frac{1}{3.5}} \quad \frac{1}{\tau_r} MI_c{}^{\frac{1}{3.5}} \\
\frac{dD_c{}^{1}}{dt} &= \frac{1}{\tau_r} D_i{}^{1} \quad \frac{1}{\tau_r} D_c{}^{1}
\end{aligned} \tag{13}$$

The definitions of all the variables in (13) and the values of the process parameters are listed in Tables 1 and 2 in [14].

As for system outputs, the reactor temperature is continuously measured on line without delay. The gas concentrations of inerts, ethylene, comonomer and hydrogen are normally sampled every $20$ min and measured by using on-line GC, which induces about $8$ min delay caused by sample preparation ($2.5$ min), sample analysis ($4$ min), and calculation ($1.5$ min). In addition, the off-line lab analysis of melt index and density is normally sampled every $40$ min with $60$ min measurement delay, which provides quality information of polyethylene [19]. Because of the difficulty in measuring the amount of active catalyst sites, it is necessary to monitor this quantity from a reliable on-line soft sensor. In addition, it is important to provide continuous and reliable estimates of the inter-sample behavior of those sampled outputs from GC and lab analysis, for quality control and monitoring purpose.

A continuous-time observer, which serves as the basis of the multi-rate multi-delay observer, will be designed by using the exact error linearization method [20] as follows

$$\dot{z}(t) = Az(t) + By(t) \tag{14}$$

where $B = \begin{bmatrix} 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \end{bmatrix}$ and $A = -0.00068$.

The immersion map $z = T(x)$ satisfies a system of linear partial differential equations, where it is possible to compute the solution $T(x)$ in the form of a multivariate Taylor series around the origin with truncation order $N = 4$.

The initial conditions of the process and the observer are given in Table 4 in [14]. The actual sampling schedule and the corresponding measurement delays are given in Table I.

The performance of the multi-rate multi-delay observer is illustrated in Fig. 2, where it is compared with a delay-free multi-rate observer with the same design parameters. Fig. 2(a) shows that the estimate from the multi-rate multi-delay observer has approximately the same convergence rate as that from the multi-rate design. Fig. 2(b)-(f) show the evolution of predicted outputs obtained from the inter-sample predictors, which reconstructs the inter-sample dynamic behavior under nonuniform sampling schedule.

## V. CONCLUSIONS

This work proposes a design method for multi-rate multi-delay observers in nonlinear systems. It is based on an available multi-rate observer design combined with dead time compensation, where asynchronous, sampled measurements, in the presence of possible measurement delays, are accounted for. Two attractive features of the proposed observer are that it inherits stability from a delay-free multi-rate observer and it can handle nonconstant, arbitrarily large delays, in the absence of measurement errors. The proposed observer has the same dimension as a multi-rate observer for a delay-free system. From the case study, we see that the multi-rate multi-delay observer can provide reliable estimation results. The presence of delay in the measurement inevitably slows down convergence of the observer.

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TABLE I

ACTUAL SAMPLING SCHEDULE AND MEASUREMENT DELAYS

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Gas chromatography | Sampling (min) | 5 | 23 | 43 | 62.5 | 81.5 | 102 | 122 | 140 | 161.5 | 179.5 | 199.5 | 219 | 238 |
| | Delay (min) | 8.0 | 8.7 | 8.5 | 7.5 | 8.0 | 8.0 | 8.2 | 7.8 | 8.5 | 8.3 | 8.0 | 8.2 | 7.7 |
| Lab analysis | Sampling (min) | 10 | 48 | 93 | 134 | 170 | 210 | 248 | 288 | | | | | |
| | Delay (min) | 60 | 56 | 62.8 | 66.3 | 54.5 | 60 | 60.5 | 66.7 | | | | | |

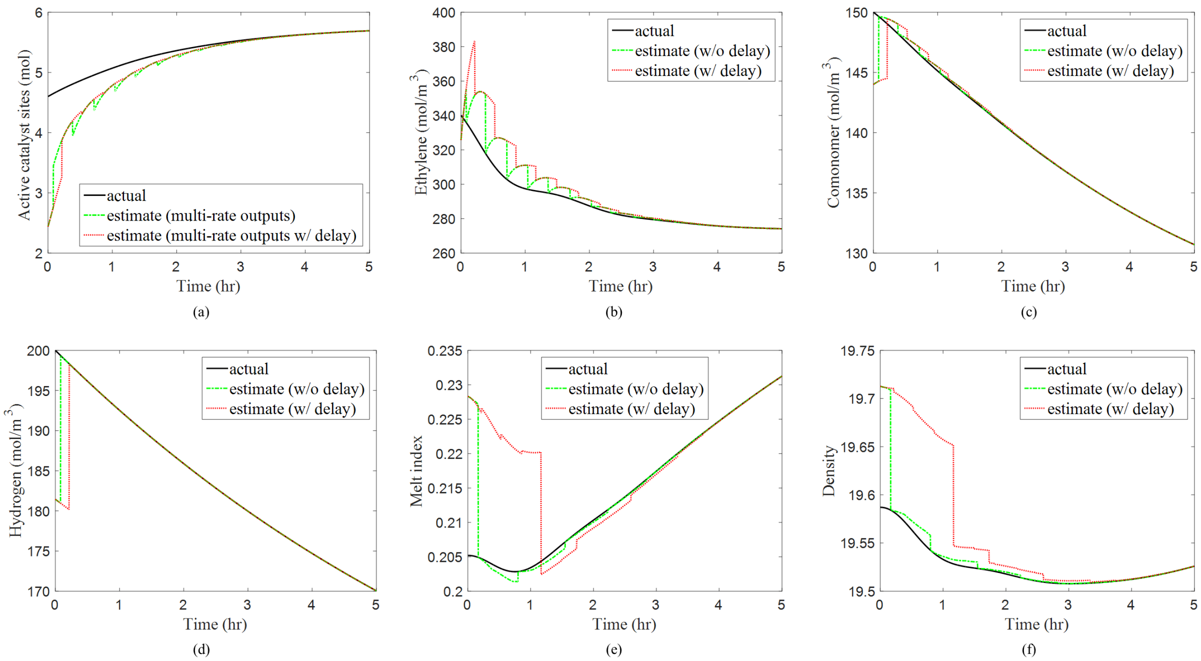


Fig. 2. Comparison of the multi-rate multi-delay observer (red) and the multi-rate observer in the absence of measurement delay (green) in the gas-phase polyethylene reactor example.

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