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Generalized constructions of Menon-Hadamard difference sets

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We revisit the problem of constructing Menon-Hadamard difference sets. In 1997, Wilson and Xiang gave a general framework for constructing Menon-Hadamard difference sets by using a combination of a spread and four projective sets of type Q in $\mathrm{PG}(3,q)$. They also found examples of suitable spreads and projective sets of type Q for $q = 5, 13, 17$. Subsequently, Chen (1997) succeeded in finding a spread and four projective sets of type Q in $\mathrm{PG}(3,q)$ satisfying the conditions in the Wilson-Xiang construction for all odd prime powers q. Thus, he showed that there exists a Menon-Hadamard difference set in groups of order $4q^4$ for all odd prime powers q. However, the projective sets of type Q found by Chen have automorphisms different from those of the examples constructed by Wilson and Xiang. In this paper, we first generalize Chen's construction of projective sets of type Q by using "semi-primitive" cyclotomic classes. This demonstrates that the construction of projective sets of type Q satisfying the conditions in the Wilson-Xiang construction is much more flexible than originally thought. Secondly, we give a new construction of spreads and projective sets of type Q in $\mathrm{PG}(3,q)$ for all odd prime powers q, which generalizes the

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examples found by Wilson and Xiang. This solves a problem left open in Section 5 of the Wilson-Xiang paper from 1997.

1. Introduction

Let G be an additively written abelian group of order v. A k-subset D of G is called a (v, k, λ) *difference set* if the list of differences "$x - y$, $x, y \in D, x \neq y$", represents each nonidentity element of G exactly λ times. In this paper, we revisit the problem of constructing Menon-Hadamard difference sets, namely those difference sets with parameters $(v, k, \lambda) = (4m^2, 2m^2 - m, m^2 - m)$, where m is a positive integer. It is well known that a Menon-Hadamard difference set generates a regular Hadamard matrix of order $4m^2$. So by constructing Menon-Hadamard difference sets in groups of order $4m^2$, we obtain regular Hadamard matrices of order $4m^2$.

The main problem in the study of Menon-Hadamard difference sets is: For each positive integer m, which groups of order $4m^2$ contain a Menon-Hadamard difference set. We give a brief survey of results on this problem in the case where the group under consideration is abelian. First we mention a product theorem of Turyn [11]: If there are Menon-Hadamard difference sets in abelian groups $H \times G_1$ and $H \times G_2$, respectively, where $|H| = 4$ and $|G_i|$, $i = 1, 2$, are squares, then there also exists a Menon-Hadamard difference set in $H \times G_1 \times G_2$. With Turyn's product theorem in hand, in order to construct Menon-Hadamard difference sets, one should start with the case where the order of the abelian group is $4q$ with q an even power of a prime. In the case where q is an even power of 2, that is, G is an abelian 2-group of order 2^{2t+2}, the existence problem was completely solved in [8] after much work was done in [5]; it was shown that there exists a Menon-Hadamard difference set in an abelian group G of order 2^{2t+2} if and only if the exponent of G is less than or equal to 2^{t+2}.

In the case where q is an even power of an odd prime, Turyn [11] observed that there exists a Menon-Hadamard difference set in $H \times (\mathbb{Z}_3)^2$; hence by the product theorem, there is a Menon-Hadamard difference set in $H \times (\mathbb{Z}_3)^{2t}$ for any positive integer t. On the other hand, McFarland [10] proved that if an abelian group of order $4p^2$, where p is a prime, contains a Menon-Hadamard difference set, then $p = 2$ or 3. After McFarland's paper [10] was published, it was conjectured [7, p. 287] that if an abelian group of order $4m^2$ contains a Menon-Hadamard difference set, then $m = 2^r 3^s$ for some nonnegative integers r and s. So it was a great surprise when Xia [13] constructed a Menon-Hadamard difference set in $H \times \mathbb{Z}_p^4$ for any odd prime p congruent to 3 modulo 4. Xia's method of construction depends on very complicated computations involving cyclotomic classes of finite fields; it was later simplified by Xiang and Chen [14] by using a character theoretic approach. Moreover, in [14], the authors also asked whether a certain family of 3-weight projective linear code exists or not, since such projective linear codes are needed for the construction of Menon-Hadamard difference set in the group $H \times (\mathbb{Z}_p)^4$, where p is a prime congruent to 1 modulo 4.

Van Eupen and Tonchev [6] found the required 3-weight projective linear codes when $p = 5$, hence constructed Menon-Hadamard difference sets in $\mathbb{Z}_2^2 \times \mathbb{Z}_5^4$, which are the first examples of abelian Menon-Hadamard difference sets in groups of order $4p^4$, where p is a prime congruent to 1 modulo 4. Inspired by these examples, Wilson and Xiang [12] gave a general framework for constructing Menon-Hadamard difference sets in the groups $H \times G$, where H is either group of order 4 and G is an elementary abelian group of order q^4, q an odd prime power, using a combination of a spread and four projective sets of type Q in $\mathrm{PG}(3, q)$. (See Section 2.2 for the definition of projective sets of type Q.) Wilson and Xiang [12] also found examples of suitable spreads and the required projective sets of type Q when $q = 5, 13, 17$. They used $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ as a model of the four-dimensional vector space $V(4, q)$ over $\mathbb{F}_q$, and considered projective sets of type Q with the automorphism

$$T' = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix},$$

where ω is a primitive element of $\mathbb{F}_{q^2}$. However, the existence of the required projective sets of type Q with this prescribed automorphism remained unsolved for $q > 17$.

Immediately after [12] appeared, Chen [4] succeeded in showing the existence of a combination of a spread and four projective sets of type Q in $\mathrm{PG}(3, q)$ satisfying the conditions in the Wilson-Xiang construction for all odd prime powers q. As a consequence, Chen [4] obtained the following theorem by applying Turyn's product theorem in [11].

Theorem 1.1. *Let p_i, $i = 1, 2, \ldots, s$, be odd primes and t_i, $i = 1, 2, \ldots, s$, be positive integers. Furthermore, let H be either group of order 4 and G_i, $i = 1, 2, \ldots, s$, be an elementary abelian group of order $p_i^{4t_i}$. Then, there exists a Menon-Hadamard difference set in $H \times G_1 \times G_2 \times \cdots \times G_s$.*

Here, Chen [4] found projective sets of type Q in $\mathrm{PG}(3, q)$ with the following automorphism

$$T = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix},$$

which is obviously different from that of the projective sets of type Q found by Wilson and Xiang [12]. Thus, the existence problem of projective sets of type Q in $\mathrm{PG}(3, q)$ with the prescribed automorphism T' remained open.

The objectives of this paper are two-fold. First, we give a generalization of Chen's construction of projective sets of type Q by using "semi-primitive" cyclotomic classes. This demonstrates that the construction of projective sets of type Q satisfying the conditions in the Wilson-Xiang construction is much more flexible than originally thought.

In particular, the proof of the candidate sets are projective sets of type Q is much simpler than that in [4]. Second, we show the existence of a combination of a spread and four projective sets of type Q with automorphism T' for all odd prime powers q. Our construction generalizes the examples found by Wilson and Xiang in [12]; this solves the problem left open in Section 5 of [12].

2. Preliminaries

2.1. Characters of finite fields

In this subsection, we collect some auxiliary results on characters of finite fields. We assume that the reader is familiar with basic theory of characters of finite fields as in [9, Chapter 5].

Let p be a prime and s, f be positive integers. We set $q = p^s$, and denote the finite field of order q by $\mathbb{F}_q$. Let $\mathrm{Tr}_{q^f/q}$ be the trace map from $\mathbb{F}_{q^f}$ to $\mathbb{F}_q$, which is defined by

$$\mathrm{Tr}_{q^f/q}(x) = x + x^q + \cdots + x^{q^{f-1}}, \quad \forall x \in \mathbb{F}_{q^f}.$$

Let ω be a fixed primitive element of $\mathbb{F}_q$, ζ_p a fixed (complex) primitive pth root of unity, and ζ_{q-1} a (complex) $q-1$th root of unity. The character $\psi_{\mathbb{F}_q}$ of the additive group of $\mathbb{F}_q$ defined by $\psi_{\mathbb{F}_q}(x) = \zeta_p^{\mathrm{Tr}_{q/p}(x)}$, $\forall x \in \mathbb{F}_q$, is called the *canonical additive character* of $\mathbb{F}_q$. Then, each additive character is given by $\psi_a(x) = \psi_{\mathbb{F}_q}(ax)$, $\forall x \in \mathbb{F}_q$, where $a \in \mathbb{F}_q$. On the other hand, each multiplicative character is given by $\chi^j(\omega^\ell) = \zeta_{q-1}^{j\ell}$, $\ell = 0, 1, \ldots, q-2$, where $j = 0, 1, \ldots, q-2$.

For a multiplicative character χ of $\mathbb{F}_q$, the character sum defined by

$$G_q(\chi) = \sum_{x \in \mathbb{F}_q^*} \chi(x)\psi_{\mathbb{F}_q}(x)$$

is called a *Gauss sum* of $\mathbb{F}_q$. Gauss sums satisfy the following basic properties: (1) $G_q(\chi)\overline{G_q(\chi)} = q$ if χ is nontrivial; (2) $G_q(\chi^{-1}) = \chi(-1)\overline{G_q(\chi)}$; (3) $G_q(\chi) = -1$ if χ is trivial.

In general, explicit evaluations of Gauss sums are difficult. There are only a few cases where the Gauss sums have been completely evaluated. The most well-known case is the *quadratic case*, i.e., the order of the multiplicative character involved is 2.

Theorem 2.1. ([9, Theorem 5.15]) *Let η be the quadratic character of $\mathbb{F}_q = \mathbb{F}_{p^s}$. Then,*

$$G_q(\eta) = (-1)^{s-1}\left(\sqrt{(-1)^{\frac{p-1}{2}}p}\right)^s.$$

The next simple case is the so-called *semi-primitive case*, where there exists an integer ℓ such that $p^\ell \equiv -1 \pmod{N}$. Here, N is the order of the multiplicative character involved. In particular, we give the following for later use.

Theorem 2.2. ([9, Theorem 5.16]) *Let χ be a nontrivial multiplicative character of $\mathbb{F}_{q^2}$ of order N dividing $q+1$. Then,*

$$G_{q^2}(\chi) = \begin{cases} q & \text{if } N \text{ odd or } \frac{q+1}{N} \text{ even}, \\ -q, & \text{if } N \text{ even and } \frac{q+1}{N} \text{ odd}. \end{cases}$$

We will also need the *Davenport-Hasse product formula*, which is stated below.

Theorem 2.3. ([2, Theorem 11.3.5]) *Let χ' be a multiplicative character of order $\ell > 1$ of $\mathbb{F}_q$. For every nontrivial multiplicative character χ of $\mathbb{F}_q$,*

$$G_q(\chi) = \frac{G_q(\chi^\ell)}{\chi^\ell(\ell)} \prod_{i=1}^{\ell-1} \frac{G_q(\chi'^i)}{G_q(\chi\chi'^i)}.$$

Let N be a positive integer dividing $q-1$. We set $C_i^{(N,q)} = \omega^i\langle\omega^N\rangle$, $0 \le i \le N-1$, which are called the Nth *cyclotomic classes* of $\mathbb{F}_q$. In this paper, we need to evaluate the (additive) character values of a union of some cyclotomic classes. In particular, the character sums defined by

$$\psi_{\mathbb{F}_q}(C_i^{(N,q)}) = \sum_{x \in C_i^{(N,q)}} \psi_{\mathbb{F}_q}(x), \quad i = 0, 1, \ldots, N-1,$$

are called the Nth *Gauss periods* of $\mathbb{F}_q$. By the orthogonality of characters, the Gauss period can be expressed as a linear combination of Gauss sums:

$$\psi_{\mathbb{F}_q}(C_i^{(N,q)}) = \frac{1}{N} \sum_{j=0}^{N-1} G_q(\chi^j)\chi^{-j}(\omega^i), \quad i = 0, 1, \ldots, N-1, \tag{2.1}$$

where χ is any fixed multiplicative character of order N of $\mathbb{F}_q$. For example, if $N = 2$, we have the following from Theorem 2.1:

$$\psi_{\mathbb{F}_q}(C_i^{(2,q)}) = \frac{-1 + (-1)^i G_q(\eta)}{2} = \frac{-1 + (-1)^{i+s-1+\frac{(p-1)s}{4}} p^{\frac{s}{2}}}{2}, \quad i = 0, 1, \tag{2.2}$$

where η is the quadratic character of $\mathbb{F}_q$. On the other hand, the Gauss sum with respect to a multiplicative character χ of order N can be expressed as a linear combination of Gauss periods:

$$G_q(\chi) = \sum_{i=0}^{N-1} \psi_{\mathbb{F}_q}(C_i^{(N,q)})\chi(\omega^i). \tag{2.3}$$

2.2. Known results on projective sets of type Q

Let $\mathrm{PG}(k-1,q)$ denote the $(k-1)$-dimensional projective space over $\mathbb{F}_q$. A set $\mathcal{S}$ of n points of $\mathrm{PG}(k-1,q)$ is called a *projective* (n,k,h_1,h_2) *set* if every hyperplane of $\mathrm{PG}(k-1,q)$ meets $\mathcal{S}$ in h_1 or h_2 points. In particular, a subset $\mathcal{S}$ of the point set of $\mathrm{PG}(3,q)$ is called *type* Q if

$$(n,k,h_1,h_2)=\Big(\frac{q^4-1}{4(q-1)},4,\frac{(q-1)^2}{4},\frac{(q+1)^2}{4}\Big).$$

In this paper, we will use the following model of $\mathrm{PG}(3,q)$: We view $\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$ as a 4-dimensional vector space over $\mathbb{F}_q$. For a nonzero vector $(x,y)\in(\mathbb{F}_{q^2}\times\mathbb{F}_{q^2})\setminus\{(0,0)\}$, we use $\langle(x,y)\rangle$ to denote the projective point in $\mathrm{PG}(3,q)$ corresponding to the one-dimensional subspace over $\mathbb{F}_q$ spanned by (x,y). Let $\mathcal{P}$ be the set of points of $\mathrm{PG}(3,q)$. Then, all (hyper)planes in $\mathrm{PG}(3,q)$ are given by

$$H_{a,b}=\{\langle(x,y)\rangle\,|\,\mathrm{Tr}_{q^2/q}(ax+by)=0\},\quad \langle(a,b)\rangle\in\mathcal{P}.$$

Let $\mathcal{S}$ be a set of points of $\mathrm{PG}(3,q)$, and define

$$E=\{\lambda(x,y)\,|\,\lambda\in\mathbb{F}_q^*,\langle(x,y)\rangle\in\mathcal{S}\}.$$

Noting that each nontrivial additive character of $\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$ is given by

$$\psi_{a,b}((x,y))=\psi_{\mathbb{F}_{q^2}}(ax+by),\quad (x,y)\in\mathbb{F}_{q^2}\times\mathbb{F}_{q^2},$$

where $(0,0)\neq(a,b)\in\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$, we have

$$\begin{aligned}\psi_{a,b}(E)&=\sum_{\lambda\in\mathbb{F}_q}\sum_{\langle(x,y)\rangle\in\mathcal{S}}\psi_{\mathbb{F}_q}(\lambda\mathrm{Tr}_{q^2/q}(ax+by))-|\mathcal{S}|\\&=q|H_{a,b}\cap\mathcal{S}|-|\mathcal{S}|.\end{aligned}$$

Hence, we have the following proposition.

Proposition 2.4. *The set $\mathcal{S}$ is a projective set of type Q in* $\mathrm{PG}(3,q)$ *if and only if* $|E|=\frac{q^4-1}{4}$ *and* $\psi_{a,b}(E)$ *take exactly two values* $\frac{q^2-1}{4}$ *and* $\frac{-3q^2-1}{4}$ *for all* $(0,0)\neq(a,b)\in\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$.

The set $E\subseteq\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$ is also called *type* Q if it satisfies the condition of Proposition 2.4.

A *spread* in $\mathrm{PG}(3,q)$ is a collection $\mathcal{L}$ of q^2+1 pairwise skew lines; equivalently, $\mathcal{L}$ can be regarded as a collection $\mathcal{K}$ of 2-dimensional subspaces of the underlying 4-dimensional vector space $V(4,q)$ over $\mathbb{F}_q$, any two of which intersect at zero only. We also call such a set $\mathcal{K}$ of 2-dimensional subspaces as a *spread* of $V(4,q)$.

The following important theorem was given by Wilson and Xiang [12].

Theorem 2.5. *Let $\mathcal{L} = \{L_i \,|\, 0 \le i \le q^2\}$ be a spread of* $\mathrm{PG}(3,q)$*, and assume the existence of four pairwise disjoint projective sets $\mathcal{S}_i$, $i = 1,2,3,4$, of type Q in* $\mathrm{PG}(3,q)$ *such that $\mathcal{S}_0 \cup \mathcal{S}_2 = \bigcup_{i=0}^{(q^2-1)/2} L_i$ and $\mathcal{S}_1 \cup \mathcal{S}_3 = \bigcup_{i=(q^2+1)/2}^{q^2} L_i$. Then there exists a Menon-Hadamard difference set in $H \times G$, where H is either group of order* 4 *and G is an elementary abelian group of order q^4.*

Remark 2.6. From Proposition 2.4 and Theorem 2.5, in order to construct a Menon-Hadamard difference set in a group of order $4q^4$, we need to find four disjoint sets $C_i \subseteq (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}) \setminus \{(0,0)\}$, $i = 0,1,2,3$, of type Q and a suitable spread $\mathcal{K} = \{K_i \,|\, 0 \le i \le q^2\}$ consisting of 2-dimensional subspaces of $V(4,q)$ such that $C_0 \cup C_2 \cup \{(0,0)\} = \bigcup_{i=0}^{(q^2-1)/2} K_i$ and $C_1 \cup C_3 \cup \{(0,0)\} = \bigcup_{i=(q^2+1)/2}^{q^2} K_i$.

We now review the construction of projective sets of type Q given by Chen [4]. Let ω be a primitive element of $\mathbb{F}_{q^2}$. Furthermore, let

$$X = \{x \in \mathbb{F}_{q^2} \,|\, \mathrm{Tr}_{q^2/q}(x) \in C_0^{(2,q)}\}, \quad X' = \{x\omega \in \mathbb{F}_{q^2} \,|\, \mathrm{Tr}_{q^2/q}(x) \in C_0^{(2,q)}\}.$$

Define

$$\begin{aligned} &X_1 = X \setminus (X \cap X'),\ X_2 = X' \setminus (X \cap X'),\\ &X_3 = X \cap X',\ X_4 = \mathbb{F}_{q^2} \setminus (X_1 \cup X_2 \cup X_3), \end{aligned}$$

and

$$\begin{aligned} C_0 &= \{(x,xy) \,|\, x \in C_0^{(2,q^2)}, y \in X_1\} \cup \{(x,xy) \,|\, x \in C_1^{(2,q^2)}, y \in X_2\}\\ &\qquad \cup \{(0,x) \,|\, x \in C_\tau^{(2,q^2)}\},\\ C_1 &= \{(x,xy) \,|\, x \in C_0^{(2,q^2)}, y \in X_3\} \cup \{(x,xy) \,|\, x \in C_1^{(2,q^2)}, y \in X_4\},\\ C_2 &= \{(x,xy) \,|\, x \in C_1^{(2,q^2)}, y \in X_1\} \cup \{(x,xy) \,|\, x \in C_0^{(2,q^2)}, y \in X_2\}\\ &\qquad \cup \{(0,x) \,|\, x \in C_{\tau+1}^{(2,q^2)}\},\\ C_3 &= \{(x,xy) \,|\, x \in C_1^{(2,q^2)}, y \in X_3\} \cup \{(x,xy) \,|\, x \in C_0^{(2,q^2)}, y \in X_4\}, \end{aligned}$$

where $\tau = 0$ or 1 according as $q \equiv 1$ or $3 \pmod 4$. It is clear that these type Q sets admit the automorphism T.

Theorem 2.7. *The sets C_i, $i = 0,1,2,3$, are type Q sets. Furthermore, these sets satisfy the conditions mentioned in Remark 2.6 with respect to the spread $\mathcal{K}$ consisting of the following* 2*-dimensional subspaces:*

$$K_y = \{(x,xy) \,|\, x \in \mathbb{F}_{q^2}\},\ y \in \mathbb{F}_{q^2}, \quad \textit{and} \quad K_\infty = \{(0,x) \,|\, x \in \mathbb{F}_{q^2}\}.$$

On the other hand, Wilson and Xiang [12] constructed Menon-Hadamard difference sets in groups of order $4q^4$ for $q = 5, 13, 17$ using the following four type Q sets:

$$\begin{aligned} C_i =& \{(0, y) \mid y \in C_{\tau_i}^{(2,q^2)}\} \cup \{(xy, xy^{-1}\omega^j) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, j \in A_i\} \\ & \cup \{(xy, xy^{-1}\omega^j) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, j \in B_i\}, \quad i = 0, 2, \\ C_i =& \{(y, 0) \mid y \in C_{\tau_i}^{(2,q^2)}\} \cup \{(xy, xy^{-1}\omega^j) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, j \in A_i\} \\ & \cup \{(xy, xy^{-1}\omega^j) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, j \in B_i\}, \quad i = 1, 3, \end{aligned}$$

for some subsets A_i, B_i, $i = 0, 1, 2, 3$, of $\{0, 1, \ldots, 2q + 1\}$ and some suitable $\tau_i \in \{0, 1\}$, $i = 0, 1, 2, 3$, and the spread $\mathcal{K}$ consisting of the following 2-dimensional subspaces:

$$K_y = \{(x, yx^q) \mid x \in \mathbb{F}_{q^2}\},\ y \in \mathbb{F}_{q^2}, \quad \text{and} \quad K_\infty = \{(0, x) \mid x \in \mathbb{F}_{q^2}\}.$$

It is clear that these type Q sets admit the automorphism T'.

3. A generalization of Chen's construction

We first fix notation used in this section. Let $q = p^s$ be an odd prime power with p a prime, and m be a fixed positive integer satisfying $2m \mid (q + 1)$. Then, there exists a minimal ℓ such that $2m \mid (p^\ell + 1)$. Write $s = \ell t$ for some $t \geq 1$. Let ω be a primitive element of $\mathbb{F}_{q^2}$. Let T_i, $i = 0, 1$, be two arbitrary subsets of $\mathbb{F}_q$, and

$$S_0 = \{x \in \mathbb{F}_{q^2} \mid \mathrm{Tr}_{q^2/q}(x) \in T_0\}, \quad S_1 = \{x \in \mathbb{F}_{q^2} \mid \mathrm{Tr}_{q^2/q}(x\omega^m) \in T_1\}. \tag{3.1}$$

Furthermore, let K be any m-subset of $\{0, 1, \ldots, 2m - 1\}$ such that $K \cap \{x + m \pmod{2m} \mid x \in K\} = \emptyset$. Define

$$A_0 = S_0 \setminus S_1, \quad A_1 = S_1 \setminus S_0, \quad D_0 = \bigcup_{i \in K} C_i^{(2m,q^2)}, \quad D_1 = \bigcup_{i \in K} C_{i+m}^{(2m,q^2)}, \tag{3.2}$$

and

$$\epsilon := \begin{cases} 1, & \text{if } (p^\ell + 1)/2m \text{ is even and } t \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Remark 3.1.

(i) The indicator function of S_i, $i = 0, 1$, is given by

$$f_{S_i}(y) = \frac{1}{q} \sum_{c \in \mathbb{F}_q} \sum_{u \in T_i} \psi_{\mathbb{F}_{q^2}}(cy\omega^{mi})\psi_{\mathbb{F}_q}(-cu), \quad i = 0, 1.$$

(ii) The size of each S_i is $q|T_i|$ since $\mathrm{Tr}_{q^2/q}$ is a linear mapping over $\mathbb{F}_q$.

(iii) The size of $S_0 \cap S_1$ is $|T_0||T_1|$; it is clear that

$$|S_0 \cap S_1| = \sum_{y \in \mathbb{F}_{q^2}} f_{S_0}(y) f_{S_1}(y)$$
$$= \frac{1}{q^2} \sum_{c,d \in \mathbb{F}_q} \sum_{u \in T_0} \sum_{v \in T_1} \sum_{y \in \mathbb{F}_{q^2}} \psi_{\mathbb{F}_{q^2}}(y(c + d\omega^m)) \psi_{\mathbb{F}_q}(-cu - dv). \quad (3.3)$$

Since $\omega^m \notin \mathbb{F}_q$, $c + d\omega^m = 0$ if and only if $c = d = 0$. Hence, the right-hand side of (3.3) is equal to $|T_0||T_1|$.

(iv) Since $2m \,|\, (q+1)$, the character values of $D_i \subseteq \mathbb{F}_{q^2}$, $i = 0, 1$, can be evaluated by using (2.1) and the Gauss sums in semi-primitive case (see, e.g., [3, Theorem 2]): for $b \in \mathbb{F}_{q^2}^*$,

$$\sum_{x \in D_\epsilon} \psi_{\mathbb{F}_{q^2}}(bx) = \begin{cases} \frac{-1-q}{2}, & \text{if } b^{-1} \in D_0, \\ \frac{-1+q}{2}, & \text{if } b^{-1} \in D_1. \end{cases}$$

The following is our main result in this section.

Theorem 3.2.

(1) *Assume that* $|T_0| = |T_1| = (q-1)/2$, *and define*

$$E_0 = \{(x, xy) \,|\, x \in D_0, y \in A_0\} \cup \{(x, xy) \,|\, x \in D_1, y \in A_1\} \cup \{(0, x) \,|\, x \in D_\epsilon\}.$$

Then E_0 *is a set of type* Q *in* $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$.

(2) *Assume that* $|T_0| = (q-1)/2$ *and* $|T_1| = (q+1)/2$, *and define*

$$E_1 = \{(x, xy) \,|\, x \in D_0, y \in A_0\} \cup \{(x, xy) \,|\, x \in D_1, y \in A_1\}.$$

Then E_1 *is a set of type* Q *in* $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$.

This theorem obviously generalizes the construction of type Q sets given by Chen [4]. Indeed, we used D_i, $i = 0, 1$, instead of $C_i^{(2,q^2)}$, $i = 0, 1$, in the definition of X and X' (see Subsection 2.2). This new construction is much more flexible than that in [4].

To prove this theorem, we will evaluate the character values $\psi_{a,b}(E_i)$, $(a, b) \in (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}) \setminus \{(0, 0)\}$, by a series of the following lemmas. We first treat the case where $b = 0$.

Lemma 3.3. *For* $b = 0$ *and* $a \neq 0$, *it holds that*

$$\psi_{a,b}(E_0) = \frac{q^2 - 1}{4}.$$

Proof. Since $|T_0| = |T_1| = (q-1)/2$, by Remark 3.1 (ii),(iii), we have $|A_0| = |A_1| = (q^2-1)/4$. Then, we have

$$\begin{aligned}\psi_{a,0}(E_0) &= \sum_{x\in D_0}\sum_{y\in A_0}\psi_{\mathbb{F}_{q^2}}(ax) + \sum_{x\in D_1}\sum_{y\in A_1}\psi_{\mathbb{F}_{q^2}}(ax) + \frac{q^2-1}{2}\\ &= \frac{q^2-1}{4}\sum_{x\in\mathbb{F}_{q^2}^*}\psi_{\mathbb{F}_{q^2}}(ax) + \frac{q^2-1}{2} = \frac{q^2-1}{4}.\end{aligned}$$

This completes the proof. □

Lemma 3.4. *For $b=0$ and $a\neq 0$, we have*

$$\psi_{a,b}(E_1) = \begin{cases} \frac{q^2-1}{4}, & \text{if } a^{-1}\in D_\epsilon,\\ \frac{-3q^2-1}{4}, & \text{otherwise.}\end{cases}$$

Proof. Since $|T_0| = (q-1)/2$ and $|T_1| = (q+1)/2$, by Remark 3.1 (ii),(iii), we have $|A_0| = (q-1)^2/4$ and $|A_1| = (q+1)^2/4$. Then, we have

$$\begin{aligned}\psi_{a,0}(E_1) &= \sum_{x\in D_0}\sum_{y\in A_0}\psi_{\mathbb{F}_{q^2}}(ax) + \sum_{x\in D_1}\sum_{y\in A_1}\psi_{\mathbb{F}_{q^2}}(ax)\\ &= \frac{(q-1)^2}{4}\sum_{x\in\mathbb{F}_{q^2}^*}\psi_{\mathbb{F}_{q^2}}(ax) + q\sum_{x\in D_1}\psi_{\mathbb{F}_{q^2}}(ax). \end{aligned}\tag{3.4}$$

Finally, by Remark 3.1 (iv), (3.4) is reformulated as

$$\psi_{a,0}(E_1) = -\frac{(q-1)^2}{4} + q\begin{cases}\frac{-1+q}{2}, & \text{if } a^{-1}\in D_\epsilon,\\ \frac{-1-q}{2}, & \text{otherwise.}\end{cases}$$

This completes the proof. □

We next treat the case where $b\neq 0$. Let f_{S_i}, $i=0,1$, be defined as in Remark 3.1 (i). Define

$$U_1 = \sum_{x\in D_0}\sum_{y\in\mathbb{F}_{q^2}}\psi_{\mathbb{F}_{q^2}}(x(a+by))f_{S_0}(y),$$

$$U_2 = \sum_{x\in D_1}\sum_{y\in\mathbb{F}_{q^2}}\psi_{\mathbb{F}_{q^2}}(x(a+by))f_{S_1}(y),$$

$$U_3 = \sum_{x\in\mathbb{F}_{q^2}^*}\sum_{y\in\mathbb{F}_{q^2}}\psi_{\mathbb{F}_{q^2}}(x(a+by))f_{S_0}(y)f_{S_1}(y).$$

Then, the character values of E_i, $i = 0, 1$, are given by

$$\psi_{a,b}(E_0) = U_1 + U_2 - U_3 + \sum_{x \in D_\epsilon} \psi_{\mathbb{F}_{q^2}}(bx) \tag{3.5}$$

and

$$\psi_{a,b}(E_1) = U_1 + U_2 - U_3. \tag{3.6}$$

Lemma 3.5. *If $b \neq 0$, it holds that*

$$U_1 = \begin{cases} -q|T_0| + q^2, & \text{if } -ab^{-1} \in S_0 \text{ and } b^{-1} \in D_0, \\ -q|T_0|, & \text{if } -ab^{-1} \notin S_0 \text{ and } b^{-1} \in D_0, \\ 0, & \text{if } b^{-1} \in D_1. \end{cases}$$

Proof. If $b \neq 0$, we have

$$U_1 = \frac{1}{q} \sum_{x \in D_0} \sum_{y \in \mathbb{F}_{q^2}} \sum_{c \in \mathbb{F}_q} \sum_{u \in T_0} \psi_{\mathbb{F}_{q^2}}(xa)\psi_{\mathbb{F}_{q^2}}((xb + c)y)\psi_{\mathbb{F}_q}(-cu). \tag{3.7}$$

If $b^{-1} \in D_1$, there are no $x \in D_0$ such that $xb + c = 0$; we have $U_1 = 0$. If $b^{-1} \in D_0$, continuing from (3.7), we have

$$\begin{aligned} U_1 &= q \sum_{c \in \mathbb{F}_q^*} \sum_{u \in T_0} \psi_{\mathbb{F}_{q^2}}(-acb^{-1})\psi_{\mathbb{F}_q}(-cu) \\ &= -q|T_0| + q \sum_{c \in \mathbb{F}_q} \sum_{u \in T_0} \psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^2/q}(-ab^{-1})c - cu) \\ &= -q|T_0| + q^2 \begin{cases} 1, & \text{if } \mathrm{Tr}_{q^2/q}(-ab^{-1}) \in T_0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This completes the proof. □

Lemma 3.6. *If $b \neq 0$, we have*

$$U_2 = \begin{cases} -q|T_1| + q^2, & \text{if } -ab^{-1} \in S_1 \text{ and } b^{-1} \in D_0, \\ -q|T_1|, & \text{if } -ab^{-1} \notin S_1 \text{ and } b^{-1} \in D_0, \\ 0, & \text{if } b^{-1} \in D_1. \end{cases}$$

Proof. If $b \neq 0$, we have

$$U_2 = \frac{1}{q} \sum_{x \in D_1} \sum_{y \in \mathbb{F}_{q^2}} \sum_{c \in \mathbb{F}_q} \sum_{u \in T_1} \psi_{\mathbb{F}_{q^2}}(xa)\psi_{\mathbb{F}_{q^2}}((xb + c\omega^m)y)\psi_{\mathbb{F}_q}(-cu). \tag{3.8}$$

If $b^{-1} \in D_1$, there are no $x \in D_1$ such that $xb + c\omega^m = 0$; hence $U_2 = 0$. If $b^{-1} \in D_0$, continuing from (3.8), we have

$$
\begin{aligned}
U_2 &= q \sum_{c \in \mathbb{F}_q^*} \sum_{u \in T_1} \psi_{\mathbb{F}_{q^2}}(-acb^{-1}\omega^m)\psi_{\mathbb{F}_q}(-cu) \\
&= -q|T_1| + q \sum_{c \in \mathbb{F}_q} \sum_{u \in T_1} \psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^2/q}(-ab^{-1}\omega^m)c - cu) \\
&= -q|T_1| + q^2 \begin{cases} 1, & \text{if } \mathrm{Tr}_{q^2/q}(-ab^{-1}\omega^m) \in T_1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

This completes the proof. □

Lemma 3.7. *If $b \neq 0$, we have*

$$
U_3 = \begin{cases} -|T_0||T_1| + q^2, & \text{if } -ab^{-1} \in S_0 \cap S_1, \\ -|T_0||T_1|, & \text{otherwise.} \end{cases}
$$

Proof. Note that $D_0 \cup D_1 = \mathbb{F}_{q^2}^*$ and $|S_0 \cap S_1| = |T_0||T_1|$. Since $b \neq 0$, we have

$$
\begin{aligned}
U_3 &= \sum_{x \in \mathbb{F}_{q^2}} \sum_{y \in \mathbb{F}_{q^2}} \psi_{\mathbb{F}_{q^2}}(x(a+by)) f_{S_0}(y) f_{S_1}(y) - |S_0 \cap S_1| \\
&= q^2 f_{S_0}(-ab^{-1}) f_{S_1}(-ab^{-1}) - |T_0||T_1|.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 3.2. In the case where $b = 0$, the statement follows from Lemmas 3.3 and 3.4. We now treat the case where $b \neq 0$. By the evaluations for U_1, U_2, U_3 in Lemmas 3.5–3.7, we have

$$
U_1 + U_2 - U_3 = \begin{cases}
-q(|T_0| + |T_1| - q) + |T_0||T_1|, & \text{if } b^{-1} \in D_0,\ -ab^{-1} \in S_0,\ -ab^{-1} \in S_1; \\
& \text{or } b^{-1} \in D_0,\ -ab^{-1} \notin S_0,\ -ab^{-1} \in S_1; \\
& \text{or } b^{-1} \in D_0,\ -ab^{-1} \in S_0,\ -ab^{-1} \notin S_1, \\
-q(|T_0| + |T_1|) + |T_0||T_1|, & \text{if } b^{-1} \in D_0,\ -ab^{-1} \notin S_0,\ -ab^{-1} \notin S_1, \\
-q^2 + |T_0||T_1|, & \text{if } b^{-1} \in D_1,\ -ab^{-1} \in S_0,\ -ab^{-1} \in S_1, \\
|T_0||T_1|, & \text{if } b^{-1} \in D_1,\ -ab^{-1} \notin S_0,\ -ab^{-1} \in S_1; \\
& \text{or } b^{-1} \in D_1,\ -ab^{-1} \in S_0,\ -ab^{-1} \notin S_1; \\
& \text{or } b^{-1} \in D_1,\ -ab^{-1} \notin S_0,\ -ab^{-1} \notin S_1.
\end{cases}
$$

(1) Since $|T_0| = |T_1| = (q-1)/2$, by Remark 3.1 (iv), we have

$$\psi_{a,b}(E_0) = U_1 + U_2 - U_3 + \sum_{x\in D_\epsilon} \psi_{\mathbb{F}_{q^2}}(bx)$$
$$= \begin{cases} \frac{-3q^2-1}{4}, & \text{if } b^{-1}\in D_0,\ -ab^{-1}\notin S_0, \text{ and } -ab^{-1}\notin S_1; \\ & \text{or if } b^{-1}\in D_1,\ -ab^{-1}\in S_0, \text{ and } -ab^{-1}\in S_1, \\ \frac{q^2-1}{4}, & \text{otherwise.} \end{cases}$$

(2) Since $|T_0| = (q-1)/2$ and $|T_1| = (q+1)/2$, we have

$$\psi_{a,b}(E_1) = U_1 + U_2 - U_3$$
$$= \begin{cases} \frac{-3q^2-1}{4}, & \text{if } b^{-1}\in D_0,\ -ab^{-1}\notin S_0, \text{ and } -ab^{-1}\notin S_1; \\ & \text{or if } b^{-1}\in D_1,\ -ab^{-1}\in S_0, \text{ and } -ab^{-1}\in S_1, \\ \frac{q^2-1}{4}, & \text{otherwise.} \end{cases}$$

This completes the proof of the theorem. □

Corollary 3.8. *Let T_i, $i = 0,1$, be arbitrary $(q-1)/2$-subsets of $\mathbb{F}_q$ and S_0, S_1, A_0, A_1 be the sets defined as in (3.1) and (3.2). Furthermore, define*

$$S_1' = \{x \in \mathbb{F}_{q^2} \mid \mathrm{Tr}_{q^2/q}(x\omega^m) \in \mathbb{F}_q \setminus T_1\}, \quad A_0' = S_0 \setminus S_1', \quad A_1' = S_1' \setminus S_0.$$

Then, the sets

$$\begin{aligned} C_0 &= \{(x,xy) \mid x\in D_0, y\in A_0\} \cup \{(x,xy)\mid x\in D_1, y\in A_1\} \cup \{(0,x)\mid x\in D_\epsilon\}, \\ C_1 &= \{(x,xy) \mid x\in D_0, y\in A_0'\} \cup \{(x,xy)\mid x\in D_1, y\in A_1'\}, \\ C_2 &= \{(x,xy) \mid x\in D_1, y\in A_0\} \cup \{(x,xy)\mid x\in D_0, y\in A_1\} \cup \{(0,x)\mid x\in D_{\epsilon+1}\}, \\ C_3 &= \{(x,xy) \mid x\in D_1, y\in A_0'\} \cup \{(x,xy)\mid x\in D_0, y\in A_1'\} \end{aligned}$$

are of type Q, where the subscript of $D_{\epsilon+1}$ is reduced modulo 2. Furthermore, these sets satisfy the conditions mentioned in Remark 2.6 with respect to the spread consisting of the following 2-dimensional subspaces:

$$K_y = \{(x,xy)\mid x\in\mathbb{F}_{q^2}\}, y\in\mathbb{F}_{q^2}, \quad \text{and} \quad K_\infty = \{(0,x)\mid x\in\mathbb{F}_{q^2}\}.$$

Proof. By Theorem 3.2, C_0 and C_1 are type Q sets. Furthermore, since $C_2 = \omega^m C_0$ and $C_3 = \omega^m C_1$, the sets C_2 and C_3 are also of type Q. Finally, C_i, $i = 0,1,2,3$, satisfy the assumption of Remark 2.6 as $C_0 \cup C_2 \cup \{(0,0)\} = \left(\bigcup_{y\in A_0\cup A_1} K_y\right) \cup K_\infty$ and $C_1 \cup C_3 \cup \{(0,0)\} = \bigcup_{y\in A_0'\cup A_1'} K_y$. □

4. A generalization of Wilson-Xiang's examples

4.1. The setting

We fix notation used in this section. Let q be a prime power and ω be a primitive element of $\mathbb{F}_{q^2}$. Let c be any fixed odd integer in $\{0, 1, \ldots, 2q+1\} = \mathbb{Z}_{2q+2}$.

Define the following subsets of $\{0, 1, \ldots, 2q+1\}$:

$$\begin{aligned} I_1 &= \{i \pmod{2(q+1)} \mid \mathrm{Tr}_{q^2/q}(\omega^i) = 0\} = \{\tfrac{q+1}{2}, \tfrac{3(q+1)}{2}\}, \\ I_2 &= \{i \pmod{2(q+1)} \mid \mathrm{Tr}_{q^2/q}(\omega^i) \in C_0^{(2,q)}\}, \\ I_3 &= \{i \pmod{2(q+1)} \mid \mathrm{Tr}_{q^2/q}(\omega^i) \in C_1^{(2,q)}\}, \\ J_i &= I_i - c \pmod{2(q+1)}, \quad i = 1, 2, 3. \end{aligned}$$

Then $|I_1| = 2$, $|I_2| = |I_3| = q$, and $I_1 \cup I_2 \cup I_3 = \mathbb{Z}_{2q+2}$. Furthermore, define

$$\begin{aligned} X_{1,c} &= (I_1 \cap J_2) \cup (I_2 \cap J_1), \\ X_{2,c} &= (I_1 \cap J_3) \cup (I_3 \cap J_1), \\ X_{3,c} &= I_2 \cap J_2, \;\; X_{4,c} = I_3 \cap J_3, \\ X_{5,c} &= (I_2 \cap J_3) \cup (I_3 \cap J_2). \end{aligned}$$

It is clear that the $X_{i,c}$'s partition $\mathbb{Z}_{2q+2}$. In the appendix, we will show that the $X_{i,c}$'s have the following properties:

- (P1) $X_{1,c} \equiv X_{2,c} + (q+1) \pmod{2(q+1)}$, $X_{3,c} \equiv X_{4,c} + (q+1) \pmod{2(q+1)}$;
- (P2) $|X_{1,c}| = |X_{2,c}| = 2$, $|X_{3,c}| = |X_{4,c}| = \frac{q-1}{2}$, $|X_{5,c}| = q-1$;
- (P3) $X_{3,c+q+1} \cup X_{4,c+q+1} = X_{5,c}$;
- (P4) $X_{1,c} + c \equiv -X_{1,c} + (q+1) \pmod{2(q+1)}$ or $X_{2,c} + c \equiv -X_{2,c} + (q+1) \pmod{2(q+1)}$ according as $q \equiv 3$ or $1 \pmod 4$;
- (P5) $|X_{1,c} \cap X_{1,c+q+1}| = 1$;
- (P6) By the properties (P2) and (P5), we can assume that $X_{1,c} = \{\alpha, \beta\}$ and $X_{1,c+q+1} = \{\alpha, \gamma\}$. Then, $\beta \equiv \gamma + (q+1) \pmod{2(q+1)}$. Furthermore, $\alpha \equiv 0 \pmod 2$ and $\beta \equiv 1 \pmod 2$ or $\alpha \equiv 1 \pmod 2$ and $\beta \equiv 0 \pmod 2$ according as $q \equiv 3$ or $1 \pmod 4$;
- (P7) Define $R_i = \bigcup_{j \in X_{i,c}} C_j^{(2(q+1),q^2)}$, $i = 1, 2, 3, 4, 5$. Then, R_i takes the character values listed in Table 1. In the language of association schemes, the Cayley graphs on $(\mathbb{F}_{q^2}, +)$ with connection sets R_i's, together with the diagonal relation arising from the connection set $R_0 = \{0\}$, form a 5-class translation association scheme. Here, $Y_{i,c}$'s are subsets of $\{0, 1, \ldots, 2q+1\}$ defined as the index sets of the dual association scheme;
- (P8) $-Y_{i,c} + c \equiv Y_{i,c} \pmod{2(q+1)}$, $i = 1, 2$;
- (P9) $-(Y_{3,c} \cup Y_{4,c}) \equiv Y_{5,c} - c \pmod{2(q+1)}$;

Table 1
The values of $\psi_{\mathbb{F}_{q^2}}(\omega^a R_i)$'s.

	R_1	R_2	R_3	R_4	R_5
$a \in Y_{1,c}$	$\frac{-2+q+G_q(\eta)}{2}$	$\frac{-2+q-G_q(\eta)}{2}$	$\frac{(q-1)(-1+G_q(\eta))}{4}$	$\frac{(q-1)(-1-G_q(\eta))}{4}$	$\frac{-q+1}{2}$
$a \in Y_{2,c}$	$\frac{-2+q-G_q(\eta)}{2}$	$\frac{-2+q+G_q(\eta)}{2}$	$\frac{(q-1)(-1-G_q(\eta))}{4}$	$\frac{(q-1)(-1+G_q(\eta))}{4}$	$\frac{-q+1}{2}$
$a \in Y_{3,c}$	$-1+G_q(\eta)$	$-1-G_q(\eta)$	$\frac{(1-G_q(\eta))^2}{4}$	$\frac{(1+G_q(\eta))^2}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{2}$
$a \in Y_{4,c}$	$-1-G_q(\eta)$	$-1+G_q(\eta)$	$\frac{(1+G_q(\eta))^2}{4}$	$\frac{(1-G_q(\eta))^2}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{2}$
$a \in Y_{5,c}$	-1	-1	$\frac{1-(-1)^{\frac{q-1}{2}}q}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{4}$	$\frac{1+(-1)^{\frac{q-1}{2}}q}{2}$

Table 2
The values of $\psi_{\mathbb{F}_{q^2}}(\omega^a R'_i)$'s.

	R'_1	R'_2	R'_3	R'_4	R'_5
$a \in X_{1,c}$	$\frac{-2+q+G_q(\eta)}{2}$	$\frac{-2+q-G_q(\eta)}{2}$	$\frac{(q-1)(-1+G_q(\eta))}{4}$	$\frac{(q-1)(-1-G_q(\eta))}{4}$	$\frac{-q+1}{2}$
$a \in X_{2,c}$	$\frac{-2+q-G_q(\eta)}{2}$	$\frac{-2+q+G_q(\eta)}{2}$	$\frac{(q-1)(-1-G_q(\eta))}{4}$	$\frac{(q-1)(-1+G_q(\eta))}{4}$	$\frac{-q+1}{2}$
$a \in X_{3,c}$	$-1+G_q(\eta)$	$-1-G_q(\eta)$	$\frac{(1-G_q(\eta))^2}{4}$	$\frac{(1+G_q(\eta))^2}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{2}$
$a \in X_{4,c}$	$-1-G_q(\eta)$	$-1+G_q(\eta)$	$\frac{(1+G_q(\eta))^2}{4}$	$\frac{(1-G_q(\eta))^2}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{2}$
$a \in X_{5,c}$	-1	-1	$\frac{1-(-1)^{\frac{q-1}{2}}q}{4}$	$\frac{1-(-1)^{\frac{q-1}{2}}q}{4}$	$\frac{1+(-1)^{\frac{q-1}{2}}q}{2}$

(P10) Define $R'_i = \bigcup_{j \in Y_{i,c}} C_j^{(2(q+1),q^2)}$, $i = 1,2,3,4,5$. Then, R'_i takes the character values listed in Table 2.

4.2. The construction

Let $X_{i,c}$, $Y_{i,c}$, R_i, R'_i, $i = 1,2,\dots,5$, be sets defined as in Subsection 4.1. Let A and B be subsets of $\{0,1,\dots,2q+1\}$ satisfying $A \cap B = X_{3,c}$ and as multisets, $A \cup B = X_{1,c} \cup X_{3,c} \cup X_{3,c}$. It follows that $(A \setminus B) \cup (B \setminus A) = X_{1,c}$.

Let $\tau = 0$ or 1 according as $q \equiv 3$ or $1 \pmod 4$. Define

$$\begin{aligned} D_0 =&\{(0,y) \mid y \in C_\tau^{(2,q^2)}\},\\ D_1 =&\{(y,0) \mid y \in C_0^{(2,q^2)}\},\\ D_2 =&\{(xy, xy^{-1}\omega^i) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, i \in A\},\\ D_3 =&\{(xy, xy^{-1}\omega^i) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, i \in B\}. \end{aligned} \qquad (4.1)$$

We denote the set of even (resp. odd) elements in any subset S of $\{0,1,\dots,2q+1\}$ by S_e (resp. S_o). The following is our main result in this section.

Theorem 4.1.

(1) *If $|A| = |B| = \frac{q+1}{2}$ and $|A_e|+|B_o| = |A_o|+|B_e|-2(-1)^{\frac{q-1}{2}}$, then $E_0 = D_0 \cup D_2 \cup D_3$ is a type Q set in $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$.*

(2) *If* $|A| = \frac{q+3}{2}$, $|B| = \frac{q-1}{2}$ *and* $|A_e| + |B_o| = |A_o| + |B_e|$, *then* $E_1 = D_1 \cup D_2 \cup D_3$ *is a type Q set in* $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$*.*

This theorem generalizes the examples of type Q sets found by Wilson-Xiang [12]. Indeed, these sets admit the automorphism T'. See Subsection 2.2.

To prove the theorem above, we will evaluate the character values of E_i, $i = 0, 1$. Define

$$V_0 = \sum_{y \in C_\tau^{(2,q^2)}} \psi_{\mathbb{F}_{q^2}}(by), \quad V_1 = \sum_{y \in C_0^{(2,q^2)}} \psi_{\mathbb{F}_{q^2}}(ay),$$

$$V_2 = \frac{1}{2} \sum_{i \in A} \sum_{y \in C_0^{(2,q^2)}} \sum_{x \in \mathbb{F}_q^*} \psi_{\mathbb{F}_{q^2}}(axy) \psi_{\mathbb{F}_{q^2}}(bxy^{-1}\omega^i),$$

$$V_3 = \frac{1}{2} \sum_{i \in B} \sum_{y \in C_1^{(2,q^2)}} \sum_{x \in \mathbb{F}_q^*} \psi_{\mathbb{F}_{q^2}}(axy) \psi_{\mathbb{F}_{q^2}}(bxy^{-1}\omega^i).$$

Noting that each element in D_2 (resp. D_3) appears exactly twice when x runs through $\mathbb{F}_q^*$ and y runs through $C_0^{(2,q^2)}$ (resp. $C_1^{(2,q^2)}$), we have $\psi_{a,b}(E_0) = V_0 + V_2 + V_3$ and $\psi_{a,b}(E_1) = V_1 + V_2 + V_3$. We will evaluate these character sums by considering two cases: (i) exactly one of a, b is zero; and (ii) $a \neq 0$ and $b \neq 0$. We first treat Case (i).

Lemma 4.2. *If exactly one of* a, b *is zero, then*

$$\psi_{a,b}(E_0) = \begin{cases} \frac{-3q^2-1}{4}, & \text{if } a = 0 \text{ and } b \in C_1^{(2,q^2)}, \\ \frac{q^2-1}{4}, & \text{otherwise.} \end{cases}$$

Proof. If $a \neq 0$ and $b = 0$, it is clear that $V_0 = \frac{q^2-1}{2}$. Furthermore, since $|A| = |B| = \frac{q+1}{2}$, we have

$$V_2 + V_3 = \frac{q+1}{4} \sum_{y \in \mathbb{F}_{q^2}^*} \sum_{x \in \mathbb{F}_q^*} \psi_{\mathbb{F}_{q^2}}(axy) = -\frac{q^2-1}{4}.$$

Hence, $\psi_{a,b}(E_0) = \frac{q^2-1}{4}$. If $a = 0$ and $b \neq 0$, we have

$$\begin{aligned} &V_0 + V_2 + V_3 \\ &= \psi_{\mathbb{F}_{q^2}}(bC_\tau^{(2,q^2)}) + \frac{q-1}{2}((|A_e| + |B_o|)\psi_{\mathbb{F}_{q^2}}(bC_0^{(2,q^2)}) + (|A_o| + |B_e|)\psi_{\mathbb{F}_{q^2}}(bC_1^{(2,q^2)})). \end{aligned} \tag{4.2}$$

Since $|A_e|+|A_o|+|B_e|+|B_o| = |A|+|B| = q+1$ and $|A_e|+|B_o| = |A_o|+|B_e|-2(-1)^{\frac{q-1}{2}}$, we have $|A_e| + |B_o| = \frac{q+1}{2} - (-1)^{\frac{q-1}{2}}$ and $|A_o| + |B_e| = \frac{q+1}{2} + (-1)^{\frac{q-1}{2}}$. Hence, (4.2) is reformulated as

$$V_0 + V_2 + V_3 = q\psi_{\mathbb{F}_{q^2}}(bC_{\tau}^{(2,q^2)}) - \Big(\frac{q-1}{2}\Big)^2.$$

Finally, by (2.2), the statement follows. □

Lemma 4.3. *If exactly one of a, b is zero, then*

$$\psi_{a,b}(E_1) = \begin{cases} \frac{-3q^2-1}{4}, & \text{if } b = 0 \text{ and } a \in C_{\tau+1}^{(2,q^2)}, \\ \frac{q^2-1}{4}, & \text{otherwise.} \end{cases}$$

Proof. If $a = 0$ and $b \neq 0$, it is clear that $V_1 = \frac{q^2-1}{2}$. Since $|A_e|+|B_o| = |A_o|+|B_e| = \frac{q+1}{2}$, we have

$$\begin{aligned} V_2 + V_3 &= \frac{q-1}{2}((|A_e| + |B_o|)\psi_{\mathbb{F}_{q^2}}(bC_0^{(2,q^2)}) + (|A_o| + |B_e|)\psi_{\mathbb{F}_{q^2}}(bC_1^{(2,q^2)})) \\ &= -\frac{q^2-1}{4}. \end{aligned}$$

Hence, $\psi_{a,b}(E_1) = \frac{q^2-1}{4}$. If $a \neq 0$ and $b = 0$,

$$V_1 + V_2 + V_3 = \psi_{\mathbb{F}_{q^2}}(aC_0^{(2,q^2)}) + \frac{q-1}{2}(|A|\psi_{\mathbb{F}_{q^2}}(aC_0^{(2,q^2)}) + |B|\psi_{\mathbb{F}_{q^2}}(aC_1^{(2,q^2)})). \quad (4.3)$$

Since $|A| = \frac{q+3}{2}$ and $|B| = \frac{q-1}{2}$, (4.3) is reformulated as

$$V_1 + V_2 + V_3 = q\psi_{\mathbb{F}_{q^2}}(aC_0^{(2,q^2)}) - \Big(\frac{q-1}{2}\Big)^2.$$

Finally, by (2.2), the statement follows. □

We next consider Case (ii), i.e., $a \neq 0$ and $b \neq 0$.

Lemma 4.4. *If $a \neq 0$ and $b \neq 0$, then*

$$\begin{aligned} V_2 + V_3 = \frac{1}{4(q+1)} \sum_{u=0,1} \sum_{h=0}^{2q+1} & G_{q^2}(\chi_{2(q+1)}^{-h}\rho^u)G_{q^2}(\chi_{2(q+1)}^{-h})\chi_{2(q+1)}^{h}(ab)\rho^u(a) \\ & \times \Big(\sum_{i\in A}\chi_{2(q+1)}^{h}(\omega^i) + \sum_{i\in B}\chi_{2(q+1)}^{h}(\omega^i)\rho^u(\omega)\Big), \end{aligned} \quad (4.4)$$

where $\chi_{2(q+1)}$ is a multiplicative character of order $2(q+1)$ of $\mathbb{F}_{q^2}$ and ρ is the quadratic character of $\mathbb{F}_{q^2}$.

Proof. Let χ be a multiplicative character of order q^2-1 of $\mathbb{F}_{q^2}$. By (2.1), we have

$$V_2 = \frac{1}{2(q^2-1)^2}\sum_{i\in A}\sum_{y\in C_0^{(2,q^2)}}\sum_{x\in\mathbb{F}_q^*}\sum_{j,k=0}^{q^2-2} G_{q^2}(\chi^{-j})\chi^j(axy)G_{q^2}(\chi^{-k})\chi^k(bxy^{-1}\omega^i)$$

$$= \frac{1}{2(q^2-1)^2}\sum_{i\in A}\sum_{j,k=0}^{q^2-2} G_{q^2}(\chi^{-j})G_{q^2}(\chi^{-k})\chi^j(a)\chi^k(b\omega^i)\chi^{j-k}(C_0^{(2,q^2)})\Big(\sum_{x\in\mathbb{F}_q^*}\chi^{j+k}(x)\Big). \tag{4.5}$$

Since $\chi^{j-k}(C_0^{(2,q^2)}) = \frac{q^2-1}{2}$ or 0 according as $j-k\equiv 0 \pmod{\frac{q^2-1}{2}}$ or not, continuing from (4.5), we have

$$V_2 = \frac{1}{4(q^2-1)}\sum_{i\in A}\sum_{u=0,1}\sum_{k=0}^{q^2-2} G_{q^2}(\chi^{-k-\frac{q^2-1}{2}u})G_{q^2}(\chi^{-k})\chi^{k+\frac{q^2-1}{2}u}(a)\chi^k(b\omega^i)$$
$$\times\Big(\sum_{x\in\mathbb{F}_q^*}\chi^{2k+\frac{q^2-1}{2}u}(x)\Big). \tag{4.6}$$

Let $\chi_{2(q+1)} = \chi^{\frac{q-1}{2}}$ and $\rho = \chi^{\frac{q^2-1}{2}}$. Since $\sum_{x\in\mathbb{F}_q^*}\chi^{2k+\frac{q^2-1}{2}u}(x) = q-1$ or 0 according as $2k\equiv 0 \pmod{q-1}$ or not, continuing from (4.6), we have

$$V_2 = \frac{1}{4(q+1)}\sum_{u=0,1}\sum_{h=0}^{2q+1} G_{q^2}(\chi_{2(q+1)}^{-h}\rho^u)G_{q^2}(\chi_{2(q+1)}^{-h})\chi_{2(q+1)}^h(ab)\rho^u(a)\sum_{i\in A}\chi_{2(q+1)}^h(\omega^i).$$

Similarly, we have

$$V_3 = \frac{1}{4(q+1)}\sum_{u=0,1}\sum_{h=0}^{2q+1} G_{q^2}(\chi_{2(q+1)}^{-h}\rho^u)G_{q^2}(\chi_{2(q+1)}^{-h})\chi_{2(q+1)}^h(ab)\rho^u(a)$$
$$\times\sum_{i\in B}\chi_{2(q+1)}^h(\omega^i)\rho^u(\omega).$$

This completes the proof of the lemma. □

Let W_0 (resp. W_1) be the contribution for $u=0$ (resp. $u=1$) in the summations of (4.4); then $V_2+V_3 = W_0+W_1$.

Lemma 4.5. *Let $r = ab \neq 0$. Then,*

$$W_0 = \begin{cases} \frac{-q^2+1}{4}, & \text{if } r \in \omega^c R_1 \text{ or } r \in \omega^c R_2, \\ \frac{q^2+1}{4} \text{ or } \frac{-3q^2+1}{4}, & \text{otherwise}, \end{cases}$$

depending on whether $q \equiv 3$ or $1 \pmod 4$.

Proof. By the definition of W_0, we have

$$W_0 = \frac{1}{4(q+1)} \sum_{h=0}^{2q+1} G_{q^2}(\chi_{2(q+1)}^{-h})^2 \chi_{2(q+1)}^h(r) \Big(\sum_{i \in A \cup B} \chi_{2(q+1)}^h(\omega^i) \Big).$$

Since $A \cup B = X_{1,c} \cup X_{3,c} \cup X_{3,c}$ as a multiset, by the property (P7), we have

$$\psi_{\mathbb{F}_{q^2}}(\omega^a \bigcup_{i \in A \cup B} C_i^{(2(q+1),q^2)}) = \begin{cases} \frac{qG_q(\eta)-1}{2} (=: c_1), & \text{if } a \in Y_{1,c} (=: Z_1), \\ \frac{-qG_q(\eta)-1}{2} (=: c_2), & \text{if } a \in Y_{2,c} (=: Z_2), \\ \frac{-1+(-1)^{\frac{q-1}{2}} q}{2} (=: c_3), & \text{if } a \in Y_{3,c} \cup Y_{4,c} (=: Z_3), \\ \frac{-1-(-1)^{\frac{q-1}{2}} q}{2} (=: c_4), & \text{if } a \in Y_{5,c} (=: Z_4). \end{cases}$$

Then, by (2.3), we have

$$\begin{aligned} G_{q^2}(\chi_{2(q+1)}^{-h}) \sum_{i \in A \cup B} \chi_{2(q+1)}^h(\omega^i) &= \sum_{a=0}^{2q+1} \psi_{\mathbb{F}_{q^2}}(\omega^a \bigcup_{i \in A \cup B} C_i^{(2(q+1),q^2)}) \chi_{2(q+1)}^{-h}(\omega^a) \\ &= \sum_{i=1}^{4} c_i \sum_{a \in Z_i} \chi_{2(q+1)}^{-h}(\omega^a). \end{aligned}$$

Then, by (2.1), we have

$$\begin{aligned} W_0 &= \sum_{i=1}^{4} \frac{c_i}{4(q+1)} \sum_{a \in Z_i} \sum_{h=0}^{2q+1} G_{q^2}(\chi_{2(q+1)}^{-h}) \chi_{2(q+1)}^h(r\omega^{-a}) \\ &= \sum_{i=1}^{4} \frac{c_i}{2} \sum_{a \in -Z_i} \psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)}). \end{aligned}$$

Since $-Y_{i,c} \equiv Y_{i,c} - c \pmod{2(q+1)}$, $i = 1, 2$, from the property (P8), we have by the property (P10) that for $i = 1, 2$

$$\sum_{a\in -Z_i}\psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)})=\begin{cases}\frac{-2+q+(-1)^{i-1}G_q(\eta)}{2}, & \text{if } r\in\omega^cR_1,\\ \frac{-2+q-(-1)^{i-1}G_q(\eta)}{2}, & \text{if } r\in\omega^cR_2,\\ -1+(-1)^{i-1}G_q(\eta), & \text{if } r\in\omega^cR_3,\\ -1-(-1)^{i-1}G_q(\eta), & \text{if } r\in\omega^cR_4,\\ -1, & \text{if } r\in\omega^cR_5.\end{cases}$$

Furthermore, since $-(Y_{3,c}\cup Y_{4,c})\equiv Y_{5,c}-c \pmod{2(q+1)}$ by the property (P9), we have

$$\sum_{a\in -Z_3}\psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)})=\sum_{a\in Y_{5,c}-c}\psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)})$$
$$=\begin{cases}\frac{1-q}{2}, & \text{if } r\in\omega^c(R_1\cup R_2),\\ \frac{1+(-1)^{\frac{q-1}{2}}q}{2}, & \text{if } r\in\omega^cR_5,\\ \frac{1-(-1)^{\frac{q-1}{2}}q}{2}, & \text{if } r\in\omega^c(R_3\cup R_4).\end{cases}$$

Similarly, we have

$$\sum_{a\in -Z_4}\psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)})=\sum_{a\in (Y_3\cup Y_4)-c}\psi_{\mathbb{F}_{q^2}}(rC_a^{(2(q+1),q^2)})$$
$$=\begin{cases}\frac{-q+1}{2}, & \text{if } r\in\omega^c(R_1\cup R_2),\\ \frac{1-(-1)^{\frac{q-1}{2}}q}{2}, & \text{if } r\in\omega^cR_5,\\ \frac{1+(-1)^{\frac{q-1}{2}}q}{2}, & \text{if } r\in\omega^c(R_3\cup R_4).\end{cases}$$

Summing up, we have

$$W_0=\begin{cases}\frac{-q^2+1}{4}, & \text{if } r\in\omega^cR_1 \text{ or } r\in\omega^cR_2,\\ \frac{q^2+1}{4}, & \text{if } r\in\omega^c(R_2\cup R_4\cup R_5) \text{ or } r\in\omega^c(R_1\cup R_3\cup R_5),\\ \frac{-3q^2+1}{4}, & \text{if } r\in\omega^cR_3 \text{ or } r\in\omega^cR_4,\end{cases}$$

according as $q\equiv 3$ or $1 \pmod 4$. This completes the proof. □

We next evaluate W_1 below.

Lemma 4.6. *Let $r=ab\neq 0$. Then,*

$$W_1=-\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{4}\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)$$

$$+\frac{(-1)^{\frac{q-1}{2}}\rho(a)q^2}{2}\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)+q+1}C_i^{(2(q+1),q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)+q+1}C_i^{(2(q+1),q^2)},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By the definition of W_1, we have

$$W_1=\frac{\rho(a)}{4(q+1)}\sum_{h=0}^{2q+1}G_{q^2}(\chi_{2(q+1)}^{-h}\rho)G_{q^2}(\chi_{2(q+1)}^{-h})\chi_{2(q+1)}^{h}(r)\times\Big(\sum_{i\in A}\chi_{2(q+1)}^{h}(\omega^i)-\sum_{i\in B}\chi_{2(q+1)}^{h}(\omega^i)\Big).\qquad(4.7)$$

By applying the Davenport-Hasse product formula (Theorem 2.3) with $\chi=\chi_{2(q+1)}^{-h}$, $\chi'=\rho$, and $\ell=2$ we have

$$G_{q^2}(\chi_{2(q+1)}^{-h})G_{q^2}(\chi_{2(q+1)}^{-h}\rho)=G_{q^2}(\rho)G_{q^2}(\chi_{q+1}^{-h}),$$

where $\chi_{q+1}=\chi_{2(q+1)}^2$ has order $q+1$. Then, (4.7) is rewritten as

$$W_1=\frac{\rho(a)}{4(q+1)}G_{q^2}(\rho)\sum_{h=0}^{2q+1}G_{q^2}(\chi_{q+1}^{-h})\chi_{2(q+1)}^{h}(r)\Big(\sum_{i\in A}\chi_{2(q+1)}^{h}(\omega^i)-\sum_{i\in B}\chi_{2(q+1)}^{h}(\omega^i)\Big).\qquad(4.8)$$

We will compute W_1 by dividing it into three parts. Let $W_{1,1},W_{1,2},W_{1,3}$ denote the contributions in the sum on the right hand side of (4.8) when $h=0,q+1$; other even h; and odd h, respectively. Then $W_1=W_{1,1}+W_{1,2}+W_{1,3}$. For $W_{1,1}$, we have

$$W_{1,1}=-\frac{\rho(a)}{4(q+1)}G_{q^2}(\rho)\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big).$$

Next, by Theorem 2.2, we have

$$W_{1,2}=\frac{\rho(a)q}{4(q+1)}G_{q^2}(\rho)\sum_{\ell=0;\ell\neq 0,\frac{q+1}{2}}^{q}\chi_{q+1}^{\ell}(r)\Big(\sum_{i\in A}\chi_{q+1}^{\ell}(\omega^i)-\sum_{i\in B}\chi_{q+1}^{\ell}(\omega^i)\Big).\qquad(4.9)$$

By the property (P2),

$$\{x\,(\mathrm{mod}\ q+1)\,|\,x\in A\setminus B\}\cap\{x\,(\mathrm{mod}\ q+1)\,|\,x\in B\setminus A\}=\emptyset.$$

Hence, continuing from (4.9), we have

$$W_{1,2}=-\frac{\rho(a)q}{4(q+1)}G_{q^2}(\rho)\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)$$

$$+\frac{\rho(a)q}{4}G_{q^2}(\rho)\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)}C_i^{(q+1,q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)}C_i^{(q+1,q^2)},\\ 0, & \text{otherwise.}\end{cases}$$

Finally, by Theorem 2.2 again, we have

$$\begin{aligned}W_{1,3}&=-\frac{\rho(a)q}{4(q+1)}G_{q^2}(\rho)\sum_{h:\,odd}\chi_{2(q+1)}^h(r)\Big(\sum_{i\in A}\chi_{2(q+1)}^h(\omega^i)-\sum_{i\in B}\chi_{2(q+1)}^h(\omega^i)\Big)\\ &=-\frac{\rho(a)q}{4(q+1)}G_{q^2}(\rho)\sum_{h=0}^{2q+1}\chi_{2(q+1)}^h(r)\Big(\sum_{i\in A}\chi_{2(q+1)}^h(\omega^i)-\sum_{i\in B}\chi_{2(q+1)}^h(\omega^i)\Big)\\ &\quad+\frac{\rho(a)q}{4(q+1)}G_{q^2}(\rho)\sum_{\ell=0}^{q}\chi_{q+1}^\ell(r)\Big(\sum_{i\in A}\chi_{q+1}^\ell(\omega^i)-\sum_{i\in B}\chi_{q+1}^\ell(\omega^i)\Big)\\ &=-\frac{\rho(a)q}{2}G_{q^2}(\rho)\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)}C_i^{(2(q+1),q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)}C_i^{(2(q+1),q^2)},\\ 0, & \text{otherwise,}\end{cases}\\ &\quad+\frac{\rho(a)q}{4}G_{q^2}(\rho)\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)}C_i^{(q+1,q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)}C_i^{(q+1,q^2)},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Summing up, we have

$$\begin{aligned}W_1&=W_{1,2}+W_{1,2}+W_{1,3}\\ &=-\frac{\rho(a)}{4}G_{q^2}(\rho)\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)\\ &\quad+\frac{\rho(a)q}{2}G_{q^2}(\rho)\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)+q+1}C_i^{(2(q+1),q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)+q+1}C_i^{(2(q+1),q^2)},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

The statement now follows from $G_{q^2}(\rho)=-(-1)^{\frac{q-1}{2}}q$. □

Remark 4.7. By Lemmas 4.5 and 4.6, we have

$$\begin{aligned}V_2+V_3&=W_0+W_1\\ &=\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{4}\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)\end{aligned}$$

$$
-\frac{(-1)^{\frac{q-1}{2}}\rho(a)q^2}{2}\cdot\begin{cases}1, & \text{if } r\in\bigcup_{i\in-(A\setminus B)+(q+1)}C_i^{(2(q+1),q^2)},\\ -1, & \text{if } r\in\bigcup_{i\in-(B\setminus A)+(q+1)}C_i^{(2(q+1),q^2)},\\ 0, & \text{otherwise},\end{cases}
$$

$$
+\begin{cases}\frac{-q^2+1}{4}, & \text{if } r\in\omega^cR_1 \text{ or } r\in\omega^cR_2,\\ \frac{-3q^2+1}{4} \text{ or } \frac{q^2+1}{4}, & \text{otherwise},\end{cases} \tag{4.10}
$$

according as $q\equiv 3$ or $1 \pmod 4$. By the property (P4), $X_{1,c}+c\equiv -((A\setminus B)\cup(B\setminus A))+(q+1) \pmod{2(q+1)}$ or $X_{2,c}+c\equiv -((A\setminus B)\cup(B\setminus A))+(q+1) \pmod{2(q+1)}$ depending on whether $q\equiv 3$ or $1 \pmod 4$. Hence, continuing from (4.10), we have

$$
\begin{aligned}
&V_2+V_3-\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{4}\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)\\
&=\frac{q^2+1}{4} \text{ or } \frac{-3q^2+1}{4}.
\end{aligned} \tag{4.11}
$$

We are now ready to prove our main theorem.

Proof of Theorem 4.1. In the case where exactly one of a,b is zero, the statement follows from Lemmas 4.2 and 4.3. We treat the case where $a\neq 0$ and $b\neq 0$.

(1) By (2.2), $V_0=\frac{-1+\rho(b)q}{2}$. Furthermore, by $|A|=|B|=\frac{q+1}{2}$ and $|A_e|+|B_o|=|A_o|+|B_e|-2(-1)^{\frac{q-1}{2}}$, we have

$$
\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{4}\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)=-\frac{\rho(b)q}{2}.
$$

Hence, by (4.11), it follows that $\psi_{a,b}(E_0)=\frac{q^2-1}{4}$ or $\frac{-3q^2-1}{4}$.

(2) By (2.2), $V_1=\frac{-1-(-1)^{\frac{q-1}{2}}\rho(a)q}{2}$. Furthermore, by $|A|=\frac{q+3}{2}$, $|B|=\frac{q-1}{2}$, and $|A_e|+|B_o|=|A_o|+|B_e|$, we have

$$
\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{4}\Big(|A|-|B|+\rho(r)(|A_e|+|B_o|-|A_o|-|B_e|)\Big)=\frac{(-1)^{\frac{q-1}{2}}\rho(a)q}{2}.
$$

Hence, by (4.11), it follows that $\psi_{a,b}(E_1)=\frac{q^2-1}{4}$ or $\frac{-3q^2-1}{4}$. □

Corollary 4.8. *Let $A=\{\beta\}\cup X_{3,c}$, $B=\{\alpha\}\cup X_{3,c}$, $A'=X_{1,c+q+1}\cup X_{3,c+q+1}$, and $B'=X_{3,c+q+1}$, where α,β are defined as in the property (P6). Then, the sets*

$$
\begin{aligned}
C_0=&\{(0,y)\,|\,y\in C_\tau^{(2,q^2)}\}\cup\{(xy,xy^{-1}\omega^i)\,|\,x\in\mathbb{F}_q^*,y\in C_0^{(2,q^2)},i\in A\}\\
&\cup\{(xy,xy^{-1}\omega^i)\,|\,x\in\mathbb{F}_q^*,y\in C_1^{(2,q^2)},i\in B\},
\end{aligned}
$$

$$
\begin{aligned}
C_1 =& \{(y,0) \mid y \in C_0^{(2,q^2)}\} \cup \{(xy, xy^{-1}\omega^i) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, i \in A'\} \\
& \cup \{(xy, xy^{-1}\omega^i) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, i \in B'\}, \\
C_2 =& \{(0,y) \mid y \in C_{\tau+1}^{(2,q^2)}\} \cup \{(xy\omega, xy^{-1}\omega^{i+q}) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, i \in A\} \\
& \cup \{(xy\omega, xy^{-1}\omega^{i+q}) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, i \in B\}, \\
C_3 =& \{(y,0) \mid y \in C_1^{(2,q^2)}\} \cup \{(xy\omega, xy^{-1}\omega^{i+q}) \mid x \in \mathbb{F}_q^*, y \in C_0^{(2,q^2)}, i \in A'\} \\
& \cup \{(xy\omega, xy^{-1}\omega^{i+q}) \mid x \in \mathbb{F}_q^*, y \in C_1^{(2,q^2)}, i \in B'\}
\end{aligned}
$$

are of type Q. Furthermore, these sets satisfy the assumptions of Remark 2.6 with respect to the spread $\mathcal{K}$ consisting of the following 2-dimensional subspaces:

$$K_y = \{(x, yx^q) \mid x \in \mathbb{F}_{q^2}\}, y \in \mathbb{F}_{q^2}, \quad \text{and} \quad K_\infty = \{(0,x) \mid x \in \mathbb{F}_{q^2}\}.$$

Proof. By the property (P6), $|A_e| + |B_o| = |A_o| + |B_e| - 2(-1)^{\frac{q-1}{2}}$ and $|A'_e| + |B'_o| = |A'_o| + |B'_e|$. Hence, by Theorem 4.1, C_0 and C_1 are type Q sets. Since $C_2 = \{(\omega x, \omega^q y) \mid (x,y) \in C_0\}$ and $C_3 = \{(\omega x, \omega^q y) \mid (x,y) \in C_1\}$, the sets C_2 and C_3 are also of type Q. Furthermore, $\bigcup_{i=1}^{3} C_i = (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}) \setminus \{(0,0)\}$ since $A \cup A' \cup (B+q+1) \cup (B'+q+1) \equiv \{0,1,\ldots,2q+1\} \pmod{2(q+1)}$ by the properties (P1), (P4), (P5) and (P6). Therefore, C_i, $i = 0,1,2,3$, satisfy the assumptions of Remark 2.6 as $C_0 \cup C_2 \cup \{(0,0)\} = \big(\bigcup_{y \in H_0} K_y\big) \cup K_\infty$ and $C_1 \cup C_3 \cup \{(0,0)\} = \bigcup_{y \in H_1} K_y$, where $H_0 = \bigcup_{i \in -(A \cup (B+q+1))} C_i^{(2(q+1),q^2)}$ and $H_1 = \bigcup_{i \in -(A' \cup (B'+q+1))} C_i^{(2(q+1),q^2)}$. □

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Appendix A

In this appendix, we prove that the sets $X_{i,c}$ and $Y_{i,c}$, $i = 1,2,3,4,5$, have the properties (P1)–(P10).

By the definition of $X_{1,c}$, we have

$$
\begin{aligned}
X_{1,c} =& (\{\tfrac{q+1}{2}, \tfrac{3(q+1)}{2}\} \cap \{i \pmod{2(q+1)} \mid \mathrm{Tr}_{q^2/q}(\omega^{i+c}) \in C_0^{(2,q)}\}) \\
& \cup (\{\tfrac{q+1}{2} - c, \tfrac{3(q+1)}{2} - c\} \cap \{i \pmod{2(q+1)} \mid \mathrm{Tr}_{q^2/q}(\omega^{i}) \in C_0^{(2,q)}\}).
\end{aligned}
$$

Hence, there are $\epsilon, \delta \in \{-1,1\}$ such that $X_{1,c} = \{\frac{q+1}{2}\epsilon, \frac{q+1}{2}\delta - c\}$. In particular, we have

$$\mathrm{Tr}_{q^2/q}(\omega^{c+\frac{q+1}{2}\epsilon}) \in C_0^{(2,q)} \quad \text{and} \quad \mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\delta - c}) \in C_0^{(2,q)}. \tag{A.1}$$

Lemma A.1. *We have* $X_{1,c} = \{\frac{q+1}{2}\epsilon, \frac{q+1}{2}\delta - c\}$ *for some* $(\epsilon, \delta) \in \{(1,1),(-1,-1)\}$ *or* $\{(-1,1),(1,-1)\}$ *according as* $q \equiv 3 \pmod 4$ *or* $q \equiv 1 \pmod 4$*.*

Proof. By (A.1), we have

$$\omega^{\frac{q+1}{2}\epsilon}(\omega^c - \omega^{cq}) \in C_0^{(2,q)} \quad \text{and} \quad \omega^{\frac{q+1}{2}\delta}(\omega^{-c} - \omega^{-cq}) \in C_0^{(2,q)}. \tag{A.2}$$

Putting $\omega^d = 1 - \omega^{c(q-1)}$, the conditions in (A.2) are rewritten as

$$\omega^{\frac{q+1}{2}\epsilon+c+d} = \omega^{2(q+1)k} \quad \text{and} \quad \omega^{\frac{q+1}{2}\delta-c+dq} = \omega^{2(q+1)\ell}$$

for some $k, \ell \in \mathbb{Z}$. Here, d is odd if $q \equiv 3 \pmod 4$, and d is even if $q \equiv 1 \pmod 4$. By multiplying these equations, we have $\omega^{\frac{q+1}{2}(\epsilon+\delta)+d(q+1)} = \omega^{2(q+1)(k+\ell)}$. Then, the statement immediately follows. □

Remark A.2. For $X_{i,c}$, $i = 1,2,3,4,5$, we observe the following facts:

(1) Since $I_2 \equiv I_3 + (q+1) \pmod{2(q+1)}$, we have $X_{1,c} \equiv X_{2,c} + (q+1) \pmod{2(q+1)}$, $X_{3,c} \equiv X_{4,c} + (q+1) \pmod{2(q+1)}$, and $X_{5,c} \equiv X_{5,c} + (q+1) \pmod{2(q+1)}$. Hence, the property (P1) follows.
(2) Since I_2 forms a $(q+1, 2, q, \frac{q-1}{2})$ relative difference set (cf. [1]), we have $|X_{3,c}| = \frac{q-1}{2}$. Then, the property (P2) follows.
(3) Since $X_{3,c+q+1} = I_2 \cap J_3$ and $X_{4,c+q+1} = I_3 \cap J_2$, we have $X_{3,c+q+1} \cup X_{4,c+q+1} = X_{5,c}$. Then, the property (P3) follows.
(4) The property (P4) directly follows from Lemma A.1.
(5) By Lemma A.1, $X_{1,c+q+1} = \{\frac{q+1}{2}\epsilon', \frac{q+1}{2}\delta' - c + q + 1\}$ for some $(\epsilon', \delta') \in \{(1,1),(-1,-1)\}$ or $\{(-1,1),(1,-1)\}$ according to whether $q \equiv 3 \pmod 4$ or $q \equiv 1 \pmod 4$. Then, it is direct to see that $|X_{1,c} \cap X_{1,c+q+1}| = 1$ and $(X_{1,c} \setminus X_{1,c+q+1}) \equiv (X_{1,c+q+1} \setminus X_{1,c}) + q + 1 \pmod{2(q+1)}$ in all cases. More precisely, $X_{1,c+q+1} = \{\frac{q+1}{2}\epsilon + q + 1, \frac{q+1}{2}\delta - c\}$ since $\frac{q+1}{2}\delta - c \in J_1 \cap I_2$. Hence, $X_{1,c} \setminus X_{1,c+q+1} = \{\frac{q+1}{2}\epsilon\}$ and $X_{1,c} \cap X_{1,c+q+1} = \{\frac{q+1}{2}\delta - c\}$. Thus, the properties (P5) and (P6) follow.

Next, we show that the $X_{i,c}$'s have property (P7).

Proposition A.3. *Let* R_i, $i = 1,2,3,4,5$, *be defined as in Subsection 4.1. Then,* R_i, $i = 1,2,3,4,5$, *take the character values listed in Table 1. In particular,* $Y_{i,c}$*'s in Table 1 are determined as follows:*

$$Y_{1,c} = \{0, c\}, \;\; Y_{2,c} = \{q+1, c+q+1\},$$

$$Y_{3,c} = \{i + c - \tfrac{q+1}{2}\delta \,|\, \mathrm{Tr}_{q^2/q}(\omega^i) \in C_0^{(2,q)}\} \cap \{i - \tfrac{q+1}{2}\epsilon \,|\, \mathrm{Tr}_{q^2/q}(\omega^i) \in C_0^{(2,q)}\},$$

$$Y_{4,c} = \{i + c - \tfrac{q+1}{2}\delta \,|\, \mathrm{Tr}_{q^2/q}(\omega^i) \in C_1^{(2,q)}\} \cap \{i - \tfrac{q+1}{2}\epsilon \,|\, \mathrm{Tr}_{q^2/q}(\omega^i) \in C_1^{(2,q)}\},$$

$$\begin{aligned}Y_{5,c} =&(\{i+c-\tfrac{q+1}{2}\delta \,|\, \mathrm{Tr}_{q^2/q}(\omega^i)\in C_0^{(2,q)}\}\cap\{i-\tfrac{q+1}{2}\epsilon \,|\, \mathrm{Tr}_{q^2/q}(\omega^i)\in C_1^{(2,q)}\})\\ &\cup(\{i+c-\tfrac{q+1}{2}\delta \,|\, \mathrm{Tr}_{q^2/q}(\omega^i)\in C_1^{(2,q)}\}\cap\{i-\tfrac{q+1}{2}\epsilon \,|\, \mathrm{Tr}_{q^2/q}(\omega^i)\in C_0^{(2,q)}\}).\end{aligned}$$

Proof. The character values $\psi_{\mathbb{F}_{q^2}}(\omega^a R_1)$, $a=0,1,\ldots,2q+1$, are evaluated as follows:

$$\begin{aligned}\psi_{\mathbb{F}_{q^2}}(\omega^a R_1) &= \psi_{\mathbb{F}_{q^2}}(\omega^{a+\frac{q+1}{2}\delta-c}C_0^{(2(q+1),q^2)})+\psi_{\mathbb{F}_{q^2}}(\omega^{a+\frac{q+1}{2}\epsilon}C_0^{(2(q+1),q^2)})\\ &=\psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\delta-c})C_0^{(2,q)})+\psi_{\mathbb{F}_q}(\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\epsilon})C_0^{(2,q)})\\ &=\begin{cases}\frac{q-1}{2}, & \text{if } a\in I_1-\frac{q+1}{2}\delta+c,\\ \frac{-1+G_q(\eta)}{2}, & \text{if } a\in I_2-\frac{q+1}{2}\delta+c,\\ \frac{-1-G_q(\eta)}{2}, & \text{if } a\in I_3-\frac{q+1}{2}\delta+c,\end{cases}+\begin{cases}\frac{q-1}{2}, & \text{if } a\in I_1-\frac{q+1}{2}\epsilon,\\ \frac{-1+G_q(\eta)}{2}, & \text{if } a\in I_2-\frac{q+1}{2}\epsilon,\\ \frac{-1-G_q(\eta)}{2}, & \text{if } a\in I_3-\frac{q+1}{2}\epsilon,\end{cases}\\ &=\begin{cases}\frac{-2+q+G_q(\eta)}{2}, & \text{if } a\in Y'_{1,c},\\ \frac{-2+q-G_q(\eta)}{2}, & \text{if } a\in Y'_{2,c},\\ -1+G_q(\eta), & \text{if } a\in Y_{3,c},\\ -1-G_q(\eta), & \text{if } a\in Y_{4,c},\\ -1, & \text{if } a\in Y_{5,c},\end{cases}\end{aligned}$$

where

$$\begin{aligned}Y'_{1,c}&=((I_1-\tfrac{q+1}{2}\delta+c)\cap(I_2-\tfrac{q+1}{2}\epsilon))\cup((I_2-\tfrac{q+1}{2}\delta+c)\cap(I_1-\tfrac{q+1}{2}\epsilon)),\\ Y'_{2,c}&=((I_1-\tfrac{q+1}{2}\delta+c)\cap(I_3-\tfrac{q+1}{2}\epsilon))\cup((I_3-\tfrac{q+1}{2}\delta+c)\cap(I_1-\tfrac{q+1}{2}\epsilon)).\end{aligned}$$

By (A.1), it is direct to see that

$$\begin{aligned}&(I_1-\tfrac{q+1}{2}\delta+c)\cap(I_2-\tfrac{q+1}{2}\epsilon)=\{c\},\quad (I_2-\tfrac{q+1}{2}\delta+c)\cap(I_1-\tfrac{q+1}{2}\epsilon)=\{0\},\\ &(I_1-\tfrac{q+1}{2}\delta+c)\cap(I_3-\tfrac{q+1}{2}\epsilon)=\{c+q+1\},\quad (I_3-\tfrac{q+1}{2}\delta+c)\cap(I_1-\tfrac{q+1}{2}\epsilon)=\{q+1\}.\end{aligned}$$

Hence, we have $Y'_{1,c}=Y_{1,c}$ and $Y'_{2,c}=Y_{2,c}$.

The character values of R_2 is determined as $\psi_{\mathbb{F}_{q^2}}(\omega^a R_2)=\psi_{\mathbb{F}_{q^2}}(\omega^{a+q+1}R_1)$.

We next evaluate $\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)$, $a=0,1,\ldots,2q+1$. By Remark 3.1 (i), the indicator function of $\{x \,|\, \mathrm{Tr}_{q^2/q}(x)\in C_0^{(2,q)}\}$ is given by

$$f(x)=\frac{1}{q}\sum_{s\in\mathbb{F}_q}\sum_{y\in C_0^{(2,q)}}\psi_{\mathbb{F}_{q^2}}(sx)\psi_{\mathbb{F}_q}(-sy).$$

Then,

$$\begin{aligned}\psi_{\mathbb{F}_{q^2}}(\omega^a R_3) &= \sum_{x\in\mathbb{F}_{q^2}} \psi_{\mathbb{F}_{q^2}}(\omega^a x)f(x)f(x\omega^c)\\ &= \frac{1}{q^2}\sum_{x\in\mathbb{F}_{q^2}}\sum_{s,t\in\mathbb{F}_q}\sum_{y,z\in C_0^{(2,q)}} \psi_{\mathbb{F}_{q^2}}(x(\omega^a+s+t\omega^c))\psi_{\mathbb{F}_q}(-sy)\psi_{\mathbb{F}_q}(-tz)\\ &= \sum_{s,t\in\mathbb{F}_q:\omega^a=s+t\omega^c}\sum_{y,z\in C_0^{(2,q)}} \psi_{\mathbb{F}_q}(sy)\psi_{\mathbb{F}_q}(tz).\end{aligned}$$

We treat the case where $a\in Y_{1,c}\cup Y_{2,c}=\{0,c,q+1,c+q+1\}$. If $a=c$, then $s=0$ and $t\in C_0^{(2,q)}$, and hence $\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)=\frac{(q-1)(-1+G_q(\eta))}{4}$. If $a=c+q+1$, then $s=0$ and $t\in C_1^{(2,q)}$, and hence $\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)=\frac{(q-1)(-1-G_q(\eta))}{4}$. If $a=0$, then $t=0$ and $s\in C_0^{(2,q)}$, and hence $\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)=\frac{(q-1)(-1+G_q(\eta))}{4}$. If $a=q+1$, then $t=0$ and $s\in C_1^{(2,q)}$, and hence $\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)=\frac{(q-1)(-1-G_q(\eta))}{4}$. Next, we treat the case where $s,t\neq 0$. Define

$$\begin{aligned}G_3 =&\{a \pmod{2(q+1)} \,|\, \omega^a=s+t\omega^c, s,t\in C_0^{(2,q)}\},\\ G_4 =&\{a \pmod{2(q+1)} \,|\, \omega^a=s+t\omega^c, s,t\in C_1^{(2,q)}\},\\ G_5 =&\{a \pmod{2(q+1)} \,|\, \omega^a=s+t\omega^c, (s,t)\in C_0^{(2,q)}\times C_1^{(2,q)} \text{ or } C_1^{(2,q)}\times C_0^{(2,q)}\}.\end{aligned}$$

Then, we have

$$\psi_{\mathbb{F}_{q^2}}(\omega^a R_3)=\begin{cases}\frac{(1-G_q(\eta))^2}{4}, & \text{if } a\in G_3,\\ \frac{(1+G_q(\eta))^2}{4}, & \text{if } a\in G_4,\\ \frac{1-(-1)^{\frac{q-1}{2}}q}{4}, & \text{if } a\in G_5.\end{cases}$$

We need to show that $G_i=Y_{i,c}$, $i=3,4,5$. Let $a\in G_3$. Then, there are some $s,t\in C_0^{(2,q)}$ such that $\omega^a=s+t\omega^c$. Taking trace of both sides of $\omega^{a+\frac{q+1}{2}\epsilon}=s\omega^{\frac{q+1}{2}\epsilon}+t\omega^{c+\frac{q+1}{2}\epsilon}$, we have $\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\epsilon})=s\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon})+t\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon+c})$. Since $\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon})=0$ and $\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon+c})\in C_0^{(2,q)}$, we obtain $\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\epsilon})\in C_0^{(2,q)}$, i.e., $a\in I_2-\frac{q+1}{2}\epsilon$. On the other hand, taking trace of both sides of $\omega^{a+\frac{q+1}{2}\delta-c}=s\omega^{\frac{q+1}{2}\delta-c}+t\omega^{\frac{q+1}{2}\delta}$, we have $\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\delta-c})=s\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\delta-c})+t\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\delta})$. Since $\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\delta})=0$ and $\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\delta-c})\in C_0^{(2,q)}$, we obtain $\mathrm{Tr}_{q^2/q}(\omega^{a+\frac{q+1}{2}\delta-c})\in C_0^{(2,q)}$, i.e., $a\in I_2+c-\frac{q+1}{2}\delta$. Thus, $a\in(I_2-\frac{q+1}{2}\epsilon)\cap(I_2+c-\frac{q+1}{2}\delta)$, and hence $G_3\subseteq Y_{3,c}$. Noting that $|G_3|=|Y_{3,c}|$, it follows that $G_3=Y_{3,c}$. Furthermore, since $G_2\equiv G_3+(q+1) \pmod{2(q+1)}$ and $G_5=\{0,1,\ldots,2q+1\}\setminus(G_3\cup G_4\cup\{0,c,q+1,c+q+1\})$, we have $G_4=Y_{4,c}$ and $G_5=Y_{5,c}$.

Finally, the character values of R_4 and R_5 are determined as $\psi_{\mathbb{F}_{q^2}}(\omega^a R_4)=\psi_{\mathbb{F}_{q^2}}(\omega^{a+q+1}R_3)$ and $\psi_{\mathbb{F}_{q^2}}(\omega^a R_5)=-1-\sum_{i=1}^4\psi_{\mathbb{F}_{q^2}}(\omega^a R_i)$. This completes the proof of the proposition. $\square$

Remark A.4. By the definition of $Y_{i,c}$, $i = 1, 2$, in Proposition A.3, it is clear that $-Y_{i,c} + c \equiv Y_{i,c} \pmod{2(q+1)}$, that is, the property (P8).

Next, we show that the $Y_{i,c}$'s have property (P9).

Proposition A.5. *We have*

$$-(Y_{3,c} \cup Y_{4,c}) + c \equiv Y_{5,c} \pmod{2(q+1)}.$$

Proof. Since $Y_{i,c} = G_i$ for $i = 3, 4, 5$ as in the proof of Proposition A.3, we have

$$\begin{aligned} Y_{3,c} \cup Y_{4,c} &= \{a \pmod{2(q+1)} \,|\, \omega^a = s + t\omega^c, (s,t) \in S \times S \text{ or } N \times N\}, \\ Y_{5,c} &= \{a \pmod{2(q+1)} \,|\, \omega^a = s + t\omega^c, (s,t) \in S \times N \text{ or } S \times N\}. \end{aligned}$$

Assume that $a \in -(Y_{3,c} \cup Y_{4,c}) + c$. There are some $s', t' \in \mathbb{F}_q$ such that $\omega^a = s' + t'\omega^c$. On the other hand, since $a \in -(Y_{3,c} \cup Y_{4,c}) + c$, $\omega^{-a+c} = s + t\omega^c$ for some $s, t \in S \times S$ or $N \times N$. Then, we have

$$(s\omega^{-c} + t)(s' + t'\omega^c) = 1. \tag{A.3}$$

By multiplying both sides of (A.3) by $\omega^{\frac{q+1}{2}\epsilon}$ and taking trace, we have

$$\begin{aligned} &ss'\mathrm{Tr}(\omega^{-c+\frac{q+1}{2}\epsilon}) + (ts' + st')\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon}) + tt'\mathrm{Tr}_{q^2/q}(\omega^{c+\frac{q+1}{2}\epsilon}) \\ &= \mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon}). \end{aligned} \tag{A.4}$$

Since $\mathrm{Tr}_{q^2/q}(\omega^{\frac{q+1}{2}\epsilon}) = 0$ by the definition of $X_{1,c}$, (A.4) is reduced to

$$-ss'u\omega^{\frac{q+1}{2}(\epsilon-\delta)} = tt'v,$$

where $u = \mathrm{Tr}_{q^2/q}(\omega^{c+\frac{q+1}{2}\epsilon})$ and $v = \mathrm{Tr}_{q^2/q}(\omega^{-c+\frac{q+1}{2}\delta})$. Here, $u, v \in C_0^{(2,q)}$ by (A.1). Furthermore, $st^{-1} \in C_0^{(2,q)}$ by the definitions of s, t, and $-\omega^{\frac{q+1}{2}(\epsilon-\delta)} \in C_1^{(2,q)}$ by the definitions of ϵ, δ. Hence, either $(s', t') \in C_0^{(2,q)} \times C_1^{(2,q)}$ or $C_1^{(2,q)} \times C_0^{(2,q)}$ holds by noting that $(s', t') = (0, 0)$ is impossible. Therefore, $a \in Y_{5,c}$, i.e., $-(Y_{3,c} \cup Y_{4,c}) + c \subseteq Y_{5,c}$, follows. Finally, since $|-(Y_{3,c} \cup Y_{4,c}) + c| = |Y_{3,c} \cup Y_{4,c}| = |Y_{5,c}|$, the statement of the proposition follows. $\square$

Finally, we show that the R_i''s have property (P10).

Proposition A.6. *Let R_i', $i = 1, 2, 3, 4, 5$, be defined as in Subsection 4.1. Then, R_i', $i = 1, 2, 3, 4, 5$, take the character values listed in Table 2.*

Proof. Since $Y_{i,c} - c + \frac{q+1}{2}\delta \equiv X_{i,c-\frac{q+1}{2}\delta+\frac{q+1}{2}\epsilon}$ by Lemma A.1, Remark A.2 (5) and the definitions of $X_{i,c}, Y_{i,c}$, $i = 1, 2, 3, 4, 5$, the statement follows from Proposition A.3. $\square$

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