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Methods

Finding Minimum Volume Circumscribing Ellipsoids Using Generalized Copositive Programming

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Abstract. We study the problem of finding the Löwner–John ellipsoid (i.e., an ellipsoid with minimum volume that contains a given convex set). We reformulate the problem as a generalized copositive program and use that reformulation to derive tractable semidefinite programming approximations for instances where the set is defined by affine and quadratic inequalities. We prove that, when the underlying set is a polytope, our method never provides an ellipsoid of higher volume than the one obtained by scaling the maximum volume-inscribed ellipsoid. We empirically demonstrate that our proposed method generates high-quality solutions and can be solved much faster than solving the problem to optimality. Furthermore, we outperform the existing approximation schemes in terms of solution time and quality. We present applications of our method to obtain piecewise linear decision rule approximations for dynamic distributionally robust problems with random recourse and to generate ellipsoidal approximations for the set of reachable states in a linear dynamical system when the set of allowed controls is a polytope.

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1. Introduction

We consider the *minimum volume ellipsoid problem* (MVEP), which can be stated as follows (Boyd and Vandenberghe 2004, Todd 2016). "Given a set $\mathcal{P} \subset \mathbb{R}^K$, find an ellipsoid $\mathcal{E}_{\text{mve}}$ with minimum volume that contains $\mathcal{P}$." In this paper, we focus on sets $\mathcal{P}$ that satisfy the following assumption.

Assumption 1. *The set $\mathcal{P}$ is compact, convex, and full-dimensional.*

Compactness guarantees the existence of a bounding ellipsoid. The convexity assumption is made without loss of generality; if the set is not convex, then we can instead consider its convex hull without affecting $\mathcal{E}_{\text{mve}}$. If $\mathcal{P}$ is not full-dimensional, then the ellipsoid $\mathcal{E}_{\text{mve}}$ is degenerate with zero volume. For sets $\mathcal{P}$ satisfying Assumption 1, such an ellipsoid, also known as the *Löwner–John ellipsoid*, is unique and affine invariant, making it an attractive outer approximation of $\mathcal{P}$ (Boyd and Vandenberghe 2004, section 8.4.1). The MVEP arises in many applications studied in the literature. Several authors discuss outer ellipsoidal approximations for the set of reachable points in control systems (Kurzhanskiĭ and Vályi 1997, Calafiore and El Ghaoui 2004), as it is easier to check whether a point lies in an ellipsoid than in the comparatively complicated reachable set. Rimon and Boyd (1997) advocate the use of $\mathcal{E}_{\text{mve}}$ for collision detection in robotics. Here, one checks whether the ellipsoids intersect as opposed to the sets that they approximate. Other applications of the MVEP include outlier detection (Silverman and Titterington 1980, Ahipaşaoğlu 2015), pattern recognition (Glineur 1998), computer graphics (Eberly 2001), and facility location (Elzinga and Hearn 1974). We refer the reader to Henk (2012) for an excellent article about the lives of the eponymous researchers Karel Löwner and Fritz John; the history of the MVEP, which dates back to late 1930s; and some important properties of Löwner–John ellipsoids.

For some sets $\mathcal{P}$, it is possible to identify $\mathcal{E}_{\text{mve}}$ in polynomial time. For example, if $\mathcal{P}$ is defined as the convex hull of a finite number of points, then the complexity of finding $\mathcal{E}_{\text{mve}}$ is polynomial in the problem size (Khachiyan 1996, Sun and Freund 2004). When $\mathcal{P}$ is a union of ellipsoids, one can employ the $\mathcal{S}$ lemma to compute $\mathcal{E}_{\text{mve}}$ in polynomial time (Yildirim 2006). However, excluding these special cases, finding $\mathcal{E}_{\text{mve}}$ is, in general, a difficult problem. For example, if $\mathcal{P}$ is a polytope defined by affine inequalities or if $\mathcal{P}$ is an intersection of ellipsoids, then finding $\mathcal{E}_{\text{mve}}$ is NP hard (Freund and Orlin 1985, Boyd and Vandenberghe 2004).

Gotoh and Konno (2006) present a constraint generation approach to solve the MVEP exactly when $\mathcal{P}$ is a polytope defined by affine inequalities. The method starts with a collection of points contained in $\mathcal{P}$ and finds the ellipsoid of minimum volume containing those points. Then, feasible points lying outside the current ellipsoid are successively generated, and the ellipsoid is updated to include the new point, until a desired optimality tolerance is reached. However, generating a point that lies in $\mathcal{P}$ but outside the current candidate ellipsoid at each iteration is very slow, as it entails solving a nonconvex optimization problem. Therefore, this approach is computationally expensive, and one has to resort to approximation schemes.

One popular approximation method for the MVEP is based on identifying and scaling the *maximum volume-inscribed ellipsoid* (MVIE; i.e., the ellipsoid with maximum volume contained in $\mathcal{P}$). In particular, it is known that scaling the MVIE around its center by a factor of K results in an ellipsoid that contains $\mathcal{P}$, thereby serving as an approximation of $\mathcal{E}_{\text{mve}}$ (John 2014). Moreover, the MVIE can be found in polynomial time if $\mathcal{P}$ is defined by affine and quadratic inequalities (Khachiyan and Todd 1993). However, this technique, which we refer to as the scaled maximum volume-inscribed ellipsoid (SMVIE) approach, produces highly suboptimal ellipsoids because of the scaling factor K. Another method for approximating the MVEP utilizes the well-known $\mathcal{S}$ procedure. Boyd et al. (1994) discuss the application of the $\mathcal{S}$ procedure to generate approximations for the MVEP when $\mathcal{P}$ is either an intersection or a Minkowski sum of ellipsoids. Finally, in a recent paper, Zhen et al. (2021) study approximations to uncertain second-order cone programs and demonstrate how this framework can be exploited to derive an approximation to the MVEP.

Several authors have identified sufficient conditions under which a convex set contains another convex set. Helton et al. (2013) discuss sufficient conditions, which guarantee that a *semidefinite-representable* set contains another such set. Kellner et al. (2013) provide slightly improved sufficient conditions compared with the ones in Helton et al. (2013). Although these articles do not focus on the MVEP specifically, their results can be used to approximate $\mathcal{E}_{\text{mve}}$ if $\mathcal{P}$ is semidefinite representable (see Appendix B). To the best of our knowledge, there are no results that provide a finite system of constraints that are necessary and sufficient to ensure that an ellipsoid contains another set. This gap in knowledge is our main focus.

In this article, we prove that checking whether an ellipsoid contains $\mathcal{P}$ is *equivalent* to solving a finite-dimensional *generalized copositive* (GC) feasibility problem. We use this result to reformulate the MVEP exactly as a GC program. This representation of the MVEP enables us to leverage state-of-the-art approximation schemes available for GC programming problems. In particular, GC programs yield a hierarchy of optimization problems, which provide an increasingly tight restriction to the original problem (Parrilo 2000, Lasserre 2001, Zuluaga et al. 2006). Although our exact reformulation holds for any $\mathcal{P}$ satisfying Assumption 1, we focus primarily on developing approximations in the case where $\mathcal{P}$ is defined by affine and convex quadratic inequalities. We demonstrate that, for these sets, the resulting approximation can be formulated as a semidefinite program (SDP), which can be solved in polynomial time. Because these SDPs are restrictions of the GC reformulation, they provide a feasible ellipsoid that contains $\mathcal{P}$. There has been previous work on developing exact copositive programming reformulations for otherwise difficult problems and using those reformulations to generate tractable approximations (Natarajan et al. 2011, Bomze 2012, Burer 2012, Burer and Dong 2012, Hanasusanto and Kuhn 2018, Prasad and Hanasusanto 2018, Mittal et al. 2020). Our results add to this literature by demonstrating the ability of generalized copositive programs to exactly model the MVEP.

We demonstrate the utility of our approximations to the MVEP in two applications. First, we consider a two-stage distributionally robust optimization (DRO) problem with *random recourse*. Such a problem is NP hard even in the absence of random recourse (Bertsimas et al. 2010). Bertsimas and Dunning (2016) study a *piecewise static* (PWS) decision rules approximation for the case of dynamic robust optimization, which leads to a tractable reformulation. Although they do not consider a DRO model, this approach can be extended to such a setting. In contrast, we focus on piecewise *linear* decision (PLD) rules approximation. In the presence of random recourse, finding the optimal PLD rule is NP hard, although feasible PLD rules can be obtained using the $\mathcal{S}$ procedure. Unfortunately, these decision rules are often of poor quality. The effectiveness of the $\mathcal{S}$ procedure in finding good PLD rules can be improved by considering an ellipsoid that contains the *support set* (i.e., the set of allowed values for the uncertain parameters). In the context of an inventory management problem, we show that the size of this ellipsoid can have a large effect on the quality of the PLD rules. We also demonstrate that the PLD rules generated using our method significantly outperform the piecewise static decision rules. Second, we utilize our method to generate high-quality ellipsoidal approximations to the set of reachable states in a linear dynamical system when the *control set* (i.e., the set of allowed controls) is a polytope. This complements the existing schemes that provide similar approximations when the control set is an ellipsoid (Kurzhanskiĭ and Vályi 1997, Calafiore and El Ghaoui 2004).

We summarize the main contributions of the article.

1. We provide necessary and sufficient finite-dimensional conic inequalities that certify whether an ellipsoid contains a set $\mathcal{P}$ satisfying Assumption 1. We use these conditions to derive a generalized copositive reformulation of the MVEP.

2. When $\mathcal{P}$ is defined by affine and convex quadratic inequalities, we derive a tractable SDP restriction to the GC reformulation, which results in a feasible ellipsoid that contains $\mathcal{P}$. We prove that the volume of the resulting ellipsoid never exceeds that of the SMVIE approach. To the best of our knowledge, our approximation is the first one to have this property. We further show that the ratio of the volume of the SMVIE to the volume of the ellipsoid generated by our method can be arbitrarily high. We also prove that both the $\mathcal{S}$ procedure (Boyd et al. 1994, section 3.7) and the approximation suggested by Zhen et al. (2021) never generate ellipsoids of lower volumes than the SMVIE approach.

3. We demonstrate through extensive numerical experiments that our method is significantly faster than solving the problem to optimality using the constraint generation technique of Gotoh and Konno (2006). The experiments further indicate that our method significantly outperforms the SMVIE approach in terms of solution quality. Also, our method outperforms the scheme that utilizes the sufficient conditions of Kellner et al. (2013) in terms of both solution time and quality.

4. We present two important applications of our approach. First, we exploit the bounding ellipsoids to obtain improved decision rule approximations to two-stage DRO models with random recourse, which have resisted effective solution schemes so far. Second, we provide ellipsoidal approximations for the set of reachable states in a linear dynamical system when the control set is a polytope.

This article is organized as follows. In Section 2, we describe the MVEP and reformulate it as an equivalent GC program. In Section 3, we use that reformulation to derive a tractable SDP that generates a near-optimal approximation to $\mathcal{E}_{\text{mve}}$ when the set is defined by affine and quadratic inequalities. In Section 4, we explain the application of our approach for obtaining improved decision rules approximation for a two-stage DRO model with random recourse. In Section 5, we present numerical experiments comparing the volumes of the ellipsoids generated by our method against those found using other approaches. We also demonstrate the efficacy of our approach in solving a distributionally robust inventory management problem. Finally, we conclude in Section 6. Auxiliary proofs and additional numerical experiments can be found in the e-companion to the paper.

1.1. Preliminaries

1.1.1. Notation. For a positive integer I, we use $[I]$ to denote the index set $\{1, 2, \ldots, I\}$. We denote the vector of ones by $\mathbf{e}$ and the identity matrix by $\mathbb{I}$; their dimensions will be clear from the context. We use $\mathbb{R}^K(\mathbb{R}^K_+)$ to denote the set of (nonnegative) vectors of length K and $\mathbb{S}^K(\mathbb{S}^K_+)$ to denote the set of all $K \times K$ symmetric (positive semidefinite) matrices. In addition, $\mathbb{S}^K_{++}$ represents the set of positive definite matrices. The functions $\text{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and the determinant of the input matrix, respectively. We define $\text{Diag}(\boldsymbol{v})$ as a diagonal matrix with vector $\boldsymbol{v}$ on its main diagonal. The symbols $\|\boldsymbol{v}\|_1$ and $\|\boldsymbol{v}\|$ denote the ℓ_1 norm and ℓ_2 norm of vector $\boldsymbol{v}$, respectively. The vertical concatenation of two scalars or vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $[\boldsymbol{u}; \boldsymbol{v}]$. For a matrix $\boldsymbol{M} \in \mathbb{R}^{I \times J}$, we use $\boldsymbol{M}_{:j} \in \mathbb{R}^I$ to denote its jth column and $\boldsymbol{M}_{i:} \in \mathbb{R}^J$ to denote the transpose of its ith row. We represent the interior and the conic hull of a set S by $\text{int}(S)$ and $\text{cone}(S)$, respectively.

1.1.2. Generalized Copositive Matrices. We use $\mathcal{C}(\mathcal{K})$ to denote the set of generalized copositive matrices with respect to cone $\mathcal{K} \subseteq \mathbb{R}^K$ (i.e., $\mathcal{C}(\mathcal{K}) = \{\boldsymbol{M} \in \mathbb{S}^K : \boldsymbol{x}^\top \boldsymbol{M} \boldsymbol{x} \geq 0 \ \forall \boldsymbol{x} \in \mathcal{K}\}$). The set of copositive matrices is a special case of such a set when $\mathcal{K} = \mathbb{R}^K_+$. We use $\mathcal{C}^*(\mathcal{K})$ to denote the set of generalized completely positive matrices with respect to cone $\mathcal{K}$ (i.e., $\mathcal{C}^*(\mathcal{K}) = \{\boldsymbol{M} \in \mathbb{S}^K : \boldsymbol{M} = \sum_{i \in [I]} \boldsymbol{x}_i \boldsymbol{x}_i^\top, \boldsymbol{x}_i \in \mathcal{K}\}$), where I is a positive integer. The cones $\mathcal{C}(\mathcal{K})$ and $\mathcal{C}^*(\mathcal{K})$ are duals of each other (Sturm and Zhang 2003). For any $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{S}^K$ and cone $\bar{\mathcal{C}} \subseteq \mathbb{S}^K$, the conic inequality $\boldsymbol{P} \succcurlyeq_{\bar{\mathcal{C}}} \boldsymbol{Q}$ indicates that $\boldsymbol{P} - \boldsymbol{Q}$ is an element of $\bar{\mathcal{C}}$. We drop the subscript and simply write $\boldsymbol{P} \succcurlyeq \boldsymbol{Q}$, when $\bar{\mathcal{C}} = \mathbb{S}^K_+$. Finally, the relation $\boldsymbol{M} \succ_{\mathcal{C}(\mathcal{K})} \mathbf{0}$ indicates that $\boldsymbol{M}$ is strictly copositive (i.e., $\boldsymbol{x}^\top \boldsymbol{M} \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathcal{K}, \boldsymbol{x} \neq \mathbf{0}$).

1.1.3. Ellipsoids. We define $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^K : \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^2 \leq 1\}$ as an ellipsoid with parameters $\boldsymbol{A} \in \mathbb{S}^K_{++}$ and $\boldsymbol{b} \in \mathbb{R}^K$. The volume of $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$, denoted by $\text{Vol}(\mathcal{E}(\boldsymbol{A}, \boldsymbol{b}))$, is proportional to $\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$. In this paper, we drop the proportionality constant and say that $\text{Vol}(\mathcal{E}(\boldsymbol{A}, \boldsymbol{b})) = 1/\det(\boldsymbol{A})$; because we use the volume as a metric for comparing different ellipsoids, doing so does not affect the results. We define the *radius* of a K-dimensional ellipsoid as $\text{Vol}(\cdot)^{1/K}$; this metric is proportional to the radius of a sphere with the same volume as that of the ellipsoid. Finally, we say the two ellipsoids are equal (i.e., $\mathcal{E}(\boldsymbol{A}_1, \boldsymbol{b}_1) = \mathcal{E}(\boldsymbol{A}_2, \boldsymbol{b}_2)$) if and only if $\boldsymbol{A}_1 = \boldsymbol{A}_2$ and $\boldsymbol{b}_1 = \boldsymbol{b}_2$.

2. Generalized Copositive Reformulation

In this section, we develop a generalized copositive reformulation for the MVEP. It is well known that $\mathcal{E}_{\text{mve}} = \mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$ if and only if $(\boldsymbol{A}, \boldsymbol{b})$ is the unique optimal solution to the following semiinfinite convex optimization problem (Todd 2016):

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A}) \\ \text{subject to} & \boldsymbol{A} \in \mathbb{S}^K, \boldsymbol{b} \in \mathbb{R}^K, Z(\boldsymbol{A}, \boldsymbol{b}) \leq 1, \end{array} \qquad (\mathcal{MVE})$$

where

$$Z(\boldsymbol{A},\boldsymbol{b}) = \sup_{\boldsymbol{x}\in\mathcal{P}} \|\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}\|^2 = \sup_{\boldsymbol{x}\in\mathcal{P}} \{\boldsymbol{x}^\top \boldsymbol{A}^2\boldsymbol{x} + 2\boldsymbol{b}^\top\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top\boldsymbol{b}\}. \tag{1}$$

The objective function of ($\mathcal{MVE}$) minimizes the logarithm of the volume, which implicitly restricts $\boldsymbol{A}$ to be positive definite. Minimizing the logarithm of the volume makes the objective function convex in $\boldsymbol{A}$. The constraint $Z(\boldsymbol{A},\boldsymbol{b}) \le 1$ forces every element of $\mathcal{P}$ to lie inside the ellipsoid. We are now ready to present the main result of this section, where we derive necessary and sufficient conditions for certifying whether an ellipsoid contains another set.

Theorem 1. *Let $\mathcal{P}$ be a set satisfying Assumption 1. Let the cone $\mathcal{K} \subseteq \mathbb{R}^{K+1}$ be defined as*

$$\mathcal{K} = \text{cone}(\{[\boldsymbol{x};1] : \boldsymbol{x}\in\mathcal{P}\}). \tag{2}$$

If $\boldsymbol{A}\in\mathbb{S}^K_{++}$ and $\boldsymbol{b}\in\mathbb{R}^K$, then the ellipsoid $\mathcal{E}(\boldsymbol{A},\boldsymbol{b})$ contains $\mathcal{P}$ if and only if there exist $\boldsymbol{F}\in\mathbb{S}^K$, $\boldsymbol{g}\in\mathbb{R}^K$, $h\in\mathbb{R}$, such that

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preccurlyeq_{\mathcal{C}(\mathcal{K})} \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & \boldsymbol{A} \\ \boldsymbol{g}^\top & h & \boldsymbol{b}^\top \\ \boldsymbol{A} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succcurlyeq \mathbf{0}. \tag{3}$$

Before proving Theorem 1, we discuss its implications. The theorem implies that the constraint $Z(\boldsymbol{A},\boldsymbol{b}) \le 1$ in ($\mathcal{MVE}$) can be replaced by the constraints in (3). Therefore, $\mathcal{E}_{\text{mve}} = \mathcal{E}(\boldsymbol{A},\boldsymbol{b})$ is the minimum volume ellipsoid if and only if $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{F},\boldsymbol{g},h)$ is the unique optimal solution to the following generalized copositive program:

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A}) \\ \text{subject to} & \boldsymbol{A}\in\mathbb{S}^K, \boldsymbol{b}\in\mathbb{R}^K, \boldsymbol{F}\in\mathbb{S}^K, \boldsymbol{g}\in\mathbb{R}^K, h\in\mathbb{R}, \\ & \text{(3) holds.} \end{array} \tag{4}$$

Remark 1. In this article, we refer to a problem with $-\log\det(\cdot)$ minimization objective and semidefinite (copositive) constraints as a "semidefinite (copositive) program," albeit with a slight abuse of terminology. The reason is that minimization of $-\log\det(\cdot)$ is equivalent to minimization of $-(\det(\cdot))^{1/K}$; the latter can be reformulated as a problem with linear objective and additional semidefinite inequality constraints (see, e.g., Ben-Tal and Nemirovski 2001, section 4.2). Some modeling frameworks, like YALMIP (Löfberg 2004), that we use for our experiments automatically carry out this conversion before sending the problem to the solver.

Next, we present the following technical lemmas, which are needed for the proof of Theorem 1.

Lemma 1. *Let $\mathcal{K}$ be the cone defined in (3). If $[\boldsymbol{x};\tau]\in\mathcal{K}$, then $\tau \ge 0$. Furthermore, $\tau = 0$ only if $\boldsymbol{x} = \mathbf{0}$.*

Proof. From the definition of $\mathcal{K}$, there exist points $\boldsymbol{x}_s \in \mathcal{P}$ and coefficients $\lambda_s \ge 0$, $s\in[S]$, such that $[\boldsymbol{x};\tau] = \sum_{s\in[S]}\lambda_s[\boldsymbol{x}_s;1]$. By comparing the last element, we get $\tau = \sum_{s\in[S]}\lambda_s \ge 0$ because $\lambda_s \ge 0$. In addition, $\tau = 0$ implies that $\lambda_s = 0$ for all $s\in[S]$, which further implies that $\boldsymbol{x} = \mathbf{0}$. □

Lemma 2. *The cone $\mathcal{K}$ defined in (2) is proper.*

Proof. The compactness of $\mathcal{P}$ implies that $\mathcal{K}$ is convex and closed. Because $\mathcal{P}$ has nonempty interior, any point $\boldsymbol{x}$ in the interior of $\mathcal{P}$ yields a point $[\boldsymbol{x};1]$ in the interior of $\mathcal{K}$; therefore, $\mathcal{K}$ has nonempty interior. Finally, to see that $\mathcal{K}$ is pointed, let $[\boldsymbol{x};\tau]\in\mathcal{K}$ and $-[\boldsymbol{x};\tau]\in\mathcal{K}$. Using Lemma 1, we have that $\tau \ge 0$ and $-\tau \ge 0$, which implies that $\tau = 0$. Again using Lemma 1, we get that $[\boldsymbol{x};\tau] = \mathbf{0}$, which implies that $\mathcal{K}$ is pointed. □

Lemma 3. *Let $\boldsymbol{M}\in\mathbb{S}^K$ be a symmetric matrix and $S\subseteq\mathbb{R}^K$ be a set with nonempty interior. If $\boldsymbol{v}^\top\boldsymbol{M}\boldsymbol{v} = 0$ for all $\boldsymbol{v}\in S$, then $\boldsymbol{M} = \mathbf{0}$.*

Proof. Let λ be an eigenvalue of $\boldsymbol{M}$ and $\boldsymbol{q}$ be the corresponding eigenvector of unit length. Because S has nonempty interior, for any $\boldsymbol{v}\in\text{int}(S)$, there exists $\bar{\tau} > 0$ such that $\boldsymbol{v}+\tau\boldsymbol{q}\in S$ for all $\tau\in[0,\bar{\tau}]$. Therefore, $(\boldsymbol{v}+\tau\boldsymbol{q})^\top\boldsymbol{M}(\boldsymbol{v}+\tau\boldsymbol{q}) = 0$ for all $\tau\in[0,\bar{\tau}]$. Furthermore,

$$\begin{aligned} (\boldsymbol{v}+\tau\boldsymbol{q})^\top\boldsymbol{M}(\boldsymbol{v}+\tau\boldsymbol{q}) &= \boldsymbol{v}^\top\boldsymbol{M}\boldsymbol{v} + 2\tau\boldsymbol{q}^\top\boldsymbol{M}\boldsymbol{v} + \tau^2\boldsymbol{q}^\top\boldsymbol{M}\boldsymbol{q} \\ &= 2\tau\lambda\boldsymbol{q}^\top\boldsymbol{v} + \tau^2\lambda\boldsymbol{q}^\top\boldsymbol{q} = \lambda\tau(2\boldsymbol{q}^\top\boldsymbol{v}+\tau). \end{aligned}$$

Thus, $\lambda\tau(2\boldsymbol{q}^\top\boldsymbol{v}+\tau) = 0$ for all $\tau\in[0,\bar{\tau}]$. Because the term $\tau(2\boldsymbol{q}^\top\boldsymbol{v}+\tau)$ is quadratic in the scalar τ, it cannot be zero for more than two values of τ. This implies that the previous equality holds for all $\tau\in[0,\bar{\tau}]$ only if $\lambda = 0$. Therefore, any eigenvalue of $\boldsymbol{M}$ is zero, which implies that $\boldsymbol{M} = \mathbf{0}$. □

Lemma 4. *Let $\mathcal{K}$ be the cone defined in (2). There exist $\boldsymbol{X}\in\mathbb{S}^K$ and $\boldsymbol{x}\in\mathbb{R}^K$ such that*

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{bmatrix} \succ_{\mathcal{C}^*(\mathcal{K})} \mathbf{0}.$$

Proof. We start by showing that $\mathcal{C}(\mathcal{K})$ is pointed. Let $\boldsymbol{M}\in\mathbb{S}^{K+1}$ be such that $\boldsymbol{M}\in\mathcal{C}(\mathcal{K})$ and $-\boldsymbol{M}\in\mathcal{C}(\mathcal{K})$. For this choice of $\boldsymbol{M}$, for all $\boldsymbol{x}\in\mathcal{K}$, we have that $\boldsymbol{x}^\top\boldsymbol{M}\boldsymbol{x} \ge 0$ and $-\boldsymbol{x}^\top\boldsymbol{M}\boldsymbol{x} \ge 0$, which imply that $\boldsymbol{x}^\top\boldsymbol{M}\boldsymbol{x} = 0$ for all $\boldsymbol{x}\in\mathcal{K}$. Because $\mathcal{K}$ has nonempty interior (by Lemma 2), Lemma 3 implies that $\boldsymbol{M} = \mathbf{0}$. Therefore, $\mathcal{C}(\mathcal{K})$ is pointed, which implies that its dual cone, $\mathcal{C}^*(\mathcal{K})$, has nonempty interior (Boyd and Vandenberghe 2004, section 2.6.1). Consider $\boldsymbol{M}\in\text{int}(\mathcal{C}^*(\mathcal{K}))$. The matrix $\boldsymbol{M}$ is positive definite (see the discussion below corollary 8.1 in Burer 2012); therefore, any element on its diagonal, which includes the bottom right component, is strictly positive. By scaling $\boldsymbol{M}$ such that the bottom right component is one, we get another matrix in the interior of $\mathcal{C}^*(\mathcal{K})$. Hence, the lemma holds. □

The following lemma is an extension of another recently proved result found in Mittal et al. (2020, lemma 4).

Lemma 5. *Let $\boldsymbol{M} \in \mathbb{S}^K$ be a symmetric matrix and $\boldsymbol{A} \in \mathbb{R}^{J\times K}$ be an arbitrary matrix. Then, for any proper cone $\mathcal{K} \subset \mathbb{R}^K$, the inequality $\boldsymbol{M} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{A}^\top \boldsymbol{A}$ is satisfied if and only if there exists a matrix $\boldsymbol{H} \in \mathbb{S}^K_+$ such that*

$$\boldsymbol{M} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{H} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{H} & \boldsymbol{A}^\top \\ \boldsymbol{A} & \mathbb{I} \end{bmatrix} \succcurlyeq \boldsymbol{0}. \tag{5}$$

Proof (⇒). The statement holds immediately by setting $\boldsymbol{H} = \boldsymbol{A}^\top \boldsymbol{A}$.

(⇐) Assume that there exists such a matrix $\boldsymbol{H} \in \mathbb{S}^K_+$. By the Schur complement, the second inequality in (5) implies that $\boldsymbol{H} \succcurlyeq \boldsymbol{A}^\top \boldsymbol{A}$, which in turn, implies that $\boldsymbol{H} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{A}^\top \boldsymbol{A}$ (because $\mathbb{S}^K_+ \subseteq \mathcal{C}(\mathcal{K})$ for any $\mathcal{K}$). Combining this with the first inequality in (5) implies that $\boldsymbol{M} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{A}^\top \boldsymbol{A}$. □

We now return to the proof of Theorem 1.

Proof of Theorem 1. The set $\mathcal{P}$ can be expressed in terms of the cone $\mathcal{K}$ as $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^K : [\boldsymbol{x}; 1] \in \mathcal{K}\}$. Therefore, we can write (1) as

$$Z(\boldsymbol{A}, \boldsymbol{b}) = \sup_{[\boldsymbol{x};1]\in\mathcal{K}} \boldsymbol{x}^\top \boldsymbol{A}^2 \boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{b}. \tag{6}$$

The Optimization Problem (6) is equivalent to the following completely positive program (Burer 2012):

$$\begin{array}{lll} Z(\boldsymbol{A}, \boldsymbol{b}) = & \sup & \operatorname{tr}(\boldsymbol{A}^2 \boldsymbol{X}) + 2\boldsymbol{b}^\top \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{b} \\ & \text{s.t.} & \boldsymbol{x} \in \mathbb{R}^K, \boldsymbol{X} \in \mathbb{S}^K, \\ & & \begin{bmatrix} \boldsymbol{X} & \boldsymbol{x} \\ \boldsymbol{x}^\top & 1 \end{bmatrix} \succcurlyeq_{\mathcal{C}^*(\mathcal{K})} \boldsymbol{0}. \end{array} \tag{7}$$

The dual of this completely positive program can be written as

$$\begin{array}{lll} Z_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{b}) = & \inf\limits_{\rho\in\mathbb{R}} & \rho \\ & \text{s.t.} & \begin{bmatrix} -\boldsymbol{A}^2 & -\boldsymbol{A}\boldsymbol{b} \\ -\boldsymbol{b}^\top \boldsymbol{A} & \rho - \boldsymbol{b}^\top \boldsymbol{b} \end{bmatrix} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0}. \end{array} \tag{8}$$

Using Lemma 4, we conclude that a Slater point exists in the Optimization Problem (7). Hence, strong duality holds, and $Z(\boldsymbol{A}, \boldsymbol{b}) = Z_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{b})$. Furthermore, there exists a dual feasible solution, which attains the value $Z(\boldsymbol{A}, \boldsymbol{b})$, because a Slater point exists in the Primal Problem (7) (Ben-Tal and Nemirovski 2001, theorem 1.4.2). Using these facts, we have that $Z(\boldsymbol{A}, \boldsymbol{b}) \le 1$ if and only if there exists a feasible solution to Problem (8) whose objective function value is at most one. Therefore, $Z(\boldsymbol{A}, \boldsymbol{b}) \le 1$ if and only if there exists $\rho \le 1$ such that

$$\begin{bmatrix} -\boldsymbol{A}^2 & -\boldsymbol{A}\boldsymbol{b} \\ -\boldsymbol{b}^\top \boldsymbol{A} & \rho - \boldsymbol{b}^\top \boldsymbol{b} \end{bmatrix} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0},$$

which in turn, holds if and only if

$$\begin{bmatrix} -\boldsymbol{A}^2 & -\boldsymbol{A}\boldsymbol{b} \\ -\boldsymbol{b}^\top \boldsymbol{A} & 1 - \boldsymbol{b}^\top \boldsymbol{b} \end{bmatrix} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0},$$

or equivalently,

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix} \succcurlyeq_{\mathcal{C}(\mathcal{K})} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}. \tag{9}$$

The Conic Inequality (9) has nonlinearity because of the terms involving the product of the decision variables $\boldsymbol{A}$ and $\boldsymbol{b}$. However, by Lemma 5, this constraint is satisfied if and only if there exist variables $\boldsymbol{F} \in \mathbb{S}^K$, $\boldsymbol{g} \in \mathbb{R}^K$, and $h \in \mathbb{R}$ such that the Constraints (3) hold. Therefore, the constraint $Z(\boldsymbol{A}, \boldsymbol{b}) \le 1$ is equivalent to Constraints (3). Hence, the claim follows. □

Theorem 1 implies that $\mathcal{E}_{\text{mve}}$ can be found by solving the GC Program (4), which is difficult in general. In the next section, we discuss tractable approximations of (4) for special cases of $\mathcal{P}$. However, before doing so, we provide some generalizations to the GC reformulation (4).

Remark 2 (Affine Mapping of a Set). Let $\mathcal{P} \subseteq \mathbb{R}^K$ be a set satisfying Assumption 1. Let $\bar{\mathcal{P}} = \boldsymbol{C}\mathcal{P} + \boldsymbol{d} = \{\boldsymbol{C}\boldsymbol{x} + \boldsymbol{d} : \boldsymbol{x} \in \mathcal{P}\} \subset \mathbb{R}^J$ be an affine mapping of $\mathcal{P}$, where $\boldsymbol{C} \in \mathbb{R}^{J\times K}$ and $\boldsymbol{d} \in \mathbb{R}^J$. In order to obtain conditions for an ellipsoid to contain $\bar{\mathcal{P}}$, note that $Z(\boldsymbol{A}, \boldsymbol{b}) = \sup_{\boldsymbol{x}\in\mathcal{P}} \|\boldsymbol{A}(\boldsymbol{C}\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{b}\|^2$. Following the steps of the proof of Theorem 1, we can see that if $\boldsymbol{A} \in \mathbb{S}^J_{++}$ and $\boldsymbol{b} \in \mathbb{R}^J$, then the ellipsoid $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$ contains $\bar{\mathcal{P}}$ if and only if there exist $\boldsymbol{F} \in \mathbb{S}^K$, $\boldsymbol{g} \in \mathbb{R}^K$, $h \in \mathbb{R}$, such that

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h - 1 \end{bmatrix} \preccurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & (\boldsymbol{A}\boldsymbol{C})^\top \\ \boldsymbol{g}^\top & h & (\boldsymbol{A}\boldsymbol{d} + \boldsymbol{b})^\top \\ \boldsymbol{A}\boldsymbol{C} & \boldsymbol{A}\boldsymbol{d} + \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succcurlyeq \boldsymbol{0},$$

where $\mathcal{K} = \operatorname{cone}(\{[\boldsymbol{x}; 1] : \boldsymbol{x} \in \mathcal{P}\})$.

Remark 3 (Union of Sets). Let $\mathcal{P} = \cup_{\ell\in[L]} \mathcal{P}_\ell$, where the set $\mathcal{P}_\ell$ satisfies Assumption 1 for all $\ell \in [L]$. The set $\mathcal{P}$ does not satisfy Assumption 1 because it may not be convex. However, it is possible to extend Theorem 1 to this case as follows. Note that an ellipsoid contains the union of sets if and only if it contains every set. We can apply Theorem 1 to every set $\mathcal{P}_\ell$ to arrive at the fact that ellipsoid $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$ contains $\mathcal{P}$ if and only if there exist $\boldsymbol{F}_\ell \in \mathbb{S}^K$, $\boldsymbol{g}_\ell \in \mathbb{R}^K$, $h_\ell \in \mathbb{R}$ $\forall \ell \in [L]$, such that

$$\begin{bmatrix} \boldsymbol{F}_\ell & \boldsymbol{g}_\ell \\ \boldsymbol{g}_\ell^\top & h_\ell - 1 \end{bmatrix} \preccurlyeq_{\mathcal{C}(\mathcal{K}_\ell)} \boldsymbol{0} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F}_\ell & \boldsymbol{g}_\ell & \boldsymbol{A} \\ \boldsymbol{g}_\ell^\top & h_\ell & \boldsymbol{b}^\top \\ \boldsymbol{A} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succcurlyeq \boldsymbol{0} \quad \forall \ell \in [L],$$

where $\mathcal{K}_\ell = \operatorname{cone}(\{[\boldsymbol{x}; 1] : \boldsymbol{x} \in \mathcal{P}_\ell\}), \ell \in [L]$.

Remark 4 (Minkowski Sum of Sets). For all $\ell \in [L]$, let the set $\mathcal{P}_\ell$ satisfy Assumption 1 and $\mathcal{K}_\ell$ be the corresponding cone defined as in (3). Let $\mathcal{P} = \{\sum_{\ell\in[L]} \boldsymbol{x}_\ell : \boldsymbol{x}_\ell \in \mathcal{P}_\ell \ \forall \ell \in [L]\}$ be the Minkowski sum of these sets. Although $\mathcal{P}$ satisfies Assumption 1, it might not have a polynomial-sized representation. As an example, if every $\mathcal{P}_\ell$ is a polytope, then $\mathcal{P}$ is a polytope defined by constraints whose number can potentially grow exponentially with L. However, we can still reformulate ($\mathcal{MVE}$) for $\mathcal{P}$ as a GC program of polynomial size as follows. Observe that

$$\begin{aligned} Z(\boldsymbol{A},\boldsymbol{b}) &= \sup_{\boldsymbol{x}_\ell \in \mathcal{P}_\ell \forall \ell \in [L]} \left\{ \left(\sum_{\ell \in [L]} \boldsymbol{x}_\ell\right)^\top \boldsymbol{A}^2 \left(\sum_{\ell \in [L]} \boldsymbol{x}_\ell\right) + 2\boldsymbol{b}^\top \boldsymbol{A} \left(\sum_{\ell \in [L]} \boldsymbol{x}_\ell\right) + \boldsymbol{b}^\top \boldsymbol{b} \right\} \\ &= \sup_{\substack{\boldsymbol{x}=[\boldsymbol{x}_1;\boldsymbol{x}_2;\cdots;\boldsymbol{x}_L],\\ \boldsymbol{x}_\ell \in \mathcal{P}_\ell \forall \ell \in [L]}} \{\boldsymbol{x}^\top \tilde{\boldsymbol{A}}\tilde{\boldsymbol{A}}^\top \boldsymbol{x} + 2\boldsymbol{b}^\top \tilde{\boldsymbol{A}}^\top \boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{b}\}, \end{aligned}$$

where $\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A} & \dots & \boldsymbol{A} \end{bmatrix}^\top \in \mathbb{R}^{LK \times K}$. By defining the cone $\mathcal{K}$ as

$$\mathcal{K} = \{[\boldsymbol{x}_1;\boldsymbol{x}_2;\cdots;\boldsymbol{x}_L;\tau] \in \mathbb{R}^{LK+1} : [\boldsymbol{x}_\ell;\tau] \in \mathcal{K}_\ell \ \ \forall \ell \in [L]\}$$

and repeating the steps in the proof of Theorem 1, we arrive at the fact that ellipsoid $\mathcal{E}(\boldsymbol{A},\boldsymbol{b})$ contains $\mathcal{P}$ if and only if there exist $\boldsymbol{F} \in \mathbb{S}^{LK}$, $\boldsymbol{g} \in \mathbb{R}^{LK}$, $h \in \mathbb{R}$ such that

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preccurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & \tilde{\boldsymbol{A}} \\ \boldsymbol{g}^\top & h & \boldsymbol{b}^\top \\ \tilde{\boldsymbol{A}}^\top & \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succcurlyeq \boldsymbol{0}.$$

In the previous three remarks, minimizing the function $-\log\det(\boldsymbol{A})$ subject to the corresponding constraints leads to a GC reformulation of ($\mathcal{MVE}$).

3. Tractable Approximations for Polytopes

In this section, we use the reformulation (4) to present tractable semidefinite programming approximations for ($\mathcal{MVE}$) in the case where the set $\mathcal{P}$ is a polytope defined as

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^K : \boldsymbol{S}\boldsymbol{x} \le \boldsymbol{t}\}, \tag{10}$$

where $\boldsymbol{S} \in \mathbb{R}^{J\times K}$ and $\boldsymbol{t} \in \mathbb{R}^J$. We start with our proposed approximation and then present theoretical comparisons with alternative approaches to approximate $\mathcal{E}_{\text{mve}}$.

Theorem 2. *Let $\mathcal{P}$ be a polytope defined as in (10) that satisfies Assumption 1. Consider any $\boldsymbol{A} \in \mathbb{S}^K_{++}$ and $\boldsymbol{b} \in \mathbb{R}^K$. Then, an ellipsoid $\mathcal{E}(\boldsymbol{A},\boldsymbol{b})$ contains $\mathcal{P}$ if there exist $\boldsymbol{N} \in \mathbb{R}^{J\times J}_+$, $\boldsymbol{F} \in \mathbb{S}^K$, $\boldsymbol{g} \in \mathbb{R}^K$, $h \in \mathbb{R}$ such that*

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preccurlyeq -\begin{bmatrix} -\boldsymbol{S}^\top \\ \boldsymbol{t}^\top \end{bmatrix} \boldsymbol{N} \begin{bmatrix} -\boldsymbol{S} & \boldsymbol{t} \end{bmatrix}, \ \text{and} \ \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & \boldsymbol{A} \\ \boldsymbol{g}^\top & h & \boldsymbol{b}^\top \\ \boldsymbol{A} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succcurlyeq \boldsymbol{0}. \tag{11}$$

Proof. For the polytope $\mathcal{P}$, the cone $\mathcal{K}$ defined in (2) can be written as $\mathcal{K} = \{[\boldsymbol{x};\tau] \in \mathbb{R}^{K+1} : \tau \ge 0, \boldsymbol{S}\boldsymbol{x} \le \tau \boldsymbol{t}\}$. We show that the Constraints (11) imply the Constraints (3). Because the second constraints in (11) and (3) are the same, we show that the first constraint of (11) implies the generalized copositive constraint in (3), which proves our claim. For any $[\boldsymbol{x};\tau] \in \mathcal{K}$, we have that

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{x} \\ \tau \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \tau \end{bmatrix} \le -\begin{bmatrix} \boldsymbol{x} \\ \tau \end{bmatrix}^\top \begin{bmatrix} -\boldsymbol{S}^\top \\ \boldsymbol{t}^\top \end{bmatrix} \boldsymbol{N} \begin{bmatrix} -\boldsymbol{S} & \boldsymbol{t} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \tau \end{bmatrix} \\ &= -(\tau \boldsymbol{t} - \boldsymbol{S}\boldsymbol{x})^\top \boldsymbol{N} (\tau \boldsymbol{t} - \boldsymbol{S}\boldsymbol{x}) \le 0, \end{aligned}$$

where the first inequality follows from the first semidefinite inequality in (11) and the final inequality holds because $\boldsymbol{N} \ge \boldsymbol{0}$ and $\tau \boldsymbol{t} - \boldsymbol{S}\boldsymbol{x} \ge \boldsymbol{0}$. Thus,

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preccurlyeq_{\mathcal{C}(\mathcal{K})} \boldsymbol{0}.$$

Hence, the claim follows. □

Theorem 2 provides a way to approximate ($\mathcal{MVE}$). We choose the ellipsoid with minimum volume among those that satisfy the conditions of Theorem 2. This can be achieved by solving the following tractable SDP:

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A}) \\ \text{subject to} & \boldsymbol{A} \in \mathbb{S}^K, \boldsymbol{b} \in \mathbb{R}^K, \boldsymbol{N} \in \mathbb{R}^{J\times J}_+, \boldsymbol{F} \in \mathbb{S}^K, \boldsymbol{g} \in \mathbb{R}^K, h \in \mathbb{R}, \\ & (11) \text{ holds.} \end{array} \tag{12}$$

If $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{N},\boldsymbol{F},\boldsymbol{g},h)$ is an optimal solution to (12), then we propose the use of the ellipsoid $\mathcal{E}_{\text{sdp}} = \mathcal{E}(\boldsymbol{A},\boldsymbol{b})$ as an approximation of $\mathcal{E}_{\text{mve}}$.

Next, we present a theoretical comparison of the quality of $\mathcal{E}_{\text{sdp}}$ with the other methods of approximating $\mathcal{E}_{\text{mve}}$. For the theoretical analysis, it is convenient to combine the two semidefinite inequalities of (11) using the Schur complement and write (12) equivalently as follows:

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A}) \\ \text{subject to} & \boldsymbol{A} \in \mathbb{S}^K, \boldsymbol{b} \in \mathbb{R}^K, \boldsymbol{N} \in \mathbb{R}^{J\times J}_+, \\ & \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{b}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix} \preccurlyeq \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{bmatrix} - \begin{bmatrix} -\boldsymbol{S}^\top \\ \boldsymbol{t}^\top \end{bmatrix} \boldsymbol{N} \begin{bmatrix} -\boldsymbol{S} & \boldsymbol{t} \end{bmatrix}. \end{array} \tag{13}$$

First, we compare $\text{Vol}(\mathcal{E}_{\text{sdp}})$ with the volume of the ellipsoid obtained by scaling the maximum volume-inscribed ellipsoid by a factor of K. We denote the latter ellipsoid by $\mathcal{E}_{\text{smvie}}$. In Theorem 3, we show that the volume of $\mathcal{E}_{\text{sdp}}$ cannot exceed the volume of $\mathcal{E}_{\text{smvie}}$. We begin with the following lemmas, which we use for proving Theorem 3.

Lemma 6. *Let $\mathcal{P}$ be a polytope defined as in (10) that satisfies Assumption 1. The optimal value of the following optimization problem is equal to* $\log(\text{Vol}(\mathcal{E}_{\text{smvie}}))$:

$$\begin{array}{ll} \text{minimize} & K\boldsymbol{\rho}^\top \boldsymbol{t} - K - \log\det\left(-\frac{1}{2}(\boldsymbol{S}^\top \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^\top \boldsymbol{S})\right) \\ \text{subject to} & \boldsymbol{\Lambda} \in \mathbb{R}^{J\times K}, \ \boldsymbol{\rho} \in \mathbb{R}^J, \\ & \boldsymbol{S}^\top \boldsymbol{\rho} = \boldsymbol{0}, \\ & \|\boldsymbol{\Lambda}_{j:}\| \le \rho_j \quad \forall j \in [J]. \end{array} \tag{14}$$

Proof. It is known that $\mathcal{E}_{\text{smvie}} = \{\boldsymbol{B}\boldsymbol{u} + \boldsymbol{d} : \|\boldsymbol{u}\| \le 1\}$ if and only if $\boldsymbol{B}$ and $\boldsymbol{d}$ are optimal in the following problem (see, e.g., Boyd and Vandenberghe 2004, section 8.4.2):

$$\begin{array}{ll} \text{maximize} & K\log\det(\boldsymbol{B}) \\ \text{subject to} & \boldsymbol{B}\in\mathbb{S}^K,\ \boldsymbol{d}\in\mathbb{R}^K \\ & \|\boldsymbol{B}\boldsymbol{S}_{j:}\| + \boldsymbol{S}_{j:}^\top\boldsymbol{d} \le t_j \quad \forall j\in[J]. \end{array} \tag{15}$$

The objective function of (15) equals the logarithm of $\mathrm{Vol}(\mathcal{E}_{\rm smvie})$. The dual of (15) is given by (14). A Slater point can be constructed in the primal problem (15) as follows. Consider a feasible solution where $\boldsymbol{B}=\kappa\mathbb{I}$ and $\boldsymbol{d}$ is any point in the interior of $\mathcal{P}$. By choosing a sufficiently small κ, the inequalities in (15) can be made strict. Therefore, strong duality holds, and the objective function of (14) is equal to the logarithm of $\mathrm{Vol}(\mathcal{E}_{\rm smvie})$. □

Lemma 7 (Horn and Johnson 1990, theorem 7.8.7). *If $M\in\mathbb{R}^{K\times K}$ is a square matrix with real entries such that $MM^\top \succ \mathbf{0}$, then*

$$\det\left(\frac{1}{2}(\boldsymbol{M}+\boldsymbol{M}^\top)\right) \le \det(\boldsymbol{M}).$$

Theorem 3. *If $\mathcal{P}$ is a polytope defined as in (11), then* $\mathrm{Vol}(\mathcal{E}_{\rm sdp}) \le \mathrm{Vol}(\mathcal{E}_{\rm smvie})$.

Proof. By Lemma 6, the logarithm of the volume of $\mathcal{E}_{\rm smvie}$ is equal to the optimal value of (14). We can compare the volumes of $\mathcal{E}_{\rm sdp}$ and $\mathcal{E}_{\rm smvie}$ by comparing the optimal values of the Minimization Problems (13) and (14). To prove the theorem, we show that any feasible solution in (14) can be used to construct a feasible solution to (13) with the same or lower objective function value. To this end, consider a solution $(\boldsymbol{\Lambda},\boldsymbol{\rho})$, which satisfies the constraints of (14). Define $\kappa=\exp(1-\boldsymbol{\rho}^\top\boldsymbol{t})$. Also, let $\boldsymbol{\Lambda}^\top\boldsymbol{S}=\boldsymbol{U}\Sigma\boldsymbol{V}^\top$ be the singular value decomposition of $\boldsymbol{\Lambda}^\top\boldsymbol{S}$, where $\boldsymbol{U},\boldsymbol{V}\in\mathbb{R}^{K\times K}$ are orthonormal matrices, and $\Sigma\in\mathbb{S}^K$ is a diagonal matrix. We note for later use that $\det(\boldsymbol{\Lambda}^\top\boldsymbol{S})=\det(\boldsymbol{U})\det(\Sigma)\det(\boldsymbol{V}^\top)=\det(\Sigma)$. Consider the following solution to (13):

$$\boldsymbol{A}=\kappa\boldsymbol{V}\Sigma\boldsymbol{V}^\top,\ \boldsymbol{b}=\kappa\boldsymbol{V}\boldsymbol{U}^\top\boldsymbol{\Lambda}^\top\boldsymbol{t},\ \boldsymbol{N}=\kappa^2(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top). \tag{16}$$

We demonstrate that this solution satisfies the constraints of (13). Note that

$$\boldsymbol{A}^2=\kappa^2\boldsymbol{V}\Sigma\boldsymbol{V}^\top\boldsymbol{V}\Sigma\boldsymbol{V}^\top=\kappa^2\boldsymbol{V}\Sigma\boldsymbol{U}^\top\boldsymbol{U}\Sigma\boldsymbol{V}^\top=\kappa^2\boldsymbol{S}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{S}$$

because $\boldsymbol{V}^\top\boldsymbol{V}=\boldsymbol{U}^\top\boldsymbol{U}=\mathbb{I}$. Similarly, $\boldsymbol{A}\boldsymbol{b}=\kappa^2\boldsymbol{S}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{t}$, and $\boldsymbol{b}^\top\boldsymbol{b}=\kappa^2\boldsymbol{t}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{t}$. Therefore,

$$\begin{aligned}\begin{bmatrix}-\boldsymbol{S}^\top\\ \boldsymbol{t}^\top\end{bmatrix}\boldsymbol{N}\begin{bmatrix}-\boldsymbol{S} & \boldsymbol{t}\end{bmatrix} &= \begin{bmatrix}\boldsymbol{S}^\top\boldsymbol{N}\boldsymbol{S} & -\boldsymbol{S}^\top\boldsymbol{N}\boldsymbol{t}\\ -\boldsymbol{t}^\top\boldsymbol{N}\boldsymbol{S} & \boldsymbol{t}^\top\boldsymbol{N}\boldsymbol{t}\end{bmatrix}\\ &=\kappa^2\begin{bmatrix}\boldsymbol{S}^\top(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top)\boldsymbol{S} & -\boldsymbol{S}^\top(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top)\boldsymbol{t}\\ -\boldsymbol{t}^\top(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top)\boldsymbol{S} & \boldsymbol{t}^\top(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top)\boldsymbol{t}\end{bmatrix}\\ &=\kappa^2\begin{bmatrix}-\boldsymbol{S}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{S} & \boldsymbol{S}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{t}\\ \boldsymbol{t}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{S} & (\boldsymbol{\rho}^\top\boldsymbol{t})^2-\boldsymbol{t}^\top\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top\boldsymbol{t}\end{bmatrix}\\ &=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & (\kappa\boldsymbol{\rho}^\top\boldsymbol{t})^2\end{bmatrix}-\begin{bmatrix}\boldsymbol{A}^2 & \boldsymbol{A}\boldsymbol{b}\\ \boldsymbol{b}^\top\boldsymbol{A} & \boldsymbol{b}^\top\boldsymbol{b}\end{bmatrix},\end{aligned}$$

where the third equality follows from the constraint $\boldsymbol{S}^\top\boldsymbol{\rho}=\mathbf{0}$. We claim that $(\kappa\boldsymbol{\rho}^\top\boldsymbol{t})^2\le 1$. To see this, first note that because the polytope $\mathcal{P}$ is nonempty, by Farkas' lemma any vector $\boldsymbol{\rho}$ satisfying $\boldsymbol{S}^\top\boldsymbol{\rho}=\mathbf{0}$ and $\boldsymbol{\rho}\ge\mathbf{0}$ also satisfies $\boldsymbol{\rho}^\top\boldsymbol{t}\ge 0$. Second, using the inequality $\exp(\nu)\ge 1+\nu$ with $\nu=\boldsymbol{\rho}^\top\boldsymbol{t}-1$, we get that $\kappa^{-1}=\exp(\boldsymbol{\rho}^\top\boldsymbol{t}-1)\ge\boldsymbol{\rho}^\top\boldsymbol{t}$, which implies that $\kappa\boldsymbol{\rho}^\top\boldsymbol{t}\le 1$. Combining these two inequalities, we get that $0\le\kappa\boldsymbol{\rho}^\top\boldsymbol{t}\le 1$, which implies that $(\kappa\boldsymbol{\rho}^\top\boldsymbol{t})^2\le 1$. Therefore, we have that

$$\begin{aligned}\begin{bmatrix}\boldsymbol{A}\\ \boldsymbol{b}^\top\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{b}\end{bmatrix} &= \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & (\kappa\boldsymbol{\rho}^\top\boldsymbol{t})^2\end{bmatrix}-\begin{bmatrix}-\boldsymbol{S}\\ \boldsymbol{t}\end{bmatrix}\boldsymbol{N}\begin{bmatrix}-\boldsymbol{S} & \boldsymbol{t}\end{bmatrix}\preccurlyeq\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}\\ &\quad -\begin{bmatrix}-\boldsymbol{S}\\ \boldsymbol{t}\end{bmatrix}\boldsymbol{N}\begin{bmatrix}-\boldsymbol{S} & \boldsymbol{t}\end{bmatrix}.\end{aligned}$$

Next, because $\boldsymbol{N}=\kappa^2(\boldsymbol{\rho}\boldsymbol{\rho}^\top-\boldsymbol{\Lambda}\boldsymbol{\Lambda}^\top)$, we have that $N_{ij}=\kappa^2\left(\rho_i\rho_j-\boldsymbol{\Lambda}_{i:}^\top\boldsymbol{\Lambda}_{j:}\right)\ge\kappa^2\left(\rho_i\rho_j-\|\boldsymbol{\Lambda}_{i:}\|\,\|\boldsymbol{\Lambda}_{j:}\|\right)\ge 0$, where the two inequalities follow from Cauchy–Schwarz and the constraint $\|\boldsymbol{\Lambda}_{j:}\|\le\rho_j$, respectively. Therefore, $\boldsymbol{N}\ge\mathbf{0}$. Next, we compare the objective values. Note that

$$\begin{aligned}-\log\det(\boldsymbol{A}) &= -\log\det(\kappa\boldsymbol{V}\Sigma\boldsymbol{V}^\top)\\ &= -\log(\kappa^K\det(\boldsymbol{V}\Sigma\boldsymbol{V}^\top))\\ &= -K\log(\kappa)-\log(\det(\boldsymbol{V})\det(\Sigma)\det(\boldsymbol{V}^\top))\\ &= K(\boldsymbol{\rho}^\top\boldsymbol{t}-1)-\log\det(\boldsymbol{\Lambda}^\top\boldsymbol{S})\le K(\boldsymbol{\rho}^\top\boldsymbol{t}-1)\\ &\quad -\log\det\left(\frac{1}{2}(\boldsymbol{\Lambda}^\top\boldsymbol{S}+\boldsymbol{S}^\top\boldsymbol{\Lambda})\right),\end{aligned}$$

where the final inequality follows from Lemma 7. Hence, the feasible solution (16) gives a lower objective function value. Thus, the claim follows. □

Corollary 1. *If the polytope $\mathcal{P}$ is a simplex, then* $\mathcal{E}_{\rm mve}=\mathcal{E}_{\rm sdp}=\mathcal{E}_{\rm smvie}$.

Proof. It is known that $\mathcal{E}_{\rm mve}=\mathcal{E}_{\rm smvie}$, if the set $\mathcal{P}$ is a simplex (Boyd and Vandenberghe 2004, section 8.4.1). Therefore, $\mathrm{Vol}(\mathcal{E}_{\rm smvie})=\mathrm{Vol}(\mathcal{E}_{\rm mve})$, which implies that $\mathrm{Vol}(\mathcal{E}_{\rm sdp})=\mathrm{Vol}(\mathcal{E}_{\rm mve})$. Because of the uniqueness of the minimum volume ellipsoid, we get that $\mathcal{E}_{\rm sdp}=\mathcal{E}_{\rm mve}$. □

In the next example, we demonstrate that the difference between the volumes of the ellipsoids $\mathcal{E}_{\rm sdp}$ and $\mathcal{E}_{\rm smvie}$ can be arbitrarily large.

Example 1 (Chipped Hypercube). Consider the polytope: $\mathcal{P}=\{\boldsymbol{x}\in\mathbb{R}^K:\mathbf{0}\le\boldsymbol{x}\le\mathbf{e},\mathbf{e}^\top\boldsymbol{x}\le\sqrt{K}\}$ formed by adding one constraint to the unit hypercube. This polytope forms a special case of (11) with $\boldsymbol{S}=[\mathbb{I};-\mathbb{I};\mathbf{e}^\top]$, and $\boldsymbol{t}=[\mathbf{e};\mathbf{0};\sqrt{K}]$. Let $R_{\rm mve}$, $R_{\rm smvie}$, and $R_{\rm sdp}$ be the radii (defined in Section 1.1) of the ellipsoids $\mathcal{E}_{\rm mve}$, $\mathcal{E}_{\rm smvie}$, and $\mathcal{E}_{\rm sdp}$, respectively. In the e-companion, we prove that $R_{\rm sdp}=O(K^{1/4})$ and $R_{\rm smvie}=\Theta(K^{1/2})$. Therefore, $R_{\rm smvie}$

grows at a strictly faster rate with the dimension K than R_{sdp}. This example demonstrates that the ratio $R_{\text{smvie}}/R_{\text{sdp}}$ can be arbitrarily high, if a large-enough K is chosen. We compute the three radii for $K = 2$ to $K = 50$ and plot the values in Figure 1(b). We observe that R_{mve} is very close to R_{sdp}, and the two appear to be growing at the same rate with K. Figure 1(a) shows the ellipsoids generated by the three methods for $K = 2$.

Next, we present the comparison of $\text{Vol}(\mathcal{E}_{\text{sdp}})$ with the volume of the ellipsoid provided by the $\mathcal{S}$ procedure described in Appendix A. However, the application of $\mathcal{S}$ procedure requires an ellipsoidal constraint in addition to the affine inequalities that define the polytope $\mathcal{P}$ (see Remark A.1 in Appendix A). This can be achieved by using any ellipsoid $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q}) = \{\boldsymbol{x} \in \mathbb{R}^K : \|\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}\|^2 \le 1\}$ that contains the polytope $\mathcal{P}$ and adding $\|\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}\|^2 \le 1$ as a redundant constraint in the definition of $\mathcal{P}$. The ellipsoid $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q})$ already serves as an approximation of $\mathcal{E}_{\text{mve}}$. We can then apply the $\mathcal{S}$ procedure in the hope of finding an ellipsoid with lower volume; we use $\mathcal{E}_{\text{sproc}}$ to denote this ellipsoid. However, in Proposition 1, we show that if the center of $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q})$ lies inside $\mathcal{P}$, then applying the $\mathcal{S}$ procedure provides no improvement and in fact, returns the ellipsoid $\mathcal{E}_{\text{sproc}} = \mathcal{E}(\boldsymbol{Q},\boldsymbol{q})$ as its unique optimal solution. This result is counterintuitive because the $\mathcal{S}$ procedure has been successfully applied in cases where $\mathcal{P}$ is defined as either the intersection or Minkowski sum of ellipsoids. Furthermore, if $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q}) = \mathcal{E}_{\text{smvie}}$ is used in the redundant quadratic constraint, then Proposition 1 implies that the $\mathcal{S}$ procedure does not improve upon $\mathcal{E}_{\text{smvie}}$ because the center of $\mathcal{E}_{\text{smvie}}$ lies inside $\mathcal{P}$. In that case, $\text{Vol}(\mathcal{E}_{\text{sdp}}) \le \text{Vol}(\mathcal{E}_{\text{smvie}}) = \text{Vol}(\mathcal{E}_{\text{sproc}})$.

Proposition 1. *Let $\mathcal{P}$ be a polytope defined as in (10) that satisfies Assumption 1, and let $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q}) = \{\boldsymbol{x} \in \mathbb{R}^K : \|\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}\|^2 \le 1\}$ be an ellipsoid containing $\mathcal{P}$ such that the center of $\mathcal{E}(\boldsymbol{Q},\boldsymbol{q})$ lies inside $\mathcal{P}$. Then, for the set $\{\boldsymbol{x} \in \mathbb{R}^K : \boldsymbol{S}\boldsymbol{x} \le \boldsymbol{t}, \|\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}\|^2 \le 1\}$, we have that $\mathcal{E}_{\text{sproc}} = \mathcal{E}(\boldsymbol{Q},\boldsymbol{q})$.*

Proof. See the e-companion. □

Finally, in Zhen et al. (2021), the authors present the following result, which can be used to approximate ($\mathcal{MVE}$).

Lemma 8 (Zhen et al. 2021, lemma 1). *Consider a polytope $\mathcal{P}$ defined as in (10). Then, $Z(\boldsymbol{A},\boldsymbol{b}) \le 1$ if there exist $\boldsymbol{V} \in \mathbb{R}^{J\times M}$ and $\boldsymbol{v} \in \mathbb{R}^J$ such that*

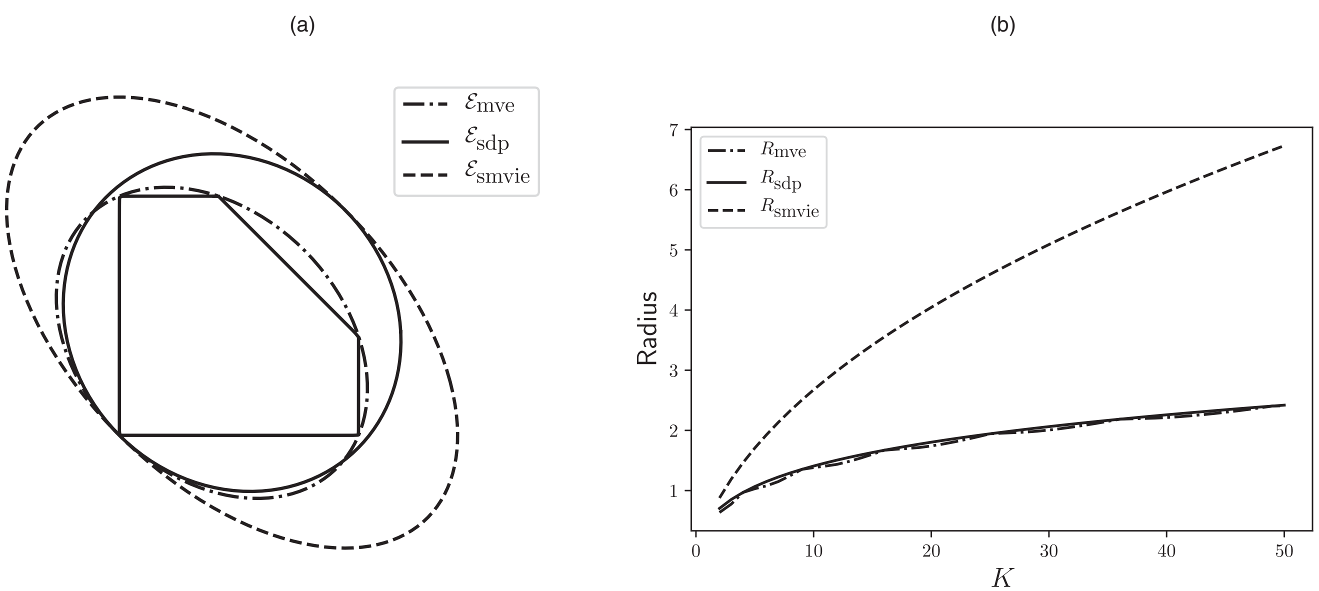
$$\begin{aligned} &\|\boldsymbol{V}^\top \boldsymbol{t} + \boldsymbol{b}\| + \boldsymbol{t}^\top \boldsymbol{v} \le 1,\\ &\boldsymbol{A} = \boldsymbol{V}^\top \boldsymbol{S},\\ &\boldsymbol{S}^\top \boldsymbol{v} = \boldsymbol{0},\\ &\|\boldsymbol{V}_{j:}\| \le v_j, \ \forall j \in [J]. \end{aligned} \tag{17}$$

Substituting these conditions for the constraint $Z(\boldsymbol{A},\boldsymbol{b}) \le 1$ in ($\mathcal{MVE}$) yields the following conservative approximation:

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A})\\ \text{subject to} & \boldsymbol{A} \in \mathbb{S}^K, \boldsymbol{b} \in \mathbb{R}^K, \boldsymbol{V} \in \mathbb{R}^{J\times K}, \boldsymbol{v} \in \mathbb{R}^J,\\ & \text{(18) holds.} \end{array} \tag{18}$$

We denote the ellipsoid generated using this approach by $\mathcal{E}_{\text{zrh}}$. In the following proposition, we show that $\text{Vol}(\mathcal{E}_{\text{zrh}})$ is never lower than $\text{Vol}(\mathcal{E}_{\text{smvie}})$. Thus, by Theorem 3, $\text{Vol}(\mathcal{E}_{\text{zrh}}) \ge \text{Vol}(\mathcal{E}_{\text{sdp}})$.

Figure 1. Chipped Hypercube Example



Notes. (a) The ellipsoids generated by the exact method and the two approximation methods for $K = 2$. (b) Radii (*i.e.*, $\text{Vol}(\cdot)^{1/K}$) of the ellipsoids generated by the three methods for different dimensions K.

Proposition 2. $\text{Vol}(\mathcal{E}_{\text{zrh}}) \ge \text{Vol}(\mathcal{E}_{\text{smvie}})$.

Proof. See the e-companion. □

3.1. Sets with Quadratic Constraints

Next, we provide a semidefinite programming approximation to ($\mathcal{MVE}$) when the set $\mathcal{P}$ is defined by affine, as well as quadratic inequalities. This generalizes the approximation (12) developed for the case of a polytope. Specifically, we consider the following set:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^K : Sx \le t, \|Q_i x + q_i\|^2 \le 1 \ \forall i \in [I] \right\}, \tag{19}$$

where $S \in \mathbb{R}^{J\times K}$, $t \in \mathbb{R}^J$, $Q_i \in \mathbb{S}^K$, and $q_i \in \mathbb{R}^K$. In the next theorem, we derive sufficient conditions that an ellipsoid $\mathcal{E}(A, b)$ contains the set $\mathcal{P}$ defined as in (19).

Theorem 4. *Let the set $\mathcal{P}$ be defined as in (19). Consider any $A \in \mathbb{S}^K_{++}$ and $b \in \mathbb{R}^K$. Then, an ellipsoid $\mathcal{E}(A, b)$ contains $\mathcal{P}$ if there exist $N \in \mathbb{R}^{J\times J}_+$, $F \in \mathbb{S}^K$, $g \in \mathbb{R}^K$, $h \in \mathbb{R}$, $\lambda_i \ge 0 \ \forall i \in [I]$, $\alpha_{ij} \in \mathbb{R}^K$, $\kappa_{ij} \in \mathbb{R} \ \forall i \in [I] \ \forall j \in [J]$ such that*

$$\begin{aligned} &\|\alpha_{ij}\| \le \kappa_{ij} \quad \forall i \in [I] \ \forall j \in [J], \\ &\begin{bmatrix} F & g \\ g^\top & h-1 \end{bmatrix} \preccurlyeq -\mathcal{S}^\top N \mathcal{S} + \sum_{i\in[I]} \lambda_i J_i - \sum_{i\in[I], j\in[J]} M_{ij}(\alpha_{ij}, \kappa_{ij}), \\ &\begin{bmatrix} F & g & A \\ g^\top & h & b^\top \\ A & b & \mathbb{I} \end{bmatrix} \succcurlyeq 0, \end{aligned} \tag{20}$$

where

$$\mathcal{S} = \begin{bmatrix} -S & t \end{bmatrix} \in \mathbb{R}^{J\times(K+1)}, \ J_i = \begin{bmatrix} Q_i^2 & Q_i^\top q_i \\ q_i^\top Q_i & q_i^\top q_i - 1 \end{bmatrix} \in \mathbb{S}^{K+1} \quad \forall i \in [I], \text{ and}$$

$$M_{ij}(\alpha, \kappa) = \begin{bmatrix} -\frac{1}{2}\left(S_{j:}\alpha^\top Q_i + Q_i \alpha S_{j:}^\top\right) & \frac{1}{2}\left(t_j Q_i \alpha - (\alpha^\top q_i + \kappa) S_{j:}\right) \\ \frac{1}{2}\left(t_j Q_i \alpha - (\alpha^\top q_i + \kappa) S_{j:}\right)^\top & (\alpha^\top q_i + \kappa) t_j \end{bmatrix} \quad \forall i \in [I] \ \forall j \in [J].$$

Proof. For the set $\mathcal{P}$, the cone $\mathcal{K}$ as defined as in (2) can be written as

$$\mathcal{K} = \left\{ [x; \tau] \in \mathbb{R}^{K+1} : \tau \ge 0, Sx \le \tau t, \|Q_i x + \tau q_i\|^2 \le \tau^2 \ \forall i \in [I] \right\}.$$

We show that the Conditions (20) imply the Conditions (3), which proves the claim. Let

$$P = \begin{bmatrix} F & g \\ g^\top & h-1 \end{bmatrix}.$$

Also, consider $[x; \tau] \in \mathcal{K}$. From the first semidefinite inequality, we have that

$$\begin{bmatrix} x \\ \tau \end{bmatrix}^\top P \begin{bmatrix} x \\ \tau \end{bmatrix} \le \begin{bmatrix} x \\ \tau \end{bmatrix}^\top \left(-\mathcal{S}^\top N \mathcal{S} + \sum_{i\in[I]} \lambda_i J_i - \sum_{i\in[I], j\in[J]} M_{ij}(\alpha_{ij}) \right) \begin{bmatrix} x \\ \tau \end{bmatrix}.$$

We show that all three terms in the expression on the right-hand side are nonpositive. The first term is nonpositive as shown in the proof of Theorem 2. Next, observe that for all $i \in [I]$, we have that $[x; \tau]^\top J_i [x; \tau] = \|Q_i x + \tau q_i\|^2 - \tau^2 \le 0$, because $[x; \tau] \in \mathcal{K}$. Also,

$$[x; \tau]^\top M_{ij}(\alpha_{ij})[x; \tau] = (\tau t_j - S_{j:}^\top x)(\tau \kappa_{ij} + \alpha_{ij}^\top (Q_i x + \tau q_i)) \ge 0.$$

The previous inequality follows because both terms in the product are nonnegative because $Sx \le \tau t$ and $\tau\kappa_{ij} + \alpha_{ij}^\top (Q_i x + \tau q_i) \ge \tau\kappa_{ij} - \|\alpha_{ij}\| \, \|Q_i x + \tau q_i\| \ge \tau\kappa_{ij} - \tau\kappa_{ij} = 0$. Hence, $[x; \tau]^\top P [x; \tau] \le 0 \ \forall [x; \tau] \in \mathcal{K}$, which implies that $P \preccurlyeq_{\mathcal{C}(\mathcal{K})} 0$. Hence, the claim follows. □

Theorem 4 implies that the following SDP serves as a restriction to ($\mathcal{MVE}$):

$$\begin{array}{ll} \text{minimize} & -\log\det(A) \\ \text{subject to} & A \in \mathbb{S}^K, b \in \mathbb{R}^K, F \in \mathbb{S}^K, g \in \mathbb{R}^K, h \in \mathbb{R}, \\ & N \in \mathbb{R}^{J\times J}_+, \lambda_i \ge 0 \ \forall i \in [I], \alpha_{ij} \in \mathbb{R}^K, \\ & \text{(20) holds.} \end{array} \tag{21}$$

Remark 5. The approximation discussed is motivated by the relaxation linearization technique discussed in Anstreicher (2009) and Sherali and Adams (2013) and SOC-RLT constraints discussed in Burer and Anstreicher (2013).

4. Application to Distributionally Robust Optimization

In this section, we demonstrate how our approximation to ($\mathcal{MVE}$) can be used to obtain good solutions to the two-stage DRO model with random recourse given by

$$\inf_{x\in\mathcal{X}} \left\{ c^\top x + \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\mathcal{R}(x, \tilde{\xi})] \right\}, \tag{22}$$

where

$$\begin{aligned} \mathcal{R}(x, \xi) = \inf_{y} \ & (D\xi + d)^\top y \\ \text{s.t.} \ & T_\ell(x)^\top \xi + h_\ell(x) \le (W_\ell \xi + w_\ell)^\top y \quad \forall \ell \in [L]. \end{aligned} \tag{23}$$

Here, $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$ represent the first- and second-stage decision variables, respectively; $\mathcal{X}$ is a set defined by tractable convex constraints on x; and $\xi \in \mathbb{R}^K$ is the vector of uncertain parameters. Also, $c \in \mathbb{R}^{N_1}, D \in \mathbb{R}^{N_2\times K}, W_\ell \in \mathbb{R}^{N_2\times K}, d \in \mathbb{R}^{N_2}$, and $w_\ell \in \mathbb{R}^{N_2}$ are problem parameters. The functions $T_\ell : \mathcal{X} \to \mathbb{R}^K$ and $h_\ell : \mathcal{X} \to \mathbb{R}$ are affine in the input parameter. We consider the following moment-based ambiguity set: $\mathcal{Q} = \{\mathbb{Q} \in \mathcal{Q}_0(\Xi) : \mathbb{E}_{\mathbb{Q}}[\tilde{\xi}] = \mu, \mathbb{E}_{\mathbb{Q}}[\tilde{\xi}\tilde{\xi}^\top] \preccurlyeq \Sigma\}$, where $\Xi = \{\xi \in \mathbb{R}^K : S\xi \le t\}$ is the bounded support set and $\mathcal{Q}_0(\Xi)$ is the set of all probability measures supported on Ξ. The objective function minimizes the sum of the first stage and the expected recourse cost, where the expectation is taken with respect to the worst case distribution among those in the ambiguity set $\mathcal{Q}$. The results presented here can be extended to other types of ambiguity sets, including the simpler case where $\mathcal{Q} = \mathcal{Q}_0(\Xi)$ (i.e., robust optimization) (Bertsimas and Dunning 2016, Xu and Burer 2018), the more sophisticated data-driven Wasserstein ambiguity set (Hanasusanto and Kuhn 2018), and the classical stochastic programming setting.

Problem (22) can be written equivalently as

$$\begin{aligned}&\inf_{\boldsymbol{x},\boldsymbol{y}(\cdot)} \boldsymbol{c}^\top \boldsymbol{x} + \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[(\boldsymbol{D}\boldsymbol{\xi}+\boldsymbol{d})^\top \boldsymbol{y}(\boldsymbol{\xi})],\\ &\text{s.t. } \boldsymbol{x}\in\mathcal{X},\\ &\qquad \boldsymbol{T}_\ell(\boldsymbol{x})^\top\boldsymbol{\xi} + h_\ell(\boldsymbol{x}) \le (\boldsymbol{W}_\ell\boldsymbol{\xi}+\boldsymbol{w}_\ell)^\top \boldsymbol{y}(\boldsymbol{\xi}) \quad \forall\, \boldsymbol{\xi}\in\Xi, \forall \ell\in[L],\end{aligned} \tag{24}$$

where the second-stage decision variable $\boldsymbol{y}$ is a function of the uncertain parameters $\boldsymbol{\xi}$. Problem (24) is difficult to solve. To generate a tractable approximation to (24), we explore the use of PLD rules. Specifically, we partition Ξ into regions $\Xi_1,\dots,\Xi_J$ and restrict $\boldsymbol{y}(\cdot)$ to be of the form $\boldsymbol{y}(\boldsymbol{\xi}) = \boldsymbol{Y}_j\boldsymbol{\xi} + \boldsymbol{y}_j$ if $\boldsymbol{\xi}\in\Xi_j$, where $\boldsymbol{Y}_j \in \mathbb{R}^{N_2\times K}$ and $\boldsymbol{y}_j \in \mathbb{R}^{N_2}$. For constructing the partitions, we start with a set of *constructor points* $\{\boldsymbol{\xi}_j\}_{j\in[J]}$ in Ξ. Then, we define the partition Ξ_j to be the set of all points in Ξ that are closer to $\boldsymbol{\xi}_j$ than any other constructor point. In other words,

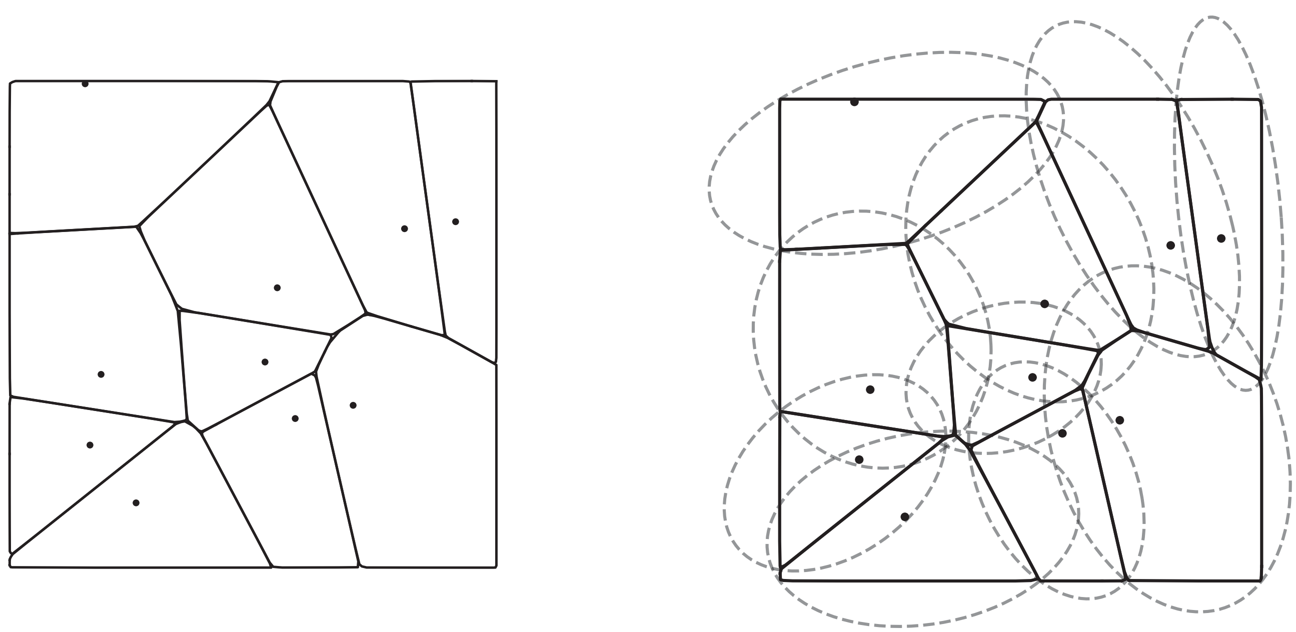
$$\begin{aligned}\Xi_j &= \{\boldsymbol{\xi}\in\mathbb{R}^K : \boldsymbol{S}\boldsymbol{\xi}\le\boldsymbol{t}, \|\boldsymbol{\xi}-\boldsymbol{\xi}_j\| \le \|\boldsymbol{\xi}-\boldsymbol{\xi}_i\|\ \forall i\in[J], i\ne j\}\\ &= \{\boldsymbol{\xi}\in\mathbb{R}^K : \boldsymbol{S}\boldsymbol{\xi}\le\boldsymbol{t}, 2(\boldsymbol{\xi}_i-\boldsymbol{\xi}_j)^\top\boldsymbol{\xi} \le \boldsymbol{\xi}_i^\top\boldsymbol{\xi}_i - \boldsymbol{\xi}_j^\top\boldsymbol{\xi}_j\ \forall i\in[J], i\ne j\}\\ &= \{\boldsymbol{\xi}\in\mathbb{R}^K : \boldsymbol{S}_j\boldsymbol{\xi}\le\boldsymbol{t}_j\},\end{aligned}$$

where the matrix $\boldsymbol{S}_j \in \mathbb{R}^{L_j\times K}$ and the vector $\boldsymbol{t}_j\in\mathbb{R}^{L_j}$ are formed by combining the linear constraints in the definition of Ξ_j. These partitions are known as *Voronoi regions*.

Because of random recourse (i.e., uncertainty in the coefficients of $\boldsymbol{y}(\cdot)$), finding the optimal PLD rule is NP hard, even if there is only one piece (Ben-Tal et al. 2004). However, we can approximate the problem of finding the optimal PLD rule using the $\mathcal{S}$ procedure. However, we need a quadratic constraint in the definition of Ξ_j for an effective application of $\mathcal{S}$ procedure (see Remark A.1 in Appendix A). To this end, let $\mathcal{E}(\boldsymbol{A}_j,\boldsymbol{b}_j)$ be an ellipsoid that contains Ξ_j. Because Ξ_j is a polytope, we can exploit the results developed in Section 3 to find $\mathcal{E}(\boldsymbol{A}_j,\boldsymbol{b}_j)$. We can write Ξ_j equivalently as $\Xi_j = \{\boldsymbol{\xi}\in\mathbb{R}^K : \boldsymbol{S}_j\boldsymbol{\xi}\le\boldsymbol{t}_j, \|\boldsymbol{A}_j\boldsymbol{\xi}+\boldsymbol{b}_j\|^2\le 1\}$. We illustrate the procedure of partitioning and covering with ellipsoids in Figure 2.

In the next proposition, we derive a tractable SDP that generates a feasible PLD rule. The optimal value of the resulting SDP approximation provides an upper bound to the optimal value of (24). In Example 2 presented after the proposition, we demonstrate that the size of the bounding ellipsoids $\mathcal{E}(\boldsymbol{A}_j,\boldsymbol{b}_j)$ can drastically impact the upper bound provided by the SDP approximation; in particular, the tighter the ellipsoids, the better the upper bound.

Figure 2. Voronoi Regions



Notes. The outer square represents the support set, and the black dots are the constructor points. The points are used to construct partitions, and an ellipsoid containing each partition is found by solving the SDP (12).

Proposition 3. *Consider the following SDP:*

$$\begin{aligned}
\inf\ & \boldsymbol{c}^\top \boldsymbol{x} + \alpha + \boldsymbol{\beta}^\top \boldsymbol{\mu} + \mathrm{tr}(\Gamma\Sigma)\\
\text{s.t.}\ & \boldsymbol{x} \in \mathcal{X},\ \Gamma \in \mathbb{S}_+^K,\ \boldsymbol{\beta} \in \mathbb{R}^K,\ \boldsymbol{\alpha} \in \mathbb{R},\\
& \boldsymbol{Y}_j \in \mathbb{R}^{N_2 \times K},\ \boldsymbol{y}_j \in \mathbb{R}^{N_2},\ \boldsymbol{\gamma}_j \in \mathbb{R}_+^{L_j},\ \delta_j \in \mathbb{R}_+ \quad \forall j \in [J],\\
& \lambda_{j\ell} \in \mathbb{R}_+,\ \boldsymbol{\rho}_{j\ell} \in \mathbb{R}_+^{L_j} \quad \forall j \in [J]\ \forall \ell \in [L],\\
& \begin{bmatrix} \Gamma & \frac{1}{2}\boldsymbol{\beta} \\ \frac{1}{2}\boldsymbol{\beta}^\top & \alpha \end{bmatrix} - \begin{bmatrix} \frac{1}{2}(\boldsymbol{D}^\top \boldsymbol{Y}_j + \boldsymbol{Y}_j^\top \boldsymbol{D}) & \frac{1}{2}(\boldsymbol{D}^\top \boldsymbol{y}_j + \boldsymbol{Y}_j^\top \boldsymbol{d}) \\ \frac{1}{2}(\boldsymbol{D}^\top \boldsymbol{y}_j + \boldsymbol{Y}_j^\top \boldsymbol{d})^\top & \boldsymbol{d}^\top \boldsymbol{y}_j \end{bmatrix}\\
& + \boldsymbol{P}_j(\boldsymbol{\gamma}_j) + \delta_j \boldsymbol{J}_j \succcurlyeq \boldsymbol{0} \quad \forall j \in [J],\\
& \begin{bmatrix} \frac{1}{2}(\boldsymbol{W}_\ell^\top \boldsymbol{Y}_j + \boldsymbol{Y}_j^\top \boldsymbol{W}_\ell) & \frac{1}{2}(\boldsymbol{W}_\ell^\top \boldsymbol{y}_j + \boldsymbol{Y}_j^\top \boldsymbol{w}_\ell) \\ \frac{1}{2}(\boldsymbol{W}_\ell^\top \boldsymbol{y}_j + \boldsymbol{Y}_j^\top \boldsymbol{w}_\ell)^\top & \boldsymbol{w}_\ell^\top \boldsymbol{y}_j \end{bmatrix}\\
& - \boldsymbol{M}_\ell(\boldsymbol{x}) + \boldsymbol{P}_j(\boldsymbol{\rho}_{j\ell}) + \lambda_{j\ell} \boldsymbol{J}_j \succcurlyeq \boldsymbol{0} \quad \forall j \in [J]\ \forall \ell \in [L], \qquad (25)
\end{aligned}$$

where

$$\boldsymbol{M}_\ell(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{0} & \frac{1}{2}\boldsymbol{T}_\ell(\boldsymbol{x}) \\ \frac{1}{2}\boldsymbol{T}_\ell(\boldsymbol{x})^\top & h_\ell(\boldsymbol{x}) \end{bmatrix},$$

$$\boldsymbol{P}_j(\boldsymbol{\rho}) = \begin{bmatrix} \boldsymbol{0} & \frac{1}{2}S_j^\top \boldsymbol{\rho} \\ \frac{1}{2}\boldsymbol{\rho}^\top S_j & -t_j^\top \boldsymbol{\rho} \end{bmatrix}, \text{ and } \quad \boldsymbol{J}_j = \begin{bmatrix} \boldsymbol{A}_j^2 & \boldsymbol{A}_j^\top \boldsymbol{b}_j \\ \boldsymbol{b}_j^\top \boldsymbol{A}_j & \boldsymbol{b}_j^\top \boldsymbol{b}_j - 1 \end{bmatrix}.$$

Let $\boldsymbol{y}(\boldsymbol{\xi}) = \boldsymbol{Y}_j \boldsymbol{\xi} + \boldsymbol{y}_j$ if $\boldsymbol{\xi} \in \Xi_j$. Then, $(\boldsymbol{x}, \boldsymbol{y}(\cdot))$ provides a feasible solution to (24)*. Furthermore, the optimal value of* (25) *provides an upper bound to the optimal value of* (24)*.*

Proof. See the e-companion. □

Example 2. Consider the following special case of (24):

$$\begin{aligned} z = \inf_{x, \boldsymbol{y}(\cdot)}\ & x \\ \text{s.t.}\ & 1 \le (\boldsymbol{\xi} + \mathbf{e})^\top \boldsymbol{y}(\boldsymbol{\xi}) \le x \quad \forall \boldsymbol{\xi} \in \Xi, \end{aligned} \qquad (26)$$

where $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^K : \boldsymbol{0} \le \boldsymbol{\xi} \le \mathbf{e}\}$ is the unit hypercube and $J = 1$. This problem is a special case of (24) with $L = 2$, $\boldsymbol{D} = \boldsymbol{0}$, $\boldsymbol{d} = \boldsymbol{0}$, $\boldsymbol{W}_1 = \mathbb{I}$, $\boldsymbol{w}_1 = \mathbf{e}$, $\boldsymbol{T}_1(x) = \boldsymbol{0}$, $h_1(x) = 1$, $\boldsymbol{W}_2 = -\mathbb{I}$, $\boldsymbol{w}_2 = -\mathbf{e}$, $\boldsymbol{T}_2(x) = \boldsymbol{0}$, and $h_2(x) = -x$. The true optimal value is $z = 1$, which is obtained by the nonlinear decision function $\boldsymbol{y}(\boldsymbol{\xi}) = (\boldsymbol{\xi} + \mathbf{e})/\|\boldsymbol{\xi} + \mathbf{e}\|^2$. In this case, $\mathcal{E}_{\text{mve}} = \{\boldsymbol{\xi} \in \mathbb{R}^K : \|\boldsymbol{\xi} - \mathbf{e}/2\|^2 \le N/4\}$. For $s \ge 0$, let $z(s)$ be the upper bound generated by the SDP approximation when $\{\boldsymbol{\xi} \in \mathbb{R}^K : \|\boldsymbol{\xi} - \mathbf{e}/2\|^2 \le N(1+s)/4\}$ is used as the bounding ellipsoid. In the e-companion, we show that

$$z(s) = \begin{cases} 9/(8-s) & \text{if } 0 \le s \le 2, \\ 1 + s/4 & \text{if } 2 \le s \le 4, \\ 2 & \text{if } 4 \le s. \end{cases}$$

Therefore, the linear decision rule (LDR) obtained with $\mathcal{E}_{\text{mve}}$ generates an objective value of $z(0) = 9/8 = 1.125$. The objective value $z(s)$ increases as the size of the ellipsoid increases. The case when s approaches ∞ corresponds to dropping the ellipsoidal constraint; in that case, we obtain an objective value of $\lim_{s\to\infty} z(s) = 2$. Hence, ignoring the ellipsoidal constraint can increase the suboptimality of the decision rules approximation from 12.5% to 100%.

Example 2 demonstrates the importance of generating good outer ellipsoids. We further elaborate on this point in Section 5.2, where we perform experiments on randomly generated instances of an inventory management model. We note that the task of finding the outer ellipsoids $\mathcal{E}(\boldsymbol{A}_j, \boldsymbol{b}_j)$ can be parallelized, which leads to a substantial reduction in the computation time.

Remark 6 (Two-Stage Stochastic Programming). In the classical stochastic programming setting, the random parameters $\tilde{\boldsymbol{\xi}}$ are assumed to be governed by a known distribution $\mathbb{P}$. The semiinfinite constraints in (24) remain unchanged and can be approximated in the same manner using the $\mathcal{S}$ procedure. On the other hand, the worst case expectation in the objective function of (24) reduces to the expectation $\mathbb{E}_{\mathbb{P}}[(\boldsymbol{D}\tilde{\boldsymbol{\xi}} + \boldsymbol{d})^\top \boldsymbol{y}(\tilde{\boldsymbol{\xi}})]$. Applying the law of total expectation and employing the proposed PLD rules, we can reformulate the expectation as

$$\begin{aligned}
& \sum_{j\in[J]} \mathbb{P}(\tilde{\boldsymbol{\xi}} \in \Xi_j) \mathbb{E}_{\mathbb{P}}\Big[(\boldsymbol{D}\tilde{\boldsymbol{\xi}} + \boldsymbol{d})^\top \boldsymbol{y}(\tilde{\boldsymbol{\xi}}) \ \big|\ \tilde{\boldsymbol{\xi}} \in \Xi_j\Big]\\
= & \sum_{j\in[J]} \mathbb{P}(\tilde{\boldsymbol{\xi}} \in \Xi_j) \mathbb{E}_{\mathbb{P}}\Big[(\boldsymbol{D}\tilde{\boldsymbol{\xi}} + \boldsymbol{d})^\top (\boldsymbol{Y}_j \tilde{\boldsymbol{\xi}} + \boldsymbol{y}_j) \ \big|\ \tilde{\boldsymbol{\xi}} \in \Xi_j\Big]\\
= & \sum_{j\in[J]} \mathbb{P}(\tilde{\boldsymbol{\xi}} \in \Xi_j) \Big(\mathrm{tr}\Big(\boldsymbol{D}^\top \boldsymbol{Y}_j \mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}^\top \mid \tilde{\boldsymbol{\xi}} \in \Xi_j]\Big)\\
& + (\boldsymbol{y}_j^\top \boldsymbol{D} + \boldsymbol{d}^\top \boldsymbol{Y}_j) \mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{\xi}} \mid \tilde{\boldsymbol{\xi}} \in \Xi_j] + \boldsymbol{d}^\top \boldsymbol{y}_j\Big).
\end{aligned}$$

This expression is affine in the decision variables $\boldsymbol{Y}_j$ and $\boldsymbol{y}_j$, $j \in [J]$. Note that the partition probabilities $\mathbb{P}(\tilde{\boldsymbol{\xi}} \in \Xi_j)$, $j \in [J]$, and conditional moments $\mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{\xi}} \mid \tilde{\boldsymbol{\xi}} \in \Xi_j]$ and $\mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}^\top \mid \tilde{\boldsymbol{\xi}} \in \Xi_j]$, $j \in [J]$, can be estimated using the Monte Carlo sampling method.

5. Numerical Experiments

In this section, we present numerical experiments that demonstrate the improved performance of our scheme for approximating ($\mathcal{MVE}$) over the existing methods. First, we show that our approach outperforms the existing approaches in terms of solution quality and computational time on randomly generated polytopes. Second, we demonstrate the efficacy of our method in generating quality solutions for a distributionally robust inventory management model. All optimization problems are solved using the YALMIP interface (Löfberg 2004) on a 16-core, 3.4-GHz computer with 32 GB RAM. We use MOSEK 8.1

to solve SDPs and CPLEX 12.8 to solve nonconvex quadratic programs to optimality.

5.1. Random Polytopes

Here, we compare our method of approximating ($\mathcal{MVE}$) with (i) the constraint generation approach (Gotoh and Konno 2006), (ii) the SMVIE approach, and (iii) the method using sufficient conditions proposed by Kellner et al. (2013). We refer to the last method as the KTT approach and denote the corresponding ellipsoid by $\mathcal{E}_{\rm ktt}$ (see Appendix B for details on the formulation).

For our experiments, we generate polytopes randomly as follows. We start with the hyperrectangle $\{x \in \mathbb{R}^K : \mathbf{0} \le x \le \mathbf{e}\}$ with center $\mathbf{c} = \mathbf{e}/2$. Then, we add M linear inequalities in the following way. For $j \in [M]$, we generate a vector $s_j \in \mathbb{R}^K$ uniformly distributed on the surface of the unit hypersphere. We generate a distance r_j uniformly at random from the interval $[-\|s_j\|_1/2, \|s_j\|_1/2]$ and add the constraint $s_j^\top(x - \mathbf{c}) \le r_j$ if $r_j > 0$ and $s_j^\top(x - \mathbf{c}) \ge r_j$ if $r_j \le 0$. Choosing r_j from the specified interval leads to a constraint that cuts the hyperrectangle (i.e., the constraint is not redundant). Also, the construction ensures that the polytope is nonempty because $\mathbf{c}$ satisfies all the constraints.

For several values of K, we solve the problem exactly and apply each approximation method on 50 randomly generated instances for $M = K, 2K, 3K$. We report the suboptimality results of the three approximation methods in Table 1. For higher values of K, for which we were not able to solve the problem exactly within 30 minutes, we report the suboptimality of the radius of $\mathcal{E}_{\rm smvie}$ and $\mathcal{E}_{\rm ktt}$ with respect to $\mathcal{E}_{\rm sdp}$ in Table 2. Finally, the solution times of different methods are reported in Table 3. We do not report the solution time of the SMVIE approach. Even for the largest problem size that we solved, the SMVIE approach produces solutions in less than two seconds, dominating every other approach.

It can be observed that the radius (and therefore, volume) of $\mathcal{E}_{\rm sdp}$ is significantly lower than that of $\mathcal{E}_{\rm smvie}$. Furthermore, the suboptimality of the radius of $\mathcal{E}_{\rm smvie}$ relative to that of $\mathcal{E}_{\rm sdp}$ increases with the dimension K (from 246% for $K = 15$ to 481% for $K = 40$). This is perhaps because the scale factor of K becomes very conservative for higher values of K. This increase in solution quality of $\mathcal{E}_{\rm sdp}$ comes at the cost of higher solution times compared with that of finding $\mathcal{E}_{\rm smvie}$.

Table 1. Random Polytopes: Mean Suboptimality of the Radii of $\mathcal{E}_{\rm sdp}$ ("Copos"), $\mathcal{E}_{\rm ktt}$ ("KTT"), and $\mathcal{E}_{\rm smvie}$ ("SMVIE") for Different Problem Sizes

	$M = K$, %			$M = 2K$, %			$M = 3K$, %		
K	Copos	KTT	SMVIE	Copos	KTT	SMVIE	Copos	KTT	SMVIE
2	3.41	4.68	34.3	5.20	6.48	32.8	5.33	6.63	31.9
5	4.88	7.02	105	9.92	13.16	91.9	13.2	16.4	93.7
10	2.53	3.72	188	7.48	9.51	176	13.6	16.9	164
15	1.29	1.84	250	5.57	7.16	230	N/A	N/A	N/A

Note. We use "N/A" when the problem cannot be solved to optimality within 30 minutes.

Table 2. Random Polytopes: Mean Suboptimality of the Radii of $\mathcal{E}_{\rm ktt}$ ("KTT") and $\mathcal{E}_{\rm smvie}$ ("SMVIE") Relative to $\mathcal{E}_{\rm sdp}$ for the Cases That Could Not Be Solved to Optimality Within 30 Minutes

	$M = K$, %		$M = 2K$, %		$M = 3K$, %	
K	KTT	SMVIE	KTT	SMVIE	KTT	SMVIE
15	0.54	246	1.50	212	2.07	191
20	0.30	310	1.01	268	1.65	245
25	0.28	357	0.66	318	—	292
30	—	401	—	364	—	329
35	—	440	—	405	—	372
40	—	481	—	447	—	414

Note. We use — for the cases when the KTT approach does not provide a solution within 30 minutes.

We also observe that the radius of $\mathcal{E}_{\rm sdp}$ is slightly better than that of $\mathcal{E}_{\rm ktt}$; the solution time, however, is significantly lower (one to two orders of magnitude). As an example, for $K = M = 30$, the KTT approach does not provide solutions within 30 minutes, whereas our method generates an solution in 13.7 seconds on average.

Finally, we observe that for small problem instances, our method finds a solution much faster than solving the problem to optimality. For higher-dimensional problems ($K > 15$), where solving the problem exactly becomes intractable, our approximation continues to provide ellipsoids of lower volume than the other approximation methods.

5.2. Risk-Averse Inventory Management

Next, we consider an inventory management problem, where we decide the purchase amount of N products before observing their demands. We incur a holding cost if we purchase more than the demand and a stockout cost if we purchase less than the demand. We assume that the demands and the stockout costs are random. The objective is to minimize the worst case conditional value at risk (CVaR) (Rockafellar and Uryasev 2000, Natarajan et al. 2009, Zhu and Fukushima 2009) of the total cost. We can write the model as follows:

$$\begin{array}{ll} \text{minimize} & \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{Q}\text{-CVaR}_\epsilon[\mathcal{R}(x, \tilde{\xi}, \tilde{s})] \\ \text{subject to} & x \in \mathbb{R}^N,\ x \ge \mathbf{0},\ \mathbf{e}^\top x \le B, \end{array}$$

where

$$\begin{aligned} \mathcal{R}(x, \xi, s) = \inf \quad & g^\top y_1 + s^\top y_2 \\ \text{s.t.} \quad & y_1 \in \mathbb{R}^N_+, y_2 \in \mathbb{R}^N_+, \\ & y_1 \ge x - \xi,\ y_2 \ge \xi - x. \end{aligned}$$

Here, the variables x, y_1, and y_2 represent the vector of purchase decisions, excess amounts, and shortfall

Table 3. Random Polytopes: Mean Solution Times (in Seconds) of the Exact Method ("Exact"), Our Proposed Method ("Copos"), and the KTT Approach ("KTT") for Different Problem Sizes

	$M = K$			$M = 2K$			$M = 3K$		
K	Exact	Copos	KTT	Exac	Copos	KTT	Exact	Copos	KTT
2	1.52	0.004	0.011	1.53	0.005	0.027	1.69	0.005	0.059
5	8.56	0.014	0.036	9.13	0.023	0.073	9.59	0.050	0.096
10	72.6	0.106	0.925	81.7	0.290	2.09	133	0.754	3.78
15	406	0.542	10.0	1,191	1.82	25.8	—	5.21	49.7
20	—	2.01	73.2	—	7.60	210	—	22.2	438
25	—	5.65	368	—	22.8	1,067	—	68.0	—
30	—	13.7	—	—	54.7	—	—	207	—
35	—	28.8	—	—	133	—	—	492	—
40	—	53.2	—	—	302	—	—	1,155	—

Note. We use — when the corresponding method does not provide a solution within 30 minutes.

amounts, respectively. The vector $g \in \mathbb{R}^N$ represents the known holding costs, and B denotes budget on the total purchase amount. Also, $\xi \in \mathbb{R}^N$ and $s \in \mathbb{R}^N$ are random parameters, which represent the vectors of demand and stockout costs, respectively. The ambiguity set $\mathcal{Q}$ is as described in Section 4. By employing the definition of CVaR, it can be shown that the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \kappa + \dfrac{1}{\epsilon} \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\tau(\tilde{\xi}, \tilde{s})] \\ \text{subject to} & \kappa \in \mathbb{R}, x \in \mathbb{R}^N, x \geq \mathbf{0}, \mathbf{e}^\top x \leq B, \\ & \left. \begin{array}{l} \tau(\xi, s) \geq 0, y_1(\xi, s) \geq \mathbf{0}, y_2(\xi, s) \geq \mathbf{0}, \\ \tau(\xi, s) \geq g^\top y_1(\xi, s) + s^\top y_2(\xi, s) - \kappa, \\ y_1(\xi, s) \geq x - \xi, \ y_2(\xi, s) \geq \xi - x \end{array} \right\} \forall (\xi, s) \in \Xi, \end{array} \tag{27}$$

which is of the form (25) (Shapiro and Kleywegt 2002, Hanasusanto et al. 2015).

We generate the parameters for this problem as follows. We use $n = 7$ products, which leads to $2N = 14$ random parameters. We choose $\Xi = \{[\xi; s] : \xi_l \leq \xi \leq \xi_u, s_l \leq s \leq s_u\}$, and $\epsilon = 5\%$. We partition Ξ into $J = 4$ regions and select the constructor points $\{[\xi_j; s_j]\}_{j \in [J]}$ by sampling uniformly at random from Ξ. We choose $B = 30$, $\xi_l = \mathbf{0}$, $\xi_u = 10\mathbf{e}$, $s_l = 8\mathbf{e}$, $s_u = 12\mathbf{e}$. For constructing the ambiguity set, we use $\mu = [\mu_\xi; \mu_s] \in \mathbb{R}^{2N}$, where $\mu_s = 10\mathbf{e}$, and every element of μ_ξ is generated uniformly from the interval $[0, 2]$. We select a random correlation matrix $C \in \mathbb{S}_+^{2N}$ with the MATLAB command "`|gallery('randcorr',2*N)|`" and set $\Sigma = \text{Diag}(\sigma)\, C\, \text{Diag}(\sigma) + \mu\mu^\top$, where $\sigma = [\sigma_\xi; \sigma_s] \in \mathbb{R}^{2N}$, $\sigma_s = \mathbf{e}/2$, and $\sigma_\xi = \mu_\xi/4$.

We approximate (27) using our proposed SDP (25), where the ellipsoids $\mathcal{E}(A_j, b_j)$ are generated using the SDP (12) developed in Section 3. We refer to this approach here as PWL. We compare the solution time and quality of the PWL approach with those of the following schemes.

- PWS (Bertsimas and Dunning 2016). Here, the second-stage decision variables are restricted to be constant within each partition (i.e., $Y_j = 0$ in Proposition 3). This approach leads to a tractable approximation and to the best of our knowledge, is the state of the art for solving DRO problems with random recourse.
- LDRs. This is similar to PWL except we do not partition the support set (i.e., $J = 1$). We compare against LDR to demonstrate the advantage of partitioning the support set.
- Ellipsoids of double radius (PWL-2). To demonstrate the importance of the size of the ellipsoid, we present comparisons against the scheme similar to PWL, except we double the radii of the ellipsoids $\mathcal{E}(A_j, b_j)$ used in PWL.

We perform the experiment on 100 randomly generated instances and present the relative objective gaps in Table 4. We also report the average solution times in Table 5. We assume that we can parallelize the task of generating the ellipsoids for each partition on four machines. Because we consider $J = 4$, for the solution time of the PWL approach, we choose the maximum among the solution times to find the four ellipsoids and add that to the solution time of solving the SDP (25).

The results indicate that we outperform the other methods in terms of the quality of the approximation. We observe that neglecting the linear term in the decision rules (i.e., using static decision rules) can lead to 75% increase in the objective value. Thus, although static decision rules lead to a tractable formulation that requires less computational time, they also generate significantly worse solutions. Furthermore, not partitioning the support set can lead to 24% higher objective values. Finally, doubling the radii of the bounding ellipsoids can increase the objective by 47%. For two-stage DRO models with random recourse, these results exhibit the importance of (i) using piecewise *linear* (PWL) instead of piecewise static decision rules, (ii) partitioning the support set, and (iii) having good ellipsoidal approximations to the partitions of the support set. The improvement in solution quality comes at the expense of increased computational time. However, if one is willing to spend computational resources, significant improvement in the solution quality can be achieved by using our method.

Table 4. Inventory Management: Objective Gaps of Other Models Relative to the PWL Model

Statistic	PWS, %	LDR, %	PWL-2, %
Mean	75.1	24.5	47.4
10th percentile	33.3	1.23	25.6
90th percentile	130	49.4	71.4

Table 5. Inventory Management: Average Solution Times of the Models (in Milliseconds)

Statistic	PWL	PWS	LDR	PWL-2
Solution time (ms)	622	91.8	219	617

6. Conclusions

In this article, we propose a GC reformulation for the minimum volume ellipsoid problem. We use that reformulation to generate tractable approximations when the set is defined by affine and quadratic inequalities. We prove that the volume of the ellipsoids that our approach provides never exceeds the volume of $\mathcal{E}_{\text{smvie}}$. Furthermore, we demonstrate empirically that our method performs better than the other competing schemes for providing approximate solutions to the MVEP, in terms of solution time and quality. Finally, we use our method to efficiently generate high-quality approximations in the context of distributional robust optimization and linear dynamical systems.

The work presented in this paper leaves room for further investigation. First, it would be interesting to study the suboptimality bounds of the radii of the ellipsoids generated by our method. In particular, for $\mathcal{E}_{\text{smvie}}$, it is known that $\text{Radius}(\mathcal{E}_{\text{smvie}}) \le K \cdot \text{Radius}(\mathcal{E}_{\text{mve}})$. It would be interesting to see if a better upper bound can be proved for the radius of $\mathcal{E}_{\text{sdp}}$. A second possible direction is to utilize the GC reformulation to generate approximation for other types of sets. Studying such approximations would add to the entire copositive programming literature and not only to the minimum volume ellipsoid problem.

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Appendix A. $\mathcal{S}$ Procedure

In this section, we discuss the $\mathcal{S}$ procedure (Boyd et al. 1994, Ben-Tal and Nemirovski 2001) and its use in approximating ($\mathcal{MVE}$).

Lemma A.1 ($\mathcal{S}$ Procedure).
Let $Q_i \in \mathbb{S}^K, q_i \in \mathbb{R}^K, r_i \in \mathbb{R}, i \in \{0\} \cup [I]$. Then, the optimal value of the nonconvex quadratic optimization problem

$$\begin{aligned} &\text{minimize} && x^\top Q_0 x + 2q_0^\top x + r_0 \\ &\text{subject to} && x \in \mathbb{R}^K, \\ &&& x^\top Q_i x + 2q_i^\top x + r_i \le 0 \quad \forall i \in [I] \end{aligned} \tag{A.1}$$

is ≥ 0 if there exist $\lambda_i \ge 0 \;\; \forall i \in [I]$ such that

$$\begin{bmatrix} Q_0 & q_0 \\ q_0^\top & r_0 \end{bmatrix} + \sum_{i\in[I]} \lambda_i \begin{bmatrix} Q_i & q_i \\ q_i^\top & r_i \end{bmatrix} \succcurlyeq \mathbf{0}. \tag{A.2}$$

The $\mathcal{S}$ procedure has been used in literature to provide sufficient conditions that certify that the optimal value of a nonconvex quadratic problem is nonnegative (Boyd et al. 1994, Ben-Tal et al. 2002, Hanasusanto and Kuhn 2018). In the following remark, we discuss a special case when we only have linear inequalities in the Optimization Problem (A.1).

Remark A.1. In the case when all the constraints are linear (i.e., $Q_i = 0, i \in [I]$), the semidefinite Constraint (A.2) reduces to

$$\begin{bmatrix} Q_0 & q_0 + \sum_{i\in[I]} \lambda_i q_i \\ q_0^\top + \sum_{i\in[I]} \lambda_i q_i & r_0 + \sum_{i\in[I]} \lambda_i r_i \end{bmatrix} \succcurlyeq \mathbf{0},$$

which implies that $Q_0 \succcurlyeq \mathbf{0}$. Therefore, if Q_0 is not positive semidefinite, then the sufficient conditions are never feasible; hence, they do not provide any certification on the optimal value of (A.1). We can overcome this limitation by adding a redundant quadratic constraint $\|Ax + b\|^2 \le 1$ to the Original Problem (A.1). Doing so does not change the optimal value of (A.1), but the sufficient Conditions (A.2) can now be written as

$$\begin{bmatrix} Q_0 & q_0 + \sum_{i\in[I]} \lambda_i q_i \\ q_0^\top + \sum_{i\in[I]} \lambda_i q_i & r_0 + \sum_{i\in[I]} \lambda_i r_i \end{bmatrix} + \mu \begin{bmatrix} A^2 & Ab \\ b^\top A & b^\top b \end{bmatrix} \succcurlyeq \mathbf{0}.$$

Because of the additional variable μ, the conditions become more flexible and might be feasible even if Q_0 fails to be positive semidefinite.

Next, we use the $\mathcal{S}$ procedure to derive an approximation to ($\mathcal{MVE}$). The constraint $Z(A, b) \le 1$ can be written as

$$\inf_{x\in\mathcal{P}} \{-x^\top A^2 x - 2b^\top Ax + 1 - b^\top b\} \ge 0.$$

Using Lemma A.1 and the definition of $\mathcal{P}$ from (19), the inequality is satisfied if there exist variables $\mu \in \mathbb{R}^J_+$ and $\lambda_i \ge 0 \;\; \forall i \in [I]$ such that

$$-\begin{bmatrix} A^2 & Ab \\ b^\top A & b^\top b - 1 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \frac{1}{2} S^\top \mu \\ \frac{1}{2}\mu^\top S & -\mu^\top t \end{bmatrix} + \sum_{i=1}^{I} \lambda_i \begin{bmatrix} Q_i^2 & Q_i q_i \\ q_i^\top Q_i & q_i^\top q_i - 1 \end{bmatrix} \succcurlyeq \mathbf{0},$$

which—using the Schur complement—is satisfied if and only if

$$\begin{bmatrix} \mathbf{0} & \frac{1}{2} S^\top \mu & A \\ \frac{1}{2}\mu^\top S & 1 - \mu^\top t & b^\top \\ A & b & \mathbb{I} \end{bmatrix} + \sum_{i=1}^{I} \lambda_i \begin{bmatrix} Q_i^2 & Q_i q_i & 0 \\ q_i^\top Q_i & q_i^\top q_i - 1 & 0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \succcurlyeq \mathbf{0}. \tag{A.3}$$

Hence, by replacing the constraint $Z(A, b) \le 1$ in ($\mathcal{MVE}$) with a stronger Constraint (A.3), we get the following conservative approximation of ($\mathcal{MVE}$):

$$\begin{aligned} &\text{minimize} && -\log\det(A) \\ &\text{subject to} && A \in \mathbb{S}^K, b \in \mathbb{R}^K, \mu \in \mathbb{R}^J_+, \lambda_i \in \mathbb{R}_+ \;\; \forall i \in [I], \\ &&& (30)\,\text{holds.} \end{aligned}$$

Appendix B. The Containment Approach of Kellner et al. (2013)

In Kellner et al. (2013), the authors provide the following sufficient conditions such that a set representable as a linear matrix inequality contains another such set.

Theorem B.1 (Kellner et al. 2013, theorem 4.3).
Let the set $S_Y = \{x \in \mathbb{R}^K : Y_0 + \sum_{k\in[K]} x_k Y_k \succcurlyeq 0\}$ *and the set* $S_Z = \{x \in \mathbb{R}^K : Z_0 + \sum_{k\in[K]} x_k Z_k \succcurlyeq 0\}$, *where* $Y_k = (Y_{ij}^k) \in \mathbb{S}^J$ *and* $Z_k \in \mathbb{S}^L$ *for all* $k \in \{0\} \cup [K]$. *Then,* $S_Y \subseteq S_Z$ *if there exist matrices* $C_{ij} \in \mathbb{R}^{L\times L}, i,j \in [J]$, *such that the following constraints hold*:

$$C = (C_{ij})_{i,j=1}^J \succcurlyeq 0, \qquad Z_0 \succcurlyeq \sum_{i,j=1}^J Y_{ij}^0 C_{ij}, \qquad Z_k = \sum_{i,j=1}^J Y_{ij}^k C_{ij} \quad \forall k \in [K]. \tag{B.1}$$

We summarize how we use this result to generate an approximation to $\mathcal{E}_{\text{mve}}$. We are interested in finding conditions under which a polytope $\mathcal{P} := \{x \in \mathbb{R}^K : Sx \le t\} = \{x \in \mathbb{R}^K : \text{Diag}(t - Sx) \succcurlyeq 0\}$ is contained in an ellipsoid $\mathcal{E}(A,b) = \{x \in \mathbb{R}^K : \|Ax + b\|^2 \le 1\} = \{x \in \mathbb{R}^K : F(x) \succcurlyeq 0\}$, where

$$F(x) = \begin{bmatrix} \mathbb{I} & Ax+b \\ (Ax+b)^\top & 1 \end{bmatrix} = \begin{bmatrix} \mathbb{I} & b \\ b^\top & 1 \end{bmatrix} + \sum_{k=1}^K x_k \begin{bmatrix} 0 & A_{k:} \\ A_{k:}^\top & 0 \end{bmatrix}.$$

Now, we can use Theorem B.1 with $S_Y = \mathcal{P}$ and $S_Z = \mathcal{E}(A,b)$ to generate constraints that ensure that $\mathcal{E}(A,b)$ contains $\mathcal{P}$. Because the matrices $Y_0 = \text{Diag}(t)$ and $Y_i = -\text{Diag}(S_i)$ are diagonal, the variables $C_{jk}, j \ne k$ do not appear in the second and third constraints of (B.1). Therefore, we can eliminate these variables from the first constraint as well, by forcing $C_{jj} \succcurlyeq 0$. In light of this observation and by redefining C_{jj} as C_j, we can rewrite the Constraints (B.1) as

$$C_j \in \mathbb{S}_+^{K+1} \quad \forall j \in [J], \qquad \begin{bmatrix} \mathbb{I} & b \\ b^\top & 1 \end{bmatrix} \succcurlyeq \sum_{j\in[J]} t_j C_j, \qquad \begin{bmatrix} 0 & A_{k:} \\ A_{k:}^\top & 0 \end{bmatrix} = \sum_{j\in[J]} -S_{jk} C_j \quad \forall k \in [K]. \tag{B.2}$$

Minimizing $-\log\det(A)$ subject to the constraints in (B.2) provides a conservative SDP approximation to ($\mathcal{MVE}$). The elimination of these redundant variables leads to a tremendous increase in the solution speed.

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