L_1-Robust Interval Observer Design for Uncertain Nonlinear Dynamical Systems

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***Abstract*—This letter presents a novel interval observer design for uncertain locally Lipschitz continuous-time (CT) and discrete-time (DT) systems with noisy nonlinear observations that is input-to-state stable (ISS) and minimizes the L_1-gain of the observer error system with respect to the uncertainties. Using mixed-monotone decompositions, the proposed observer is correct and positive by construction without the need for additional constraints/assumptions. This, in turn, allows us to directly leverage techniques for positive systems to design an ISS and L_1-robust interval observer via mixed-integer (linear) programs instead of semi-definite programs with linear matrix inequalities. Further, our observer design offers additional degrees of freedom that may serve as a surrogate for coordinate transformations. Finally, we demonstrate the effectiveness of our proposed observer on some CT and DT systems.**

***Index Terms*—Observers for nonlinear systems, uncertain systems.**

I. Introduction

STATE estimation is an important problem in most engineering applications such as autonomous vehicles and power systems, either for the purpose of monitoring or for decision-making and control. One such estimator design for uncertain systems is called interval observers that provide interval-valued estimates of the states, which are particularly useful when system and sensing uncertainties are set-valued or non-stochastic, or when their distributions are unknown.

Literature Review: An extensive body of research literature is available on the design of set-valued/interval observers for diverse classes of systems, e.g., linear, nonlinear, cooperative/monotone, mixed-monotone and Metzler dynamics [1]–[6]. The core idea in most of the proposed interval observers in that literature is to design appropriate observer gains such that the observer error dynamics are both Schur/Hurwitz stable and positive/cooperative. Taking all these constraints into consideration usually leads to theoretical and computational complexities. For specific classes of systems, such difficulties have been resolved by leveraging interval arithmetic-based approaches [7], by transforming it into a positive system [8] and/or by applying (time-varying/invariant) state transformations, e.g., [3], prior to designing the interval observer.

On the other hand, for more general classes of nonlinear systems, various types of bounding mappings/decomposition functions [9] have been leveraged to cast the observer design problem into semi-definite programs (SDPs)/optimization problems with LMI constraints [1], [3], [10]–[13]. However, the obtained LMIs might still be restrictive and solutions may not exist for some systems, i.e., the LMIs might be infeasible for some systems, due to several imposed conditions and upper bounding. To tackle this problem, coordinate transformations have been proposed to relax the design conditions and to facilitate obtaining feasible observer gains [14], [15]. However, unfortunately, the coordinate transformation and observer gains cannot be simultaneously synthesized/designed.

More recently, in the context of uncertain systems, other design criteria such as robustness of the estimates against noise/uncertainty have also been taken into consideration. In this regard, the design problem can often be translated to solving SDPs subject to a combination of framer satisfaction, stabilizability and noise attenuation/mitigation constraints, which could potentially lead to conservatism and computational complexity [3], [8], [11], including [2] that considers L_1/L_2 performance for continuous-time linear parameter varying (LPV) systems and our previous work in [16] that minimizes the L_2-gain of the observer error system using an $\mathcal{H}_\infty$-optimal observer. In this letter, we propose a novel interval observer design that minimizes the L_1-gain of the error system for nonlinear systems using a mixed-integer (linear) optimization framework.

Contributions: Inspired by L_1-norm minimization techniques for stabilizing positive systems [17] and leveraging *remainder-form mixed-monotone decomposition functions*, we synthesize L_1-robust and input-to-state stable (ISS) interval observers for a very broad range of locally Lipschitz bounded-error nonlinear CT and DT systems in a unified framework. Moreover, we introduce additional degrees of freedom in the design procedure, as a surrogate to coordinate transformations, where the matrices corresponding to the extra degrees of

Manuscript received 21 March 2022; revised 30 May 2022; accepted 7 June 2022. Date of publication 22 June 2022; date of current version 5 July 2022. This work was supported in part by NSF under Grant CNS-1932066 and Grant CNS-1943545. Recommended by Senior Editor C. Prieur. *(Corresponding author: Sze Zheng Yong.)*

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Digital Object Identifier 10.1109/LCSYS.2022.3185555

freedom can be synthesized/designed simultaneously with the observer gains, as opposed to most designs in the literature. In addition, we show that by construction, i.e., without imposing any additional assumption, the observer error system is positive and the true state trajectory of the system is guaranteed to be framed by the states of the proposed observer for all possible realizations of interval-valued noise/disturbances and initial states. Further, we show that the stability of the framers can be guaranteed by solving mixed-integer (linear) programs to minimize the L_1-norm of the observer error system, which can lead to less conservative framers when compared to SDP-based methods, since our approach does not involve extra positivity constraints while SDP-based methods often require them.

II. Preliminaries

Notation: $\mathbb{R}^n$, $\mathbb{R}^n_{>0}$, $\mathbb{R}^{n\times p}$, $\mathbb{N}_n$ and $\mathbb{N}$ denote the n-dimensional Euclidean space, positive vectors of size n, matrices of size n by p, natural numbers up to n and natural numbers, respectively. For a vector $v \in \mathbb{R}^n$, its vector p-norm is given by $\|v\|_p \triangleq (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$, while for a matrix $M \in \mathbb{R}^{n\times p}$, M_{ij} represents its j-th column and i-th row entry, $\mathrm{sgn}(M)$ represents its element-wise signum function, $M^{\oplus} \triangleq \max(M, \mathbf{0}_{n\times p})$, $M^{\ominus} \triangleq M^{\oplus} - M$, and $|M| \triangleq M^{\oplus} + M^{\ominus}$ is its element-wise absolute value. Moreover, M^{d} is a diagonal matrix with the diagonal elements of a square matrix $M \in \mathbb{R}^{n\times n}$, $M^{\mathrm{nd}} \triangleq M - M^{\mathrm{d}}$ is the matrix with only its off-diagonal elements, and $M^{\mathrm{m}} \triangleq M^{\mathrm{d}} + |M^{\mathrm{nd}}|$ is the "Metzlerized" matrix.[1] Further, all vector and matrix inequalities are element-wise inequalities, and the matrices of zeros and ones of dimension $n \times p$ are denoted as $\mathbf{0}_{n\times p}$ and $\mathbf{1}_{n\times p}$, respectively. Finally, a function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, positive definite (i.e., $\alpha(x) = 0$ for $x = 0$; $\alpha(x) > 0$ otherwise), and strictly increasing, and is of class $\mathcal{K}_\infty$ if it is also unbounded, while $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\lambda(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\lambda(s, t)$ decreases to zero as $t \to \infty$.

Definition 1 (Interval): An (n-dimensional) interval $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^n$ is the set of vectors $z \in \mathbb{R}^{n_z}$ such that $\underline{z} \leq z \leq \overline{z}$. A similar definition applies to matrix intervals.

Definition 2 (Jacobian Sign-Stability): A vector-valued function $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ is Jacobian sign-stable (JSS), if in its domain $\mathcal{Z}$, the entries of its Jacobian matrix do not change signs, i.e., if one of the following hold:

$$J^f_{ij}(z) \geq 0 \quad \text{or } J^f_{ij}(z) \leq 0$$

for all $z \in \mathcal{Z}, \forall i \in \mathbb{N}_p, \forall j \in \mathbb{N}_{n_z}$, where $J^f(z)$ represents the Jacobian matrix of the mapping f evaluated at $z \in \mathcal{Z}$.

Proposition 1 (Jacobian Sign-Stable Decomposition [13, Proposition 2]): For a mapping $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$, if $J^f(z) \in [\underline{J}^f, \overline{J}^f]$ for all $z \in \mathcal{Z}$,, where $\underline{J}^f, \overline{J}^f \in \mathbb{R}^{p\times n_z}$ are known matrices, then the function f can be decomposed to a JSS mapping $\mu(\cdot)$ and a (remainder) affine mapping Hz (that is also JSS), in an additive *remainder-form*:

$$\forall z \in \mathcal{Z}, f(z) = Hz + \mu(z), \tag{1}$$

where the matrix $H \in \mathbb{R}^{p\times n_z}$, satisfies

$$\forall (i, j) \in \mathbb{N}_p \times \mathbb{N}_{n_z}, H_{ij} = \underline{J}^f_{ij} \ \vee H_{ij} = \overline{J}^f_{i,j}. \tag{2}$$

[1]A Metzler matrix is a square matrix in which all the off-diagonal components are nonnegative (equal to or greater than zero).

Definition 3 (Mixed-Monotonicity and Decomposition Functions [18, Definition 1], [19, Definition 4]): Consider the uncertain dynamical system with initial state $x_0 \in \mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}^n$ and process noise $w_t \in \mathcal{W} \triangleq [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$:

$$x_t^+ = g(z_t) \triangleq g(x_t, w_t), z_t \triangleq [x_t^\top \ w_t]^\top, \tag{3}$$

where $x_t^+ \triangleq x_{t+1}$ if (3) is a DT system and $x_t^+ \triangleq \dot{x}_t$ if (3) is a CT system. Moreover, $g : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^n$ is the vector field with augmented state $z_t \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{W} \subset \mathbb{R}^{n_z}$ as its domain, where $\mathcal{X}$ is the entire state space and $n_z = n + n_w$.

Suppose (3) is a DT system. Then, a mapping $g_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^p$ is a DT mixed-monotone decomposition function with respect to g, if i) $g_d(z, z) = g(z)$, ii) g_d is monotone increasing in its first argument, i.e., $\hat{z} \geq z \Rightarrow g_d(\hat{z}, z') \geq g_d(z, z')$, and iii) g_d is monotone decreasing in its second argument, i.e., $\hat{z} \geq z \Rightarrow g_d(z', \hat{z}) \leq g_d(z', z)$.

On the other hand, if (3) is a CT system, a mapping $g_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^p$ is a CT mixed-monotone decomposition function with respect to g, if i) and iii) for the DT system hold, while ii) is modified to the following: ii') g_d is monotone increasing in its first argument with respect to "off-diagonal" arguments, i.e., $\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_{n_z} \wedge (i \neq j), \hat{z}_j \geq z_j, \hat{z}_i = z_i \Rightarrow g_{d,i}(\hat{z}, z') \geq g_{d,i}(z, z')$.

Proposition 2 (Tight and Tractable Decomposition Functions for JSS Mappings [13, Proposition 4]): Let $\mu : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ be a JSS mapping on its domain. Then, it admits a tight decomposition function for each $\mu_i(\cdot)$, $i \in \mathbb{N}_p$ as follows:

$$\mu_{d,i}(z_1, z_2) = \mu_i(D^i z_1 + (I_{n_z} - D^i) z_2), \tag{4}$$

for any ordered $z_1, z_2 \in \mathcal{Z}$, where D^i is a binary diagonal matrix determined by which vertex of the interval $[z_2, z_1]$ or $[z_1, z_2]$ that maximizes (if $z_2 \leq z_1$) or minimizes (if $z_2 > z_1$) the function $\mu_i(\cdot)$ that can be found in closed-form as:

$$D^i = \mathrm{diag}(\max(\mathrm{sgn}(\overline{J}^{\mu}_i), \mathbf{0}_{1,n_z})). \tag{5}$$

Consequently, a tight and tractable remainder-form decomposition function for the system in (3) can be found by applying Proposition 2 to the Jacobian sign-stable decomposition from Proposition 1, which is discussed in more detail in [13].

Definition 4 (Embedding System): For an n-dimensional system (3) with any decomposition function $g_d(\cdot)$, its *embedding system is defined as the following* $2n$-dimensional dynamical system with initial condition $\begin{bmatrix}\overline{x}_0^\top & \underline{x}_0^\top\end{bmatrix}^\top$:

$$\begin{bmatrix}\underline{x}_t^+ \\ \overline{x}_t^+\end{bmatrix} = \begin{bmatrix}\underline{g}_d(\begin{bmatrix}(\underline{x}_t)^\top & \underline{w}^\top\end{bmatrix}^\top, \begin{bmatrix}(\overline{x}_t)^\top & \overline{w}^\top\end{bmatrix}^\top) \\ \overline{g}_d(\begin{bmatrix}(\overline{x}_t)^\top & \overline{w}^\top\end{bmatrix}^\top, \begin{bmatrix}(\underline{x}_t)^\top & \underline{w}^\top\end{bmatrix}^\top)\end{bmatrix}. \tag{6}$$

Note that by [9, Proposition 3], the solution to an embedding system (6) with decomposition function g_d corresponding to the system dynamics in (3) has a *state framer property*, i.e., it is guaranteed to frame the unknown state trajectory x_t of (3): $\underline{x}_t \leq x_t \leq \overline{x}_t$ for all $t \in \mathbb{T}$.

III. Problem Formulation

System Dynamics: The uncertain/noisy discrete-time (DT) or continuous-time (CT) nonlinear systems considered in this letter is given as:

$$\mathcal{G} : \begin{cases} x_t^+ = \hat{f}(x_t, u_t) + W w_t \triangleq f(x_t) + W w_t, \\ y_t = \hat{h}(x_t, u_t) + V v_t \triangleq h(x_t) + V v_t, \end{cases} \tag{7}$$

for all $t \in \mathbb{T}$, where $x_t^+ = x_{t+1}, \mathbb{T} = \{0\} \cup \mathbb{N}$ if $\mathcal{G}$ is a DT system and $x_t^+ = \dot{x}_t, \mathbb{T} = \mathbb{R}_{\geq 0}$, if $\mathcal{G}$ is a CT system. Moreover, $x_t \in \mathcal{X} \subset \mathbb{R}^n$, $w_t \in \mathcal{W} \triangleq [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$, $v_t \in \mathcal{V} \triangleq [\underline{v}, \overline{v}] \subset \mathbb{R}^{n_v}$, $u_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are state, process noise, measurement noise, known control input and output measurement signals, respectively. $\hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the nonlinear state vector field and $\hat{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ denotes observation/constraint functions, from which the mappings/functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are well-defined, since the input signal $u_t \in \mathbb{R}^m$ is known. Moreover, the noise matrices W and V are known. Our goal is to estimate the state trajectories of the plant $\mathcal{G}$ in (7), when the initial state satisfies $x_0 \in \mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathcal{X}$. Further, we assume the following:

Assumption 1: The disturbance/noise bounds $\underline{w}$, $\overline{w}$, $\underline{v}$, and $\overline{v}$] as well as the signals y_t (output) and u_t (input, if any) are known at all times. Moreover, the initial state x_0 is such that $x_0 \in \mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$ with known bounds $\underline{x}_0$ and $\overline{x}_0$.

Assumption 2: The mappings/functions $f(\cdot)$ and $h(\cdot)$ are known, locally Lipschitz, differentiable and mixed-monotone in their domain.[2] Moreover, the lower and upper bounds of their Jacobian matrices, $\underline{J}^f, \overline{J}^f \in \mathbb{R}^{n \times n_z}$ and $\underline{J}^h, \overline{J}^h \in \mathbb{R}^{l \times n_\zeta}$ are known, where $n_z = n + n_w$ and $n_\zeta = n + n_v$.

Next, the notions of *framers*, *stability* and *correctness* used throughout this letter are formally defined.

Definition 5 (Correct Interval Framers and Framer Errors): Let Assumptions 1–2 hold. Given the nonlinear plant $\mathcal{G}$ in (7), $\overline{x}, \underline{x} : \mathbb{T} \to \mathbb{R}^n$ are called upper and lower framers for the system states, if $\underline{x}_t \leq x_t \leq \overline{x}_t, \ \forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}$. Further, $\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$ is called the *framer error* at time t. Any dynamical system whose states are correct framers for the system states of the plant $\mathcal{G}$, i.e., with $\varepsilon_t \geq 0, \forall t \in \mathbb{T}$, is called a *correct* interval framer for system (7).

Definition 6 (L_1-Robust Interval Observer): An interval framer $\hat{\mathcal{G}}$ is L_1-robust and optimal, if the L_1-gain of the framer error system $\tilde{\mathcal{G}}$, defined below, is minimized:

$$\|\tilde{\mathcal{G}}\|_{L_1} \triangleq \sup_{\|\Delta\|_{L_1}=1} \|\varepsilon\|_{L_1}, \tag{8}$$

where $\|v\|_{L_1} \triangleq \int_0^\infty \|v_t\|_1 dt$ is the L_1 signal norm for $v \in \{\varepsilon, \Delta\}$, and ε_t and $\Delta_t = \Delta \triangleq \begin{bmatrix} \Delta w^\top & \Delta v^\top \end{bmatrix}^\top$ are the framer error and combined noise signals, respectively, with $\Delta w \triangleq \overline{w} - \underline{w}$ and $\Delta v \triangleq \overline{v} - \underline{v}$.

The observer design problem can be stated as follows:

Problem 1: Given the nonlinear system in (7), as well as Assumptions 1–2, design a correct and L_1-robust interval observer (cf. Definition 6) whose framer error (cf. Definition 5) is input-to-state stable (ISS),[3] i.e.,

$$\|\varepsilon_t\|_2 \leq \beta(\|\varepsilon_0\|_2, t) + \rho(\|\Delta\|_{L_\infty}), \forall t \in \mathbb{T}, \tag{9}$$

where β and ρ are functions of classes $\mathcal{KL}$ and $\mathcal{K}_\infty$, respectively, with the L_∞ signal norm $\|\Delta\|_{L_\infty} = \sup_{s \in [0,\infty)} \|\Delta_s\|_2 = \|\Delta\|_2$ and Δ_s defined as in (8).

[2]Both assumptions of locally Lipschitz continuity and differentiability are primarily for ease of exposition and can be relaxed to a much weaker continuity assumption (cf. [9] for more details).

[3]If desired, we can replace the 2-norm in the original ISS definition with the 1-norm that is more aligned with L_1-robustness, and the satisfaction of the former would also imply the latter by norm equivalence.

IV. Proposed Interval Observer

A. Interval Observer Design

Before presenting the proposed interval observer design, we first provide an equivalent representation of the system dynamics for the plant $\mathcal{G}$ in (7).

Lemma 1: Consider plant $\mathcal{G}$ in (7) and suppose that Assumptions 1–2 hold. Let $L, N \in \mathbb{R}^{n \times l}$ and $T \in \mathbb{R}^{n \times n}$ be arbitrary matrices that satisfy $T + NC = I_n$. Then, the system dynamics (7) can be equivalently written as

$$\begin{aligned} \xi_t^+ &= (TA - LC - NA_2)x_t + T\phi(x_t) - N\rho(x_t, w_t) \\ &\quad + (TW - NB_2)w_t - L\psi(x_t) + L(y_t - Vv_t), \\ x_t &= \xi_t + Ny_t - NVv_t, \end{aligned} \tag{10}$$

where $A \in \mathbb{R}^{n \times n}$, $C, A_2 \in \mathbb{R}^{l \times n}$, and $B_2 \in \mathbb{R}^{l \times n_w}$ are chosen such that the following decompositions hold $\forall x \in \mathcal{X}$, $w \in \mathcal{W}$ (cf. Definition 2 and Proposition 1):

$$\begin{aligned} f(x) = Ax + \phi(x), \ \ h(x) = Cx + \psi(x), \\ \psi^+(x, w) = A_2 x + B_2 w + \rho(x, w), \end{aligned} \tag{11}$$

such that ϕ, ψ, ρ are JSS, with $\psi^+(x, w) = \dot{\psi}(x, w) = \frac{\partial \psi}{\partial x}(f(x) + Ww)$ if $\mathcal{G}$ is a CT system and $\psi^+(x, w) = \psi(x^+) = \psi(f(x) + Ww)$ if $\mathcal{G}$ is a DT system.

Proof: We begin by defining an auxiliary state $\xi_t \triangleq x_t - Ny_t + NVv_k$. Then, from (7) and (11), we have $\xi_t = x_t - N(y_t - Vv_k) = x_t - N(Cx_t + \psi(x_t))$, and moreover, by choosing N to satisfy $T + NC = I_n$, we obtain $\xi_t = Tx_t - N\psi(x_t)$ that has the following dynamics:

$$\begin{aligned} \xi_t^+ &= Tx_t^+ - NC\psi^+(x_t, w_t) \\ &= T(Ax_t + \phi(x_t) + Ww_t) - N(A_2 x_t + B_2 w_t + \rho(x_t, w_t)), \end{aligned}$$

with x_t^+ from (7) and $f(x_t)$ and $\psi^+(x_t, w_t)$ from (11). Finally, adding a 'zero term' $L(y_t - Cx_t - \psi(x_t) - Vv_t) = 0$ (cf. (7) and (11)) to the above yields (10), where x_t can be recovered from the definition of ξ_t. ■

Then, using the equivalent system in (10), we propose a unified interval observer $\hat{\mathcal{G}}$ based on the construction of an embedding system (cf. Definition 4) to address Problem 1:

$$\begin{aligned} \underline{\xi}_t^+ &= M^\uparrow \underline{x}_t - M^\downarrow \, \overline{x}_t + Ly_t + T^\oplus \phi_d(\underline{x}_t, \overline{x}_t) - T^\ominus \phi_d(\overline{x}_t, \underline{x}_t) \\ &\quad + (LV)^\oplus \overline{v} + (LV)^\ominus \underline{v} + (TW - NB_2)^\oplus \underline{w} \\ &\quad - (TW - NB_2)^\ominus \overline{w} - L^\oplus \psi_d(\overline{x}_t, \underline{x}_t) + L^\ominus \psi_d(\underline{x}_t, \overline{x}_t) \\ &\quad - N^\oplus \rho_d(\overline{x}_t, \overline{w}, \underline{x}_t, \underline{w}) + N^\ominus \rho_d(\underline{x}_t, \underline{w}, \overline{x}_t, \overline{w}) \\ &\quad + MNy - (MNV)^\oplus \underline{v} + (MNV)^\ominus \overline{v}, \\ \overline{\xi}_t^+ &= M^\uparrow \overline{x}_t - M^\downarrow \, \underline{x}_t + Ly_t + T^\oplus \phi_d(\overline{x}_t, \underline{x}_t) - T^\ominus \phi_d(\underline{x}_t, \overline{x}_t) \\ &\quad + (LV)^\oplus \underline{v} + (LV)^\ominus \overline{v} + (TW - NB_2)^\oplus \overline{w} \\ &\quad - (TW - NB_2)^\ominus \underline{w} - L^\oplus \psi_d(\underline{x}_t, \overline{x}_t) + L^\ominus \psi_d(\overline{x}_t, \underline{x}_t) \\ &\quad - N^\oplus \rho_d(\underline{x}_t, \underline{w}, \overline{x}_t, \overline{w}) + N^\ominus \rho_d(\overline{x}_t, \overline{w}, \underline{x}_t, \underline{w}) \\ &\quad + MNy - (MNV)^\oplus \overline{v} + (MNV)^\ominus \underline{v}, \\ \underline{x}_t &= \underline{\xi}_t + Ny_t - (NV)^\oplus \overline{v} + (NV)^\ominus \underline{v}, \\ \overline{x}_t &= \overline{\xi}_t + Ny_t - (NV)^\oplus \underline{v} + (NV)^\ominus \overline{v}, \end{aligned} \tag{12}$$

where $\overline{\xi}_t, \underline{\xi}_t, \overline{\xi}_t^+, \underline{\xi}_t^+ \in \mathbb{R}^n$ are auxiliary variables, $M \triangleq TA - LC - NA_2$ and if $\mathcal{G}$ is a CT system, then

$$\overline{x}_t^+ \triangleq \dot{\overline{x}}_t, \ \underline{x}_t^+ \triangleq \dot{\underline{x}}_t, \ M^\uparrow \triangleq M^{\mathrm{d}} + M^{\mathrm{nd},\oplus}, \ M^\downarrow \triangleq M^{\mathrm{nd},\ominus}, \tag{13}$$

and if $\mathcal{G}$ is a DT system, then

$$\overline{x}_t^+ \triangleq \overline{x}_{t+1}, \ \underline{x}_t^+ \triangleq \underline{x}_{t+1}, \ M^\uparrow \triangleq M^\oplus, \ M^\downarrow \triangleq M^\ominus. \tag{14}$$

Further, $\phi_d, \psi_d : \mathbb{R}^{2n} \to \mathbb{R}^n$ and $\rho_d : \mathbb{R}^{2l} \to \mathbb{R}^l$ are tight mixed-monotone decomposition functions of ϕ, ψ, and ρ, respectively (cf. (11), Definition 3), which are JSS and thus, can be computed using (4) and (5). Finally, $N, L \in \mathbb{R}^{n\times l}$ and $T \in \mathbb{R}^{n\times n}$ are the observer gain matrices to be designed with T and N satisfying $T + NC = I_n$. The detailed derivation of the observer design $\hat{\mathcal{G}}$ in (12) and its desired properties are given in the next subsections.

Remark 1: Note that the above observer structure is inspired by [10] to introduce additional degrees of freedom, where we now have three to-be-designed observer gains N, T, L, in contrast to only one observer gain L in our previous work [16]. While this is very helpful from a performance perspective, it is found in Section V-A to be not an exact substitute for coordinate transformations that may help to make the observer gain design problem in Theorem 2 feasible, presumably because the latter are nonlinear operations and the proposed structure only introduces linear terms. In this case, a coordinate transformation can be applied in a straightforward manner, similar to [3, Sec. V] (omitted for brevity; cf. [20] and references therein for more discussions).

B. Observer Correctness (Framer Property)

In this subsection, we show that by construction, the proposed interval observer $\hat{\mathcal{G}}$ in (12) is a correct framer for the system $\mathcal{G}$ (and equivalently, for (10) in Lemma 1) in the sense of Definition 5 for both the CT (13) and DT (14) cases.

Theorem 1 (Correctness): Consider the nonlinear plant $\mathcal{G}$ in (7) and suppose Assumptions 1 and 2 hold. Then, $\underline{x}_t \le x_t \le \overline{x}_t, \forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}$, where x_t and $[\underline{x}_t^\top \overline{x}_t^\top]^\top$ are the state vectors in $\mathcal{G}$ and $\hat{\mathcal{G}}$ at time $t \in \mathbb{T}$, respectively. In other words, the dynamical system (12) constructs a correct interval framer for the nonlinear plant $\mathcal{G}$ in (7).

Proof: We show that by framing all the constituent items in the right hand side of system (10) (which is equivalent to the original system (7)), using Proposition 1 and [21, Lemma 1], we obtain (12). Hence, (12) is a framer system for the original plant $\mathcal{G}$, since it is straightforward to see that the summation of the embedding systems/framers of constituent systems constructs an embedding system for the summation of constituent systems. In order to compute framers for the constituent systems in the right hand side of (10), we split them into three groups, as follows. i) Known/certain terms that are independent of state and noise and so, their upper and lower bounds are equal to their original known value; ii) Linear terms with respect to the state and noise, which can be upper and lower framed by applying [21, Lemma 1]. Note that there is a subtle difference in computing the upper and lower framing/embedding systems for the CT and DT cases, which is reflected in the definition of $M^\uparrow, M^\downarrow$ in (13) and (14) (cf. Definition 4); iii) Nonlinear terms in the state and noise that can be upper and lower framed by leveraging Propositions 1 and 2 as well as [21, Lemma 1].

Summing up the constituent embedding systems/framers in i)-iii) yields the embedding system (12). Finally, the correctness property follows from the framer property of embedding systems [9, Proposition 3]. ∎

C. L_1-Robust Observer Design

In addition to the correctness property, it is important to guarantee the stability of the proposed framer, i.e., we aim to design the observer gains T, N, and L to obtain input-to-state stability (ISS) (cf. (9)) and L_1-robustness.

Theorem 2 (L_1-Robust and L_1-ISS Observer Design): If Assumptions 1–2 hold for the nonlinear system $\mathcal{G}$ in (7), then the correct interval framer $\hat{\mathcal{G}}$ proposed in (12) is L_1-robust (cf. Definition 6), if there exist $Q, \tilde{T} \in \mathbb{R}^{n\times n}$, $\tilde{N}, N, \tilde{L} \in \mathbb{R}^{n\times l}$, $p \in \mathbb{R}^n_{>0}$, and $\gamma > 0$ that solve the following mixed-integer program (MIP):

$$
\begin{aligned}
&(\gamma^*, p^*, Q^*, \tilde{T}^*, \tilde{L}^*, \tilde{N}^*, N^*) \\
&\quad \in \underset{\{\gamma,p,Q,\tilde{T},\tilde{L},\tilde{N},N\}}{\arg\min} \ \gamma \\
&\quad \text{s.t. } \mathbf{1}_{1\times n}\begin{bmatrix}\Omega & \Lambda & \Upsilon\end{bmatrix} < \begin{bmatrix}\sigma & \gamma\mathbf{1}_{1\times n_w} & \gamma\mathbf{1}_{1\times n_v}\end{bmatrix}, \\
&\qquad \tilde{T} + \tilde{N}C = Q,\ \tilde{N} = \Gamma,\ p > 0,\ \gamma > 0,
\end{aligned} \tag{15}
$$

where $Q \triangleq diag(p)$ denotes a matrix whose diagonal entries are the elements of p, $\Lambda \triangleq |\tilde{T}W - \tilde{N}B_2| + |\tilde{N}|\overline{F}^\rho_w$, and

(i) for a DT system $\mathcal{G}$: $\sigma \triangleq p^\top - \mathbf{1}_{1\times n}$, $\Omega \triangleq |M| + |\tilde{T}|\overline{F}^\phi_x + |\tilde{N}|\overline{F}^\rho_x + |\tilde{L}|\overline{F}^\psi_x$, $\Upsilon \triangleq |\tilde{L}V| + |\tilde{N}V|$, and $\Gamma \triangleq \tilde{N}$;

(ii) for a CT system $\mathcal{G}$: $\sigma \triangleq -\mathbf{1}_{1\times n}$, $\Omega \triangleq M^m + |\tilde{T}|\overline{F}^\phi_x + |\tilde{N}|\overline{F}^\rho_x + |\tilde{L}|\overline{F}^\psi_x$, $\Upsilon \triangleq |\tilde{L}V| + Z$, and $\Gamma \triangleq QN$ if $V \ne 0$ and $\Gamma \triangleq \tilde{N}$ otherwise,

with $M \triangleq TA - LC - NA_2$ and $Z \triangleq (|M| - M^m)|NV|$. Furthermore, in both cases, $\overline{F}^\phi_x, \overline{F}^\psi_x, \overline{F}^\rho_x, \overline{F}^\rho_w$ are computed from the JSS functions ϕ, ψ and ρ as follows:

$$
\overline{F}^\mu \triangleq [\overline{F}^\mu_x \ \overline{F}^\mu_w] \triangleq (\overline{J}^\mu)^\oplus + (\underline{J}^\mu)^\ominus, \tag{16}
$$

with $\overline{J}^\mu = \overline{J}^f - H$, $\underline{J}^\mu = \underline{J}^f - H$, $\overline{F}^\mu_x \in \mathbb{R}^{n\times n}$ and $\overline{F}^\mu_w \in \mathbb{R}^{n\times n_w}$ (cf. Proposition 1 and [16, Lemma 3]).

Then, the corresponding L_1-robust stabilizing observer gains T^*, L^*, and N^* can be obtained as $T^* = (Q^*)^{-1}\tilde{T}^*, L^* = (Q^*)^{-1}\tilde{L}^*$ and $N^* = (Q^*)^{-1}\tilde{N}^*$. Moreover, the interval observer is ISS, i.e., it satisfies (9).

Proof: We start by deriving the framer error ($\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$) from (12), before proving that the DT and CT error systems satisfy the condition in (8) and (9), i.e., $\hat{\mathcal{G}}$ is L_1-robust and ISS. To do this, we define $\Delta s \triangleq \overline{s} - \underline{s}$, $\Delta^\mu_d \triangleq \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s}) - \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s})$ for all $s \in \{w, v\}$ and $\mu \in \{\phi, \psi, \rho\}$.

First, from (12), the framer error is $\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t = \overline{\xi}_t - \underline{\xi}_t + |NV|\Delta v$. Then, the DT observer error dynamics $\tilde{\mathcal{G}}$ obtained from (12) and (14) can be written as:

$$
\begin{aligned}
\varepsilon_t^+ &= |TA - LC - NA_2|\varepsilon_t + |T|\Delta^\phi_d + |N|\Delta^\rho_d + |L|\Delta^\psi_d \\
&\quad + |TW - NB_2|\Delta w + (|LV| + |NV|)\Delta v + |MNV|\Delta v \\
&\le (|TA - LC - NA_2| + |T|\overline{F}^\phi_x + |N|\overline{F}^\rho_x + |L|\overline{F}^\psi_x)\varepsilon_t \\
&\quad + (|TW - NB_2| + |N|\overline{F}^\rho_w)\Delta w + (|LV| + |NV|)\Delta v,
\end{aligned} \tag{17}
$$

where the inequality holds since Δ^μ_d, $\mu \in \{\phi, \psi, \rho\}$ satisfy $\Delta^\mu_d \le \overline{F}^\mu_x\varepsilon_t + \overline{F}^\mu_w\Delta w$, with $\overline{F}^\mu_x, \overline{F}^\mu_w$ defined in (16) by [16, Lemma 3] and their pre-multiplier matrices $|\cdot|$ are non-negative. Further, by the *Comparison Lemma* [22, Lemma 3.4], the actual framer error system is stable if the comparison system on the right hand side in (17) is stable.

Defining $\tilde{A} \triangleq |TA - LC - NA_2| + |T|\overline{F}^\phi_x + |N|\overline{F}^\rho_x + |L|\overline{F}^\psi_x$, $\tilde{B} \triangleq \big[(|TW - NB_2| + |N|\overline{F}^\rho_w) \quad (|LV| + |NV|)\big] \ge 0$, $\tilde{C} \triangleq I_{n\times n} \ge 0$, and $\tilde{D} \triangleq \mathbf{0}_{n\times n} \ge 0$, the DT comparison system with $z_t = \varepsilon_t$ can be written as:

$$
\varepsilon_t^+ \le \tilde{A}\varepsilon_t + \tilde{B}[(\Delta w)^\top (\Delta v)^\top]^\top, z_t = \tilde{C}\varepsilon_t + \tilde{D}[(\Delta w)^\top (\Delta v)^\top]^\top. \tag{18}
$$

Since $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are non-negative, the error system in (18) is positive. Then, by [23, Th. 1], the comparison system (18) is asymptotically stable with $\gamma > 0$ as the L_1-gain (cf. Definition 8), if there exists $p \in \mathbb{R}^n_{>0}$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^\top \begin{bmatrix} p \\ \mathbf{1}_{n\times 1} \end{bmatrix} < \begin{bmatrix} p \\ \gamma \mathbf{1}_{(n_w+n_v)\times 1} \end{bmatrix}. \quad (19)$$

Next, by defining $Q = Q^\top \triangleq diag(p) > 0$, $\tilde{T} = QT$, $\tilde{N} = QN$, and $\tilde{L} = QL$, we have $p = Q\mathbf{1}_{n\times 1}$ and $\tilde{C}^\top \mathbf{1}_{n\times 1} = \mathbf{1}_{n\times 1}$. Further defining $\Omega \triangleq Q\tilde{A}$ and $\begin{bmatrix} \Lambda & \Upsilon \end{bmatrix} \triangleq Q\tilde{B}$, we obtain that (19) is equivalent to the inequality constraint in (15). Similarly, we can pre-multiply $T + NC = I_n$ by the invertible Q matrix to obtain the equality constraint in (15). Hence, by solving the MILP in (15) results in observer gains $T^* = (Q^*)^{-1}\tilde{T}^*$, $N^* = (Q^*)^{-1}\tilde{N}^*$ and $L^* = (Q^*)^{-1}\tilde{L}^*$ that result in a L_1-robust comparison system (18), i.e., it satisfies (8) with γ^*. Moreover, since the comparison system is linear, asymptotically stability also implies that it is ISS [24]. Consequently, by the Comparison Lemma [22, Lemma 3.4], the actual DT framer error system on the left hand side of (17) is also L_1-robust and ISS.

The CT case is similar to the DT case, where from (12) and (13), we obtain CT observer error dynamics $\tilde{\mathcal{G}}$:

$$\begin{aligned} \dot{\varepsilon}_t &= (TA - LC - NA_2)^m \varepsilon_t + |T|\Delta_d^\phi + |N|\Delta_d^\rho + |L|\Delta_d^\psi \\ &\quad + |TW - NB_2|\Delta w + |LV|\Delta v + |MNV|\Delta v \\ &\le ((TA - LC - NA_2)^m + |T|\overline{F}_x^\phi + |N|\overline{F}_x^\rho + |L|\overline{F}_x^\psi)\varepsilon_t \\ &\quad + (|TW - NB_2| + |N|\overline{F}_w^\rho)\Delta w + (|LV| + Z)\Delta v, \end{aligned}$$

with $Z \triangleq (|M| - M^m)|NV| \ge 0$. Further, defining $\tilde{A} \triangleq (TA - LC - NA_2)^m + |T|\overline{F}_x^\phi + |N|\overline{F}_x^\rho + |L|\overline{F}_x^\psi$, $\tilde{B} \triangleq \begin{bmatrix} (|TW - NB_2| + |N|\overline{F}_w^\rho) & (|LV| + Z) \end{bmatrix} \ge 0$, $\tilde{C} \triangleq I_{n\times n} \ge 0$, and $\tilde{D} \triangleq \mathbf{0}_{n\times n} \ge 0$, the CT comparison system with $z_t = \varepsilon_t$ is given by (18). Since $\tilde{A}$ is Metzler and $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are non-negative, the error comparison system is positive. Then, by [17, Lemma 1], the CT comparison system is asymptotically stable with L_1-gain, $\gamma > 0$, if there exists $p \in \mathbb{R}^n_{>0}$ such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^\top \begin{bmatrix} p \\ \mathbf{1}_{n\times 1} \end{bmatrix} < \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \gamma \mathbf{1}_{(n_w+n_v)\times 1} \end{bmatrix}. \quad (20)$$

By defining Q, $\tilde{T}$, $\tilde{N}$, and $\tilde{L}$ similar to the DT case, $\tilde{M} \triangleq QM$, and $\tilde{Z} \triangleq QZ$, the inequality in (20) is equivalent to the inequality condition (15). The equality constraint can also be obtained as in the DT case. Finally, a similar argument to the DT case implies that the CT interval observer is L_1-robust and is also ISS. ■

Note that in some cases, a coordinate transformation may help to make the MIP in (15) feasible (cf. Remark 1). Further, we found that it is often helpful to multiply Υ by a factor representing the ratio of the magnitudes of the measurement to process noise signals to penalize their effects equally.

Remark 2: In the CT case (only), the presence of the term $\tilde{Z} = (|\tilde{M}| - \tilde{M}^m)|NV|$ leads to bilinear constraints, but the MIP remains solvable with off-the-shelf solvers, e.g., Gurobi [25]. Nonetheless, in the absence of measurement noise, i.e., $V = 0$, or by choosing $N = 0$ (at the cost of losing the extra degrees of freedom with T and N), the MIP reduces to a mixed-integer linear program (MILP) similar to the DT case.

Remark 3: We have a mixed-integer problem in (15) due to the presence of terms[4] involving absolute values $|M|$ and "Metzlerization" $M^m = M^d + |M^{nd}|$. If desired, extra positivity constraints can be imposed (i.e., by setting $M \ge 0$, $M^{nd} \ge 0$ and replacing $|M|$, $|M^{nd}|$ with M, M^{nd}), similar to the literature on SDP/LMI-based interval observer designs, to obtain a linear program. This addition is found to sometimes not incur any conservatism (e.g., in the DT example in Section V-B) but the problem becomes infeasible in others (e.g., in the CT example in Section V-A.

V. Illustrative Examples

In this section, we consider both CT and DT examples to demonstrate the effectiveness of our approaches. The MILPs in (15) are solved using YALMIP [26] and Gurobi [25].

A. CT System Example

Let us consider the CT system in [14, eq. (30)]:

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1, \quad \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2 + w_2, \\ \dot{x}_3 &= -a_3(a_2 x_1 + x_2) + \frac{a_1}{b_1}(a_4 \sin(x_1) + \cos(x_1)x_2) - a_4 x_3 + w_3, \end{aligned}$$

with output $y = x_1$ (i.e., $V = 0$ and we have an MILP formulation (cf. Remark 3)) and the following parameters: $a_1 = 35.63, b_1 = 15, a_2 = 0.25, a_3 = 36, a_4 = 200, \mathcal{X}_0 = [19.5, 9] \times [9, 11] \times [0.5, 1.5], \mathcal{W} = [-0.1, 0.1]^3$. The problem in (15), even with the additional degrees of freedom, as well as the observer designs in [14], [16] are infeasible without a state transformation (also cf. Remark 1). Hence, similar to [16, Sec. V-A], we consider a similarity transformation $z = Sx$ with $S = \begin{bmatrix} 20 & 0.1 & 0.1 \\ 0 & 0.01 & 0.06 \\ 0 & -10 & -0.4 \end{bmatrix}$, and added and subtracted $5y$ from the dynamics of $\dot{x}_1$. Further, adding positivity constraints to cast (15) as an LP led to infeasibility; thus, the MILP formulation is less conservative (cf. Remark 3).

By solving (15), we obtain the following observer gains: $T = \begin{bmatrix} -1.569 & 4.138 & -0.021 \\ 0.016 & 0.973 & 0.0001 \\ 50.200 & -80.860 & 1.421 \end{bmatrix}$, $N = \begin{bmatrix} 51.386 \\ -0.330 \\ -1004.014 \end{bmatrix}$, $L = \begin{bmatrix} 7869.338 \\ 0.186 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 1 & 0 \\ 0 & 0.25 & 15 \\ 0 & 36 & 200 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_2 = \emptyset$, and $B_2 = \emptyset$. From Figure 1 (only results for x_3 is depicted for brevity; other states show the same trends), we observe that the state framers obtained by our proposed approach, $\underline{x}, \overline{x}$, are tighter than those obtained by both the interval observers in [16], $\underline{x}^{\mathcal{H}_\infty}, \overline{x}^{\mathcal{H}_\infty}$, and in [14], $\underline{x}^{DMN}, \overline{x}^{DMN}$. Moreover, the framer error $\varepsilon_t = \overline{x}_t - \underline{x}_t$ is observed to converge exponentially to a steady-state value.

B. DT System Example

Next, we consider the noisy DT Hénon chaos system [16, Sec. V-B]: $x_{t+1} = Ax_t + r[1 - x_{t,1}^2] + Bw_t$, $y_t = x_{t,1} + v_t$, where $A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}$, $B = I$, $r = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$, $\mathcal{X}_0 = [-2, 2] \times$

[4] Note that absolute values are internally converted into a mixed-integer formulation in off-the-shelf tools, e.g., YALMIP [26], where a binary variable is introduced to indicate if $|x| = x$ *or* $|x| = -x$.

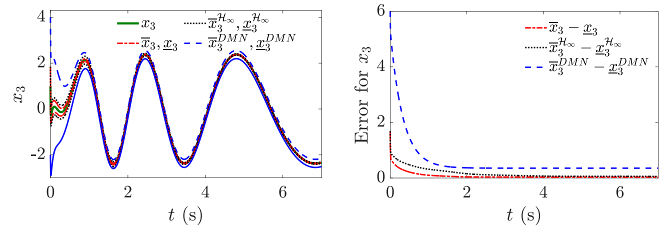

Fig. 1. CT Example: State, x_3, and its upper and lower framers and error of our proposed observer, $\overline{x}_3, \underline{x}_3, \varepsilon_3$, and by the observers in [16], $\overline{x}_3^{\mathcal{H}_\infty}, \underline{x}_3^{\mathcal{H}_\infty}, \varepsilon_3^{\mathcal{H}_\infty}$, and [14], $\overline{x}_3^{DMN}, \underline{x}_3^{DMN}, \varepsilon_3^{DMN}$.

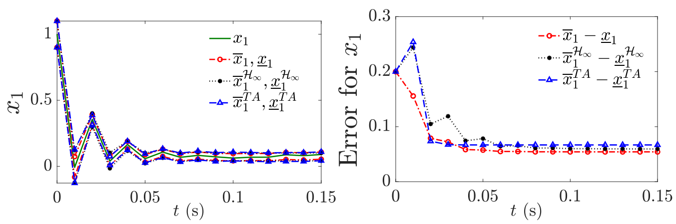

Fig. 2. DT Example: State, x_1, and its upper and lower framers and error of our proposed observer, $\overline{x}_1, \underline{x}_1, \varepsilon_1$, and by the observers in [16], $\overline{x}_1^{\mathcal{H}_\infty}, \underline{x}_1^{\mathcal{H}_\infty}, \varepsilon_1^{\mathcal{H}_\infty}$, and [3], $\overline{x}_1^{TA}, \underline{x}_1^{TA}, \varepsilon_1^{TA}$.

$[-1, 1]$, $\mathcal{W} = [-0.01, 0.01]^2$ and $\mathcal{V} = [-0.025, 0.025]$. Note that in this example, the problem in (15) is feasible without any coordinate transformation and further, additional positivity constraints can be added to cast (15) as an LP without any performance loss (cf. Remark 3). By solving (15), we obtain the following observer gains: $T = \begin{bmatrix} 0.5448 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 0.4552 \\ 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $A_2 = \emptyset$, and $B_2 = \emptyset$. From Figure 2 (x_2 omitted for brevity), the estimates from our proposed approach are tighter than the methods in [3] and [16] that instead minimizes the $\mathcal{H}_\infty$-gain.

VI. Conclusion and Future Work

In this letter, we presented a new interval observer design for uncertain locally Lipschitz CT and DT systems with nonlinear noisy observations. In particular, the proposed observer is correct and positive by construction without the need for additional constraints/assumptions by leveraging mixed-monotone decompositions/embedding systems. Moreover, we showed that the design is input-to-state stable (ISS) and minimizes the L_1-gain of the observer error system. In contrast to most existing interval observers, our design involves mixed-integer (linear) programs instead of semi-definite programs with linear matrix inequalities, and offers additional degrees of freedom that can be simultaneously designed. Our proposed observers for both CT and DT systems were further shown to be effective and to outperform most existing designs. As future work, we will consider extensions of our framework to hybrid systems with jumps and systems with unknown inputs.

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