$\mathcal{H}_\infty$-Optimal Interval Observer Synthesis for Uncertain Nonlinear Dynamical Systems via Mixed-Monotone Decompositions

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***Abstract*—This letter introduces a novel $\mathcal{H}_\infty$-optimal interval observer synthesis for bounded-error/uncertain locally Lipschitz nonlinear continuous-time (CT) and discrete-time (DT) systems with noisy nonlinear observations. Specifically, using mixed-monotone decompositions, the proposed observer is correct by construction, i.e., the interval estimates readily frame the true states without additional constraints or procedures. In addition, we provide sufficient conditions for input-to-state (ISS) stability of the proposed observer and for minimizing the $\mathcal{H}_\infty$ gain of the framer error system in the form of semi-definite programs (SDPs) with Linear Matrix Inequalities (LMIs) constraints. Finally, we compare the performance of the proposed $\mathcal{H}_\infty$-optimal interval observers with some benchmark CT and DT interval observers.**

***Index Terms*—Estimation, observers for nonlinear systems, uncertain systems, interval observers.**

I. Introduction

ENGINEERING applications, e.g., monitoring, system identification, control synthesis, and fault detection often require knowledge of system states. However, due to the presence of noise/uncertainties and/or inaccuracies in sensor measurements, system states are usually not exactly known. This has motivated the design of state observers to estimate system states using uncertain/noisy observations and system dynamics. In particular, for bounded-error settings, i.e., when uncertainties are set-valued (and distribution-free), *interval observer* designs have recently gained much attention due to their simple principles and computational efficiency [1].

Recent years have produced an extensive body of seminal literature on the design of interval/set-valued observers for several classes of systems, e.g., linear, cooperative/monotone, Metzler and mixed-monotone dynamics [1]–[5]. It has been noted that the design of interval observers that must simultaneously satisfy correctness (framer property) and stability/convergence is not a trivial task, even for linear systems [2]. Thus, especially when the system dynamics is nonlinear, either relatively restrictive assumptions on system properties were required to guarantee the applicability of the proposed approaches, or monotone systems properties [6] need to be directly imposed to satisfy positivity/cooperative behavior of the error dynamics.

This challenge has been addressed for specific system classes by leveraging Müller's theorem or interval arithmetic-based approaches [7], transformation to a positive system before designing an observer (only for linear systems) [8] or applying time-invariant/varying state transformations [3]. On the other hand, the work in [9] leveraged *bounding functions* to design interval observers for a class of continuous-time nonlinear systems under some relatively restrictive assumptions on the nonlinear dynamics, without providing a systematic approach to compute the bounding functions nor necessary/sufficient conditions for their existence. More recently, bounding/mixed-monotone decomposition functions were applied in [3] to design interval observers for nonlinear discrete-time dynamics, where conservative additive terms were added to the error dynamics to guarantee its positivity. Moreover, to best of our understanding, the resulting Linear Matrix Inequalities (LMIs) do not include the required conditions to guarantee that the computed bounding functions are decomposition functions.

Decomposition functions were also applied in the authors' previous work [4], [5] to design interval observers for nonlinear discrete-time systems under the (restrictive) assumption of global Lipschitz continuity as well as additional sufficient structural system properties to guarantee stability. Further, the applied decomposition functions were not necessarily the tightest. In our preliminary work [10], we proposed an interval observer design for noiseless nonlinear CT and DT systems based on tight remainder-form decomposition functions [11]. Our goal is to extend the design in [10] to noisy uncertain nonlinear dynamics in this letter.

In particular, to consider bounded noise as an extension to our preliminary work [10], notions of *mixed-monotonicity* and *embedding systems* for uncertain systems are required. Moreover, in the presence of noise, a different notion of stability than was considered in [10], specifically, input-to-state

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stability, needs to be proven. Furthermore, in addition to stability, a noise attenuation/optimality criterion is considered that requires additional constraints to make the corresponding SDP solvable. Hence, this letter tackles these challenges by proposing a unified framework to synthesize $\mathcal{H}_\infty$-optimal and input-to-state stable (ISS) interval observers for a very broad range of locally Lipschitz bounded-error nonlinear CT and DT systems with noisy nonlinear observation functions. Using *remainder-form mixed-monotone decomposition functions* [11], we show that the states of the designed observer frame the true state trajectory of the system for all possible realizations of interval-valued noise/disturbance and initial states, i.e., the proposed observer is correct-by-construction, without imposing additional constraints or assumptions. Further, we formulate semi-definite programs (SDPs) with LMI constraints for both CT and DT cases to minimize the $\mathcal{H}_\infty$-gain of the framer error (interval width) system and to ensure input-to-state stability of the correct-by-construction framers, which are solved offline to find the stabilizing observer gains.

Notation: $\mathbb{R}^n, \mathbb{R}^{n\times p}, \mathbb{D}_n, \mathbb{N}, \mathbb{N}_n, \mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the n-dimensional Euclidean space and the sets of n by p matrices, n by n diagonal matrices, natural numbers, natural numbers up to n, nonnegative and positive real numbers, respectively, while $\mathbb{M}_n$ denotes the set of all n by n Metzler matrices, i.e., square matrices whose off-diagonal elements are nonnegative. For $M \in \mathbb{R}^{n\times p}$, M_{ij} denotes M's entry in the i'th row and the j'th column, $M^\oplus \triangleq \max(M, \mathbf{0}_{n,p})$, $M^\ominus \triangleq M^\oplus - M$ and $|M| \triangleq M^\oplus + M^\ominus$, where $\mathbf{0}_{n,p}$ is the zero matrix in $\mathbb{R}^{n\times p}$, while $\text{sgn}(M) \in \mathbb{R}^{n\times p}$ is the element-wise sign of M with $\text{sgn}(M_{ij}) = 1$ if $M_{ij} \geq 0$ and $\text{sgn}(M_{ij}) = -1$, otherwise. Further, if $p = n$, M^{d} denotes a diagonal matrix whose diagonal coincides with the diagonal of M, $M^{\text{nd}} \triangleq M - M^{\text{d}}$ and $M^{\text{m}} \triangleq M^{\text{d}} + |M^{\text{nd}}|$, while $M \succ 0$ and $M \prec 0$ (or $M \succeq 0$ and $M \preceq 0$) denote that M is positive and negative (semi-) definite, respectively. Further, a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, where $0 \in S$, is positive definite if $f(x) > 0$ for all $x \in S\backslash\{0\}$, and $f(0) = 0$. Finally, an interval $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^n$ is the set of all real vectors $z \in \mathbb{R}^{n_z}$ that satisfies $\underline{z} \leq z \leq \overline{z}$ (component-wise), where $\|\overline{z} - \underline{z}\|_\infty \triangleq \max_{i\in\{1,\ldots,n_z\}} |z_i|$ is interval width of $\mathcal{I}$.

II. Problem Formulation

System Assumptions: Consider the following uncertain nonlinear continuous-time (CT) or discrete-time (DT) system:

$$\mathcal{G} : \begin{cases} x_t^+ = \hat{f}(x_t, w_t, u_t) \triangleq f_t(x_t, w_t), \\ y_t = \hat{h}(x_t, v_t, u_t) \triangleq h_t(x_t, v_t), \end{cases} x_t \in \mathcal{X}, t \in \mathbb{T}, \quad (1)$$

where $x_t^+ = \dot{x}_t, \mathbb{T} = \mathbb{R}_{\geq 0}$ if $\mathcal{G}$ is a CT system, and $x_t^+ = x_{t+1}, \mathbb{T} = \{0\}\cup\mathbb{N}$, if $\mathcal{G}$ is a DT system. Moreover, $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ ($\mathcal{X}$ can be unbounded), $w_t \in \mathcal{W} \triangleq [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$, $v_t \in \mathcal{V} \triangleq [\underline{v}, \overline{v}] \subset \mathbb{R}^{n_v}$, $u_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are continuous state, process noise, measurement disturbance, known (control) input and output (measurement) signals. Further, $\hat{f} : \mathbb{R}^n \times \mathbb{R}^{n_w} \times \mathbb{R}^s \to \mathbb{R}^n$ and $\hat{h} : \mathbb{R}^n \times \mathbb{R}^{n_v} \times \mathbb{R}^s \to \mathbb{R}^l$ are nonlinear state vector field and observation/constraint mapping, respectively, from which, the time-varying mappings $f_t : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ and $h_t : \mathbb{R}^n \times \mathbb{R}^{n_v} \to \mathbb{R}^l$ are well-defined since the input signal u_t is known. We are interested to estimate the trajectories of (1), initialized in an interval $\mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathcal{X}$. Moreover, we assume the following:

Assumption 1: The initial state x_0 satisfies $x_0 \in \mathcal{X}_0 = [\underline{x}_0, \overline{x}_0]$, where $\underline{x}_0$ and $\overline{x}_0$ are known initial state bounds.

Assumption 2: $f_t(\cdot)$ and $h_t(\cdot)$ are known, differentiable and locally Lipschitz mappings in their domain, with *a priori* known upper and lower uniform[1] bounds for their Jacobian matrices, $\overline{J}^f, \underline{J}^f \in \mathbb{R}^{n\times(n+n_w)}, \overline{J}^h, \underline{J}^h \in \mathbb{R}^{l\times(n+n_v)}$.

Both assumptions of locally Lipschitz continuity and differentiability are made for ease of exposition and can be relaxed to weaker assumptions (cf. [11]). Further, by Assumption 2, each of the mappings f_t and h_t can be decomposed into two functions via the following proposition. (Note that throughout the rest of this letter, to simplify the notations, we drop the explicit time-dependency of f_t, h_t and their Jacobian matrices when they are clear from context.)

Proposition 1 (JSS Decomposition, [10, Proposition 2]): Let $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^n$ satisfies Assumption 2, uniformly in $\mathcal{Z}$. Then, $f(z)$ can be decomposed into a (remainder) affine mapping Hz and a mapping $\mu(z)$, in an additive form:

$$\forall z \in \mathcal{Z}, f(z) = \mu(z) + Hz, \quad (2)$$

where H is a matrix in $\mathbb{R}^{n\times n_z}$, that satisfies the following

$$\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_{n_z}, H_{ij} = \overline{J}^f_{ij} \text{ or } H_{ij} = \underline{J}^f_{ij}. \quad (3)$$

Further, μ is a Jacobian sign-stable (JSS) [12] mapping in $\mathcal{Z}$, by construction, i.e., its Jacobian matrix entries do not change signs and have constant signs over $\mathcal{Z}$. In other words, for each $(i, j) \in \mathbb{N}_n \times \mathbb{N}_{n_z}$, either of the following hold:

$$\forall z \in \mathcal{Z}, J^\mu_{ij}(z) \geq 0 \text{ (positive JSS)},$$
$$\forall z \in \mathcal{Z}, J^\mu_{ij}(z) \leq 0 \text{ (negative JSS)},$$

where $J^\mu(z)$ denotes the Jacobian matrix of μ at $z \in \mathcal{Z}$.

Assumption 3: $\mathcal{X}_0, \mathcal{W}, \mathcal{V}$ and the values of the input u_t and output/measurement y_t signals are known/given at all times.

Further, we formally define the notions of *framers*, *correctness* and *stability* that are used throughout this letter.

Definition 1 (Correct Interval Framers): Suppose Assumptions 2 and 3 hold. Given the nonlinear plant (1), the mappings/signals $\overline{x}, \underline{x}$:$\mathbb{T} \to \mathbb{R}^n$ are called upper and lower framers for the states of System (1), if

$$\forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}, \ \underline{x}_t \leq x_t \leq \overline{x}_t. \quad (4)$$

In other words, starting from the initial interval $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, the true state of the system in (1), x_t, is guaranteed to evolve within the interval flow-pipe $[\underline{x}_t, \overline{x}_t]$, for all $(t, w_t, v_t) \in \mathbb{T} \times \mathcal{W} \times \mathcal{V}$. Finally, any dynamical system whose states are correct framers for the states of the plant $\mathcal{G}$, i.e., any (tractable) algorithm that returns upper and lower framers for the states of plant $\mathcal{G}$ is called a *correct* interval framer for system (1).

Definition 2 (Framer Error): Given state framers $\underline{x}_t \leq \overline{x}_t$, $\varepsilon : \mathbb{T} \to \mathbb{R}^n$, denoting the interval width of $[\underline{x}_t, \overline{x}_t]$, is called the framer error. It can be easily verified that correctness (cf. Definition 1) implies that $\varepsilon_t \geq 0, \forall t \in \mathbb{T}$.

Definition 3 (Input-to-State Stability & Interval Observer): An interval framer is input-to-state stable (ISS), if the framer error (cf. Definition 2) is bounded as follows:

$$\forall t \in \mathbb{T}, \|\varepsilon_t\|_2 \leq \beta(\|\varepsilon_0\|_2, t) + \rho(\|\Delta\|_{\ell_\infty}), \quad (5)$$

[1]The assumption of uniform Jacobian bounds for all values of u_t may sometimes be conservative, unless the input domain is small or if the system is autonomous (with no inputs). Note, however, that this assumption can be relaxed for systems with additive inputs u_t.

where $\Delta \triangleq [\Delta w^\top \ \Delta v^\top]^\top \triangleq [(\overline{w}-\underline{w})^\top \ (\overline{v}-\underline{v})^\top]^\top$, β and ρ are functions of classes[2] $\mathcal{KL}$ and $\mathcal{K}_\infty$, respectively, and $\|\Delta\|_{\ell_\infty} \triangleq \sup_{t\in[0,\infty]} \|\Delta_t\|_2 = \|\Delta\|_2$ is the ℓ_∞ signal norm. An ISS interval framer is called an interval observer.

Definition 4 ($\mathcal{H}_\infty$-Optimal Interval Observer Synthesis): An interval framer design $\hat{\mathcal{G}}$ is $\mathcal{H}_\infty$-optimal if the $\mathcal{H}_\infty$ gain of the framer error system $\tilde{\mathcal{G}}$, i.e., $\|\tilde{\mathcal{G}}\|_{\mathcal{H}_\infty} \triangleq \sup_{\Delta\neq 0} \frac{\|\varepsilon\|_{\ell_2}}{\|\Delta\|_{\ell_2}}$, is minimized, where $\|s\|_{\ell_2} \triangleq \sqrt{\int_0^\infty \|s_t\|_2^2 dt}$ is the ℓ_2 signal norm for $s \in \{\varepsilon, \Delta\}$.

The observer design problem can be stated as follows:

Problem 1: Given the nonlinear system in (1), as well as Assumptions 2 and 3, synthesize an ISS and $\mathcal{H}_\infty$-optimal interval observer (cf. Definitions 1–4).

III. Proposed Interval Observer

A. Decomposition Functions and Embedding Systems

Our observer design approach hinges upon constructing framers for system (1) through designing *embedding systems* based on the literature on mixed-monotonicity, e.g., [10], [11], [13], [14]. Note that embedding systems can be constructed based on any *decomposition/inclusion function*, including interval arithmetic-based decomposition/inclusion functions, and are well-known bounding tools in the literature to frame state trajectories of dynamical systems [12]–[16]. In this letter, we choose to leverage *remainder-from mixed-monotone decomposition functions* [11], due to their tractability and consistency with decomposability of our dynamics based on Assumption 2. For the sake of completeness and before proposing our observer structure, we first formally define the notions of mixed-monotonicity and embedding systems, as follows.

Definition 5 (Mixed-Monotone Decomposition Functions [13, Definition 1], [12, Definition 4]): Consider the nonlinear system (1) and let $\mathcal{Z} \triangleq \mathcal{X} \times \mathcal{W}, n_z \triangleq n + n_w$.

Suppose (1) is a DT system. Then, a mapping $f_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ is a DT mixed-monotone decomposition function with respect to f, if i) $f_d(z,z) = f(z)$, ii) f_d is monotone increasing in its first argument, i.e., $\hat{z} \geq z \Rightarrow f_d(\hat{z}, z') \geq f_d(z, z')$, and iii) f_d is monotone decreasing in its second argument, i.e., $\hat{z} \geq z \Rightarrow f_d(z', \hat{z}) \leq f_d(z', z)$.

On the other hand, if (1) is a CT system, a mapping $f_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ is a CT mixed-monotone decomposition function with respect to f, if i) $f_d(z,z) = f(z)$, ii) f_d is monotone increasing in its first argument with respect to "off-diagonal" arguments, i.e., for each f_i, $i \in \mathbb{N}_n$, we have $\hat{z} \geq z \wedge \hat{z}_i = z_i \Rightarrow f_{d,i}(\hat{z}, z') \geq f_{d,i}(z, z')$, and iii) f_d is monotone decreasing in its second argument, i.e., $\hat{z} \geq z \Rightarrow f_d(z', \hat{z}) \leq f_d(z', z)$.

Note that locally Lipschitz vector fields (cf. Assumption 2) have been proven to be mixed-monotone, i.e., they admit mixed-monotone decomposition functions [13]. Moreover, as shown in [11, Corollary 2], if $f : [\underline{z}, \overline{z}] \subset \mathbb{R}^{n_z} \to \mathbb{R}^n$ is a JSS mapping, then $f_{d,i}$ for each $f_i, i \in \mathbb{N}_n$ is *tight*, i.e., $f_{d,i}(\underline{z}, \overline{z}) = \min_{z\in[\underline{z},\overline{z}]} f_i(z), f_{d,i}(\overline{z}, \underline{z}) = \max_{z\in[\underline{z},\overline{z}]} f_i(z)$. Moreover, in this case, $f_{d,i}(\cdot,\cdot)$ can be tractably computed through [10, Proposition 4] for each f_i, $i \in \mathbb{N}_n$, as follows:

$$f_{d,i}(z_1, z_2) = f_i(D^i z_1 + (I_{n_z} - D^i) z_2), \tag{6}$$

for any ordered $z_1, z_2 \in \mathcal{Z}$, where $D^i \in \mathbb{D}_{n_z}$ is a binary diagonal matrix determined by which vertex of the interval $[z_2, z_1]$ that maximizes or $[z_1, z_2]$ that minimizes the JSS function $f_i(\cdot)$ and can be computed as follows:

$$D^i = \mathrm{diag}(\max(\mathrm{sgn}(\overline{J}_i^f), \mathbf{0}_{1,n_z})). \tag{7}$$

Definition 6 (Bounded-Error Embedding Systems [11], [13]): For an n-dimensional system (1) with any mixed-monotone decomposition function $f_d(\cdot)$, its *embedding system is a* $2n$-dimensional system with initial state $[\underline{x}_0^\top \ \overline{x}_0^\top]^\top$:

$$\begin{bmatrix} \underline{x}_t^+ \\ \overline{x}_t^+ \end{bmatrix} = \begin{bmatrix} f_d([(\underline{x}_t)^\top \ \underline{w}^\top]^\top, [(\overline{x}_t)^\top \ \overline{w}^\top]^\top) \\ f_d([(\overline{x}_t)^\top \ \overline{w}^\top]^\top, [(\underline{x}_t)^\top \ \underline{w}^\top]^\top) \end{bmatrix}. \tag{8}$$

The following proposition characterizes the relationship between embedding systems and state framers that will be used in the next subsections.

Proposition 2 (State Framer Property [11, Proposition 3]): Let system (1) be mixed-monotone with respect to f_d with an embedding system (8). Then, for all $t \in \mathbb{T}$, $R^f(t, \mathcal{X}_0, \mathcal{W}) \subset \mathcal{X}_t \triangleq [\underline{x}_t, \overline{x}_t]$, where $R^f(t, \mathcal{X}_0, \mathcal{W}) \triangleq \{\Phi_f(t, x_0, w) \mid x_0 \in \mathcal{X}_0, \forall t \in \mathbb{T}, \forall w \in \mathcal{W}\}$ is the reachable set at time t of (1) when initialized within $\mathcal{X}_0$, $\Phi_f(t, x_0, w)$ is the true state trajectory function of system (1) and $(\underline{x}_t, \overline{x}_t)$ is the solution to the embedding system (8), with $\mathbb{T} \in \mathbb{R}_{\geq 0}$ for CT systems and $\mathbb{T} \in \{0\} \cup \mathbb{N}$ for DT systems. Consequently, the system state trajectory $x_t = \Phi_f(t, x_0, w)$ satisfies $\underline{x}_t \leq x_t \leq \overline{x}_t, \forall t \geq 0, \forall w \in \mathcal{W}$, i.e., it is *framed* by $\mathcal{X}_t \triangleq [\underline{x}_t, \overline{x}_t]$.

B. Interval Observer Design

Given the nonlinear plant $\mathcal{G}$, in order to address Problem 1, we propose an interval observer (cf. Definition 1) for $\mathcal{G}$ through the following dynamical system $\hat{\mathcal{G}}$:

$$\begin{aligned}
\underline{x}_t^+ &= (A-LC)^{\uparrow}\underline{x}_t - (A-LC)^{\downarrow}\overline{x}_t + \phi_d(\underline{x}_t, \underline{w}, \overline{x}_t, \overline{w}) \\
&\quad - L^{\oplus}\psi_d(\overline{x}_t, \overline{v}, \underline{x}_t, \underline{v}) + L^{\ominus}\psi_d(\underline{x}_t, \underline{v}, \overline{x}_t, \overline{v}) + Ly_t \\
&\quad + B^{\oplus}\underline{w} - B^{\ominus}\overline{w} + (LD)^{\ominus}\underline{v} - (LD)^{\oplus}\overline{v}, \\
\overline{x}_t^+ &= (A-LC)^{\uparrow}\overline{x}_t - (A-LC)^{\downarrow}\underline{x}_t + \phi_d(\overline{x}_t, \overline{w}, \underline{x}_t, \underline{w}) \\
&\quad - L^{\oplus}\psi_d(\underline{x}_t, \underline{v}, \overline{x}_t, \overline{v}) + L^{\ominus}\psi_d(\overline{x}_t, \overline{v}, \underline{x}_t, \underline{v}) + Ly_t \\
&\quad + B^{\oplus}\overline{w} - B^{\ominus}\underline{w} + (LD)^{\ominus}\overline{v} - (LD)^{\oplus}\underline{v},
\end{aligned} \tag{9}$$

where if $\mathcal{G}$ is a CT system, then

$$\begin{aligned}
\overline{x}_t^+ &\triangleq \dot{\overline{x}}_t, \ (A-LC)^{\uparrow} \triangleq (A-LC)^{\mathrm{d}} + (A-LC)^{\mathrm{nd}\oplus}, \\
\underline{x}_t^+ &\triangleq \dot{\underline{x}}_t, \ (A-LC)^{\downarrow} \triangleq (A-LC)^{\mathrm{nd}\ominus},
\end{aligned} \tag{10}$$

and if $\mathcal{G}$ is a DT system, then

$$\begin{aligned}
\overline{x}_t^+ &\triangleq \overline{x}_{t+1}, \ (A-LC)^{\uparrow} \triangleq (A-LC)^{\oplus}, \\
\underline{x}_t^+ &\triangleq \underline{x}_{t+1}, \ (A-LC)^{\downarrow} \triangleq (A-LC)^{\ominus}.
\end{aligned} \tag{11}$$

Moreover, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times n_v}$ are chosen such that the following decompositions hold (cf. Proposition 1): $\forall x, w, v \in \mathcal{X} \times \mathcal{W} \times \mathcal{V}$,

$$\begin{cases} f(x,w) = Ax + Bw + \phi(x,w), \\ h(x,v) = Cx + Dv + \psi(x,v), \end{cases} \ s.t.\ \phi, \psi \text{ are JSS.} \tag{12}$$

[2]A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing and is of class $\mathcal{K}_\infty$ if it is also unbounded. Moreover, $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\lambda(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\lambda(s, t)$ decreases to zero as $t \to \infty$.

Furthermore, $\phi_d : \mathbb{R}^{2n_z} \to \mathbb{R}^n$ and $\psi_d : \mathbb{R}^{2n_\zeta} \to \mathbb{R}^l$ are tight mixed-monotone decomposition functions of ϕ and ψ, respectively (cf. Definition 5 and Proposition 2). Finally, $L \in \mathbb{R}^{n\times l}$ is the observer gain matrix, designed via Theorem 2, such that the proposed observer $\hat{\mathcal{G}}$ possesses the desired properties discussed in the following subsections.

C. Observer Correctness (Framer Property)

Our strategy is to design a *correct-by-construction* interval observer for plant $\mathcal{G}$. To accomplish this goal, first, note that from (9) and (12) we have $y_t - Cx_t - Dv_t - \psi(x_t, v_t) = 0$, and so $L(y_t - Cx_t - Dv_t - \psi(x_t, v_t)) = 0$, for any $L \in \mathbb{R}^{n\times l}$. Adding this "zero" term to the right hand side of (1) and applying (12) yield the following equivalent system to $\mathcal{G}$:

$$x_t^+ = (A - LC)x_t + Bw_t - LDv_t + \phi(x_t, w_t) - L\psi(x_t, v_t) + Ly_t. \tag{13}$$

From now on, we are interested in computing embedding systems, in the sense of Definition 6, for the system in (13), so that by Proposition 2, the state trajectories of (13) are "framed" by the state trajectories of the computed embedding system. To do so, we split the right hand side of (13) (except for Ly_t that is a known signal) into two constituent systems: the affine constituent $(A-LC)x_t+Bw_t - LDv_t$, and the nonlinear constituent, $\phi(x_t, w_t) - L\psi(x_t, v_t)$. Then, we compute embedding systems for each constituent, separately. Finally, we add the computed embedding systems to construct an embedding system for (13). The following theorem addresses the *correctness* of the proposed observer.

Theorem 1 (Correct Interval Framer): Consider the nonlinear plant $\mathcal{G}$ in (1) and suppose Assumptions 1–3 hold. Then, the dynamical system $\hat{\mathcal{G}}$ in (9) constructs a correct interval framer for the nonlinear plant $\mathcal{G}$, i.e., $\forall t \in \mathbb{T}, \forall w_t \in \mathcal{W}, \forall v_t \in \mathcal{V}, \underline{x}_t \le x_t \le \overline{x}_t$, where x_t and $[\underline{x}_t^\top \overline{x}_t^\top]^\top$ are the state vectors in $\mathcal{G}$ and $\hat{\mathcal{G}}$ at time $t \in \mathbb{T}$, respectively.

Proof: First, note that the affine constituent system $f_\ell(\xi_t) = (A - LC)x_t + Bw_t - LDv_t$ (with $\xi \triangleq [x^\top \; w^\top \; v^\top]^\top$) admits the following tight decomposition function:

$$\begin{aligned} f_{\ell d}(\xi_1, \xi_2) &= (A - LC)^{\uparrow}x_1 - (A - LC)^{\downarrow}x_2 \\ &\quad + B^{\oplus}w_1 - B^{\ominus}w_2 + (LD)^{\ominus}v_1 - (LD)^{\oplus}v_2, \end{aligned} \tag{14}$$

where $(A-LC)^{\uparrow}$ and $(A-LC)^{\downarrow}$ are given in (10) and (11) for CT and DT systems, respectively. This follows from a similar reasoning to the proof of [10, Lemma 1], with the slight modification of considering the extra noise terms $B^{\oplus}w_1 - B^{\ominus}w_2$ that is non-decreasing in w_1 and non-increasing in w_2 due to the nonnegativity of $B^{\oplus}$ and $B^{\ominus}$, as well as $(LD)^{\ominus}v_1 - (LD)^{\oplus}v_2$ that is non-decreasing in v_1 and non-increasing in v_2 due to the nonnegativity of $(LD)^{\oplus}$ and $(LD)^{\ominus}$. Next, consider the nonlinear constituent system $f_\nu(x_t, w_t, v_t) = \phi(x_t, w_t) - L\psi(x_t, v_t)$. We show that f_ν admits the following decomposition function:

$$\begin{aligned} f_{\nu d}(x_1, w_1, v_1, x_2, w_2, v_2) &= \phi_d(x_1, w_1, x_2, w_2) \\ &\quad - L^{\oplus}\psi_d(x_2, v_2, x_1, v_1) + L^{\ominus}\psi_d(x_1, v_1, x_2, v_2), \end{aligned} \tag{15}$$

where ϕ_d, ψ_d are decomposition functions for the mappings ϕ, ψ. $f_{\nu d}$ is increasing in ξ_1 since it is a summation of three increasing mappings in ξ_1, including $\phi_d(x_1, w_1, x_2, w_2)$ (a decomposition function that by construction is increasing in (x_1, w_1) and hence, in $\zeta_1 = (x_1, w_1, v_1)$), $-L^{\oplus}\psi_d(x_2, v_2, x_1, v_1)$ (a multiplication of the nonpositive matrix $-L^{\oplus}$ and the decomposition function $\psi_d(x_2, v_2, x_1, v_1)$ that is decreasing in (x_1, v_1) and hence, in ξ_1 by construction) and $L^{\ominus}\psi_d(x_1, v_1, x_2, v_2)$ (a multiplication of the nonnegative matrix $L^{\ominus}$ and the decomposition function $\psi_d(x_1, v_1, x_2, v_2)$ that is itself increasing in (x_1, v_1) and hence, in ξ_1 by construction). A similar reasoning shows that $f_{\nu d}$ is decreasing in ξ_2. Moreover, $f_{\nu d}(\xi, \xi) = \phi_d(x, w, x, w) - L^{\oplus}\psi_d(x, v, x, x) + L^{\ominus}\psi_d(x, v, x, v) = \phi(x, w) - L\psi(x, v) = f_\nu(x, w, v) = f_\nu(\xi)$. Finally, it is straightforward to show that the summation of decomposition functions of constituent systems is also a decomposition function of the summation of the constituent systems. Hence, $f_d(\xi_1, \xi_2) \triangleq f_{\ell d}(\xi_1, \xi_2) + f_{\nu d}(\xi_1, \xi_2) + Ly$ is a decomposition function for (13) and equivalently for (1), where $f_{\ell d}, f_{\nu d}$ are given in (14), (15), respectively. So, the $2n$-dimensional system $[(\underline{x}_t^+)^\top \; (\overline{x}_t^+)^\top]^\top = [f_d^\top(\underline{\xi}_t, \overline{\xi}_t) \; f_d^\top(\overline{\xi}_t, \underline{\xi}_t)]^\top$, initialized at $[\underline{x}_0^\top \; \overline{x}_0^\top]^\top$, is an embedding system for (1), and $\underline{x}_t \le x_t \le \overline{x}_t$, by Proposition 2. ∎

D. ISS and $\mathcal{H}_\infty$-Optimal Observer Design

In addition to the correctness property, it is important to guarantee the stability of the proposed framer, i.e., we aim to design the observer gain L to ensure input-to-state stability (ISS) of the observer error, $\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$ (cf. Definitions 2 and 3). Before introducing our observer design, we first find some upper bounds for the interval widths of the JSS functions in terms of the interval widths of their domains via the following lemma, whose proof is a slight modification of the proof of [10, Lemma 3], with the difference that here, the domain is the augmentation of the state and the noise.

Lemma 1 (JSS Function Interval Width Bounding): Let $f : \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{W} \subset \mathbb{R}^{n_z} \to \mathbb{R}^n$ be a mapping that satisfies the assumptions in Proposition 1 and hence, can be decomposed in the form of (2). Let $\mu_d \triangleq [\mu_{d,1} \ldots \mu_{d,n}]^\top : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ be the tight decomposition function for the JSS mapping $\mu(\cdot)$, given in (6). Then, for any interval domain $\underline{z} \le z \triangleq [x^\top \; w^\top]^\top \le \overline{z}$ in $\mathcal{Z}$, the following inequality holds:

$$\Delta_d^{\mu} \triangleq \mu_d(\overline{z}, \underline{z}) - \mu_d(\underline{z}, \overline{z}) \le \overline{F}^{\mu}\varepsilon, \tag{16}$$

where $\overline{F}^{\mu} \triangleq [\overline{F}_x^{\mu} \; \overline{F}_w^{\mu}] \triangleq (\overline{J}^{\mu})^{\oplus} + (\underline{J}^{\mu})^{\ominus}$, with $\varepsilon \triangleq \overline{z} - \underline{z}$, $\overline{J}^{\mu} = \overline{J}^f - H$, $\underline{J}^{\mu} = \underline{J}^f - H$, $\overline{F}_x^{\mu} \in \mathbb{R}^{n\times n}$ and $\overline{F}_w^{\mu} \in \mathbb{R}^{n\times n_w}$.

Now, equipped with the tools in Lemma 1, we derive sufficient LMIs to synthesize the stabilizing observer gain L for both DT and CT systems through the following theorem.

Theorem 2 (ISS and $\mathcal{H}_\infty$-Optimal Observer Design): Consider the nonlinear plant $\mathcal{G}$ in (1) and suppose Assumptions 2 and 3 hold. Then, the proposed correct interval framer $\hat{\mathcal{G}}$ in (9) is ISS, and hence, is an interval observer in the sense of Definition 3, and also is $\mathcal{H}_\infty$-optimal (cf. Definition 4), if there exist matrices $\mathbb{R}^{n\times n} \ni P \succ \mathbf{0}_{n,n}$, $G \in \mathbb{R}^{n\times l}_{\ge 0}$ and $\gamma \in \mathbb{R}_{>0}$ that solve the following problem:

$$(\gamma^*, P^*, G^*) \in \arg\min_{\{\gamma, P, G\}} \gamma \;\text{ s.t. } \Gamma \prec 0, \tag{17}$$

where

(i) if $\mathcal{G}$ is a CT system, then

$$\Gamma \triangleq \begin{bmatrix} \Omega & \Lambda & I \\ \Lambda^\top & -\gamma I & 0 \\ I & 0 & -\gamma I \end{bmatrix}, -GC \in \mathbb{M}_n, P \in \mathbb{D}_n, GD \ge 0, \tag{18}$$

with $\Omega \triangleq ((A^m) + \overline{F}_x^{\phi})^\top P + P(A^m + \overline{F}_x^{\phi}) + (-C + \overline{F}_x^{\psi})^\top G^\top + G(-C + \overline{F}_x^{\psi})$, and

(ii) if $\mathcal{G}$ is a DT system, then

$$\Gamma \triangleq -\begin{bmatrix} P & \Omega & \Lambda & 0 \\ \Omega^\top & P & 0 & I \\ \Lambda^\top & 0 & \gamma I & 0 \\ 0 & I & 0 & \gamma I \end{bmatrix}, GC \geq 0, -P \in \mathbb{M}_n, GD \geq 0, \quad (19)$$

with $\Omega \triangleq P(|A| + \overline{F}_x^{\phi}) + G(C + \overline{F}_x^{\psi})$.

Furthermore, in both cases, $\Lambda \triangleq P[\overline{F}_w^{\phi} + |B|\ 0] + G[0\ \overline{F}_v^{\psi} + D]$ and $\overline{F}_x^{\phi}, \overline{F}_x^{\psi}, \overline{F}_w^{\phi}, \overline{F}_v^{\psi}$ are computed by applying Lemma 1 on the JSS functions ϕ and ψ, respectively. Finally, the corresponding optimal stabilizing observer gain L^* can be obtained as $L^* = (P^*)^{-1}G^*$.

Proof: Starting from (9), we first derive the framer error ($\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$) dynamics $\tilde{\mathcal{G}}$. Then, we show that the provided conditions in (18) and (19) are sufficient for stability of the error system in the CT and DT cases, respectively. To do so, define $\Delta_d^{\mu} \triangleq \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s}) - \mu_d(\overline{x}, \overline{s}, \underline{x}, \underline{s})$ and $\Delta s \triangleq \overline{s} - \underline{s}$, $\forall \mu \in \{\phi, \psi\}$, $s \in \{w, v\}$.

Now, considering the CT case, from (9) and (10), we obtain the observer error dynamics:

$$\begin{aligned} \dot{\varepsilon}_t &= ((A-LC)^{\rm d} + |(A-LC)^{\rm nd}|)\varepsilon_t + \Delta_d^{\phi} + |L||\Delta_d^{\psi}| + |B|\Delta w + |LD|\Delta v \\ &\leq (A^{\rm m} + (-LC)^{\rm m} + \overline{F}_x^{\phi} + |L|\overline{F}_x^{\psi})\varepsilon_t + \delta^{w,v}(L), \end{aligned} \quad (20)$$

where $\delta^{w,v}(L) \triangleq (\overline{F}_w^{\phi} + |B|)\Delta w + (|L|\overline{F}_v^{\psi} + |LD|)\Delta v$ and $\overline{F}_s^{\mu}$, $\forall \mu \in \{\phi, \psi\}, s \in \{w, v\}$ is given in (16). The inequality holds by Lemma 1, [9, Lemma 1], and the facts that for any $M, N \in \mathbb{R}^{n \times n}$, $(M+N)^{\rm d} = M^{\rm d} + N^{\rm d}$, $(M+N)^{\rm nd} = M^{\rm nd} + N^{\rm nd}$, $|M+N| \leq |M| + |N|$ by triangle inequality and the fact that $\varepsilon_t \geq 0$ by the correctness property (Lemma 1). Now, note that by the *Comparison Lemma* [17, Lemma 3.4] and positivity of the system in (20), stability of the system in (20) implies stability for the actual error system. To show the former, we require the following: G and P are nonnegative and diagonal matrices, respectively. This forces P and its inverse to be diagonal matrices with strictly positive diagonal elements, and since G is forced to be non-negative, $L = P^{-1}G$ must be nonnegative, and hence $|L| = L$. Moreover, $-GC$ is Metzler, which results in $-LC = -P^{-1}GC$ being Metzler, since it is a product of a diagonal and positive matrix P^{-1} and a Metzler matrix $-GC$. Thus, $(-LC)^{\rm m} = -LC$. Further, since GD is nonnegative, then $LD = P^{-1}GD$ is a product of two nonnegative matrices P^{-1} and GD and so, $|LD| = LD$, and the system in (20) becomes the linear comparison system

$$\dot{\varepsilon}_t \leq (A^{\rm m} - LC + \overline{F}_{\phi} + L\overline{F}_{\psi})\varepsilon_t + L(\overline{F}_v^{\psi} + D)\Delta v + (\overline{F}_w^{\phi} + |B|)\Delta w, \quad (21)$$

where by [18, Sec. 9.2.2], solving the SDP in (17)–(18) results in the optimal observer gain $L^* = (P^*)^{-1}G^*$, in the $\mathcal{H}_\infty$ sense, i.e., with an $\mathcal{H}_\infty$ gain of γ^* (cf. Definition 4). This implies that the above linear comparison system (21) satisfies the following asymptotic gain (AG) property [19]:

$$\limsup_{t\to\infty} \|\varepsilon_t\|_2 \leq \rho(\|\tilde{\Delta}\|_{\ell_\infty}), \ \forall \varepsilon_0, \forall \tilde{\Delta} \in [\Delta w^\top \ \Delta v^\top]^\top, \quad (22)$$

where $\tilde{\Delta}$ is any realization of the augmented noise interval width and ρ is any class $\mathcal{K}_\infty$ function that is lower bounded by $\gamma^* \tilde{\Delta}$. On the other hand, by setting $\Delta = 0$, the LMIs in (18) reduce to their noiseless counterparts in [10, eq. (19)]. Hence, by [10, Th. 2], the comparison system (21) is 0-stable (0-GAS), which in addition to the AG property (22) is equivalent to the ISS property for (21) by [19, Th. 1-e]. Hence, the designed CT observer is also ISS.

For the DT case, from (9) and (11) and by a similar reasoning to the CT case, we obtain

$$\begin{aligned} \varepsilon_{t+1} &= |A - LC|\varepsilon_t + \Delta_d^{\phi} + |L||\Delta_d^{\psi}| + |B|\Delta w + |LD|\Delta v \\ &\leq (|A| + |LC| + \overline{F}_x^{\phi} + |L|\overline{F}_x^{\psi})\varepsilon_t + \delta^{w,v}(L). \end{aligned} \quad (23)$$

In addition, we enforce $-P$ to be Metzler, as well as G and GC to be nonnegative. Consequently, since P is positive definite, P becomes a nonsingular M-matrix, i.e., a square matrix whose negation is Metzler and whose eigenvalues have nonnegative real parts, and hence is inverse-positive [20, Th. 1], i.e., $P^{-1} \geq 0$. Therefore, $L = P^{-1}G \geq 0$ and $LC = P^{-1}(GC) \geq 0$, because they are matrix products of nonnegative matrices, P^{-1}, G and P^{-1}, GC, respectively. Finally, by a similar argument as in the CT case, LD is nonnegative. Hence, $|L| = L$, $|LC| = LC$, $|LD| = LD$, and so, the system in (23) becomes

$$\varepsilon_{t+1} \leq (|A| + LC + \overline{F}_x^{\phi} + L\overline{F}_x^{\psi})\varepsilon_t + L(\overline{F}_v^{\psi} + D)\Delta v + (\overline{F}_w^{\phi} + |B|)\Delta w,$$

for which the solution to the the SDP in (17) and (19) provides the $\mathcal{H}_\infty$-optimal observer gain $L^* = (P^*)^{-1}G^*$, by [18, Sec. 9.2.3]. Furthermore, a similar argument as in the CT case implies that the DT observer is also ISS. ■

Finally, note that if the LMIs in (18) or (19) are infeasible, a coordinate transformation can be applied in a straightforward manner, similar to [3, Sec. V] (omitted due to space limitations; interested readers are referred to [21] and references therein for a more detailed discussion on how to choose the right transformation matrix), which is also helpful for making the LMIs in Theorem 2 feasible, as observed in Section IV-A.

IV. Illustrative Examples

The effectiveness of our observer design is illustrated for CT and DT systems (using SeDuMi [22] to solve the LMIs).

A. CT System Example

Consider the CT system in [23, Sec. IV, eq. (30)]:

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1, \quad \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2 + w_2, \\ \dot{x}_3 &= -a_3(a_2 x_1 + x_2) + \frac{a_1}{b_1}(a_4 \sin(x_1) + \cos(x_1)x_2) - a_4 x_3 + w_3, \end{aligned}$$

with output $y = x_1, a_1 = 35.63, b_1 = 15, a_2 = 0.25, a_3 = 36, a_4 = 200, \mathcal{X}_0 = [19.5, 9] \times [9, 11] \times [0.5, 1.5], \mathcal{W} = [-0.1, 0.1]^3$. Without a coordinate transformation, the LMIs in (18) as well as the approach in [23] were infeasible. However, with a coordinate transformation $z = Tx$ with $T = \begin{bmatrix} 20 & 0.1 & 0.1 \\ 0 & 0.01 & 0.06 \\ 0 & -10 & -0.4 \end{bmatrix}$ (similar to [3, Sec. V]) and adding and subtracting $5y$ to the dynamics of $\dot{x}_1$, the state framers returned by our approach, $\underline{x}, \overline{x}$ are initially tighter than the ones obtained by the interval observer in [23], $\underline{x}^{DMN}, \overline{x}^{DMN}$, and they are comparable after the transients, as shown in Figure 1 (similar trends are observed for x_1, x_2; omitted for brevity). Furthermore, the framer error is initially smaller with

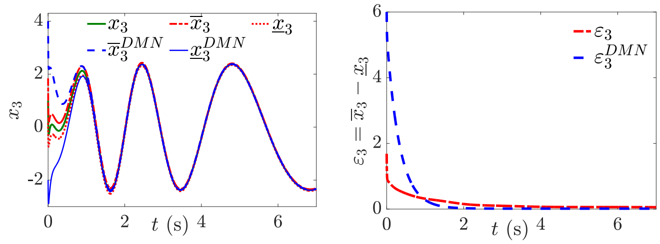

Fig. 1. State, x_3, as well as its upper and lower framers and error returned by our proposed observer, $\overline{x}_3, \underline{x}_3, \varepsilon_3$, and by the observer in [23], $\overline{x}_3^{DMN}, \underline{x}_3^{DMN}, \varepsilon_3^{DMN}$ for the CT System example.

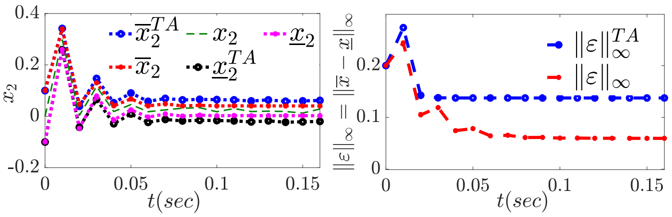

Fig. 2. State, x_2, and its upper and lower framers, returned by our proposed observer, $\overline{x}_2, \underline{x}_2$, and by the observer in [3], $\overline{x}_2^{TA}, \underline{x}_2^{TA}$ (left) and norm of framer error (right) for the DT System example.

our approach when compared to the one in [23] but converges/decays slower to a steady-state value than the framer error with the approach in [23].

B. DT System Example

Consider a noisy variant of the Hénon chaos system [24]:

$$x_{t+1} = Ax_t + r[1 - x_{t,1}^2] + Bw_t, \quad y_t = x_{t,1} + v_t,$$

where $A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}$, $B = I$, $r = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$, $\mathcal{X}_0 = [-2, 2] \times [-1, 1]$, $\mathcal{W} = [-0.01, 0.01]^2$ and $\mathcal{V} = [-0.1, 0.1]$. Using the solutions to the corresponding LMIs in (19), it can be observed from Figure 2 that the interval estimates for x_2 are tighter than the ones returned by the approach in [3] (similarly for x_1, omitted for brevity). Moreover, the depicted error plots demonstrate the convergence of the error sequence to steady state (i.e., ISS) and show smaller errors for the proposed approach when compared to the one in [3].

V. Conclusion and Future Work

A novel unified approach to synthesize interval-valued observers for bounded-error locally Lipschitz nonlinear continuous-time (CT) and discrete-time (DT) systems with nonlinear noisy observations was presented. The proposed observer was shown to be correct by construction using mixed-monotone decompositions, i.e., the true state trajectory of the system is guaranteed to be framed by the states of the observer without the need for additional constraints or assumptions. Moreover, we provide semi-definite programs for both CT and DT cases to find input-to-state stabilizing observer gains that are proven to be optimal in the sense of $\mathcal{H}_\infty$. Finally, simulation results demonstrated the better performance of the proposed interval observers when compared to some benchmark CT and DT interval observers. Designing hybrid interval observers and considering unbounded unknown inputs will be considered in our future work.

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