# ASYMPTOTIC PROPERTIES OF MAXIMAL $p$-CORE $p'$-PARTITIONS

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ABSTRACT. For primes $p$, we study the maximal possible size of a $p$-core $p'$-partition (a partition with no hook lengths or parts divisible by $p$). McDowell recently proved that the maximum is attained by a unique partition, say $\Lambda_p$. Using his graph-theoretic description of $\Lambda_p$, we prove for $p > 10^6$ that
$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p},$$
which shows that $|\Lambda_p| \sim p^6/24$ as $p \to \infty$.

## 1. INTRODUCTION

A partition is called $p$-core if none of its hook lengths are divisible by $p$; a partition is called a $p'$-partition (or $p$-regular) if none of its parts are divisible by $p$. In this paper, we study the largest possible size of a $p$-core $p'$-partition.

Interest in $p$-core $p'$-partitions arises from studying zeros in the character table of the symmetric group $S_n$. In particular, McSpirit and Ono [3] studied entries of the character table indexed by two $p$-core partitions $\lambda$ and $\mu$. Using the result that $\chi_\lambda(\mu) = 0$ whenever $\mu$ has a part which is not a hook length in $\lambda$, they answered a question of McKay by studying pairs $(\lambda, \mu)$ of $p$-core partitions, where $\mu$ is also a $p'$-partition (see [3, Theorem 1.3]). For each prime $p$, they proved that there are finitely many $p$-core $p'$-partitions — in particular, they showed that any such partition $\mu$ must satisfy
$$|\mu| \le \frac{1}{24}(p^6 - 2p^5 + 2p^4 - 3p^2 + 2p).$$
Using this result, they concluded that for all sufficiently large $n$, any entry in the character table of $S_n$ indexed by two $p$-core partitions must be 0, providing a lower bound for the number of zeros indexed by $p$-core partitions.

The problem of determining the largest possible size of a $p$-core $p'$-partition was recently further studied by McDowell [2], who proved that any $p$-core $p'$-partition $\lambda$ must satisfy

(1) $$|\lambda| \le \frac{1}{24}(p^6 - 4p^5 + 5p^4 + 12p^3 - 42p^2 + 52p - 24),$$

and that there exists a $p$-core $p'$-partition $\lambda$ of size
$$|\lambda| = \frac{1}{96}(p^6 + 6p^4 - 12p^3 + 89p^2 - 120p - 48).$$

Furthermore, McDowell obtained an elegant graph-theoretic criterion describing the maximal $p$-core $p'$-partition (see [2, Theorem 4.8]), and showed that such a partition is unique for every odd prime $p$. We denote this unique partition by $\Lambda_p$.

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In view of these results, it is natural to study the asymptotics as $p \to \infty$ of $|\Lambda_p|$. Our work makes use of McDowell's description to improve the upper and lower bounds above. Our main result explicitly restricts $|\Lambda_p|$ to a narrow interval around $\frac{1}{24}p^6$.

**Theorem 1.** *For all primes $p > 10^6$, we have*

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

**Remark.** The range $p > 10^6$ was chosen so that the constants in the theorem above are nice.

This paper is organized as follows. First, we describe the characterization given by McDowell, which represents $\Lambda_p$ in terms of the longest walk on the additive residue graph $\mathcal{G}_p$. The challenge is to recover the size of $\Lambda_p$ from this description. After recalling this criterion, we establish a few lemmas about the sizes of individual pieces of this longest walk. Finally, we combine these lemmas to establish the upper and lower bounds on $|\Lambda_p|$.

## 2. A Graph-Theoretic Interpretation and Proof of Theorem 1

2.1. **Abacus Notation and a Graph-Theoretic Interpretation.** To describe $p$-core $p'$-partitions, we use *abacus notation* on a $p$-abacus, as described in [1, Chapter 2]. The abacus has $p$ vertical runners, labelled 0 through $p - 1$; positions on the abacus are ordered from left to right and top to bottom (starting with position 0), so that the $i$th row on runner $j$ is position $(i - 1)p + j$.

On this abacus, we place some *beads*, and call positions without beads *gaps*; we require position 0 to be a gap. Any such abacus corresponds to the partition where each bead contributes a part equal to the number of gaps preceding the bead. Under this correspondence, every partition can be represented by a unique configuration on the abacus.

**Lemma 2** ([1, Lemma 2.7.13]). *A partition is $p$-core if and only if all beads of its corresponding abacus are at the top of their runners (in other words, there is no bead with a gap above it).*

Given any $p$-core partition, for each $1 \leq i \leq p - 1$, we define the $i$th *bead multiplicity* $b_i$ to be the number of beads on runner $i$. By Lemma 2, the list of bead multiplicities uniquely determines a $p$-core partition.

Given the bead multiplicities of a partition, it is possible to explicitly compute the size of the partition.

**Lemma 3** ([2, Lemma 2.2]). *If $\lambda$ is a $p$-core partition with bead multiplicities $(b_1, b_2, \ldots, b_{p-1})$, then*

$$|\lambda| = -\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right)^2 + \frac{p}{2}\sum_{i=1}^{p-1} b_i^2 + \sum_{i=1}^{p-1}\left(i - \frac{p-1}{2}\right)b_i.$$

**Remark.** The lemma above is a minor reformulation of [2, Lemma 2.2].

It is possible to further describe the abacus configuration corresponding to a maximal $p$-core $p'$-partition.

**Lemma 4** ([2, Lemma 2.3]). *The abacus corresponding to any maximal $p$-core $p'$-partition has all beads at the right end of their rows (in other words, there is no bead with a gap to its right).*

We call an abacus *aligned* if it has the properties given in Lemmas 2 and 4, i.e., if all beads are topmost in their runners and rightmost in their rows. Given any aligned abacus, in a row with $i$ gaps followed by $p - i$ beads, all beads will contribute a part of the same size; this size is exactly $i$ greater than the parts contributed by the beads in the preceding row.

McDowell provided a graph-theoretic interpretation of this characterization. Consider the *additive residue graph* $\mathcal{G}_p$, where the vertices are the residues modulo $p$, and for each residue $x$ and every $1 \leq i \leq p - 1$, there is an edge from $x$ to $x + i$ modulo $p$, labelled $i$. Then every aligned abacus corresponds to a walk on the additive residue graph, where we start at 0, and for every row of $i$ gaps and $p - i$ beads, we walk along the unique edge labelled $i$ from the current vertex. The labels of the edges on this walk must be nondecreasing; conversely, any walk on $\mathcal{G}_p$ starting at 0 which has nondecreasing edge labels corresponds to a unique aligned abacus.

For every row of the abacus, the parts contributed by this row modulo $p$ are the residue visited at the end of its corresponding edge. In particular, the partition corresponding to an aligned abacus is a $p'$-partition if and only if the walk never returns to 0. In light of this property, we call a walk on $\mathcal{G}_p$ *valid* if it begins at 0, has nondecreasing edge labels, and never returns to 0.

McDowell proved the following interpretation of maximal $p$-core $p'$-partitions in terms of walks on $\mathcal{G}_p$.

**Theorem 5** ([2, Theorem 4.8]). *There is a unique $p$-core $p'$-partition of maximal size, which corresponds to the unique longest valid walk on $\mathcal{G}_p$.*

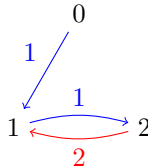Furthermore, McDowell also proved a characterization of the longest valid walk on $\mathcal{G}_p$.

**Theorem 6** ([2, Theorem 4.6]). *There is a unique longest valid walk on $\mathcal{G}_p$, and for every $1 \leq i \leq p - 1$, this walk must have an edge labelled $i$ incident to the vertex $p - 1$. In other words, the longest walk must consist of the following segments:*

(1) *Starting from 0, take $p - 1$ edges labelled 1 to reach $p - 1$.*
(2) *For each $1 \leq i \leq p - 2$, starting from $p - 1$, take some number (possibly 0) of edges labelled $i$, and then some number (possibly 0) of edges labelled $i + 1$, to return to $p - 1$ (without reaching 0).*
(3) *Starting from $p - 1$, take $p - 2$ edges labelled $p - 1$ to reach 1.*

**Example.** *The maximal 3-core 3'-partition is $\Lambda_3 = (4, 2, 2, 1, 1)$, which corresponds to the following abacus.*



*The corresponding walk on $\mathcal{G}_3$ is $(0, 1, 2, 1)$, consisting of two edges labelled 1 and one edge labelled 2.*



We will use the characterization of the walk corresponding to $\Lambda_p$ given by Theorems 5 and 6 to bound $|\Lambda_p|$.

2.2. **Some Lemmas.** We establish some notation to describe the walk in Theorem 6. For each $1 \leq i \leq p - 2$, define $x_i^{\max}$ and $y_i^{\max}$ to be the inverses of $i$ and $-(i + 1) \bmod p$ respectively (such that $x_i^{\max}$ and $y_i^{\max}$ are

in $\{1, 2, \ldots, p-1\}$) — in other words, $x_i^{\max}$ is the number of steps required to walk from $p-1$ to $0$ using edges labelled $i$, and $y_i^{\max}$ is the number of steps required to walk from $0$ to $p-1$ using edges labelled $i+1$. Then the segment described in (2) corresponding to $i$ avoids $0$ if and only if it consists of strictly less than $x_i^{\max}$ edges labelled $i$ and strictly less than $y_i^{\max}$ edges labelled $i+1$.

Define $(x_i, y_i)$ to be the solution to

$$ix + (i+1)y \equiv 0 \pmod{p}$$

over positive integers $x \le x_i^{\max}$ and $y \le y_i^{\max}$, with minimal $x + y$ (which exists because $(x_i^{\max}, y_i^{\max})$ is a solution, and is unique because $x + y$ uniquely determines $x$ and $y$ mod $p$). Then the segment of the walk described in (2) corresponding to $i$ must consist of exactly $x_i^{\max} - x_i$ edges labelled $i$, followed by $y_i^{\max} - y_i$ edges labelled $i+1$ — this is because the segment must consist of $x_i^{\max} - x$ edges labelled $i$ and $y_i^{\max} - y$ edges labelled $i+1$ for some $x$ and $y$ satisfying these conditions, and its length is maximized when $x + y$ is minimized. The reason to consider the *subtractions* $x_i$ and $y_i$ from the upper bounds on the path lengths (given by the fact that the path cannot visit $0$) rather than the lengths themselves is that proving $|\Lambda_p| \sim \frac{1}{24} p^6$ essentially amounts to showing that the subtractions are usually small; these subtractions turn out to be easier to work with than the original lengths.

In order to use this characterization to bound $|\Lambda_p|$, we first establish some results about the size of $(x_i, y_i)$ over all possible $i$.

**Lemma 7.** *For any $1 \le i \le p-2$, either*

$$i \equiv -\frac{s}{r+s} \pmod{p}$$

*for some relatively prime $0 < r, s < \sqrt{p}$, or*

$$i \equiv \frac{s}{r-s} \pmod{p}$$

*for some relatively prime $0 < r, s < \sqrt{p}$ which are not both $1$. Furthermore, there is at most one such pair $(r, s)$ with $i \equiv -\frac{s}{r+s} \pmod{p}$, and at most one such pair with $i \equiv \frac{s}{r-s} \pmod{p}$.*

*Proof.* To prove the existence of such a pair $(r, s)$, consider the residues

$$ia + (i+1)b \pmod{p}$$

across all pairs of integers $(a, b)$ with $0 < a, b < \sqrt{p} + 1$. Since there are strictly more than $p$ such pairs, some residue must be repeated; therefore we have

$$i(a_1 - a_2) + (i+1)(b_1 - b_2) \equiv 0 \pmod{p}$$

for some $(a_1, b_1) \ne (a_2, b_2)$. We cannot have $a_1 = a_2$ or $b_1 = b_2$, as one equality would imply the other (since $i$ and $i+1$ are nonzero mod $p$), so $0 < |a_1 - a_2|, |b_1 - b_2| < \sqrt{p}$.

If $a_1 - a_2$ and $b_1 - b_2$ have the same sign, then we have

$$ir + (i+1)s \equiv 0 \pmod{p}$$

and therefore $i \equiv -\frac{s}{r+s} \pmod{p}$ for some $0 < r, s < \sqrt{p}$. Meanwhile, if $a_1 - a_2$ and $b_1 - b_2$ have opposite sign, then we have

$$ir - (i+1)s \equiv 0 \pmod{p}$$

and therefore $i \equiv \frac{s}{r-s} \pmod{p}$ for some $0 < r, s < \sqrt{p}$. In either case, if $r$ and $s$ are not relatively prime, we can divide both by $\gcd(r, s)$ to get the desired result.

To prove uniqueness, note that if there were two pairs $(r_1, s_1)$ and $(r_2, s_2)$ corresponding to the same value of $i$ (in either case), then we would have

$$\frac{r_1}{s_1} \equiv \frac{r_2}{s_2} \pmod{p},$$

which would imply $r_1 s_2 - s_1 r_2 \equiv 0 \pmod{p}$. But we have

$$|r_1 s_2 - s_1 r_2| < \max(r_1 s_2, s_1 r_2) < p,$$

so we must have $r_1 s_2 = s_1 r_2$; then since $\gcd(r_1, s_1) = \gcd(r_2, s_2) = 1$, this implies $(r_1, s_1) = (r_2, s_2)$. □

Let $S$ denote the set of residues $1 \leq i \leq p - 2$ for which $i \equiv \frac{s}{r-s} \pmod{p}$ for some relatively prime $0 < r, s < \sqrt{p}$, and let $T$ denote the set of residues $1 \leq i \leq p - 2$ which cannot be written in this form. Then by Lemma 7, every residue $i \in T$ can instead be written as $i \equiv -\frac{s}{r+s} \pmod{p}$ for some relatively prime $0 < r, s < \sqrt{p}$.

**Lemma 8.** *If $i \in S$ with $i \equiv \frac{s}{r-s} \pmod{p}$, then*

$$\frac{p}{\max(r, s)} < x_i + y_i < \frac{p}{\max(r, s)} + \max(r, s) - 1.$$

*Proof.* We can rewrite the equation $ix + (i + 1)y \equiv 0 \pmod{p}$ as

$$sx + ry \equiv 0 \pmod{p}.$$

Assume without loss of generality that $s > r$ (the proof of the case $s < r$ is symmetric). Then we must have

$$s(x + y) > sx + ry \geq p$$

and therefore $x + y > \frac{p}{s}$ for any positive integer solution $(x, y)$. On the other hand, by the Chinese Remainder Theorem there exists some $0 < a < rs$ such that $s \mid (p - a)$ and $r \mid a$; then

$$(x, y) = \left( \frac{p - a}{s}, \frac{a}{r} \right)$$

is a solution, and satisfies $x < \frac{p}{s}$ and $y \leq s - 1$.

It remains to check that this solution satisfies $x \leq x_i^{\max}$ and $y \leq y_i^{\max}$. To check the first statement, we have $x_i^{\max} \equiv \frac{r-s}{s} \pmod{p}$, so

$$sx_i^{\max} - r + s \equiv 0 \pmod{p}.$$

Since the left-hand side is positive, it must then be at least $p$, which implies

$$x_i^{\max} \geq \frac{p + r - s}{s} > \frac{p}{s} - 1.$$

Then since $x$ and $x_i^{\max}$ are both integers and $x < x_i^{\max} - 1$, we must have $x \leq x_i^{\max}$.

Similarly, we have $y_i^{\max} \equiv -\frac{r-s}{r} \pmod{p}$, so

$$ry_i^{\max} + r - s \equiv 0 \pmod{p}.$$

If $r = 1$, then $y_i^{\max} = s - 1 \geq y$. Otherwise, we cannot have $ry_i^{\max} + r - s = 0$ (as this implies $r \mid s$, but $r$ and $s$ are relatively prime), so $ry_i^{\max} + r - s \geq p$, and

$$y_i^{\max} \geq \frac{p + s - r}{r} > \frac{p}{r} > s.$$

This means

$$x_i + y_i \leq x + y < \frac{p}{s} + s - 1,$$

as desired. □

**Remark.** In fact, it is always true that $(x_i, y_i) = (x, y)$ for the solution described above — we must have $sx_i + ry_i = p$, since otherwise $x_i + y_i > \frac{2p}{s} > \frac{p}{s} + s$. Then $(x_i, y_i) = (\frac{p-a}{s}, \frac{a}{r})$ for some positive integer $a$, and $x_i + y_i$ is minimized when $a$ is minimized, which occurs at our choice of $a$. Note that the stronger bound

$$x_i + y_i < \frac{p}{\max(r, s)} + \max(r, s) - \min(r, s)$$

is true as well, using the fact that $x_i + y_i = \frac{p}{s} + a \left( \frac{1}{r} - \frac{1}{s} \right) < \frac{p}{s} + s - r$. However, this does not have a large impact on the resulting bounds, so we use the weaker bound to simplify computations.

**Lemma 9.** *If $i \in T$ with $i \equiv -\frac{s}{r+s} \pmod{p}$, then $x_i + y_i \leq r + s$.*

*Proof.* First, the equation $ix + (i+1)y \equiv 0 \pmod{p}$ can be rewritten as

$$-sx + ry \equiv 0 \pmod{p},$$

so $(x, y) = (r, s)$ is a solution. To check that it satisfies $x \leq x_i^{\max}$, we have $x_i^{\max} \equiv -\frac{r+s}{s} \pmod{p}$, so

$$sx_i^{\max} + r + s \equiv 0 \pmod{p}.$$

But if $x_i^{\max} \leq r - 1$, then we would have

$$sx_i^{\max} + r + s \leq rs + r < \sqrt{p}(\sqrt{p} - 1) + \sqrt{p} = p,$$

since we cannot have $r = s = \lfloor\sqrt{p}\rfloor$ (as $r$ and $s$ are relatively prime); this is a contradiction. The same proof shows that we cannot have $y_i^{\max} \leq s - 1$.

This implies $(r, s)$ is a valid solution for $(x, y)$, which means

$$x_i + y_i \leq r + s,$$

as desired. $\qquad\square$

**Remark.** It is again true that we always have $(x_i, y_i) = (r, s)$. To prove this, assume for contradiction there is a solution $(x, y)$ with $x + y < r + s$. Then $x$ and $y$ cannot both be less than $\sqrt{p}$, as otherwise by uniqueness in Lemma 7, we would have $\frac{1}{\gcd(x,y)}(x, y) = (r, s)$. On the other hand, we must have $x < r + \sqrt{p}$ and $y < s + \sqrt{p}$ (as otherwise $x$ or $y$ would be greater than $r + s$). Then considering the equation

$$i(x - r) + (i+1)(y - s) \equiv 0 \pmod{p}$$

gives that $i \in S$ (since $x - r$ and $y - s$ have opposite sign, and $0 < |x - r|, |y - s| < \sqrt{p}$), contradiction.

Finally, we will also need the following number theoretic lemma (which arises from the fact that the residues $i \in S$ correspond exactly to pairs of distinct relatively prime $(r, s)$ with $0 < r, s < \sqrt{p}$).

**Lemma 10.** *For any positive integer $n$, we have*

$$\sum_{m=1}^{n} \frac{\varphi(m)}{m} > \frac{3}{5}n - 6,$$

*where $\varphi$ denotes the Euler totient function.*

**Remark.** In general, one can obtain better lower bounds than Lemma 10. However, for our purposes, we are content with this simple argument.

*Proof.* By using the fact that $\varphi = \mu * \mathrm{id}$ (where $*$ denotes Dirichlet convolution, $\mu$ denotes the Möbius function, and id denotes the identity map sending every positive integer to itself) and swapping the order of summation, we have

$$\sum_{m=1}^{n} \frac{\varphi(m)}{m} = \sum_{m=1}^{n} \frac{1}{m} \sum_{d \mid m} \mu(d) \cdot \frac{m}{d}$$

$$= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \cdot \left\lfloor \frac{n}{d} \right\rfloor$$

$$> n \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=1}^{n} \frac{1}{d} - n \sum_{d=n+1}^{\infty} \frac{1}{d^2}.$$

To calculate the first term, let 1 denote the function sending every positive integer to 1. Then $(\mu * 1)(n)$ is 1 if $n = 1$ and 0 otherwise, so $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \cdot \sum_{d=1}^{\infty} \frac{1}{d^2} = 1$, and therefore

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

This provides the bound

$$\sum_{m=1}^{n} \frac{\varphi(m)}{m} > n \cdot \frac{6}{\pi^2} - \log n - 2 > \frac{3}{5}n - 6. \qquad \square$$

2.3. **Proof of Theorem 1.** Suppose that $p > 10^6$ is prime. We use the above lemmas to establish bounds on $|\Lambda_p|$.

For convenience, we first define a few more pieces of data associated with the abacus corresponding to $|\Lambda_p|$. Recall that $b_1, \ldots, b_{p-1}$ denote the bead multiplicities of $\Lambda_p$. By Lemma 4, we can also define the *row multiplicities* $m_1, \ldots, m_{p-1}$, where $m_i$ is the number of rows consisting of $i$ gaps followed by $p - i$ beads. Then we have

$$b_i = m_1 + m_2 + \cdots + m_i$$

for all $1 \le i \le p - 1$. By Theorems 5 and 6, we have $m_1 = p - 1$, $m_{p-1} = p - 2$, and for all $2 \le i \le p - 2$,

$$m_i = y_{i-1}^{\max} - y_{i-1} + x_i^{\max} - x_i = p - y_{i-1} - x_i.$$

As noted by McDowell, the row multiplicities have useful symmetry — we have $(x_i, y_i) = (y_{p-1-i}, x_{p-1-i})$ for all $1 \le i \le p - 2$ (since the conditions used to define them are identical), which means $m_i = m_{p-i}$ for all $2 \le i \le p - 2$ [2, Corollary 4.9].

Now define $d_i := p - m_i$ for all $1 \le i \le p - 1$, so that $d_1 = 1$, $d_{p-1} = 2$, and $d_i = y_{i-1} + x_i$ for all $2 \le i \le p - 2$. Finally, define $c_i := d_1 + d_2 + \cdots + d_i$, so that

$$b_i = m_1 + \cdots + m_i = ip - c_i,$$

and define

$$c := \sum_{i=1}^{p-2}(x_i + y_i) = 2 + \sum_{i=2}^{p-2}(y_{i-1} + x_i).$$

Then we have $c_{p-1} = c + 1$, while the symmetry property implies that

$$c_i + c_{p-1-i} = c$$

for all $1 \le i \le p - 2$.

Intuitively, the variables $d_i$ and $c_i$ represent how far $m_i$ and $b_i$ are from their crude upper bounds — in particular, McDowell obtained the upper bound (1) by using the fact that $m_i \le p - 2$ for all $i \ge 2$. Our strategy is therefore to show that these subtractions are small. More precisely, we use the previous lemmas to establish upper and lower bounds on $c$ which are both on the order of $p\sqrt{p}$, and then translate these bounds on $c$ to bounds on $|\Lambda_p|$ via Lemma 3.

**Lemma 11.** *We have[1] that $c < \frac{11}{3}p\sqrt{p}$.*

*Proof.* Separate the sum as

$$c = \sum_{i=1}^{p-2}(x_i + y_i) = \sum_{i \in S}(x_i + y_i) + \sum_{i \in T}(x_i + y_i).$$

For the sum over $i \in S$, if $i \equiv \frac{s}{r-s} \pmod{p}$ as given by Lemma 7 with $\max(r, s) = m$, then we have

$$x_i + y_i \le \frac{p}{m} + m - 1$$

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[1]This lemma is true for all $p \ge 17$.

by Lemma 8. For each $1 \leq m < \sqrt{p}$, there are less than $2m$ pairs $(r, s)$ with $\max(r, s) = m$, so we have

$$\sum_{i \in S} (x_i + y_i) < \sum_{m=1}^{\lfloor \sqrt{p} \rfloor} 2m \left( \frac{p}{m} + m - 1 \right)$$

$$< 2p\sqrt{p} + 4 \sum_{m=1}^{\lfloor \sqrt{p} \rfloor} \binom{m}{2}$$

$$= 2p\sqrt{p} + 4 \binom{\lfloor \sqrt{p} \rfloor + 1}{3}$$

$$< \frac{8}{3} p\sqrt{p}.$$

Meanwhile, for the sum over $i \in T$, if $i \equiv -\frac{s}{r+s} \pmod{p}$ as given by Lemma 7, then we have $x_i + y_i \leq r + s$ by Lemma 9. Since $\gcd(r, s) = 1$ for any such pair $(r, s)$, for any fixed $s > 2$ there are less than $\sqrt{p} - 1$ values of $r$ (as $r$ cannot equal $s$); meanwhile for $s = 2$ there are less than $\sqrt{p} - 2$ values of $r$ (as $r$ cannot equal 2 or 4, which are both less than $\sqrt{p}$), and for $s = 1$ there are less than $\sqrt{p}$ values of $r$. Combining these estimates, we have

$$\sum_{i \in T} (x_i + y_i) \leq \sum_{(r,s)} (r + s) < 2(\sqrt{p} - 1) \cdot \frac{\sqrt{p}(\sqrt{p} + 1)}{2} < p\sqrt{p}.$$

Combining the two sums gives $c < \frac{11}{3} p\sqrt{p}$. □

We now obtain a lower bound on $c$.

**Lemma 12.** *We have that $c > \frac{6}{5} p\sqrt{p} - 16p$.*

*Proof.* We have

$$c \geq \sum_{i \in S} (x_i + y_i) > \sum_{(r,s)} \frac{p}{\max(r, s)}$$

by Lemma 8. For each $2 \leq m < \sqrt{p}$, there are $2\varphi(m)$ pairs of relatively prime distinct $(r, s)$ with $0 < r, s < \sqrt{p}$ and $\max(r, s) = m$, and by uniqueness in Lemma 7, each corresponds to a different value of $i \in S$. This means

$$c > \sum_{m=2}^{\lfloor \sqrt{p} \rfloor} \frac{p}{m} \cdot 2\varphi(m) > 2p \left( \frac{3}{5} \lfloor \sqrt{p} \rfloor - 7 \right) > \frac{6}{5} p\sqrt{p} - 16p,$$

using the estimate given by Lemma 10. □

Finally, we will also need the following upper bound on an intermediate sum.

**Lemma 13.** *We have[2] that $c_{\lfloor p/18 \rfloor} < \frac{2}{5} p\sqrt{p} + p$.*

*Proof.* We have

$$c_{\lfloor p/18 \rfloor} = 1 + \sum_{i=2}^{\lfloor p/18 \rfloor} (y_{i-1} + x_i) < \sum_{i=1}^{\lfloor p/18 \rfloor} (x_i + y_i).$$

By Lemmas 8 and 9, if $i \in T$ then $x_i + y_i < 2\sqrt{p}$; on the other hand, if $i \in S$ is written as $i \equiv \frac{r}{r-s} \pmod{p}$ with $\max(r, s) = m$, then $x_i + y_i < \frac{p}{m} + m - 1$. For each $1 \leq m < \sqrt{p}$, there are at most $m$ pairs $(r, s)$ with $\max(r, s) = m$ corresponding to indices $i < \frac{p}{18}$, since swapping $r$ and $s$ maps $i \mapsto p - 1 - i$.

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[2]This lemma is true for all $p > 256$.

Let $t := \lfloor \sqrt{p}/3 \rfloor$. Then we can bound one entry $x_i + y_i$ above by $p$, two entries by $\frac{p}{2} + 1$, three by $\frac{p}{3} + 2$, and so on, up to $t$ entries by $\frac{p}{t} + (t-1)$ — note that $1 + 2 + \cdots + t > \frac{t^2}{2} > \frac{p}{18}$, and $\frac{p}{t} + (t-1) > 2\sqrt{p}$. This means

$$\sum_{i=1}^{\lfloor p/18 \rfloor} (x_i + y_i) < \sum_{m=1}^{t} m \left( \frac{p}{m} + m - 1 \right)$$
$$= tp + 2 \binom{t+1}{3}$$
$$< p \left( \frac{\sqrt{p}}{3} + 1 \right) + \frac{1}{3} \left( \frac{\sqrt{p}}{3} + 1 \right)^3.$$

This expression is less than $\frac{2}{5} p\sqrt{p} + p$, as $p > 10^6 > 256$. □

Now we are in a position to establish the claimed inequalities in Theorem 1.

*Proof of Theorem 1.* In order to establish both bounds, we use the formula

$$(2) \qquad |\Lambda_p| = -\frac{1}{2} \left( \sum_{i=1}^{p-1} b_i \right)^2 + \frac{p}{2} \sum_{i=1}^{p-1} b_i^2 + \sum_{i=1}^{p-1} \left( i - \frac{p-1}{2} \right) b_i$$

given by Lemma 3. For both directions, we first have

$$(3) \qquad \sum_{i=1}^{p-1} b_i = \sum_{i=1}^{p-1} (ip - c_i) = \frac{1}{2} p^2 (p-1) - \frac{1}{2} pc - 1,$$

using the fact that $c_i + c_{p-1-i} = c$ for all $1 \le i \le p-2$, and $c_{p-1} = c + 1$.

We first prove the lower bound

$$|\Lambda_p| > \frac{1}{24} p^6 - p^5 \sqrt{p}.$$

For the first term of (2), we use the bound

$$-\frac{1}{2} \left( \sum_{i=1}^{p-1} b_i \right)^2 > -\frac{1}{2} \left( \frac{p^3 - pc}{2} \right)^2 = -\frac{1}{8} p^6 + \frac{1}{4} p^4 c - \frac{1}{8} p^2 c^2$$

from (3). For the second term of (2), we use the bound

$$b_i^2 = (ip - c_i)^2 > i^2 p^2 - 2ipc$$

for all $1 \le i \le p-1$ (this is clear for $1 \le i \le p-2$ as $c_i \le c$, while for $i = p-1$, it follows from the fact that $(c+1)^2 > 2p(p-1)$, as $p > 10^6 > 250$, using the bound in Lemma 12), which gives

$$\frac{p}{2} \sum_{i=1}^{p-1} b_i^2 > \frac{1}{12} p^4 (p-1)(2p-1) - \frac{1}{2} p^3 (p-1)c.$$

Finally, the third term in (2) must be positive, as for each $1 \le i \le \frac{p-1}{2}$ we have $b_{p-1-i} \ge b_i$, which means

$$\left( i - \frac{p-1}{2} \right) b_i + \left( p - 1 - i - \frac{p-1}{2} \right) b_{p-1-i} \ge 0.$$

Combining these bounds and using (2), we have

$$|\Lambda_p| > \frac{1}{24} p^6 - \frac{1}{4} p^4 c - \frac{1}{8} p^2 c^2 - \frac{1}{4} p^5.$$

Using the bound $c < \frac{11}{3} p\sqrt{p} < 4p\sqrt{p}$, we have

$$|\Lambda_p| > \frac{1}{24} p^6 - \frac{11}{12} p^5 \sqrt{p} - \frac{9}{4} p^5 > \frac{1}{24} p^6 - p^5 \sqrt{p},$$

where the second inequality holds as $p > 10^6 > 729$.

We now prove the upper bound

$$|\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

The first term of (2) is given by

$$-\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right)^2 < -\frac{1}{8}(p^3 - pc - p^2 - 2)^2$$

$$< -\frac{1}{8}p^6 + \frac{1}{4}p^4c - \frac{1}{8}p^2c^2 + \frac{1}{4}p^5$$

by (3). To bound the second term of (2), we pair terms — for each $1 \le i \le \frac{p-1}{2}$, we have

$$b_i^2 + b_{p-1-i}^2 = (ip - c_i)^2 + ((p-1-i)p - c_{p-1-i})^2$$

$$< i^2p^2 + (p-1-i)^2p^2 - p(c - 2c_i)(p-1-2i) - cp(p-1) + c^2,$$

where we used the fact that $c_i^2 + c_{p-1-i}^2 < (c_i + c_{p-1-i})^2 = c^2$. For all $1 \le i \le \frac{p-1}{2}$, we have $c - 2c_i \ge 0$ and $p - 1 - 2i \ge 0$. Furthermore, for all $i < \frac{p}{18}$, by Lemmas 12 and 13 we have

$$c - 2c_i > \frac{2}{5}p\sqrt{p} - 18p,$$

and $p - 1 - 2i \ge \frac{8p}{9} - 1$. Finally, we have $b_{p-1}^2 < p^2(p-1)^2$. Combining these bounds gives

$$\sum_{i=1}^{p-1} b_i^2 < \frac{p^3(p-1)(2p-1)}{6} - \frac{p-2}{2} \cdot cp(p-1) - p\left(\frac{1}{18}p - 1\right)\left(\frac{2}{5}p\sqrt{p} - 18p\right)\left(\frac{8}{9}p - 1\right) + \frac{p-2}{2} \cdot c^2$$

$$< \frac{1}{3}p^5 - \frac{1}{2}p^3c - \frac{1}{60}p^4\sqrt{p} + \frac{1}{2}pc^2 + \frac{1}{4}p^4 + p^2c + \frac{1}{6}p^3,$$

using the bounds $\frac{1}{18}p - 1 > \frac{1}{20}p$ and $\frac{8}{9}p - 1 > \frac{5}{6}p$, which hold as $p > 10^6 > 180$. So the second term in (2) is bounded above by

$$\frac{p}{2}\sum_{i=1}^{p-1} b_i^2 < \frac{1}{6}p^6 - \frac{1}{4}p^4c - \frac{1}{120}p^5\sqrt{p} + \frac{1}{4}p^2c^2 + \frac{1}{8}p^5 + \frac{1}{2}p^3c + \frac{1}{12}p^3.$$

Finally, the third term in (2) is bounded above by

$$\sum_{i=1}^{p-1}\left(i - \frac{p-1}{2}\right)b_i < \sum_{i=1}^{p-1} i \cdot ip < \frac{1}{3}p^4.$$

Combining these bounds and using (2), we have

$$|\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{120}p^5\sqrt{p} + \frac{1}{8}p^2c^2 + \frac{5}{8}p^5 + \frac{1}{2}p^3c + \frac{1}{3}p^4 + \frac{1}{12}p^3.$$

Using the bound $c < 4p\sqrt{p}$ then gives

$$|\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{120}p^5\sqrt{p} + \frac{23}{8}p^5 + 2p^4\sqrt{p} + \frac{1}{3}p^4 + \frac{1}{12}p^3 < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p},$$

where the second inequality holds as $p > 10^6$.                                                                    $\square$

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