An elementary alternative to ECH capacities

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The ECH capacities are a sequence of numerical invariants of symplectic four-manifolds which give (sometimes sharp) obstructions to symplectic embeddings. These capacities are defined using embedded contact homology, and establishing their basic properties currently requires Seiberg-Witten theory. In this note we define a new sequence of symplectic capacities in four dimensions using only basic notions of holomorphic curves. The new capacities satisfy the same basic properties as ECH capacities and agree with the ECH capacities for the main examples for which the latter have been computed, namely convex and concave toric domains. The new capacities are also useful for obstructing symplectic embeddings into closed symplectic four-manifolds. This work is inspired by a recent preprint of McDuff-Siegel (1) giving a similar elementary alternative to symplectic capacities from rational SFT.

symplectic embeddings | symplectic capacities | ECH capacities

We define a *symplectic capacity* to be a function c which maps some set of symplectic manifolds (possibly noncompact, disconnected, and/or with boundary or corners) to $[0,\infty]$. We assume the following two properties:

(Monotonicity) If (X,ω) and (X',ω') are symplectic manifolds of the same dimension for which c is defined, and if there exists a symplectic embedding $\varphi:(X,\omega)\to(X',\omega')$, then $c(X,\omega)\le c(X',\omega')$.

(Conformality) If $r>0$ then $c(X,r\omega)=rc(X,\omega)$.

Various symplectic capacities are used to study symplectic embedding problems. In particular, symplectic capacities give obstructions to symplectic embeddings via the Monotonicity property, because under the hypotheses of this property, if $c(X,\omega)>c(X',\omega')$, then a symplectic embedding $(X,\omega)\to(X',\omega')$ cannot exist. See e.g. (2) for a survey of symplectic capacities.

Perhaps the most basic example of a symplectic capacity is the *Gromov width* c_{Gr}. For $a>0$, define the ball

$$B^{2n}(a)=\{z\in\mathbb{C}^n \mid \pi|z|^2\le a\}$$

with the restriction of the standard symplectic form $\sum_{i=1}^n dx_i\,dy_i$ on $\mathbb{C}^n=\mathbb{R}^{2n}$. If $\dim(X)=2n$, then $c_{\mathrm{Gr}}(X,\omega)$ is defined to be the supremum over a such that there exists a symplectic embedding $B^{2n}(a)\to(X,\omega)$. The celebrated Gromov nonsqueezing theorem (3) is equivalent to the statement that the cylinder

$$Z^{2n}(a)=\{z\in\mathbb{C}^n \mid \pi|z_1|^2\le a\}$$

has Gromov width equal to a.

While the Gromov width has a very simple definition, it is difficult to use by itself for studying symplectic embedding problems, since it is defined in terms of symplectic embeddings. In general, there is a gap we would like to bridge between (1) symplectic capacities with simple geometric definitions that can be hard to compute, such as the Gromov width; and (2) symplectic capacities defined using Floer-theoretic or related machinery which are more computable, but whose definition requires substantial technical work.

One example of the latter type of capacity is the sequence of Ekeland-Hofer capacities defined using variational methods in (4), or the conjecturally equivalent capacities defined in (5) using positive S^1-equivariant symplectic homology.

Another example, which is the focus of the present paper, is the sequence of ECH capacities introduced in (6); see the expositions in (7, 8) and the review below. Let (X,ω) be a symplectic four-manifold, not necessarily closed or connected. The ECH capacities of (X,ω) are a sequence of real numbers

$$0=c_0^{\mathrm{ECH}}(X,\omega)<c_1^{\mathrm{ECH}}(X,\omega)\le c_2^{\mathrm{ECH}}(X,\omega)\le\cdots\le+\infty.$$

Monotonicity of ECH capacities means that if (X',ω') is another symplectic four-manifold, and if there exists a symplectic embedding $(X,\omega)\to(X',\omega')$, then

$$c_k^{\mathrm{ECH}}(X,\omega)\le c_k^{\mathrm{ECH}}(X',\omega') \qquad [1]$$

for all k. This obstruction is known to be sharp in some cases. For example, McDuff (9) showed that if X and X' are open ellipsoids in $\mathbb{R}^4$ with the restriction of the standard symplectic form, then there exists a symplectic embedding $X\to X'$ if and only if $c_k^{\mathrm{ECH}}(X)\le c_k^{\mathrm{ECH}}(X')$ for all k. More generally, Cristofaro-Gardiner (10) showed that this sharpness result extends to the case when X is an open "concave toric domain", and X' is a "convex toric domain", in $\mathbb{R}^4$; see the definitions below. The ECH capacities are defined using embedded contact homology (8), and the proof of the symplectic embedding obstruction in Eq. (1) uses cobordism maps on embedded

Significance Statement

Symplectic geometry is the basic geometry underlying classical mechanics. The Gromov nonsqueezing theorem from the 1980s showed that it is a subtle problem to determine when one domain in phase space can be embedded into another while preserving the symplectic structure. Since then various "symplectic capacities" have been developed to study this question. In particular the ECH capacities give sometimes sharp results in the four-dimensional case. This article introduces a new sequence of symplectic capacities which have roughly the same power as the ECH capacities, but which are defined in a more elementary way. Variants of this construction are expected to lead to further progress on understanding symplectic embeddings.

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contact homology, which currently need to be defined using Seiberg-Witten theory*.

More recently, Siegel (13) used rational symplectic field theory (SFT) (14) to define a set of symplectic capacities which are well suited to studying stabilized symplectic embedding problems. These capacities are not yet rigorously defined because the technical foundations of rational SFT are still a work in progress. However McDuff-Siegel (1) showed that the key applications of Siegel's capacities can be proved rigorously, using a replacement of some of Siegel's capacities by an alternate set of capacities with a more elementary definition directly in terms of holomorphic curves with local tangency constraints.

More generally, one can hope that capacities extracted from Floer theories can be understood geometrically without passing through Floer theory, or at least can be replaced by more elementary capacities with the same applications. Roughly speaking, following the idea of the McDuff-Siegel capacities, the elementary capacities that we have in mind are answers to versions of the following question: *What is the minimal energy for which holomorphic curves satisfying certain conditions are guaranteed to exist?*

The purpose of this article is to pursue this direction for the ECH capacities. Namely we give an elementary definition of a sequence of symplectic capacities for symplectic four-manifolds, which we denote by c_k, which are defined directly in terms of holomorphic curves constrained to pass through k points. We show that the capacities c_k have the same basic properties as ECH capacities and agree with them in important examples. In particular, this allows some of the applications of ECH capacities to be re-proved without using Seiberg-Witten theory. The capacities c_k also give good obstructions to symplectic embeddings into some closed symplectic four-manifolds with $b_2^+ = 1$ such as $\mathbb{C}P^2$ or $S^2 \times S^2$, whose ECH capacities are not known. At the end, we define an even simpler sequence of capacities $\widehat{c}_k$ in any dimension, which conjecturally agree with the capacities c_k in the main four-dimensional cases.

Definition of the capacities c_k

We begin by recalling some basic definitions.

Let Y be a three-manifold and let λ be a contact form on Y. Let $\xi = \operatorname{Ker}(\lambda)$ denote the associated contact structure, and let R denote the associated Reeb vector field. Define an *orbit set* to be a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the α_i are distinct simple Reeb orbits, and the m_i are positive integers. Define the *symplectic action* of the orbit set α by

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda.$$

The contact form λ is *nondegenerate* if every Reeb orbit (simple or multiply covered) is nondegenerate, i.e. the linearized return map does not have 1 as an eigenvalue.

We say that an almost complex structure J on $\mathbb{R} \times Y$ is *λ-compatible* if $J\partial_s = R$, where s denotes the $\mathbb{R}$ coordinate; J sends the contact structure ξ to itself, rotating positively in the sense that $d\lambda(v, Jv) > 0$ for every nonzero $v \in \xi$; and J is $\mathbb{R}$-invariant.

*Heuristically one might expect to define such a cobordism map just by counting holomorphic curves. Although this is possible in some special cases (11, 12), in general there are severe transversality difficulties with multiply covered curves; see (8, §5.5) for explanation.

We define a four-dimensional *Liouville domain* to be a compact symplectic four-manifold (X, ω) with boundary Y such that there exists a primitive of ω which restricts to a contact form λ on Y, for which the contact orientation of Y agrees with the boundary orientation of ∂X. A basic example is a star-shaped domain in $\mathbb{R}^4$. Here a "star-shaped domain" is a compact domain in $\mathbb{R}^{2n}$ with smooth boundary which is transverse to the radial vector field, with the restriction of the standard symplectic form. We say that the Liouville domain (X, ω) is *nondegenerate* if the contact form λ on Y is nondegenerate; this notion does not depend on the choice of primitive of ω.

Given a Liouville domain as above, and given $\varepsilon > 0$, a choice of primitive of ω determines a neighborhood N_ε of Y in X, and an identification

$$N_\varepsilon \simeq (-\varepsilon, 0] \times Y, \qquad [2]$$

under which $\omega|_{N_\varepsilon}$ is identified with $d(e^s\lambda)$, where s denotes the $(-\varepsilon, 0]$ coordinate. Using this identification, we can glue to obtain a smooth manifold

$$\overline{X} = X \cup_Y ([0, \infty) \times Y), \qquad [3]$$

which we call the "symplectization completion" of X. This manifold has a symplectic form $\overline{\omega}$ which agrees with ω on X and with $d(e^s\lambda)$ on $[0, \infty) \times Y$. Strictly speaking, this completion depends on the choice of primitive of ω, which we suppress from the notation.

We say that an almost complex structure J on $\overline{X}$ is *cobordism-compatible* if $J|_X$ is ω-compatible, and if $J|_{[0,\infty)\times Y}$ is the restriction of a λ-compatible almost complex structure on $\mathbb{R} \times Y$.

Define an *admissible symplectic four-manifold* to be a (possibly disconnected) compact symplectic four-manifold (X, ω) such that each component is either closed or a nondegenerate Liouville domain. Define $\overline{X}$ to be the union of the closed components and the symplectization completions of the Liouvile domain components. Define $\mathcal{J}(\overline{X}, \omega)$ to be the set of almost complex structures on $\overline{X}$ which are ω-compatible on the closed components and cobordism-compatible on the completed Liouville domain components.

Let $J \in \mathcal{J}(\overline{X}, \omega)$. We consider holomorphic maps

$$u : (\Sigma, j) \longrightarrow (\overline{X}, J)$$

where Σ is a punctured compact Riemann surface (possibly disconnected), such that for each puncture in Σ, there is a Reeb orbit γ on ∂X and a neighborhood of the puncture mapping asymptotically to $[0, \infty) \times \gamma$ as $s \to \infty$. To avoid trivialities we assume that the restriction of u to each component of the domain Σ is nonconstant. Let $\mathcal{M}^J(\overline{X})$ denote the set of J-holomorphic maps as above, modulo reparametrization by biholomorphic maps $(\Sigma', j') \xrightarrow{\simeq} (\Sigma, j)$. If $x_1, \ldots, x_k \in X$ are distinct points, let $\mathcal{M}^J(\overline{X}; x_1, \ldots, x_k)$ denote the set of $u \in \mathcal{M}^J(\overline{X})$ such that $x_1, \ldots, x_k \in u(\Sigma)$.

Define the *energy* $\mathcal{E}(u)$ as follows. If Σ is connected and u maps to a closed component of X, then $\mathcal{E}(u) = \int_\Sigma u^*\omega$. If Σ is connected and u maps to a completed Liouville domain component, then $\mathcal{E}(u)$ is the sum over the punctures of Σ of the symplectic actions of the corresponding Reeb orbits. If Σ is disconnected, then $\mathcal{E}(u)$ is the sum of the energies of the connected components.

Definition 1. Let (X,ω) be an admissible symplectic four-manifold and let k be a nonnegative integer. Define

$$c_k(X,\omega) = \sup_{\substack{J\in\mathcal{J}(\overline{X})\\ x_1,\ldots,x_k\in X \text{ distinct}}} \inf_{u\in\mathcal{M}^J(\overline{X};x_1,\ldots,x_k)} \mathcal{E}(u) \in [0,\infty]. \qquad [4]$$

Remark 2. A key observation, which avoids various technical difficulties, is that in Eq. (4), we can restrict attention to holomorphic curves u that *do not have any multiply covered components*[†]. This is because we can always replace a multiply covered component by the underlying somewhere injective curve to reduce energy without invalidating the point constraints.

The following lemma will be proved below:

Lemma 3. *Let (X,ω) and (X',ω') be admissible symplectic four-manifolds and let k be a nonnegative integer. If there exists a symplectic embedding $\varphi:(X,\omega)\to(X',\omega')$, then*

$$c_k(X,\omega)\le c_k(X',\omega').$$

To extend the definition of c_k to more general symplectic four-manifolds, we use the following basic procedure; compare (6, §4.2).

Definition 4. Let (X',ω') be any symplectic four-manifold (possibly noncompact, disconnected, and/or with boundary or corners) and let k be a nonnegative integer. Define

$$c_k(X',\omega') = \sup\{c_k(X,\omega)\}$$

where the supremum is over admissible symplectic four-manifolds (X,ω) for which there exists a symplectic embedding $\varphi:(X,\omega)\to(X',\omega')$.

It follows from Lemma 3 that Definition 4 agrees with Definition 1 when (X',ω') is already an admissible symplectic four-manifold.

Remark 5. The definition of c_k is inspired by the paper of McDuff-Siegel (1), which gives a similar elementary definition of a sequence of symplectic capacities $\widetilde{\mathfrak{g}}_k$, as an alternative to symplectic capacities that were defined in (13) using rational SFT (14). The capacities $\widetilde{\mathfrak{g}}_k$ are defined for symplectic manifolds of any dimension using genus zero holomorphic curves that are constrained to have contact of order k with a local divisor.

Some variants of Definition 1 are possible. For example one could require each component of the domain of u to have genus zero; the resulting capacities may be related to the capacities $\widetilde{\mathfrak{g}}_k$.

Proof of the monotonicity lemma

To begin discussing the basic properties of the capacities c_k, we now prove Lemma 3.

The following notation will be useful. Let (X,ω) be an admissible symplectic four-manifold and let $\alpha=\{(\alpha_i,m_i)\}$ be an orbit set for $Y=\partial X$. Let $H_2(X,\alpha)$ denote the set of relative homology classes of 2-chains Z in X with $\partial Z=\alpha$. This set is an affine space over $H_2(X)$.

[†] We say that $u:\Sigma\to\overline{X}$ "has no multiply covered components" if the restriction of the map u to each component of the domain Σ is not multiply covered (which means that it must be somewhere injective), and no two components of Σ have the same image under u.

Given $J\in\mathcal{J}(\overline{X},\omega)$, let $\mathcal{M}^J(\overline{X},\alpha;x_1,\ldots,x_k)$ denote the set of holomorphic curves in $\mathcal{M}^J(\overline{X};x_1,\ldots,x_k)$ such that for each i, there are punctures asymptotic to covers of α_i with total multiplicity m_i, and there are no other punctures. Note that each $u\in\mathcal{M}^J(\overline{X},\alpha;x_1,\ldots,x_k)$ has a well-defined relative homology class $[u]\in H_2(X,\alpha)$.

Proof of Lemma 3. For $\varepsilon>0$, let N_ε denote the neighborhood of ∂X in Eq. (2). The time ε flow of the Liouville vector field (coming from the primitive of ω) defines a symplectomorphism

$$(X\setminus N_\varepsilon,\omega|_{X\setminus N_\varepsilon})\simeq(X,e^{-\varepsilon}\omega). \qquad [5]$$

It follows from Definition 1 that c_k satisfies the Conformality property (Eq. (9) below), so we deduce from Eq. (5) that

$$c_k(X\setminus N_\varepsilon,\omega|_{X\setminus N_\varepsilon}) = e^{-\varepsilon}c_k(X,\omega) \qquad [6]$$

Consequently, by replacing X with $X\setminus N_\varepsilon$ for $\varepsilon>0$ small if necessary, we can assume without loss of generality that $\varphi(X)\subset\operatorname{int}(X')$.

Now fix $x_1,\ldots,x_k\in X$ distinct, $J\in\mathcal{J}(\overline{X},\omega)$, and $\varepsilon>0$. To prove the lemma, we need to show that there exists $u\in\mathcal{M}^J(\overline{X};x_1,\ldots,x_k)$ with

$$\mathcal{E}(u)<c_k(X',\omega')+\varepsilon.$$

We will use a "neck stretching" argument.

Write $Y=\partial X$ and let λ denote the contact form on Y. Since $\varphi(X)\subset\operatorname{int}(X')$, there exists a neighborhood $\mathcal{U}$ of $\varphi(Y)$ in $X'\setminus\varphi(\operatorname{int}(X))$ and an identification

$$(\mathcal{U},\omega'|_{\mathcal{U}})\simeq([0,\delta)\times Y, d(e^s\lambda))$$

for some $\delta>0$, where s denotes the $[0,\delta)$ coordinate. For each $R>0$, we can choose an almost complex structure $J_R\in\mathcal{J}(\overline{X'},\omega')$ such that φ extends to a biholomorphism

$$\varphi_R:(X\cup_Y([0,R)\times Y),J)\xrightarrow{\simeq}(\varphi(X)\cup\mathcal{U},J_R). \qquad [7]$$

We can further assume that J_R is independent of R outside of $\varphi(X)\cup\mathcal{U}$.

By the definition of c_k, for each R we can choose

$$u_R\in\mathcal{M}^{J_R}(\overline{X'};\varphi(x_1),\ldots,\varphi(x_k))$$

with

$$\mathcal{E}(u_R)<c_k(X',\omega')+\varepsilon. \qquad [8]$$

Let u_R^φ denote the intersection of the curve u_R with $\varphi(X)\cup\mathcal{U}$, composed with φ_R^{-1}. We now want to argue that there is a sequence $R_i\to\infty$ such that the intersections of the curves $u_{R_i}^\varphi$ converge in some sense to the desired curve u. This task is complicated by the fact that we do not have an a priori bound on the genus of the components of the domains of the curves u_R, so we cannot directly use SFT compactness as in (15, 16).

Fortunately, there is a local version of Gromov compactness using currents which does not require any genus bound. This was proved in the four-dimensional case by Taubes (17, Prop. 3.3), and an updated version which works in arbitrary dimension was proved by Doan-Wapulski (18, Prop. 1.9). By this local Gromov compactness and the energy bound Eq. (8), as applied in (19, §9.4), we can find a sequence $R_i\to\infty$ such that the curves $u_{R_i}^\varphi$ converge as currents to a proper holomorphic map u to $\overline{X}$ which passes through the points $x_1,\ldots,x_k$, is asymptotic as a current as $s \to \infty$ to an orbit set for Y, and has energy less than $c_k(X', \omega') + \varepsilon$. A priori, components of the domain of u may have infinite genus, and to complete the proof of the lemma we need to arrange that they are punctured compact Riemann surfaces.

We can pass to a subsequence so that there is a single orbit set α' for $\partial X'$ such that each u_{R_i} is in the moduli space $\mathcal{M}^{J_{R_i}}(\overline{X'}, \alpha'; \varphi(x_1), \ldots, \varphi(x_k))$, because there are only finitely many orbit sets for $\partial X'$ with action less than $c_k(X', \omega') + \varepsilon$. When applying Gromov compactness above, we can further use the arguments in (19, §9.4) to chase down the rest of the energy of the holomorphic curves u_{R_i} and pass to a subsequence such that the relative homology class $[u_{R_i}] \in H_2(X', \alpha')$ does not depend on i.

By Remark 2, we can assume that each u_{R_i} has no multiply covered components. Since we are in four dimensions, the relative adjunction formula of (20, Prop. 4.9) and the asymptotic writhe bound of (20, Lem. 4.20) imply that there is a lower bound on the Euler characteristic of the domain of u_{R_i} depending only on the orbit set α' and the relative homology class $[u_{R_i}]$. We can also assume that the domain of each u_{R_i} has at most k components, since otherwise some components can be discarded without violating the requirement to pass through the points $x_1, \ldots, x_k$. Consequently we obtain an i-independent upper bound on the genus of each component of the domain of u_{R_i}.

We can then pass to a subsequence such that the components of the domain of u_{R_i} can be numbered so that the j^{th} component is a punctured compact Riemann surface with the genus and number of punctures not depending on i, and the sequence of j^{th} components with the restrictions of the maps u_{R_i} converges as $i \to \infty$ to a component of u whose domain is also a punctured compact Riemann surface. ☐

Properties of the capacities c_k

Theorem 6. *The capacities c_k of four-dimensional symplectic manifolds have the following properties:*

(Conformality) If $r > 0$ then

$$c_k(X, r\omega) = r c_k(X, \omega). \qquad [9]$$

(Increasing)

$$0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq +\infty.$$

(Disjoint Union)

$$c_k\left(\coprod_{i=1}^m (X_i, \omega_i)\right) = \max_{k_1 + \cdots + k_m = k} \sum_{i=1}^m c_{k_i}(X_i, \omega_i).$$

(Sublinearity)

$$c_{k+l}(X, \omega) \leq c_k(X, \omega) + c_l(X, \omega).$$

(Monotonicity) If there exists a symplectic embedding $\varphi : (X, \omega) \to (X', \omega')$, then

$$c_k(X, \omega) \leq c_k(X', \omega').$$

(C^0-Continuity) For each k, the capacity c_k defines a continuous function on the set of star-shaped domains in $\mathbb{R}^4$ with respect to the Hausdorff metric on compact sets.

(Spectrality) If (X, ω) is a four-dimensional Liouville domain with boundary Y, then for each k with $c_k(X, \omega) < \infty$, there exists an orbit set α in Y, which is nullhomologous in X, with $c_k(X, \omega) = \mathcal{A}(\alpha)$.

(ECH Index) If X is a nondegenerate star-shaped domain in $\mathbb{R}^4$, then $c_k(X) < \infty$, and in the Spectrality property, we can choose α so that its ECH index[‡] satisfies $I(\alpha) \geq 2k$.

(Ball)

$$c_k(B^4(a)) = da$$

where d is the unique nonnegative integer with

$$d^2 + d \leq 2k \leq d^2 + 3d.$$

(Asymptotics) If $X \subset \mathbb{R}^4$ is a compact domain with smooth boundary, then

$$c_k(X) = 2\,\mathrm{vol}(X)^{1/2} k^{1/2} + O(k^{1/4}).$$

Proof. For admissible symplectic four-manifolds, the Conformality, Increasing, Disjoint Union, and Sublinearity properties follow immediately from Definition 1. It then follows from Lemma 3 and Definition 4 that these properties, as well as the Monotonicity property, also hold for general symplectic four-manifolds.

The C^0-Continuity property follows from Conformality and Monotonicity, since if two star-shaped domains are close in the Hausdorff metric, then each is contained in the scaling of the other by a number slightly larger than 1. Note here that if X is a star-shaped domain and $r > 0$, then Conformality implies that $c_k(rX) = r^2 c_k(X)$.

To prove the Spectrality property, suppose first that (X, ω) is a nondegenerate Liouville domain with $c_k(X, \omega) < \infty$. It follows from the definition of c_k that there is an orbit set α with $c_k(X, \omega) = \mathcal{A}(\alpha)$, because in Eq. (4), for every curve u, the energy $\mathcal{E}(u)$ is the action of some orbit set α, and the set of all such actions is discrete. Also α is nullhomologous in X because there is a holomorphic curve in $\overline{X}$ asymptotic to it.

If (X, ω) is a degenerate Liouville domain, then the Spectrality property follows by approximating with nondegenerate Liouville domains and using Eq. (6) and Monotonicity as in the proof of C^0 continuity.

To prove the ECH Index property, first note that $c_k(X) < \infty$ by Monotonicity and the upper bound on c_k of a ball proved in Eq. (13) below. Recall from Remark 2 that in Eq. (4), we can restrict attention to holomorphic curves that do not have any multiply covered components. Let $\mathcal{M}^J_*(\overline{X}, \alpha; x_1, \ldots, x_k)$ denote the set of curves in $\mathcal{M}^J(\overline{X}, \alpha; x_1, \ldots, x_k)$ without multiply covered components. The hypothesis that X is nondegenerate implies that the set of symplectic actions of orbit sets in ∂X is discrete, so we can rewrite Eq. (4) as

$$c_k(X) = \max_{\substack{J \in \mathcal{J}(\overline{X}) \\ x_1, \ldots, x_k \in X \text{ distinct}}} \min\left\{\mathcal{A}(\alpha) \;\middle|\; \mathcal{M}^J_*\left(\overline{X}, \alpha; x_1, \ldots, x_k\right) \neq \emptyset\right\}. \qquad [10]$$

If $u \in \mathcal{M}^J_*(\overline{X}, \alpha; x_1, \ldots, x_k)$, then it follows from the ECH index inequality, see e.g. (8, §3.4), that

$$\mathrm{ind}(u) \leq I(\alpha). \qquad [11]$$

[‡] See e.g. (21, Def. 5.2) for the definition of the ECH index of α. The definition there is stated for ECH generators (a special kind of orbit set, see Remark 7), but is valid for arbitrary orbit sets.

Here $\mathrm{ind}(u)$ denotes the Fredholm index of u, which for generic $J \in \mathcal{J}(\overline{X})$ is the dimension of the component of the moduli space $\mathcal{M}^J_*(\overline{X},\alpha)$ containing u. In particular, if $J \in \mathcal{J}(\overline{X})$ and $x_1,\ldots,x_k \in X$ are generic, then for any $u \in \mathcal{M}^J_*(\overline{X},\alpha;x_1,\ldots,x_k)$, the dimension of the component of the latter moduli space containing u is $\mathrm{ind}(u) - 2k \geq 0$, so if the latter moduli space is nonempty then by Eq. (11) we have $I(\alpha) \geq 2k$. It follows that for generic $J \in \mathcal{J}(\overline{X})$ and $x_1,\ldots,x_k \in X$, the minimum in Eq. (10) has the form $\mathcal{A}(\alpha)$ where $I(\alpha) \geq 2k$. By Gromov compactness as in the proof of Lemma 3, the maximum in Eq. (10) must be realized by generic $J \in \mathcal{J}(\overline{X})$ and $x_1,\ldots,x_k \in X$.

To prepare for the proof of the Ball property, if $a,b>0$, define the ellipsoid

$$E(a,b) = \left\{ z \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}.$$

Calculations e.g. in (8, §3.7) show that for any ellipsoid $E(a,b)$ with a/b irrational, there are just two simple Reeb orbits, which have symplectic action a and b, and the ECH index defines a bijection from the set of orbit sets to the set of nonnegative even integers. Furthermore the symplectic action is an increasing function of the ECH index.

To prove the Ball property, by the Conformality property we can assume that $a=1$. Let $\varepsilon > 0$ be irrational and consider the ellipsoid

$$E(1-\varepsilon,1) \subset E(1,1) = B^4(1).$$

For a given nonnegative integer d, if ε is sufficiently small, then by the previous paragraph, the orbit set of ECH index d^2+d has symplectic action $d(1-\varepsilon)$. Taking $\varepsilon \to 0$, it follows from the ECH index and Monotonicity properties that

$$c_{(d^2+d)/2}(B^4(1)) \geq d. \qquad [12]$$

To complete the proof of the Ball property, by the Increasing property, we need to show that

$$c_{(d^2+3d)/2}(B^4(1)) \leq d. \qquad [13]$$

By Monotonocity, it is enough to show that

$$c_{(d^2+3d)/2}(\mathbb{C}P^2,\omega_{FS}) \leq d. \qquad [14]$$

Here ω_{FS} denotes the Fubini-Study form on $\mathbb{C}P^2$, normalized so that a line has symplectic area 1. To prove Eq. (14), write $k = (d^2+3d)/2$; it is enough to show that for any $J \in \mathcal{J}(\mathbb{C}P^2,\omega_{FS})$ and any $x_1,\ldots,x_k \in \mathbb{C}P^2$, there exists a J-holomorphic curve, possibly with disconnected domain, of total degree d passing through the points $x_1,\ldots,x_k$. For a given J, for generic $x_1,\ldots,x_k$ this was shown by Gromov (3, §0.2.B) (it also follows from Taubes's "Seiberg-Witten = Gromov" theorem as explained in the proof of Theorem 17 below), and for arbitrary $x_1,\ldots,x_k$ it follows from Gromov compactness.

Finally, the Asymptotics property was shown for ECH capacities in (21, Thm. 1.1). The proof there just uses the Monotonicity and Disjoint Union properties for ECH capacities and the formula for the ECH capacities of a cube. Theorem 9 below implies that for a cube, the ECH capacities and the capacities c_k agree. Hence the Asymptotics property also holds for the capacities c_k. □

Remark 7. The properties of the capacities c_k in Theorem 6, aside from the Sublinearity property, are also known to hold for ECH capacities. These properties of ECH capacities were proved in (6), except for the Asymptotics property, which is a later refinement proved in (21).

For the ECH capacities, a slighty stronger version of the ECH Index property follows from the definition of ECH capacities reviewed in Eq. (20) below: namely one can arrange that $I(\alpha) = 2k$, and furthermore that the orbit set α is an ECH generator. Here we say that an orbit set $\alpha = \{(\alpha_i, m_i)\}$ is an *ECH generator* if $m_i = 1$ whenever α_i is hyperbolic (meaning that the linearized return map has real eigenvalues).

Remark 8. Some applications of ECH capacities only need the properties in Theorem 6, and thus can be re-proved using the capacities c_k. For example, Irie (22) proved a C^∞ closing lemma for Reeb vector fields on closed three-manifolds, using the asymptotics of the ECH spectrum (23). In the case of S^3 with the standard contact structure, which corresponds to star-shaped hypersurfaces in $\mathbb{R}^4$, the ECH spectrum agrees with the ECH capacities of the corresponding star-shaped domain, and Irie's proof of the closing lemma works using only the C^0-Continuity, Spectrality, and Asymptotics properties in Theorem 6.

Computation for convex toric domains

We now show that for "convex toric domains", the capacities c_k agree with a known combinatorial formula for their ECH capacities[§]. In fact, the capacities c_k for these examples are uniquely determined by the properties in Theorem 6.

Let Ω be a compact domain in $\mathbb{R}^2_{\geq 0}$. Define the *toric domain*

$$X_\Omega = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi\left(|z_1|^2, |z_2|^2\right) \in \Omega \right\}.$$

Define a (four-dimensional) *convex toric domain* to be a toric domain X_Ω as above such that the set

$$\widehat{\Omega} = \left\{ \mu \in \mathbb{R}^2 \;\middle|\; (|\mu_1|,|\mu_2|) \in \Omega \right\}$$

is convex[¶]. Define a (four-dimensional) *concave toric domain* to be a toric domain X_Ω such that the set $\mathbb{R}^2_{\geq 0} \setminus \Omega$ is convex.

If X_Ω is a four-dimensional convex toric domain, let $\|\cdot\|^*_\Omega$ denote the norm on $\mathbb{R}^2$ defined by

$$\|v\|^*_\Omega = \max\left\{ \langle v, w\rangle \;\middle|\; w \in \widehat{\Omega} \right\}.$$

If $\gamma : [\alpha,\beta] \to \mathbb{R}^2$ is a continuous, piecewise differentiable curve, define its Ω-*length* by

$$\ell_\Omega(\gamma) = \int_\alpha^\beta \|J\gamma'(t)\|^*_\Omega \, dt \qquad [15]$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Define a *convex integral path* to be a polygonal path Λ in the nonnegative quadrant from the point $(0,b)$ to the point $(a,0)$, for some nonnegative integers a and b, with vertices at lattice points, such that the region bounded by Λ and the line segments from $(0,0)$ to $(a,0)$ and from $(0,0)$ to $(0,b)$ is convex. Define $\widehat{\mathcal{L}}(\Lambda)$ to be the number of lattice points in this region, including lattice points on the boundary.

[§] This formula appears in (24, Prop. 5.6). It is a specialization of a result in (10, Cor. A.12) computing the ECH capacities of a more general notion of "convex toric domain".

[¶] This is slightly misleading terminology, as a "convex toric domain" is not the same thing as a toric domain that is convex; see (25, §2) for clarification.

Theorem 9. *If X_Ω is a four-dimensional convex toric domain, then*

$$c_k(X_\Omega) = \min\{\ell_\Omega(\Lambda) \mid \widehat{\mathcal{L}}(\Lambda) = k+1\} \qquad [16]$$

where the minimum is over convex integral paths Λ.

Proof. It is shown in (24, Lem. 5.4) that given $L, \varepsilon > 0$, there is a nondegenerate star-shaped domain X' with $\operatorname{dist}_{C^0}(X', X_\Omega) < \varepsilon$ with the following property: Every orbit set α for X' with action $\mathcal{A}(\alpha) < L$ determines a convex integral path Λ such that $|\mathcal{A}(\alpha) - \ell_\Omega(\Lambda)| < \varepsilon$ and the ECH index $I(\alpha) \le 2(\widehat{\mathcal{L}}(\Lambda) - 1)$. It then follows from the C^0-Continuity and ECH Index properties in Theorem 6 that

$$c_k(X_\Omega) \ge \min\{\ell_\Omega(\Lambda) \mid \widehat{\mathcal{L}}(\Lambda) \ge k+1\}. \qquad [17]$$

We now prove the reverse inequality. Let $a > 0$ be the smallest real number such that $X_\Omega \subset B^4(a)$. In (10, §2.2), a "negative weight sequence" is defined; this is a nonincreasing (possibly finite) sequence of positive real numbers $(a_1, a_2, \ldots)$. It has the property that there is a symplectic embedding

$$X_\Omega \sqcup \coprod_i \operatorname{int}(B^4(a_i)) \longrightarrow B^4(a)$$

which fills the volume of $B^4(a)$. It follows from the Disjoint Union property that

$$c_k(X_\Omega) \le \inf_{l \ge 0} \left(c_{k+l}(B^4(a)) - c_l\left(\coprod_{i \le l} B^4(a_i) \right) \right).$$

Furthermore, c_k agrees with c_k^{ECH} for a disjoint union of balls by the Disjoint Union and Ball properties, so we can rewrite the above inequality as

$$c_k(X_\Omega) \le \inf_{l \ge 0} \left(c_{k+l}^{\mathrm{ECH}}(B^4(a)) - c_l^{\mathrm{ECH}}\left(\coprod_{i \le l} B^4(a_i) \right) \right). \qquad [18]$$

Finally, a combinatorial calculation in (10, §A.3) shows that the right hand side of Eq. (18) is less than or equal to the right hand side of Eq. (17).

To complete the proof, we observe that

$$\min\{\ell_\Omega(\Lambda) \mid \widehat{\mathcal{L}}(\Lambda) \ge k+1\} = \min\{\ell_\Omega(\Lambda) \mid \widehat{\mathcal{L}}(\Lambda) = k+1\},$$

as explained in (10, §A.3). □

Remark 10. By Theorem 9 and (24, Prop. 5.6), the capacities c_k agree with the ECH capacities for convex toric domains. It follows from the Monotonicity property that all obstructions to symplectic embeddings between convex toric domains coming from ECH capacities can be recovered using the capacities c_k.

Remark 11. Going beyond ECH capacities, it is shown in (24, Thm. 1.19) that if X_Ω and $X_{\Omega'}$ are four-dimensional convex toric domains, and if there exists a symplectic embedding $X_\Omega \to X_{\Omega'}$, then a certain combinatorial criterion holds. This leads to stronger symplectic embedding obstructions in some cases where ECH capacities do not give sharp obstructions, for example to symplectically embedding a polydisk into a ball or ellipsoid; see (24, 26, 27).

The proof of (24, Thm. 1.19) rests on the existence of an ECH index 0 holomorphic curve with certain properties in (the completion of) a symplectic cobordism between the (perturbed) boundaries of X_Ω and $X_{\Omega'}$, which is produced using Seiberg-Witten theory. One can re-prove the existence of such a curve using the methods of this paper, namely by using the existence of curves in $\overline{X_{\Omega'}}$ with point constraints in the image of X_Ω, as guaranteed by the capacities c_k, and then neck stretching as in the proof of Lemma 3.

Comparison with ECH capacities

Aside from the examples of toric domains, we do not know to what extent c_k agrees with c_k^{ECH}, but we do have the following general fact, whose proof (and statement) use Seiberg-Witten theory:

Theorem 12. *Let X be a four-dimensional Liouville domain and let k be a nonnegative integer. Then*

$$c_k(X) \le c_k^{\mathrm{ECH}}(X).$$

To prepare for the proof of Theorem 12, we now recall the definition of the ECH capacities c_k^{ECH}, for the simplest case of four-dimensional nondegenerate Liouville domains with connected boundary.

Let Y be a closed oriented three-manifold and let λ be a nondegenerate contact form on Y. The following is an outline of the definition of the *embedded contact homology* $ECH(Y, \lambda)$. We define $ECC(Y, \lambda)$ to be the free $\mathbb{Z}/2$-module[‖] generated by the ECH generators; see Remark 7. For a generic λ-compatible almost complex structure J on $\mathbb{R} \times Y$, the ECH differential

$$\partial_J : ECC(Y, \lambda) \longrightarrow ECC(Y, \lambda)$$

is defined as follows. If α and β are ECH generators, then the coefficient of β in $\partial_J \alpha$, which we denote by $\langle \partial_J \alpha, \beta \rangle \in \mathbb{Z}/2$, is a mod 2 count of "$J$-holomorphic currents" $\mathcal{C}$ in $\mathbb{R} \times Y$, modulo $\mathbb{R}$ translation, that are asymptotic to α as $s \to +\infty$ and to β as $s \to -\infty$, and that have ECH index $I(\mathcal{C}) = 1$. See (8, §3) for detailed definitions. It is shown in (29) that $\partial_J^2 = 0$. We define $ECH(Y, \lambda)$ to be the homology of the chain complex $(ECC(Y, \lambda), \partial_J)$.

It follows from the definition of λ-compatible almost complex structure that the ECH differential decreases symplectic action:

$$\langle \partial_J \alpha, \beta \rangle \ne 0 \Longrightarrow \mathcal{A}(\alpha) > \mathcal{A}(\beta). \qquad [19]$$

As a result, for each $L \in \mathbb{R}$, the ECH generators with action less than L span a subcomplex of $(ECC(Y, \lambda), \partial_J)$. We define the *filtered ECH*, which we denote by $ECH^L(Y, \lambda)$, to be the homology of this subcomplex.

It was shown by Taubes (30) that $ECH(Y, \lambda)$ is isomorphic to a version of Seiberg-Witten Floer cohomology defined by Kronheimer-Mrowka (31). Taubes's isomorphism was used in (32, Thm. 1.3) to show that $ECH(Y, \lambda)$ and $ECH^L(Y, \lambda)$ do not depend on J; that is, the homologies for different choices of J are canonically isomorphic to each other.

There is also a map

$$U : ECH^L(Y, \lambda) \longrightarrow ECH^L(Y, \lambda)$$

induced by a chain map which counts J-holomorphic currents with ECH index 2 passing through a base point in $\mathbb{R} \times Y$. This map does not depend on the choice of base point when Y is connected; otherwise it depends on a choice of connected component of Y. See (33, §2.5) for more details.

‖ It is also possible to define ECH with integer coefficients (28, §9).

Now let (X,ω) be a four-dimensional nondegenerate Liouville domain with connected boundary Y and associated contact form λ. In this case the k^{th} ECH capacity is defined by

$$c_k^{\mathrm{ECH}}(X,\omega) = \inf\left\{L \geq 0 \;\middle|\; \exists \eta \in ECH^L(Y,\lambda) : U^k\eta = [\emptyset]\right\}. \qquad [20]$$

Here $[\emptyset]$ is the homology class in $ECH^L(Y,\lambda)$ of the empty set of Reeb orbits, which is a cycle by Eq. (19). Note that by Eq. (21) below, the existence of an exact filling of Y (namely the Liouville domain X) implies that the class $[\emptyset] \neq 0$ in $ECH^L(Y,\lambda)$.

Proof of Theorem 12. Let Y denote the boundary of X. For brevity we just explain the case when Y is connected; the general case follows by a similar argument using the more general definition of ECH capacities in (6, Def. 4.3).

Since c_k and c_k^{ECH} both satisfy Conformality and Monotonicity, by a continuity argument using Eq. (6) and the analogous equation for c_k^{ECH}, we can assume without loss of generality that X is nondegenerate.

Let λ denote the contact form on Y. As explained for example in (6, Thm. 2.3), for each $L \geq 0$, the exact filling X of Y induces a cobordism map

$$\Phi : ECH^L(Y,\lambda) \longrightarrow \mathbb{Z}/2, \qquad [21]$$

defined using Seiberg-Witten theory, which sends $[\emptyset]$ to 1.

Now suppose that $J \in \mathcal{J}(\overline{X})$ and $x_1,\ldots,x_k \in X$. Heuristically one might expect that if J and $x_1,\ldots,x_k$ are generic, then the composition

$$\Phi \circ U^k : ECH^L(Y,\lambda) \longrightarrow \mathbb{Z}/2 \qquad [22]$$

is induced by a cocycle

$$\phi : ECC^L(Y,\lambda) \longrightarrow \mathbb{Z}/2$$

that counts J-holomorphic curves in $\overline{X}$ with ECH index $2k$ passing through $x_1,\ldots,x_k$. What one can actually prove, as in the "holomorphic curves axiom" for ECH cobordism maps in (32, Thm. 1.9) and the comparison of U maps in (34, Thm. 1.1), is the following. For any $J \in \mathcal{J}(\overline{X})$ and any $x_1,\ldots,x_k \in X$ (not necessarily generic), the composition in Eq. (22) is induced by a (noncanonical) cocycle ϕ with the following property: If α is an ECH generator and $\phi(\alpha) \neq 0$, then there exists a "broken J-holomorphic current" in $\overline{X}$ passing through $x_1,\ldots,x_k$. This last statement implies that there is an orbit set α' with $\mathcal{A}(\alpha') \leq \mathcal{A}(\alpha)$ and a holomorphic curve in $\mathcal{M}^J(\overline{X},\alpha';x_1,\ldots,x_k)$.

Now suppose that $L > c_k^{\mathrm{ECH}}(X)$. Then by the definition of ECH capacities in Eq. (20), there exists $\eta \in ECH^L(Y,\lambda)$ with $U^k\eta = [\emptyset]$. It follows that $(\Phi\circ U^k)(\eta) = 1$. By the previous paragraph, for any $J \in \mathcal{J}(X)$ and any $x_1,\ldots,x_k \in X$, there exists an ECH generator α' with $\mathcal{A}(\alpha') < L$ such that $\mathcal{M}^J(X,\alpha';x_1,\ldots,x_k) \neq \emptyset$. It then follows from Eq. (10) that $c_k(X) \leq L$. Since $L > c_k^{\mathrm{ECH}}(X)$ was arbitrary, the theorem follows. □

Remark 13. One can understand the inequality in Theorem 12 as follows: The number $c_k(X)$ measures the minimal energy of holomorphic curves in $\overline{X}$ through k points that are guaranteed to exist, for whatever reason. On the other hand, $c_k^{\mathrm{ECH}}(X)$ measures the energy of certain holomophic curves in $\overline{X}$ through k points that are guaranteed to exist for ECH reasons.

Remark 14. There exist examples of Liouville domains and positive integers k for which c_k is strictly less than c_k^{ECH}. An example is given by the unit cotangent bundle $D^*S^2(4\pi)$, where $S^2(a)$ denotes the 2-sphere with the round metric of area a. It follows from results in (35, 36) that there exist symplectic embeddings

$$\mathrm{int}(P(2\pi,2\pi)) \longrightarrow \mathrm{int}(D^*S^2(4\pi)) \longrightarrow S^2(2\pi)\times S^2(2\pi).$$

Here the left hand side is a polydisk; see equation Eq. (24) for the notation. We will see in Remark 19 below that the capacities c_k are the same for $P(2\pi,2\pi)$ and $S^2(2\pi)\times S^2(2\pi)$, so by Monotonicity they are also the same for $D^*S^2(4\pi)$. However the ECH capacities $c_k^{\mathrm{ECH}}(D^*S^2(4\pi))$ are computed in (35) and found to be larger for some k.

The main reason for the discrepancy is the following: The Spectrality property in Theorem 6 asserts that c_k of a Liouville domain X with boundary Y is the action of an orbit set which is nullhomologous in X. However by the definition of the ECH capacities in Eq. (20), $c_k^{\mathrm{ECH}}(X)$ is the action of an orbit set which is nullhomologous in Y, a more restrictive condition.

Additional computations using Seiberg-Witten theory

We now compute some additional examples of the capacities c_k using Seiberg-Witten theory (which could perhaps be avoided with more work).

If X_Ω is a four-dimensional concave toric domain as defined above, define an "anti-norm" on $\mathbb{R}^2$ by

$$[v]_\Omega = \min\{\langle(|v_1|,|v_2|),w\rangle \mid w \in \partial_+\Omega\}$$

where $\partial_+\Omega$ denotes the closure of the portion of $\partial\Omega$ not on the axes. If γ is a continuous, piecewise differentiable curve in $\mathbb{R}^2$, now define its Ω-length as in Eq. (15), but replacing the norm $\|\cdot\|$ by the anti-norm $[\cdot]$.

Define a *concave integral path* to be a polygonal path Λ in the nonnegative quadrant from the point $(0,b)$ to the point $(a,0)$, for some nonnegative integers a and b, with vertices at lattice points, which is the graph of a convex function. Define $\check{\mathcal{L}}(\Lambda)$ to be the number of lattice points in the region bounded by Λ and the axes, this time (in contrast to the case of convex toric domains) not including lattice points on Λ.

Theorem 15. *If X_Ω is a four-dimensional concave toric domain, then*

$$c_k(X_\Omega) = \max\{\ell_\Omega(\Lambda) \mid \check{\mathcal{L}}(\Lambda) = k\} \qquad [23]$$

where the maximum is over concave integral paths Λ.

Remark 16. It is shown in (37, Thm. 1.21) that the same formula holds for the ECH capacities $c_k^{\mathrm{ECH}}(X_\Omega)$.

Proof of Theorem 15. In (37, §1.3), see also (21, §1.3), a "weight expansion" of X_Ω is defined; this is a nonincreasing (possibly finite) sequence of positive real numbers $(a_1,a_2,\ldots)$. There is a symplectic embedding

$$\coprod_i \mathrm{int}\, B^4(a_i) \longrightarrow X_\Omega$$

which fills the volume of X_Ω. It follows from the Monotonicity property that

$$c_k(X_\Omega) \geq c_k\left(\coprod_{i\leq k} B^4(a_i)\right).$$

By the Ball and Disjoint Union properties, we have

$$c_k\left(\coprod_{i\le k} B^4(a_i)\right) = c_k^{\mathrm{ECH}}\left(\coprod_{i\le k} B^4(a_i)\right).$$

It is shown in (37, §2) by a combinatorial calculation that

$$c_k^{\mathrm{ECH}}\left(\coprod_{i\le k} B^4(a_i)\right) \ge \max\{\ell_\Omega(\Lambda) \mid \check{\mathcal{L}}(\Lambda) = k\}.$$

By Remark 16 and Theorem 12, the above inequalities are equalities. □

We now consider some closed symplectic manifolds. Given $a > 0$, let $\mathbb{C}P^2(a)$ denote $\mathbb{C}P^2$ with the Fubini-Study form, scaled so that a line has symplectic area a. Let $S^2(a)$ denote S^2 with a symplectic form of area a.

Theorem 17. *Let $a, b > 0$ and let k be a nonnegative integer.*

(a) $c_k(\mathbb{C}P^2(a)) = da$ where d is the unique nonnegative integer with $d^2 + d \le 2k \le d^2 + 3d$.

(b) $c_k(S^2(a) \times S^2(b)) = \min\{am + bn \mid m, n \in \mathbb{Z}_{\ge 0},\ (m+1)(n+1) \ge k+1\}$.

To prepare for the proof of this theorem, if (X, ω) is a closed symplectic four-manifold with $b_2^+(X) = 1$, and if $A \in H_2(X)$, let $SW(X, \omega, A) \in \mathbb{Z}/2$ denote the mod 2 Seiberg-Witten invariant of X, for the spin-c structure determined by ω and A, in the symplectic chamber; see the review in (38, §2). Define the ECH index

$$I(A) = A \cdot A + \langle c_1(TX), A\rangle \in \mathbb{Z}.$$

Lemma 18. *Let (X, ω) be a closed symplectic four-manifold*[**] *with $b_2^+(X) = 1$ and let $A \in H_2(X)$. If $SW(X, \omega, A) \ne 0$ and $I(A) = 2k$, then $c_k(X, \omega) \le \langle[\omega], A\rangle$.*

Proof. If $J \in \mathcal{J}(X, \omega)$ and $x_1, \ldots, x_k \in X$ are generic, then it follows from Taubes's "Seiberg-Witten = Gromov" theorem (39) that there exists a J-holomorphic curve (possibly with disconnected domain) in the homology class A passing through the points $x_1, \ldots, x_k$. Thus

$$\inf_{u \in \mathcal{M}^J(X; x_1, \ldots, x_k)} \mathcal{E}(u) \le \langle[\omega], A\rangle$$

when $J, x_1, \ldots, x_k$ are generic. A Gromov compactness argument shows that the supremum in the definition of $c_k(X, \omega)$ in Eq. (4) is realized for generic $J, x_1, \ldots, x_k$. □

Proof of Theorem 17. (a) Let d be the integer in the statement of the theorem. Then by Eq. (12) and the Conformality, Monotonicity, and Increasing properties, we have $c_k(\mathbb{C}P^2(a)) \ge da$. On the other hand, by Eq. (14) and the Conformality, Monotonicity, and Increasing properties, we have $c_k(\mathbb{C}P^2(a)) \le da$. The latter inequality also follows from Lemma 18 and the Increasing property, because if $A \in H_2(\mathbb{C}P^2)$ is d times the homology class of a line, then $I(A) = d^2 + 3d$, and as reviewed in (38, §2.4) we have $SW(A) \ne 0$.

(b) Let L denote the right hand side of the equation in (b). If m and n are nonnegative integers, and if $A = (m, n) \in H_2(S^2 \times S^2)$, then $I(A) = 2(mn + m + n)$. As reviewed in (38, §2.4), we have $SW(A) \ne 0$. It follows from Lemma 18 and the Increasing property that

$$c_k(S^2(a) \times S^2(b)) \le L.$$

To prove the reverse inequality, consider the polydisk

$$P(a, b) = \left\{z \in \mathbb{C}^2 \;\middle|\; \pi|z_1|^2 \le a,\ \pi|z_2|^2 \le b\right\}. \qquad [24]$$

A calculation using Theorem 9 shows that

$$c_k(P(a, b)) = L.$$

Since the interior of $P(a, b)$ symplectically embeds into $S^2(a) \times S^2(b)$, we are done by Monotonicity. □

Remark 19. Theorem 17 shows that the capacities c_k are the same for $\mathbb{C}P^2(a)$ and the ball $B^4(a)$; and likewise they are the same for $S^2(a) \times S^2(b)$ and the polydisk $P(a, b)$. This means that if the capacities c_k obstruct a symplectic embedding of a symplectic four-manifold (X, ω) into $B^4(a)$ or $P(a, b)$ respectively, then a symplectic embedding of (X, ω) into $\mathbb{C}P^2(a)$ or $S^2(a) \times S^2(b)$ respectively is not possible either. The same statement is true for the ECH capacities c_k^{ECH} when X is a star-shaped domain by (38, Thm. 1.4).

An even simpler definition of capacities

To conclude, we now define an even simpler series of symplectic capacities, for symplectic manifolds of any dimension.

If (X, ω) is a symplectic manifold, let $\mathcal{J}(X, \omega)$ denote the set of ω-compatible almost complex structures on X. Given $J \in \mathcal{J}(X, \omega)$, let $\mathcal{P}^J(X)$ denote the set of proper holomorphic maps

$$u : (S, j) \longrightarrow (X, J)$$

where (S, j) is a one-dimensional complex manifold (not necessarily compact or connected), and we assume that the restriction of u to each component of S is nonconstant. Note that regarded as a two-dimensional real manifold, S does not have boundary. Given u as above, define the *energy*

$$\mathcal{E}(u) = \int_S u^*\omega \in [0, \infty].$$

Note that the energy is well-defined because $u^*\omega$ is pointwise nonnegative. If $x_1, \ldots, x_k \in X$ are distinct, let $\mathcal{P}^J(X; x_1, \ldots, x_k)$ denote the set of proper maps u as above such that $x_1, \ldots, x_k \in u(S)$.

Definition 20. Let (X, ω) be a compact symplectic manifold (possibly disconnected and/or with boundary), and let k be a nonnegative integer. Define

$$\widehat{c}_k(X, \omega) = \sup_{\substack{J \in \mathcal{J}(X,\omega) \\ x_1, \ldots, x_k \in \operatorname{int}(X) \text{ distinct}}} \inf_{u \in \mathcal{P}^J(\operatorname{int}(X); x_1, \ldots, x_k)} \mathcal{E}(u) \in [0, \infty]. \qquad [25]$$

Remark 21. It follows immediately from the definition that the capacities $\widehat{c}_k$ satisfy the Conformality, Increasing, Disjoint Union, and Sublinearity properties in Theorem 6.

We can also quickly show that they satisfy Monotonicity under symplectic embeddings $\varphi : (X, \omega) \to (X', \omega')$ between symplectic manifolds of the same dimension, without using

[**] If $b_2^+(X) > 1$ then the lemma is also true (now the Seiberg-Witten invariant does not depend on a choice of chamber), but vacuous, because in this case one of the corollaries of Taubes's "Seiberg-Witten = Gromov" theorem in (39) is that $SW(X, \omega, A) \ne 0$ implies $I(A) = 0$.

Gromov compactness. This is because since X is compact, any $J \in \mathcal{J}(X,\omega)$ can be extended to $J' \in \mathcal{J}(X',\omega')$ with $J'|_{\varphi(X)} = \varphi_* J$.

One can now further deduce that each $\widehat{c}_k$ is a C^0-continuous function on the set of star-shaped domains in $\mathbb{R}^{2n}$.

Remark 22. When $k=1$, the capacity $\widehat{c}_1$ is very similar[††] to the "symplectic width" defined by Gromov in (40, §4.1). In particular, $\widehat{c}_1(B^{2n}(a)) = a$. The symplectic width should not be confused with the Gromov width c_{Gr} defined in the introduction. The Monotonicity property of $\widehat{c}_1$ implies that $c_{\mathrm{Gr}} \le \widehat{c}_1$.

In a sense the capacities $\widehat{c}_k$ are more natural than the c_k, because for domains that are not Liouville domains, they are defined directly, without taking a supremum over symplectic embeddings as in Definition 4. However the price for this is that we have to consider holomorphic curves without nice boundary conditions, which makes computations more difficult.

Remark 23. Suppose that $\dim(X) = 4$. If X is closed, then $\widehat{c}_k(X,\omega) = c_k(X,\omega)$ by definition. If (X,ω) is a Liouville domain, then we have

$$\widehat{c}_k(X,\omega) \le c_k(X,\omega). \qquad [26]$$

This is because if $\varepsilon > 0$, then any almost complex structure $J \in \mathcal{J}(X,\omega)$ can be extended to an $\overline{\omega}$-compatible almost complex structure on $\overline{X}$ whose restriction to $[\varepsilon,\infty)\times Y$ agrees with an $e^{\varepsilon}\lambda$-compatible almost complex structure on $\mathbb{R}\times Y$. It follows from this as in Eq. (6) that

$$\widehat{c}_k(X,\omega) \le e^{\varepsilon} c_k(X,\omega).$$

We can choose $\varepsilon > 0$ arbitrarily small, and this proves Eq. (26).

We conjecture that in fact $\widehat{c}_k(X,\omega) = c_k(X,\omega)$ when (X,ω) is a four-dimensional Liouville domain.

Example 24. The simplest example of $\widehat{c}_k$ that we do not know how to compute is $\widehat{c}_3$ of a four-dimensional ball. We currently just know that

$$\frac{3}{2} \le \widehat{c}_3(B^4(1)) \le 2.$$

Here the first inequality holds because three copies of $\mathrm{int}(B^4(1/2))$ can be symplectically embedded into $B^4(1)$, and the second inequality holds because $\mathrm{int}(B^4(1))$ can be symplectically embedded into $\mathbb{C}P^2(1)$.

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1. D McDuff, K Siegel, Symplectic capacities, unperturbed curves, and convex toric domains. *arXiv:2111.00515* (2021).
2. K Cieliebak, H Hofer, J Latschev, F Schlenk, Quantitative symplectic geometry in *Dynamics, Ergodic Theory, and Geometry*. (Cambridge University Press), pp. 1–44 (2007).
3. M Gromov, Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82**, 307–347 (1985).
4. I Ekeland, H Hofer, Symplectic topology and Hamiltonian dynamics, II. *Math. Zeitschrift* **203**, 553–567 (1990).
5. J Gutt, M Hutchings, Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebr. Geom. Topol.* **18**, 3537–3600 (2018).
6. M Hutchings, Quantitative embedded contact homology. *J. Differ. Geom.* **88**, 231–266 (2011).
7. M Hutchings, Recent progress on symplectic embedding problems in four dimensions. *PNAS* **108**, 8093–8099 (2011).
8. M Hutchings, Lecture notes on embedded contact homology in *Contact and symplectic topology, Bolyai Soc. Math. Stud. 26.* (Springer), pp. 389–484 (2014).
9. D McDuff, The Hofer conjecture on embedding symplectic ellipsoids. *J. Differ. Geom.* **88**, 519–532 (2011).
10. D Cristofaro-Gardiner, K Choi, Symplectic embeddings of concave toric domains into convex ones. *J. Differ. Geom.* **112**, 199–232 (2019).
11. C Gerig, Taming the pseudoholomorphic beasts in $R \times (S^1 \times S^2)$. *Geom. Topol.* **24**, 1791–1839 (2020).
12. J Rooney, Cobordism maps in embedded contact homology. *arXiv:1912.01048* (2019).
13. K Siegel, Higher symplectic capacities. *arXiv:1902.01490* (2019).
14. Y Eliashberg, A Givental, H Hofer, Introduction to symplectic field theory in *Geom. Funct. Anal. Special Volume, Part II.* pp. 560–673 (2000).
15. F Bourgeois, Y Eliashberg, H Hofer, K Wysocki, E Zehnder, Compactness results in symplectic field theory. *Geom. Topol.* **7**, 799–888 (2003).
16. K Cieliebak, K Mohnke, Compactness for punctured holomorphic curves. *J. Symplectic Geom.* **3**, 589–654 (2005).
17. CH Taubes, The structure of pseudoholomorphic subvarieties for a degenerate almost complex structure and symplectic form on $S^1 \times B^3$. *Geom. Topol.* **2**, 221–332 (1998).
18. A Doan, T Walpuski, Castelnuovo's bound and rigidity in almost complex geometry. *Adv. Math.* (to appear).
19. M Hutchings, An index inequality for embedded pseudoholomorphic curves in symplectizations. *J. Eur. Math. Soc.* **4**, 313–361 (2002).
20. M Hutchings, The embedded contact homology index revisited in *New Perspectives and Challenges in Symplectic Field Theory, CRM Proc. Lecture Notes, vol. 49.* (American Mathematical Society), pp. 263–297 (2009).
21. M Hutchings, ECH capacities and the Ruelle invariant. *J. Fixed Point Theory Appl.* (to appear).
22. K Irie, Dense existence of periodic orbits and ECH spectral invariants. *J. Mod. Dyn.* **9**, 357–363 (2015).
23. D Cristofaro-Gardiner, M Hutchings, V Ramos, The asymptotics of ECH capacities. *Invent. Math.* **199**, 187–214 (2015).
24. M Hutchings, Beyond ECH capacities. *Geom. Topol.* **20**, 1085–1126 (2016).
25. J Gutt, M Hutchings, V Ramos, Examples around the strong Viterbo conjecture. *J. Fixed Point Theory Appl.* (to appear).
26. K Christianson, J Nelson, Symplectic embeddings of four-dimensional polydisks into balls. *Algebr. Geom. Topol.* **18**, 2151–2178 (2018).
27. L Digiosia, J Nelson, H Ning, M Weiler, Y Yang, Symplectic embeddings of four-dimensional polydisks into half integer ellipsoids. *arXiv:2010.06687* (2020).
28. M Hutchings, CH Taubes, Gluing pseudoholomorphic curves along branched covered cylinders II. *J. Symplectic Geom.* **7**, 29–133 (2009).
29. M Hutchings, CH Taubes, Gluing pseudoholomorphic curves along branched covered cylinders I. *J. Symplectic Geom.* **5**, 43–137 (2007).
30. CH Taubes, Embedded contact homology and Seiberg-Witten Floer cohomology I. *Geom. Topol.* **14**, 2497–2581 (2010).
31. PB Kronheimer, TS Mrowka, *Monopoles and three-manifolds.* (Cambridge University Press), (2008).
32. M Hutchings, CH Taubes, Proof of the Arnold chord conjecture in three dimensions, II. *Geom. Topol.* **17**, 2601–2688 (2013).
33. M Hutchings, CH Taubes, The Weinstein conjecture for stable Hamiltonian structures. *Geom. Topol.* **13**, 901–941 (2009).
34. CH Taubes, Embedded contact homology and Seiberg-Witten Floer cohomology V. *Geom. Topol.* **14**, 2961–3000 (2010).
35. B Ferreira, VGB Ramos, Symplectic embeddings into disk cotangent bundles. *arXiv:2110.02312* (2021).
36. J Oakley, M Usher, On certain Lagrangian submanifolds of $S^2 \times S^2$ and CP^n. *Algebr. Geom. Topol.* **16**, 149–209 (2016).
37. K Choi, D Cristofaro-Gardiner, D Frenkel, M Hutchings, V Ramos, Symplectic embeddings into four-dimensional concave toric domains. *J. Topol.* **7**, 1054–1076 (2014).
38. J Chaidez, B Wormleighton, ECH embedding obstructions for rational surfaces. *arXiv:2008.10125* (2020).
39. CH Taubes, *Seiberg-Witten and Gromov invariants for symplectic 4-manifolds.* (International Press), (2000).
40. M Gromov, Soft and hard symplectic geometry in *Proceedings of the ICM, Berkeley.* (1986).

[††] The only difference is that Gromov uses tame rather than compatible almost complex structures.