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Model reference adaptive anti-windup compensation

Jorge Sofrony, Matthew C. Turner and Christopher M. Richards

Abstract— **This paper proposes an anti-windup mechanism for a model reference adaptive control scheme subject to actuator saturation constraints. The proposed compensator has the same architecture as well known non-adaptive schemes, which rely on the assumption that the system model is known fairly accurately. This is in contrast to the adaptive nature of the controller, which assumes that the system (or parts of it) is unknown. The approach proposed here uses of an "estimate" of the system matrices for the anti-windup compensator formulation and modifies the adaptation laws that update the controller gains. It will be observed that if the (unknown) ideal control gain is reached, a type of "model recovery anti-windup" formulation is obtained. In addition, it is shown that if the *ideal* control signal eventually lies within the control constraints, then, under certain conditions, the system states will converge to those of the reference model as desired. The paper highlights the main challenges involved in the design of anti-windup compensators for model-reference adaptive control systems and demonstrates its success via a flight control simulation.**

I. INTRODUCTION

Model reference adaptive controller (MRAC) is well known in the control community, with a large body of work being devoted to its development [1], [2]. The main idea behind MRAC is to use a reference model, chosen by the designer, in order to generate a state tracking error which is then used to govern adaptation of the control gains. MRAC has become one of the preferred adaptive control architectures and there is compelling evidence of successful deployment on real systems - see for example [3], [4], [5], [6]. Unfortunately, MRAC systems are vulnerable to the effects of unmatched uncertainty, disturbances and unmodeled actuator dynamics, hence different robustifying modifications have been proposed (see for example [1]). A further "uncertainty" present in all real applications is actuator saturation, and this appears also to be pernicious to adaptive control systems. In essence, the saturation nonlinearity corrupts the mechanisms by which the control gains are updated [7], in addition to the traditional wind-up effects that occur in many constrained systems [8], [9].

Several researchers have highlighted the impact of saturation in the adaptation process, and various papers have attempted to address the issue (for example [10], [11], [12], [13], [14], [6]). Most of the work presented is focused on addressing the input constraints using approaches that diverge from the more traditional anti-windup philosophy used in linear control systems. The advantage of the "traditional" anti-windup (AW) approach is its two-stage philosophy [15], [16], [17]: when no saturation is present, the compensator is inactive and the baseline (nominal) controller stabilizes the closed-loop and guarantees acceptable performance; in the event of input saturation, an additional element (the anti-windup compensator) becomes active and improves performance and enhances stability properties. For linear systems, the development of most anti-windup schemes [16], [17], [18], [8], [9] requires *knowledge of the model of the plant*; in MRAC the model is assumed to be *unknown* so the generalisation to the adaptive case is not trivial.

Consequently, anti-windup for adaptive controllers lacks thorough investigation, due to its complexity in demonstrating stability and correct adaptation of the controller gains. However some work exists, notably the work [19] where an indirect adaptive control is developed; the pseudo-hedging technique described in [20]; the sliding mode technique given in [21] for systems with rate-limits; the output feedback adaptive controller with AW in [22]; the adaptive scalar AW gain for chaotic systems in [23]; and most recently the application of the approach of [18] to systems with inertia variations [24]. Most of these schemes have drawbacks.

Additionally, recent work on *positive μ modification* [25], which relies on a modified reference model that includes information about the saturating input, exhibits AW-like behaviour. In fact, under certain assumptions, the error between the ideal model state and the plant state will converge, provided that the ideal steady-state control signal is within the control bounds [26], [27]. However, the structure of this scheme is quite different from standard anti-windup schemes; the main contribution of this paper is to formulate and solve a "model reference anti-windup"(MRAW) problem for MRAC schemes.

A. Notation

A positive (negative) square matrix P is denoted as $P > 0$ ($P < 0$). The Hermitian of a square matrix is defined as

$$He\{A\} = A' + A.$$

A' denotes the transpose of a matrix A, and $\mathbf{tr}(A)$ its trace. A signal $x(t)$ is said to belong to $\mathcal{L}_2$ if

$$\|x(t)\|_2 := \left(\int_0^\infty \|x(t)\|^2 dt\right)^{\frac{1}{2}} < \infty$$

where $\|x\|$ denotes the euclidean norm of the vector. A signal $x(t)$ is said to belong to $\mathcal{L}_\infty$ if

$$\|x(t)\|_\infty := \sup_{t\geq 0} \max_i |x_i(t)| < \infty$$

J. Sofrony is with Faculty of Mechanical Engineering and Mechatronics, Faculty of Engineering, Universidad Nacional de Colombia, Bogota, Colombia jsofronye@unal.edu.co

M.C. Turner is with the School of Electronics and Computer Science, University of Southhampton, Southhampton, UK m.c.turner@soton.ac.uk

C.M. Richards is with the Department of Mechanical Engineering, University of Louisville, USA chris.richards@louisville.edu

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The scalar saturation function, $\mathrm{sat}(.) : \mathbb{R} \to [\underline{u}_i,\ \bar{u}_i]$, is defined as

$$\mathrm{sat}_i(u_i) = \begin{cases} u_i & \text{if} \quad \underline{u}_i < u_i < \bar{u}_i \\ \bar{u}_i & \text{if} \quad u_i \geq \bar{u}_i \\ \underline{u}_i & \text{if} \quad u_i \leq \underline{u}_i \end{cases}$$

The values $\bar{u}_i$ and $\underline{u}_i$ are the upper and lower limits respectively; if $\bar{u}_i = -\underline{u}_i$, the saturation is said to be symmetric. The vector saturation function $\mathrm{sat}(.) : \mathbb{R}^m \to \mathbb{R}^m$ is simply

$$\mathrm{sat}(u) = [\mathrm{sat}_1(u_1)\ \dots\ \mathrm{sat}_m(u_m)]'$$

Extensive use is made of the deadzone function, $\mathrm{Dz}(u)$ which can be defined via the identity

$$\mathrm{sat}(u) + \mathrm{Dz}(u) = u$$

Both saturation and deadzone function are globally Lipschitz with unity gain, such that the following property holds:

$$\|\psi_i(u_1 + u_2) - \psi(u_1)\| \leq \|u_2\| \quad \forall u_1,\ u_2 \in \mathbb{R}$$

B. Preliminaries

Fact 1: The saturation and deadzone functions are slope restricted, viz

$$0 \leq \frac{\sigma_i(u_1) - \sigma_i(u_2)}{u_1 - u_2} \leq 1 \quad \forall u_1, u_2 \in \mathbb{R}$$

where $\sigma_i(\cdot)$ is the i'th component of either the saturation or the deadzone function. □

Fact 1 implies that the saturation and deadzone functions both satisfy the following incremental sector condition

$$(\sigma(u_1) - \sigma(u_2))' W \big(u_1 - u_2 - \sigma(u_1) + \sigma(u_2)\big) \geq 0 \quad (1)$$

for all diagonal matrices $W > 0$ and all $u_1, u_2 \in \mathbb{R}^m$. The following lemma from [26] is also required.

Lemma 1: Consider the dynamics

$$\dot{x} = Ax + B\lambda \mathrm{sat}(u) \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$ and λ is a positive scalar. If A is Hurwitz, then the state $x(t)$ is bounded for all $u(t) \in \mathbb{R}^{n_u}$.

II. Model Reference Adaptive Control

Consider the linear-time-invariant (LTI) plant

$$G \sim \begin{cases} \dot{x} &= Ax + Bu \\ y &= x \end{cases} \quad (3)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$. The signal y is an objective signal for performance purposes. The state is assumed to be available to the controller.

Assumption 1: Matrix A is unknown but Hurwitz, while matrix B is completely known.

The plant is stabilised via a (state feedback) MRAC controller where the reference model is given by

$$G_m \sim \{\dot{x}_m = A_m x_m + B_m r \quad (4)$$

It is assumed that the reference model is compatible with the plant structure, hence the following *Model Matching Conditions* are assumed to be true:

Assumption 2 (Model matching conditions): There exist matrices $K_x^* \in \mathbb{R}^{n_u\times m}$ and $K_r^* \in \mathbb{R}^{n_u\times n_r}$ such that

$$A_m = A + BK_x^* \qquad B_m = BK_r^* \quad (5)$$

It is important to note that K_x^* and K_r^* are not known and are assumed to be the "ideal" feedback gains. The selection of a reference model that guarantees the existence of such gains is restrictive and is one of the main limitations in the applicability of MRAC strategies. If no saturation is present, the standard MRAC controller (6)-(7), below, ensures that the tracking error $e(t) = x_m(t) - x(t)$ converges to zero and all controller gains are bounded [1].

$$u = \hat{K}_x x + \hat{K}_r r \quad (6)$$

$$\mathcal{A} \sim \begin{cases} \dot{\hat{K}}_x' &= \Gamma_x x (e' P_1 B) \\ \dot{\hat{K}}_r' &= \Gamma_r r (e' P_1 B) \end{cases} \quad (7)$$

The symmetric positive definite matrix P_1 is obtained from the solution, for some $Q_1 > 0$, of the Lyapunov equation

$$A_m' P_1 + P_1 A_m + Q_1 = 0 \quad (8)$$

Following the previous discussion, saturation of the control signal can be highly detrimental to closed-loop system performance/stability and thus the above strategy requires modification in its is presence.

III. Main Results

Now consider the plant with input saturation,

$$\dot{x} = Ax + B\mathrm{sat}(u) \quad (9)$$

The main results in this paper show how an anti-windup compensator can be used to modify the nominal adaptive control algorithms (6)-(7) so that stability and convergence of the tracking error is guaranteed for this plant. As with standard AW schemes, the proposed AW compensator modifies the closed-loop system's response, during periods of saturation or during the recovery from it. It is emphasized that the system matrix A is *unknown*, and thus the model recovery structure, that relies on a coprime factorization of the plant [18], *cannot* be directly implemented. Having this in mind, the following AW compensator structure is presented:

$$\Sigma \sim \begin{cases} \dot{x}_{aw} &= (A_m - B\hat{K}_x)x_{aw} + BFx_{aw} + BDz(u) \\ v_1 &= Fx_{aw} \\ v_2 &= x_{aw} \end{cases} \quad (10)$$

The compensator has two outputs, namely v_1, v_2. These are the compensation signals that modify the output and input of the controller respectively. The (compensated) control signal is then defined as the difference of an adaptive control law plus terms emanating from the anti-windup compensator

$$\begin{aligned} u &= \hat{K}_x(x + v_2) + \hat{K}_r r - v_1 \\ &= \hat{K}_x(x + x_{aw}) + \hat{K}_r r - Fx_{aw} \end{aligned} \quad (11)$$

where F is the anti-windup "gain" matrix from (10) and the adaptive gain matrices are updated in the same way as described in (7) except, using the modified error state-vector

$$e = \underbrace{x + x_{aw}}_{x_l} - x_m \quad (12)$$

Since it is assumed that A is unknown, an "estimate" $A_m - B\hat{K}_x(t))$ is used, see (10), which may converge to A under the model matching conditions in Assumption 2.

The closed-loop system with anti-windup is depicted in Figure 1. The saturation nonlinearity and the uncertain nature of the system (A is unknown) have two main effects on the proposed model recovery AW and MRAC configurations: (i) initially, the AW is unaware of the true system dynamics and, hence the decoupling of the closed-loop system into nominal closed-loop dynamics and nonlinear dynamics (as presented by [17]) is not directly achievable (see Figure 2); and (ii) the AW states feed into the model reference error dynamics, hence boundedness of the states and controller gain estimates is more involved. These two issues add complexity to the proof of stability and asymptotic convergence to the origin of the tracking error.

The main results are presented as two propositions. The first proposition ensures that the error $e(t) = x_l(t) - x_m(t)$ decays, asymptotically, to zero and that the adaptive gains are bounded; the second provides conditions under which $x_{aw}(t)$ will also decay asymptotically to zero and hence ensure that $x(t)$ approaches the ideal reference model states as $t \to \infty$.

Proposition 1: Let Assumptions 1 and 2 be satisfied and consider the interconnection of the plant (9), the reference model (4), the control law (11), the adaptive laws (7) and the anti-windup compensator (10). Additionally, assume $r \in \mathcal{L}_\infty$. Then the error defined in equation (12) is such that $\lim_{t\to\infty} e(t) = 0$ and the adaptive gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$ are bounded.

Proof: Using the dynamics (9), (10) and (4) it follows that

$$\begin{aligned}\dot{e} &= Ax + B\mathrm{sat}(u) - A_m x_m - B_m r \\ &\quad + (A_m - B\hat{K}_x + BF)x_{aw} + B\mathrm{Dz}(u) \\ &= Ax + B\big(\hat{K}_x(x + x_{aw}) + \hat{K}_r r - F x_{aw}\big) - B\mathrm{Dz}(u) \\ &\quad - A_m x_m - B_m r + (A - B\hat{K}_x + BF)x_{aw} + B\mathrm{Dz}(u) \\ &= Ax + B\hat{K}_x x + B\hat{K}_r r - A_m x_m - B_m r + A_m x_{aw} \\ &= A_m e + B(\hat{K}_x - K_x^*)x + B(\hat{K}_r - K_r^*)r\end{aligned}$$

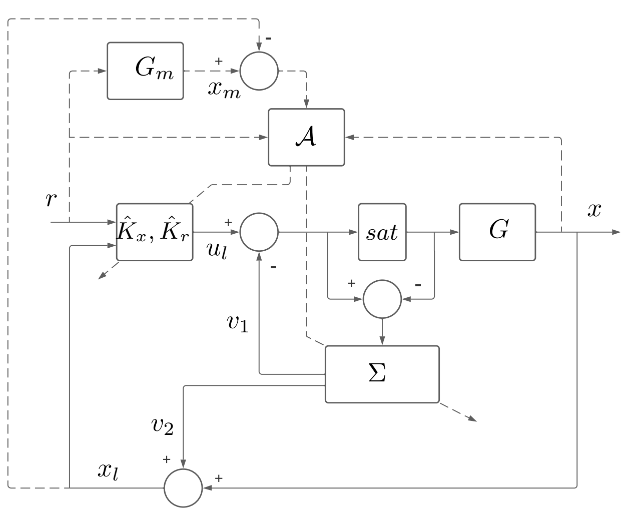

Fig. 1. Closed-loop saturated system

where the matching condition Assumption 2 has been used, along with the control law (11). Defining $\Delta K_x(t) = \hat{K}_x(t) - K_x^*$ and $\Delta K_r(t) = \hat{K}_r(t) - K_r^*$ then yields

$$\dot{e} = A_m e + B\Delta K_x x + B\Delta K_r r$$

Forming the Lyapunov function

$$V_1 = e' P_1 e + \mathbf{tr}[\Delta K_x \Gamma_x^{-1} \Delta K_x'] + \mathbf{tr}[\Delta K_r \Gamma_r^{-1} \Delta K_r']$$

it follows in the same way as standard MRAC, using the adaptive laws (7), that $\dot{V}_1 = -e' Q_1 e$ This enables one to conclude that $e(t)$, $\Delta K_x(t)$ and $\Delta K_r(t)$ are bounded, and then that $\hat{K}_x(t)$ and $\hat{K}_r(t)$ are also bounded.

Now, from Barbalat's lemma, it follows that if $\dot{V}_1(t)$ is uniformly continuous, then $\lim_{t\to\infty} V_1(t) = 0$ and hence, since $Q_1 > 0$, then $\lim_{t\to\infty} e(t) = 0$. Note that $\dot{V}_1(t)$ is uniformly continuous if $\ddot{V}_1(t)$ is bounded, where

$$\ddot{V}_1(t) = -2e'Q\big(A_m e + B\Delta K_x x + B\Delta K_r r\big) \tag{13}$$

By Lemma 1, $x(t)$ is bounded, so since $e(t)$, ΔK_x, ΔK_r and r are all bounded, uniform continuity is established, from which one infers convergence of $e(t)$. □□

Although the above proposition guarantees that the adaptive laws will be bounded and convergence of the error

$$e(t) = x_l(t) - x_m(t) = x(t) - x_{aw}(t) - x_m(t)$$

will be achieved, note that the error is constructed as the difference between some compensated states x_l and the reference model. Due to the presence of the $x_{aw}(t)$ state vector, it is not clear that, as $t \to \infty$ that $x(t) \to x_m(t)$. If it can be proved that, under certain conditions, $x_{aw}(t)$ itself converges to the origin, then $x_l(t) = x(t)$ and $x(t)$ will be guaranteed to converge to $x_m(t)$. Proposition 2 below gives conditions under which this will be achieved.

Proposition 2: Under the assumptions of Proposition 1, $\lim_{t\to\infty} x(t) = x_m(t)$ if there exists a scalar $\eta > 0$ such that the following conditions are satisfied:

1) $\hat{K}_x'(t)\hat{K}_x(t) \leq \eta \quad \forall t \geq 0$
2) There exist matrices $P_1 > 0$, diagonal $W > 0$ and F of suitable dimensions, and a scalar ϵ such that the following matrix Ψ is negative semi-definite where

$$\Psi = \begin{bmatrix} He\{P_1(A_m + BF)\} + \frac{1}{\epsilon}P_1 BB' P_1 + \eta\epsilon I & P_1 B - F'W \\ \star & -2W \end{bmatrix} \tag{14}$$

3) $\mathrm{Dz}(K_x^{ss} x_m + K_r^{ss} r) \in \mathcal{L}_2$ where K_x^{ss} and K_r^{ss} are the steady state values of the adaptive gains $\hat{K}_x$ and $\hat{K}_r$ respectively, i.e. $\lim_{t\to\infty} \hat{K}_{(x,r)} = K_{(x,r)}^{ss}$.

In essence, this proposition ensures convergence of the actual state $x(t)$ to the desired state $x_m(t)$ if: the bound on the adaptive control gain $\hat{K}_x$ is known (condition 1); the matrix P_1 satisfies a stronger condition than the Lyapunov equation, which involves knowledge of the bound on $\hat{K}_x$ (condition 2); and a certain signal belongs to $\mathcal{L}_2$ (condition 3). This final condition can be interpreted as requiring a fictitious control law (i.e. $K_x^{ss} x_m + K_r^{ss} r$), where the adaptive controller gains are replaced by their steady state values, to be below the saturation bounds as time approaches infinity.

This same idea has been used extensively in linear anti-windup research ([16]) and is a logical condition for which one would expect acceptable performance from a system equipped with anti-windup. The condition is a little more complex in the case of an adaptive control law, since K_x^{ss} and K_r^{ss} are not known in advance. Observe that the magnitude of such control gains will depend on "how far" the plant and the reference model are from each other (i.e. how far A_m is from A and B_m is from B), hence one might expect saturations events to cease if the open-loop and the desired closed-loop dynamics are not too far apart.

Proof of Proposition 2: As argued earlier, since $\lim_{t\to\infty} e(t) = 0$, then $\lim_{t\to\infty} x(t) = x_m(t)$, if $\lim_{t\to\infty} x_{aw}(t) = 0$. Therefore, consider

$$\dot{x}_{aw} = (A_m - B\hat{K}x + BF)x_{aw} + B\mathrm{Dz}(u)$$

and note that u can be re-written as

$$u = \underbrace{\hat{K}_x(x + x_{aw}) + \hat{K}_r r}_{u_0} - Fx_{aw}$$

Then, defining $\phi(u_0, x_{aw}) = \mathrm{Dz}(u_0 - Fx_{aw}) - \mathrm{Dz}(u_0)$ and adding and subtracting $B\mathrm{Dz}(u_0)$, yields

$$\dot{x}_{aw} = (A_m - B\hat{K}x + BF)x_{aw} + B\phi(u_0, x_{aw}) + B\mathrm{Dz}(u_0)$$

Since $\mathrm{Dz}(\cdot)$ is a slope-restricted nonlinearity (Fact 1), it follows from inequality (1) that, for all diagonal $W > 0$,

$$\phi(u_0, x_{aw})'W(-Fx_{aw} - \phi(u_0, x_{aw})) \geq 0 \quad (15)$$

Next, choosing a Lyapunov function $V(x_{aw}) = x_{aw}'P_1x_{aw}$, its derivative is bounded by

$$\begin{aligned}\dot{V}(x_{aw}) &\leq 2x_{aw}'P_1[(A_m + BF)x_{aw} - B\hat{K}_x x_{aw} + B\phi \\ &\quad + B\mathrm{Dz}(u_0)] + 2\phi'W(-Fx_{aw} - \phi) \\ &= \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix}' \begin{bmatrix} He\{P_1(A_m + BF)\} & P_1B - F'W \\ \star & -2W \end{bmatrix} \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix} \\ &\quad + 2x_{aw}'P_1B\hat{K}_x x_{aw} + 2x_{aw}'P_1B\mathrm{Dz}(u_0)\end{aligned}$$

However, for some $\epsilon > 0$, if Condition 1 of the proposition holds, then

$$2x_{aw}'P_1B\hat{K}_x x_{aw} \leq x_{aw}\Big(\frac{1}{\epsilon}P_1B'BP_1 + \epsilon\eta I\Big)x_{aw}$$

which implies that

$$\dot{V}(x_{aw}) \leq \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix}' \Psi \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix} + 2x_{aw}'P_1B\mathrm{Dz}(u_0) \quad (16)$$

Therefore, if Condition 2 of the proposition holds, one infers that there exist positive scalars c_1 and c_2 such that

$$\dot{V}(x_{aw}) \leq -c_1\|x_{aw}\|^2 + c_2\|x_{aw}\|\|\mathrm{Dz}(u_0)\| \quad (17)$$

Now observe that by using the Lipschitz property of the deadzone, then $\|\mathrm{Dz}(u_0)\|$ may be rewritten as

$$\begin{aligned}\|\mathrm{Dz}(u_0)\| &= \|\mathrm{Dz}(\hat{K}_x(x_m + e) + \hat{K}_r r)\| \\ &\leq \|\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r)\| + \|\hat{K}_x e\| \\ &\quad + \|(\hat{K}_x - K_x^{ss})x_m + (\hat{K}_r - K_r^{ss}r)\|\end{aligned}$$

Thus, since $\lim_{t\to\infty} e(t) = 0$, $\lim_{t\to\infty} \hat{K}_{(x,r)} = K_{(x,r)}^{ss}$ and x_m, r and $\hat{K}_x$ are bounded it follows that $\mathrm{Dz}(u_0) \in \mathcal{L}_2$ if $\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r) \in \mathcal{L}_2$. Applying the Comparison Lemma to equation (17), it therefore follows that $x_{aw} \to 0$ as $t \to \infty$ if $\mathrm{Dz}(u_0) \in \mathcal{L}_2$, which holds if $\mathrm{Dz}(K_{x,ss}x_m + K_{r,ss}r) \in \mathcal{L}_2$, which is exactly Condition 3 in the proposition. □□

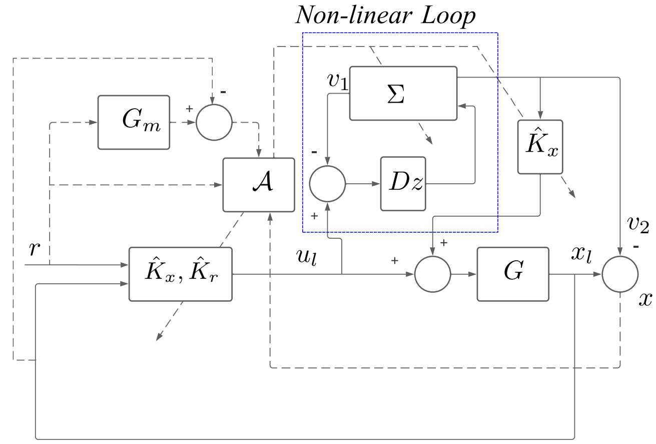

Fig. 2. Alternative representation of the closed-loop system

A. Solving the matrix inequality

Condition 2 of Proposition (2) contains a nonlinear matrix inequality which is difficult to solve. To circumvent this difficulty, the matrix inequality in (14) can be "linearised" via the Schur complement and some similarity transformations.

Lemma 1: There exist matrices $P_1 > 0$, diagonal $W > 0$ and F and a scalar $\epsilon > 0$, such that $\Psi < 0$ (see equation (14)), if there exist matrices $Q > 0$, diagonal $U > 0$ and L, and a scalar $\tilde{\epsilon} > 0$ such that the following linear matrix inequality is satisfied,

$$\begin{bmatrix} He\{A_mQ + BL\} + \tilde{\epsilon}BB' & BU - L' & Q \\ \star & -2W & 0 \\ \star & \star & -\tilde{\epsilon}\frac{1}{\eta} \end{bmatrix} < 0 \quad (18)$$

In particular, the relationship between the matrices is $Q = P_1^{-1}$ $U = W^{-1}$, $\tilde{\epsilon} = \epsilon^{-1}$ and $F = LQ^{-1}$.

This inequality has a similar form to the matrix inequality required to be solved for the linear anti-windup problem discussed in [18]; the extra terms involving $\tilde{\epsilon}$ and η capture the fact that the matrix A is not known and hence an estimate must be used. Clearly, the bound on η influences the solution of the matrix inequality and the anti-windup gain F returned.

B. Estimating η

A key element of Proposition 2 is the estimation of η, since this is used in the matrix inequality from which the anti-windup gain F is computed. Furthermore, η must be chosen correctly since if it is under-estimated, the stability results will actually be local rather than global.

From the proof of Proposition 1 it is known that $\dot{V}_1(e, \Delta K_x, \Delta K_r) \leq 0$ which implies that

$$\begin{aligned}&\mathbf{tr}(\Delta K_x(t)\Gamma_x^{-1}\Delta Kx(t)) \leq e(0)'Pe(0) + \\ &\quad \mathbf{tr}(\Delta K_x(0)\Gamma_x^{-1}\Delta K_x(0)') + \mathbf{tr}(\Delta K_r(0)\Gamma_r^{-1}\Delta K_r(0)')\end{aligned}$$

Assuming that $e(t)$ is initially zero and that $\hat{K}_x(0) = 0$ and $\hat{K}_r(0) = 0$ also, the above inequality simplifies to

$$\begin{aligned}\mathbf{tr}(\hat{K}_x\Gamma^{-1}\hat{K}_x) \leq & \mathbf{tr}(K_r^*\Gamma_r^{-1}K_r^*) + \delta\mathbf{tr}(\hat{K}_x\Gamma^{-1}\hat{K}_x)) \\ & + \frac{1}{\delta}\mathbf{tr}(K_x^*\Gamma_x^{-1}K_x^*)\end{aligned}$$

for some nonnegative $\delta < 1$. Thus,

$$\mathbf{tr}(\hat{K}_x'\hat{K}_x) \leq \frac{\mathbf{tr}(\Gamma_x^{-1})\|K_r^*\|^2 + \frac{1}{\delta}\mathbf{tr}(\Gamma_r^{-1})\|K_x^*\|^2}{(1-\delta)\mathbf{tr}(\Gamma_x^{-1})}$$

The left hand side is simply $\|\hat{K}_x\|^2$ and thus the right hand side can be used to obtain η, under the conditions assumed on the initial values of $e(0)$ and $\hat{K}_x(0)$ and $\hat{K}_r(0)$. This implies that η can be calculated provided the bounds on $\|K_x^*\|$ and $\|K_r^*\|$ can themselves be estimated adequately.

IV. Simulation Example

Consider a flight control application where the plant is the longitudinal attitude dynamics of the JAXA μ-Pal experimental aircraft [28]. The aircraft is trimmed at flight condition, levelled wings straight flight, of altitude $1524\,m$, velocity $VTAS = 66.5\,m/s$, angle of attack $\alpha = 4.98\,deg$. The linearised plant has state-space model:

$$A = \begin{bmatrix} -0.0175 & 0.173 & -9.77 & -5.63 \\ -0.192 & -1.09 & -0.846 & 64.6 \\ 0 & 0 & 0 & 1 \\ 0.0081 & -0.0738 & 0.0062 & -1.9 \end{bmatrix} \quad B = \begin{bmatrix} -0.428 \\ 4.91 \\ 0 \\ 4.22 \end{bmatrix}$$

The states are given by $x = [u_x,\ u_z,\ \theta,\ q]^T$, where the first two states are the translational velocity in the x and z directions respectively, and the last two are pitch angle θ, and pitch rate q. The control signal is the deflection angle of the elevator, which is assumed to be (symmetrically) saturated with $\bar{u} = 50$. In this example we assume that B is known, but A is unknown. Access to all the states is assumed.

Define the reference model matrices A_m and B_m as:

$$A_m = \begin{bmatrix} -0.03 & 0.166 & 12.56 & 37.29 \\ -0.052 & -1.02 & -1554.7 & -427.82 \\ 0 & 0 & 0 & 1 \\ 0.128 & -0.0142 & -1335.49 & -425.12 \end{bmatrix} B_m = \begin{bmatrix} -138.1 \\ 1584.2 \\ 0 \\ 1361.6 \end{bmatrix}$$

The MRAC parameters were chosen as $\Gamma_x = \mathrm{diag}\{1, 1, 10, 10\}$ and $\Gamma_r = 1$, and the solution for the associated Lyapunov is obtained with $Q_1 = 10I$. The AW gain F was obtained from Lemma 1 with $\eta = 100$, which is the maximum before the LMI solution becomes infeasible, and applying Lemma 1. The matrix obtained is

$$F = 10^4 \begin{bmatrix} 0.436 & 0.0112 & -5.5556 & 0.0102 \end{bmatrix}$$

The system is subject to a reference pulse with magnitude of $30\,\mathrm{deg}$ and duration of 100 seconds, and its nominal response (no saturation) is presented in Figure 3 (with zero initial conditions for the controller gains). The MRAC system provides a well-damped transient response and ensures a small tracking error. Nonetheless, the control signal saturates severely (with commands of the order of magnitude of 10^3 degrees) during the attitude acquisition stage, which means that some detriment to stability and performance may be present. Indeed, the saturated system with no AW augmentation remains stable but shows a loss of performance, with the system unable to track the reference model and exhibiting large oscillations (see Figure 4); the control signal is severely saturated, presents high frequency dynamics and is unable to recover from saturation . The closed-loop response of the system with AW compensation recovers system performance, presenting a clear reduction in oscillation and enhancement of steady state tracking error, and the control signal recovers linear dynamics during period of no saturation of the nominal system (see Figure 5)

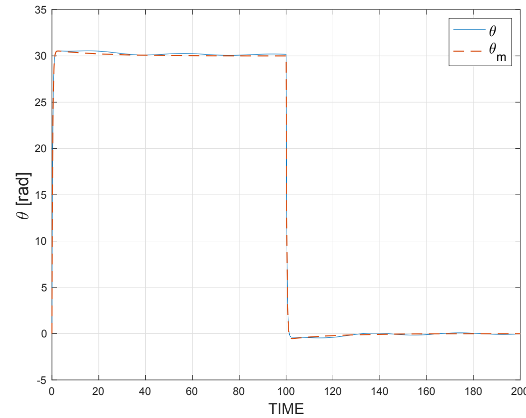

Fig. 3. Pitch attitude response of closed-loop system with no saturation

The results show how the adaptive MRAC compensator is able to achieve stability and retain performance even during periods of saturation. It is noted that for this reference signal, the "ideal" control signal u_l converges to a value that is within the control constraints, i.e. in steady state $\mathrm{Dz}(u_l) = 0$. Further simulations (not shown) with this model, reveal that as the reference amplitude increases, and as control signal saturation becomes more severe, the response of the system without anti-windup degrades further. However, this can be arrested substantially by the inclusion of anti-windup.

V. Conclusions

This paper has proposed a full-order, model recovery anti-windup compensator for MRAC schemes. Traditional MRAW anti-windup uses a copy of the plant in order to achieve stability and system decoupling properties. In this paper, an "estimate" of the plant's A-matrix has been used instead, by making use of the model matching conditions (5) and an adaptation law that guarantees that the "ideal" behaviour is recovered. In fact, it was shown that if a fictitious linear control signal lies within the constraint set, the nominal MRAC formulation is recovered. The effects of disturbances, measurement noise and unmodelled dynamics have not been accounted for, hence future work must address the addition of the so-called robust adaptive control modifications and their implications for the anti-windup scheme under consideration.

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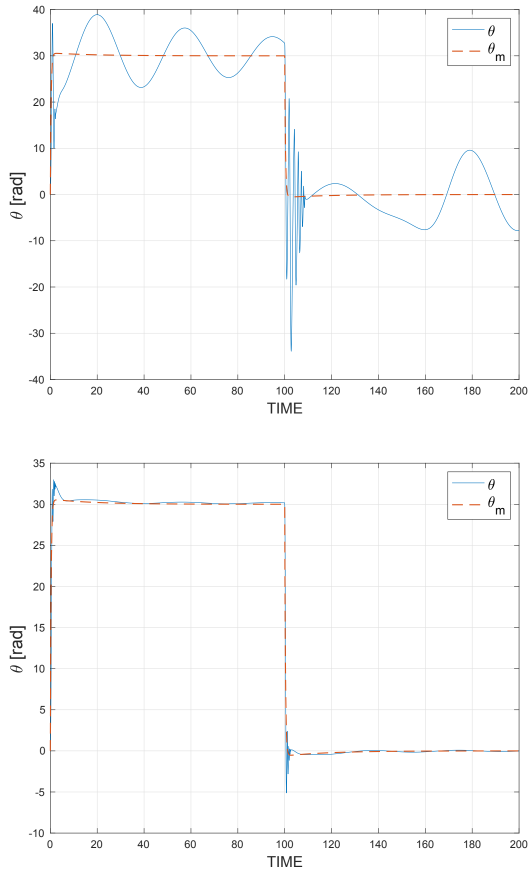

Fig. 4. Closed-loop System with saturation: top, system with no AW augmentation; bottom, closed-loop system response with AW compensation

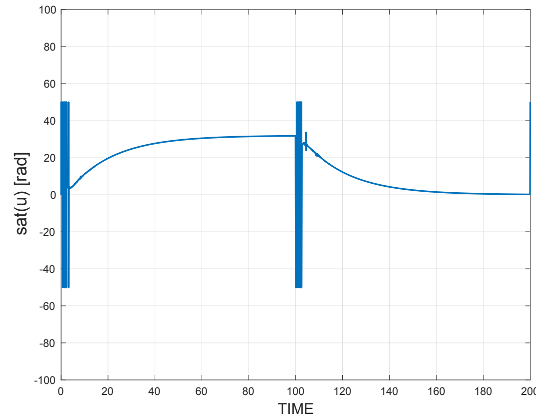

Fig. 5. Closed-loop System with saturation: top, system with no AW augmentation; bottom, closed-loop system response with AW compensation

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