Journal of Elasticity
https://doi.org/10.1007/s10659-022-09948-7

CLASSROOM NOTE

# An Elementary Derivation of the Maximum Shear Stress in a Three Dimensional State of Stress

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Received: 14 September 2022 / Accepted: 16 October 2022


**Abstract**
The maximum shear stress associated with a 3D stress state is a widely used quantity in solid mechanics. While the expression of this quantity in terms of principal stresses is given in most mechanics classes, its derivation is far less common. In this classroom note, an elementary derivation of the maximum shear stress is given that avoids vector calculus, Lagrange multipliers, and the full framework necessary for Mohr's graphical derivation.

**Keywords** Stress analysis · Critical plane analysis · Mohr's circle · Tresca yield surface · Plasticity · Plastic slip

**Mathematics Subject Classification** 15-01

## 1 Introduction

A central result of continuum mechanics is the existence of a traction vector [1, 2], $\vec{t}$, acting on a plane defined by its outward normal unit vector, $\hat{n}$, which can be written in terms of a symmetric and real valued 2nd rank stress tensor, $\sigma$,

$$\vec{t} = \sigma \hat{n}. \tag{1}$$

In many contexts, such as metal plasticity [3–6], it is useful to decompose $\vec{t}$ into its orthogonal shear and normal components

$$\vec{t} = t_s \hat{s} + t_n \hat{n}, \tag{2}$$

and determine the maximum absolute value of the shear traction, denoted here as $\max_{\hat{n}}(|t_s|)$, over all unit vectors, $\hat{n}$, for a given $\sigma$. The expression for this maximum shear stress is given in most mechanics textbooks as a function of the maximum and minimum eigenvalues (principal stresses) of $\sigma$,

$$\max_{\hat{n}}(|t_s|) = \frac{p_{max} - p_{min}}{2}. \tag{3}$$

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The above expression is graphically apparent from the 3D version of Mohrs circle [4, 7] and can be derived with 3D vector calculus by finding the stationary points of $|t_s|$ with respect to $\hat{n}$ subject to the constraint that $|\hat{n}| = 1$, as is done in Lai, Ruben, and Krempl [8]. In most textbooks, the complexities associated with the latter and the time required to introduce the framework of the former are avoided. In these cases, 3 is often only partially developed, derived in 2D, or given as an axiom.

This can leave a student with an unsatisfied feeling as other important characteristics of the stress state are directly derived in 3D without calculus, using basic linear algebra operations. Examples include the largest signed normal traction, $\max_{\hat{n}}(t_n) = p_{max}$, and the least signed normal traction, $\min_{\hat{n}}(t_n) = p_{min}$. Thus, shouldn't the maximum shear stress (3) not also follow directly from simple linear algebra?

To address this point, a derivation of $\max_{\hat{n}}(|t_s|)$ that avoids calculus and graphical demonstration is given below in 3D.

## 2 Derivation

We start by considering the dot products of $\vec{t}$ with $\hat{n}$ and itself. In relation to the orthogonal decomposition of (2), these dot products can be expressed as

$$\vec{t} \cdot \hat{n} = t_n \tag{4}$$

and

$$\vec{t} \cdot \vec{t} = t_n^2 + t_s^2. \tag{5}$$

In an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ that diagonalizes the stress tensor in terms of eignvalues according to $p_{max} \geq p_{mid} \geq p_{min}$, we have

$$\sigma = \begin{bmatrix} p_{max} & 0 & 0 \\ 0 & p_{mid} & 0 \\ 0 & 0 & p_{min} \end{bmatrix}, \tag{6}$$

and the dot products can be expressed as

$$t_n = p_{max} n_1^2 + p_{mid} n_2^2 + p_{min} n_3^2 \tag{7}$$

and

$$t_s^2 + t_n^2 = p_{max}^2 n_1^2 + p_{mid}^2 n_2^2 + p_{min}^2 n_3^2, \tag{8}$$

where $n_1$, $n_2$, and $n_3$ represent the components of the unit vector $\hat{n}$ in the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Combining the above two equations with the constraint

$$1 = n_1^2 + n_2^2 + n_3^2, \tag{9}$$

we arrive at three independent linear equations with respect to $n_1^2$, $n_2^2$, and $n_3^2$. Eliminating $n_1^2$ and $n_3^2$ from (7), (8), and (9) we find

$$t_s^2 + t_n^2 - t_n(p_{min} + p_{max}) + p_{min} p_{max} = (p_{min} - p_{mid})(p_{max} - p_{mid}) n_2^2. \tag{10}$$

The development and utilization of (10) is viewed as the key step in this derivation. If instead, $n_2^2$ had been eliminated from (7), (8), and (9), the subsequent upper bound on $t_s^2$ does not follow.

Combining the two $t_n$ terms via completion of the square and isolating the $t_s^2$ term in (10), we obtain equivalently

$$t_s^2 = \frac{1}{4}(p_{max} - p_{min})^2 - \left(t_n - \frac{1}{2}(p_{max} + p_{min})\right)^2 + (p_{min} - p_{mid})(p_{max} - p_{mid})n_2^2, \quad (11)$$

which is valid for all unit vectors $\hat{n}$. Now observe that the second and third terms on the rhs of (11) are each less than or equal to zero. Thus, we see that

$$t_s^2 \leq \frac{1}{4}(p_{max} - p_{min})^2 \quad \forall\, \hat{n}, \qquad (12)$$

and we may conclude that equality holds if and only if both the second and third terms on the rhs of (11) are identically zero.

To complete the proof of (3), we now need only show that there is at least one unit vector $\hat{n}$ for which equality holds. To show this, let us suppose that $n_2 = 0$ in (11) and use (7) to conclude that the second term on the rhs of (11) will vanish if and only if

$$p_{max}n_1^2 + p_{min}n_3^2 = \frac{1}{2}(p_{max} + p_{min}), \qquad (13)$$

which, according to (9), is to hold for $n_1^2 + n_3^2 = 1$. Eliminating $n_3^2$ from (13), we then find that $n_1^2(p_{max} - p_{min}) = (p_{max} - p_{min})/2$, and there are two cases to consider.

Case (i) $p_{max} \neq p_{min}$: Here, $n_1^2 = n_3^2 = 1/2$ and, with $n_2 = 0$, we find that $\hat{n} = (\hat{e}_1 + \hat{n}_3)/\sqrt{2}$. Thus, (11) shows that equality holds in (12) for this $\hat{n}$, and (3) is proved.

Case (ii) $p_{max} = p_{min}$: Here, we have the case where all principal stresses are equal and (13) is identically satisfied because we have taken $n_2 = 0$ and (9) requires that $n_1^2 + n_3^2 = 1$. Thus, (11) shows that equality holds in (12), with the rhs identically zero, for any $\hat{n}$ of the form $\hat{n} = n_1\hat{e}_1 + n_3\hat{e}_3$ with $n_1^2 + n_3^2 = 1$, and (3) is proved for any such $\hat{n}$.

The interested student may gain further insight by examining (11) more fully to determine the complete sets of unit vectors $\hat{n}$ that lead to equality in (12) for each of the four cases of principal stresses: (i) $p_{\max} \neq p_{\text{mid}} \neq p_{\min}$; (ii) $p_{\max} = p_{\text{mid}} \neq p_{\min}$; (iii) $p_{\max} \neq p_{\text{mid}} = p_{\min}$; and (iv) $p_{\max} = p_{\text{mid}} = p_{\min}$.

**Acknowledgements** The author gratefully acknowledges technical input from Roger Fosdick and support from the Office of Naval Research # N000142012484, the National Science Foundation # 1922081, and Sandia National Laboratories # 2304832.

**Author contributions** This is a single author paper.

## Declarations

**Competing interests** The authors declare no competing interests.

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