The Derby Game: An Ordering-based Colonel Blotto Game

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The Colonel Blotto game and its variants are a common approach to study competitive allocation of *interchangeable* resources (e.g., soldiers, money, or votes). We introduce a new variant of Blotto, which we call the Derby game, to study competitive allocation of *non-interchangeable* resources, such as campaign surrogates in politics or skilled workers in companies, and analyze its Nash equilibria. While Derby games unsurprisingly admit no pure Nash equilibria, our main results surprisingly show that Nash equilibria generically exist *where one player plays a pure strategy*, and we give necessary and sufficient conditions for such equilibria.

CCS Concepts: • **Theory of computation** → **Algorithmic game theory**; **Solution concepts in game theory**.

Additional Key Words and Phrases: Resource allocation games; Colonel Blotto games; Nash equilibria

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1 INTRODUCTION

Much research has been devoted to answering questions regarding competitive allocation of resources, based on scenarios like:

- (Politics) During an election, how should a candidate spend their funds to target various voters/voting districts in order to win more votes than their opposition?
- (Business development) How should a company divide its product development workforce across various product areas to best compete with other companies in these areas and maximize its future market share?
- (Entertainment) How should a television network divide its total budget across its various shows in order to compete for viewership with the same-time-slot shows of a rival network?

The principal tools in answering such questions are the Colonel Blotto game and its variants, which model the above scenarios as a multiplayer game in which each player is given some number of soldiers which they must distribute across some number of battlefields. In the original game, a player wins a battlefield if they allocate more soldiers to that battlefield than their competitors, and a player wins the game if they win a plurality of battlefields, though many variants have changed how player utility is calculated.

One limitation of Blotto and its existing variants, however, is that the games are restricted to scenarios regarding the competitive allocation of *interchangeable* resources – the particular choice

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of which soldiers are allocated to a given battlefield should not matter, merely the number of soldiers each player has allocated.

This has traditionally made Blotto-like games very well-suited to modeling scenarios like the first and third bullet points above (where units of currency are identical), but somewhat less suited to modeling scenarios like the second bullet point (where employees are not necessarily identical). Going even further, it is easy to imagine scenarios where every resource is unique and non-interchangeable (i.e., nonfungible):

- (Politics) During an election, each candidate may have many campaign surrogates, each surrogate with different reputation and public sway. On a given week, where should the candidate send each surrogate to campaign, given that their opposition will also be sending surrogates of their own?
- (Politics) During an election, on which two-week news cycles should a candidate plan to release various pieces of scandalous information about their opponent, knowing that their opponent will also be releasing scandalous pieces of information about them?
- (Business Development) How should a company assign its variously skilled vice-presidents or project managers across its product development teams, given that companies making competing products also have variously skilled vice-presidents and project managers that they too will be assigning? Should a company necessarily place its best people on its version of a given product if their competition already has more skilled/experienced people working on a version of the same product?
- (Entertainment) How should a television network assign its TV shows to various time slots to best compete for viewership against rival networks (given some measure of show quality, like pilot ratings)?

For these scenarios, every resource being allocated is different from every other resource — such scenarios cannot be modeled by Blotto or other existing resource allocation games.

To start filling this gap, we introduce a new variant of Blotto, which we call the Derby game[1]. A Derby game consists of a number of *rounds* and each player is equipped with the same number of *resources*. Players assign each resource to a particular round; a player wins round i and gets a payoff of w_i, the weight of round i, if they play the better resource (according to a preference order over resources) in that round, and 0 otherwise. The total payoff is the sum of payoffs for each round. This game is suited to modeling many scenarios regarding the competitive allocation of non-interchangeable resources, including those listed above. We formally define Derby games in Section 2.

1.1 Results

We analyze Nash equilibria in Derby games. While Derby games unsurprisingly admit no pure Nash equilibria, our main results surprisingly show that Nash equilibria generically exist *where one player plays a pure strategy* (and we'll refer to such equilibria as *half-pure*). Specifically, we show the following:

- As a warmup, we consider unweighted Derby games. Here, we fully characterize all Nash equilibria as "effectively uniform," and show that no half-pure Nash equilibria exist (Section 3, Theorem 3.10).
- We provide necessary and sufficient conditions for a half-pure Nash equilibrium to exist in a Derby game (Section 5, Corollary 5.16).

[1]We chose this name to pay homage to Tian Ji's horse-racing strategy, an ancient Chinese parable closely related to our work. The parable is told in more detail in Section 2.

- These necessary and sufficient conditions can already be satisfied by the simplest non-trivial Derby game, which we study as a special case in Section 4.
- Our main technical hammer is Theorem 5.4, which we term the *Narrow Wins Theorem*. Intuitively, if Player B's best resource B_1 loses only to Player A's best resource A_1, it seems as though any best response of Player A must only play resource A_1 in rounds where there is a non-zero probability of playing B_1. This turns out to be true (Corollary 5.7). More generally, if the ordering of resources satisfies $A_i > B_i > A_{i+1}$ (with no resources in between), intuitively it seems as though any best response of Player A must only play resource A_i in rounds where there is a non-zero probability of playing B_i (otherwise, it seems as though Player A should just play A_{i+1} instead). This intuition is not quite correct, but the Narrow Wins Theorem establishes that any Nash equilibrium violating this intuition is quite structured. In general, the Narrow Wins Theorem provides strong conclusions on the structure of Nash equilibria of Derby games that follow downstream from repeated iterations of this simple intuition (Section 5.3, Corollary 5.16). In particular, while we apply the Narrow Wins Theorem to identify necessary and sufficient conditions for the existence of half-pure Nash equilibria, the Narrow Wins Theorem will be a generally useful tool for any study of Derby games.

To summarize, our main contribution is the introduction of Derby games, the study of their Nash equilibria, and in particular necessary and sufficient conditions for half-pure Nash equilibria to exist. We also provide a technical hammer for analyzing equilibria of Derby games, the Narrow Wins Theorem.

In Section 2, we introduce Derby games. Section 3 considers unweighted Derby games as a warmup, and establishes that half-pure Nash equilibria never exist. Section 4 considers the simplest non-trivial weighted case, and finds necessary and sufficient conditions for half-pure Nash equilibria. Section 5 provides our main result: necessary and sufficient conditions for half-pure Nash equilibria in the general case, and our main technical hammer: the Narrow Wins Theorem. Section 6 discusses related work, and Section 7 provides concluding thoughts.

2 DERBY GAMES

Derby games are a generalization of a game described in the parable of Tian Ji's horse races against King Wei [13]: we will briefly summarize this parable, then describe our generalization and give its formalization, and finally end this section with an example Derby game.

In the ancient Chinese state of Qi, a general named Tian Ji often competes in horse racing competitions with the King of Qi, King Wei. In one such competition, Tian Ji and King Wei each bring three horses, in order to have three rounds of racing one horse each (never reusing a horse). Both players have a fast horse, a medium horse, and a slow horse, but all of Tian Ji's horses are slower than the same class of King Wei's horses (but are still faster than successive classes).

Tian Ji knows that if he plays his fast horse against King Wei's fast horse, and his medium horse against King Wei's medium horse, and his slow horse against King Wei's slow horse, he will lose all three races, so he turns to his strategist, Sun Bin, for advice. Sun Bin thinks for a moment before offering up the following strategy: *Play your slow horse against King Wei's fast horse, your fast horse against King Wei's medium horse, and your medium horse against King Wei's slow horse.* With this strategy, Tian Ji wins the overall competition, since his slow horse loses, but the other two horses are able to win.

We generalize the game from this parable in three ways: (1) we consider an arbitrary number of rounds, (2) we allow rounds to have different weights, and (3) we consider an arbitrary total order on the horses of both players, rather than the alternating order described above. Formally:

Definition 2.1 (Derby game). A Derby game is defined by a tuple $(m, F, G, >, \vec{w})$ where

#	Channel A	Box Television	#
A1	Formula 1 Racing	Box News	B1
A2	Trendy: bomber jackets to bowties	Good Food, hosted by Ribo Flavin	B2
A3	Opening with A. Anderssen	The Number of the Counting	B3
A4	Paper: an in-depth video series	History of Ancient Civilizations	B4
A5	Keeping it simple with Nona Belgroup	Basically Besties: Boozy Bougie Brunches	B5

(a) Shows planned by each TV network.

A1 > B1 > B2 > B3 > A3 > A2 > A4 > B5 > B4 > A5

(b) Expected performance based on show pilots.

Fig. 1. A Derby game for scheduling TV shows.

- $m \in \mathbb{N}$ is the number of rounds
- F and G are the sets of resources (or horses) of players A and B respectively, where $|F| = |G| = m$
- $> \subseteq (F \cup G)^2$, the victory relation, is a strict total order over $F \cup G$.
- $\vec{w} \in \mathbb{R}^m_{>0}$ is a vector of round weights. WLOG, we require that $w_1 \geq w_2 \geq \cdots \geq w_m$.

In a game, A and B play *schedules*, bijections $p\colon [m] \to F$ and $q\colon [m] \to G$ respectively, which map each round to the resource the player plays in that round. Then for a round i the player who played the winning resource—A if $p(i) > q(i)$, otherwise B—wins the round and gets a payoff of w_i. The total payoff of each player is given by

$$U_A(p, q) = \sum_{i=1}^{m} \mathbb{1}[p(i) > q(i)]w_i \quad ; \quad U_B(p, q) = \sum_{i=1}^{m} w_i - U_A(p, q)$$

An important feature of this definition is that it requires allocating exactly one resource to each round – unlike Blotto, which also supports many or zero allocated resources. This restriction is required in order for us to model the competition from the parable, and indeed, many of the scenarios listed in the introduction. We now take one such scenario from the introduction and use it as our running example.

Example 2.2 (Running example). Consider two television networks, Channel A (A) and Box Television (B), as shown in Figure 1. Channel A and Box are both preparing schedules for the various TV shows they intend to run this season. Both know that whichever network has the better show at a given time slot will get the lion's share of viewers (and by extension the lion's share of advertising dollars). Moreover, after running pilot episodes for each of their to-be-scheduled shows last season, both A and Box know (1) what shows the other will schedule and (2) approximately how well the show will be received. The game describing this example would be $(5, \{\mathsf{A1}, \ldots, \mathsf{A5}\}, \{\mathsf{B1}, \ldots, \mathsf{B5}\}, >, \vec{w})$, where $\vec{w}$ could represent the average viewership of each time slot.

Naturally, one might ask, does Sun Bin's strategy generalize to this example and other Derby games — if all rounds have the same weight, and Box Television's schedule is fixed, is it a good strategy for Channel A to schedule their k best shows against Box's k worst, *narrowly winning* those time slots by sacrificing A's remaining $m - k$ shows against Box's $m - k$ best shows? This is indeed the case:

Remark 2.3 (Optimality of best-vs-worst). *For a Derby game, if all rounds have the same weight and player B's schedule is fixed, consider a schedule for player A such that (1) for some constant $K \in [m]$, A plays their best resource against B's $(m - K + 1)$th best resource, their second best resource* *against B's $(m-K+2)th$ best resource, and so on until A plays their Kth best resource against B's worst (mth-best) resource, (2) A wins these K rounds while losing the remaining $m-K$ rounds, and (3) K is the largest such constant where (1) and (2) hold. Then this schedule gives at least as much payoff as any other schedule A can play.*

We show this more formally in Appendix A, but the idea is to start by finding conditions under which taking a schedule for player A and swapping two resources produces a schedule that is at least as good as the original. Then repeated swaps that respect such conditions can be used to produce the strategy above.

A more interesting scenario occurs when *neither* player's strategy is fixed – when both Channel A and Box Television are free to choose and update their schedules. In the coming sections we will analyze the Nash Equilibria of such games, even seeing a recurrence of Sun Bin's best-vs-worst strategy in the more complex game (see Corollary 5.5).

3 UNWEIGHTED DERBY GAMES

We begin by considering instances of the Derby game where all round weights are equal, i.e., games of the form $(m, F, G, >, \vec{1})$. In our running example, this would be the case if all time slots have roughly the same viewership, e.g., perhaps if each of the five time slots corresponds with the same time on a different weekday.

Since neither player is fixed to any particular schedule, each player plays some strategy. A pure strategy of player A or player B corresponds to a schedule selected by that player, while a mixed strategy is a distribution over schedules. We wish to understand the equilibria of this game.

Remark 3.1. *In an unweighted Derby game, there are no strictly/weakly dominant strategies: either (1) A gets the same payoff regardless of how A responds to B's schedule, or (2) A's best responding pure strategy depends on the schedule that B uses (it is the optimal response against B's strategy described in Remark 2.3).*

Since neither player has a dominant strategy, we next consider Nash Equilibria of the game. Because the game is finite, there is guaranteed to be at least one Nash Equilibrium:

Lemma 3.2. *Every unweighted Derby game has a Nash equilibrium where both players play schedules uniformly at random.*

Proof. If player A plays uniformly at random, every schedule of player B has the same utility. This is because (1) every round has the same weight, and (2) every round has the same distribution of resources A can play, ensuring the payoff for B is always the total expected return of B's resources against a random resource of A. The argument for player A is similar. □

This gives one Nash equilibrium, but we now seek to characterize all Nash equilibria. Our approach will be to first reintroduce a notion of interchangability to Derby games, and then to use that notion to give necessary and sufficient conditions for a pair of strategies to be at equilibrium.

3.1 Reintroducing interchangability

Note that with the victory relation in Figure 1 (b), the resources B_1, B_2, B_3 give the same payoff regardless of which of A's resources they play against: we might say they are interchangeable, or *equivalent*. More generally:

Definition 3.3 (Resource equivalence). For a (possibly weighted) Derby game $(m, F, G, >, \vec{w})$, two resources $f_1, f_2 \in F$ are *equivalent* if for all resources $g \in G$, $f_1 > g \iff f_2 > g$ (and similarly for two resources in G). We use this to partition F and G into *equivalence classes* of resources,

$A_1 \cup \cdots \cup A_{n_1} = F$ and $B_1 \cup \cdots \cup B_{n_2} = G$, where two resources are in the same equivalence class A_x iff they are equivalent (and similarly for B_x). We also lift the total order $>$ such that for all $C_x, C_y \in \{A_1, \ldots, A_{n_1}, B_1, \ldots, B_{n_2}\}$, $C_x > C_y$ denotes that $\forall f_1 \in C_x . \forall f_2 \in C_y . f_1 > f_2$.

Corollary 3.4 (Resources equivalent iff consecutive). *For a Derby game $(m, F, G, >, \vec{w})$, two resources $f_1, f_2 \in F$ are in the same equivalence class iff there is no $g \in G$ such that $f_1 > g > f_2$ or $f_2 > g > f_1$. (And similarly with two resources in G.) This means we can represent the equivalence classes of resources in F and G as $A_1 > B_1 > A_2 > B_2 > \cdots > A_n > B_n$, for some n, with allowance that A_1, B_n may be empty.*

We can use this to express the Derby game and strategies for the Derby game in a canonical form, hiding the redundant strategies produced by swapping equivalent resources.

Definition 3.5 (Canonical Derby game). For a Derby game $(m, F, G, >, \vec{w})$, we denote its canonical form by the tuple $(m, n, \vec{A}, \vec{B}, \vec{w})$ where

- m and $\vec{w}$ are as before
- $n \in \mathbb{N}$ is the number of equivalence classes F, G are both partitioned into,
- $\vec{A} = (A_1, \ldots, A_n), \vec{B} = (B_1, \ldots, B_n)$ are vectors of the equivalence classes of F and G resp., where A_1 and B_n are potentially empty,
- It holds that $A_1 > B_1 > A_2 > B_2 > \cdots > A_n > B_n$,

We also reserve the lowercase of each equivalence class to denote its size, i.e. $a_x = |A_x|$ and $b_x = |B_x|$.

This canonicalization of the Derby game is why we consider Derby games a variant of Blotto – like in Blotto we have competitive allocation of interchangeable resources, although resources are only interchangeable from the perspective of the other player, and that too only if they are of the same equivalence class.

Definition 3.6 (Effective Strategy). For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$, we define effective strategies of A and B as an $m \times n$ matrices $P, Q \in [0, 1]^{m \times n}$, such that

- each cell P_{ix} or Q_{ix} represents the marginal probability of playing a resource from the xth equivalence class (A_x or B_x) on round i
- columns of P, Q sum to facet counts: $\forall x. \sum_{i=1}^{m} P_{ix} = a_x$, $\forall x. \sum_{i=1}^{m} Q_{ix} = b_x$
- rows of P, Q sum to 1: $\forall i. \sum_{x=1}^{n} P_{ix} = 1$, $\forall i. \sum_{x=1}^{n} Q_{ix} = 1$

We also lift payoff to effective strategies: $U_A(P, Q) = \sum_{i=1}^{m} \sum_{x=1}^{n} \sum_{y=x}^{n} w_i P_{ix} Q_{iy}$. (Note that we sum from $y = x$ since A_x wins against $B_x, \ldots, B_n$.)

Remark 3.7. *Effective strategies in the canonical game abstract mixed strategies in the original game: (1) every mixed strategy be mapped to effective strategies in a utility-preserving way (if p, q are mixed strategies mapping to effective strategies P, Q, then $u_A(p, q) = u_A(P, Q)$), but also (2) every effective strategy corresponds to at least one mixed strategy.*

We show the above remark formally in Appendix B. This feature is another difference between Derby games and Blotto (and many Blotto variants), in which modeling strategies using independent marginal distributions over rounds does not guarantee that the budget will never be exceeded.

Example 3.8. The television network running example, $(5, \{\mathsf{A1}, \ldots, \mathsf{A5}\}, \{\mathsf{B1}, \ldots, \mathsf{B5}\}, >, \vec{w})$, has the canonical form $(5, 3, \{A_1, A_2, A_3\}, \{B_1, B_2, B_3\}, \vec{w})$, where $(A_1, A_2, A_3) = (\{\mathsf{A1}\}, \{\mathsf{A2}, \mathsf{A3}, \mathsf{A4}\}, \{\mathsf{A5}\})$ and $(B_1, B_2, B_3) = (\{\mathsf{B1}, \mathsf{B2}, \mathsf{B3}\}, \{\mathsf{B4}, \mathsf{B5}\}, \emptyset)$.

Henceforth, we no longer need to work with schedules – we will use $(m, n, \vec{A}, \vec{B}, \vec{w})$ to describe a Derby game and use effective strategies instead of strategies (soon dropping the "effective" for brevity). Instead, we will say a pure effective strategy has either 0 or 1 in every cell, while a mixed effective strategy can have any value in $[0, 1]$.

3.2 Characterizing all Nash Equilibria of the Unweighted Game

Earlier, we remarked that all strategies can be mapped to effective strategies that generalize them. The following is the effective strategy that generalizes the uniform strategy mentioned in Lemma 3.2 (see Appendix B for the exact mapping):

Definition 3.9 (Effectively uniform strategy). The *effectively uniform strategy* plays each equivalence class with probability proportional to the size of the equivalence class, i.e. for a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$, the effectively uniform strategies for players A and B, $\text{Uniform}(\vec{A}), \text{Uniform}(\vec{B}) \in [0, 1]^{m\times n}$, have the form $\text{Uniform}(\vec{A})_{ix} = a_x/m, \text{Uniform}(\vec{A})_{ix} = b_x/m$.

Theorem 3.10 (NE iff effectively uniform). *For any unweighted Derby game, $(m, n, \vec{A}, \vec{B}, \vec{\mathbf{1}})$, the only Nash Equilibrium occurs when both players play the (effectively) uniform strategy, i.e. at $(\mathit{Uniform}(\vec{A}), \mathit{Uniform}(\vec{B}))$.*

Proof sketch. (Full proof in Appendix B.) Because effective strategies preserve utility, we know from Lemma 3.2 that there is a Nash equilibrium when both players play effectively uniform strategies. We show this is the only equilibrium by contradiction: suppose there is another Nash equilibrium (P, Q), where P is not $\text{Uniform}(\vec{A})$. Because Derby games are constant-sum, this implies $(P, \text{Uniform}(\vec{B}))$ is also an equilibrium.[2] However, because P is not uniform, there must be two rounds i, j and some resource A_x that is played less frequently on round i and more frequently on round j compared to if P was uniform. We can then construct strategies Q, R for player B that give player B different expected payoffs: let Q, R be pure strategies that differ only in that Q plays a resource from B_x on round i and B_{x-1} on round j, while B plays a resource from B_x on j and a resource from B_{x-1} on i. One can show that Q has higher payoff than R for player B, so $\text{Uniform}(\vec{B})$, which plays both strategies with equal probability, cannot be a best response to P. This would imply $(P, \text{Uniform}(\vec{B}))$ is not a Nash equilibrium, completing the contradiction and establishing that the only Nash Equilibrium is effectively uniform. □

Definition 3.11 (Non-trivial Derby game). We say a Derby game $(m, n, \vec{A}, \vec{B}, \vec{\mathbf{1}})$ is trivial if at least one player is restricted to only one effective strategy. This occurs when a player only has one equivalence class of resources. We call all other Derby games *non-trivial*.

Remark 3.12. *No non-trivial unweighted Derby game has a Nash equilibrium in which a player plays a pure (effective) strategy (i.e. an effective strategy whose cells are either* 0 *or* 1*), since in a non-trivial Derby game, pure strategies are not effectively uniform.*

Our use of equivalence classes of resources and effective strategies has allowed us to show that there is effectively only one Nash equilibrium for nontrivial instances of the unweighted game, and in future, it will be fundamental to understanding the weighted Derby game. Before we tackle the full complexity of the weighted game, we will first consider a slightly restricted version where rounds have arbitrary weights, but the players are limited to only two equivalence classes of resources.

[2]This follows from the following well-known fact about constant-sum games (stated formally in e.g., Theorem 1.11 in [9]): if (P, Q) is a Nash equilibrium, then P is a best response to *every* Q' that is part of a Nash equilibrium

4 BINARY DERBY GAMES

In a binary Derby game, each player has two equivalence classes of resources. Recall (Definition 3.11) that a Derby game is trivial if at least one player has only one equivalence class of resources. Hence, a binary Derby game can be thought of as the simplest non-trivial Derby game. The binary Derby game is similar to the Boolean Blotto game proposed in prior work [3]; resources in A_1 can represent bids of 1 and resources in A_2 can represent bids of 0. While in Boolean Blotto players who both bid the same value split the value of the battlefield equally, in binary Derby games there are no ties as the victory relation is a total order. In other words, ties are broken deterministically in favor of player A ($A_1 > B_1$ and $A_2 > B_2$). This makes the optimal strategies and Nash equilibria in binary Derby games significantly different from Boolean Blotto. In two player Boolean Blotto, placing bids on the highest valued rounds is a dominant pure strategy for both players, but this is not the case in binary Derby games. In fact, we will show binary Derby games have no pure strategy Nash equilibria (Lemma 4.3).

We begin by defining the *support* of an equivalence class in a strategy P as the set of rounds in which P places resources in that equivalence class with non-zero probability.

Definition 4.1 (Support of an equivalence class). Given a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$, we define the support of an equivalence class C_x with index x in a strategy $P \in [0,1]^{m\times n}$ to be the set of rounds where a resource in that equivalence class might be played,

$$\mathrm{supp}_C(P, x) = \{i \mid P_{ix} > 0\}$$

Since the equivalence class A_1 contains the best resources, player A wins all rounds in which it plays A_1. Also, since player A wins all ties ($A_1 > B_1$ and $A_2 > B_2$), A has no incentive to play A_1 in a round if B_1 is not played there as it can win that round by playing a worse resource in A_2. This leads us to a necessary condition for a Nash equilibrium:

Lemma 4.2. *For a non-trivial binary Derby game $(m, n, \langle A_1, A_2\rangle, \langle B_1, B_2\rangle, \vec{w})$, if (P, Q) is a Nash equilibrium, it must be that if A plays A_1 with some probability in a given round, then B also plays B_1 with some probability in that round:* $\mathrm{supp}_A(P, 1) \subseteq \mathrm{supp}_B(Q, 1)$.

Proof. The proof is by contradiction. Suppose $\mathrm{supp}_A(P, 1) \not\subseteq \mathrm{supp}_B(Q, 1)$. This implies there is some round j in which A plays A_1 sometimes but B never plays B_1. Note that A can win round j by always playing A_2, which means A can do strictly better by playing a different strategy P' which shifts some probability mass of A_1 from j to a different round, k, in which A sometimes loses. We complete the proof by showing that such a round k exists since the game is non-trivial and Q is a best response to P.

Since the game is non-trivial, $|A_1| < m$ and so there exist rounds where A sometimes plays A_2 (*i.e.* $\mathrm{supp}_A(P, 2) \neq \emptyset$). Note that there must be some round $k \in \mathrm{supp}_A(P, 2)$ such that $q_{k1} > 0$ (*i.e.* B plays B_1 against A_2 in round k with non-zero probability and so A sometimes loses) in order for Q to be optimal against P. If this was not the case, then it means B always plays B_2 in all rounds where A sometimes plays A_2, making B lose all rounds (B always loses in rounds where A never plays A_2) and get a utility of 0, which means Q cannot be optimal against P. □

As a consequence of Lemma 4.2, there is no Nash equilibrium in which both players play pure strategies.

Lemma 4.3. *For a non-trivial binary Derby game $(m, n, \langle A_1, A_2\rangle, \langle B_1, B_2\rangle, \vec{w})$, if both A and B play pure strategies then they cannot be in equilibrium.*

Proof. Let A play a pure strategy P. Since A_1 is played with probability 1 in rounds in $\mathrm{supp}_A(P, 1)$, B will always lose in those rounds and get a utility of 0. Hence, B's best response pure strategy Q will place B_1 in as many rounds outside $\text{supp}_A(P, 1)$ as possible. For a non-trivial Derby game, $b_1 < m$ which means Q will leave out some round in $\text{supp}_A(P, 1)$, which means $\text{supp}_A(P, 1) \nsubseteq \text{supp}_B(Q, 1)$. By Lemma 4.2, P and Q cannot be at equilibrium. □

Interestingly, it is sometimes possible for a *half-pure* Nash equilibrium to exist in which only player A plays a pure strategy. Theorem 4.5 establishes a necessary and sufficient condition for such an equilibrium to exist. The following lemma establishes a necessary condition.

Lemma 4.4. *For a binary Derby game $(m, n, \langle A_1, A_2\rangle, \langle B_1, B_2\rangle, \vec{w})$, if A plays a pure strategy P in a Nash equilibrium, then (1) the number of resources in B_1 must satisfy $b_1 > m - a_1$ and (2) the support of B_1 in B's strategy Q, $\text{supp}_B(Q, 1) = [m]$.*

Proof. The proof is similar to the proof of Lemma 4.3. By Lemma 4.2, a necessary condition for a Nash equilibrium is $\text{supp}_A(P, 1) \subseteq \text{supp}_B(Q, 1)$ *i.e.* B must play B1 in all rounds in $\text{supp}_A(P, 1)$. Since B gets a utility of 0 for all rounds in $\text{supp}_A(P, 1)$, any best responding pure strategy of B will play B_1 in all rounds in $[m] \setminus \text{supp}_A(P, 1)$. For the condition in Lemma 4.2 to be satisfied, B needs to have enough resources in B_1 to play in all rounds in $[m] \setminus \text{supp}_A(P, 1)$ as well as in some rounds in $\text{supp}_A(P, 1)$. This gives the condition $b_1 > m - a_1$ (where $\text{supp}_A(P, 1) = a_1$ as P is a pure strategy). B's optimal mixed strategy Q is composed of pure strategy best responses which play B_1 in different rounds in $\text{supp}_A(P, 1)$ so that $\text{supp}_B(Q, 1) = [m] \setminus \text{supp}_A(P, 1) \cup \text{supp}_A(P, 1) = [m]$. □

The necessary condition in Lemma 4.4 can be strengthened to give a necessary and sufficient condition for the existence of a half-pure Nash equilibrium in which A plays a pure strategy.

Theorem 4.5. *For a binary Derby game $(m, n, \langle A_1, A_2\rangle, \langle B_1, B_2\rangle, \vec{w})$, there is a half-pure Nash Equilibrium where A plays a pure strategy P iff*

$$b_1 \geq m - a_1 + \sum_{i=1}^{a_1} \frac{w_{(a_1+1)}}{w_i}.$$

where P places resources in A_1 in the highest weight rounds.

Proof sketch. Let A play a pure strategy P with $\text{supp}_A(P, 1) = S$ and let B's strategy be Q. For (P, Q) to be a Nash equilibrium, (1) Q must be a best response for B against P and (2) P must be a best response for A against Q. We use (2) and the fact that A plays a pure strategy to derive a sufficient condition. We then combine it with the necessary condition $b_1 > m - a_1$ from Lemma 4.4 to obtain the necessary and sufficient condition above. The complete proof is in Appendix C. □

Thus, we have seen that adding weights to the Derby game has greatly reduced the symmetry of the game, and we even find interesting equilibria where one player plays a pure strategy. In this section we found necessary and sufficient conditions for player A playing a pure strategy, and we have paved the way for very similar analysis in the general game.

5 WEIGHTED DERBY GAMES

We now analyze the weighted Derby Game without restricting each player to two equivalence classes of resources. Consider the TV broadcasting example from sections 2 and 3, in which Channel A and Box Television are both preparing schedules for the various TV shows they intend to run this season. The added complication we consider in this section is that not every time slot is equally valuable: even if Channel A knows it can win the 3-4am time slot if it schedules its best show there, it may not be worth the relatively low ad revenue from winning that time slot. This gives us an example of the full weighted game: $(m, n, \vec{A}, \vec{B}, \vec{w})$ where m is the number of time slots, $\vec{A}, \vec{B}$ are the equivalence classes of shows each network can schedule, and $\vec{w}$ is the expected increase in ad revenue from winning each time slot.

5.1 Nash Equilibria

Channel A and Box Television are both interested in best-response scheduling strategies to the scheduling strategy of the other. Because the weighted Derby game is a two-player constant-sum game, one standard approach to finding an optimal strategy for either player would be to apply the minimax theorem to design a linear program (LP) giving said optimum [1]. While an LP is useful for finding equilibria of a specific Derby game, there is further structure to the equilibria of Derby games under particular resource and weight constraints, which we will soon explore. As a small teaser, consider the following example:

Example 5.1. In the running example, Channel A and Box Television are competing in a Derby game $(5, 3, \{A_1, A_2, A_3\}, \{B_1, B_2, B_3\}, \vec{w})$ where $(a_1, a_2, a_3) = (1, 3, 1)$ and $(b_1, b_2, b_3) = (3, 2, 0)$. Suppose also that $\vec{w} = (6, 6, 5, 2, 2)$. Our results will show that this game has a Nash equilibrium (P, Q) where

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ is a pure strategy!}$$

Moreover, note that Box Television is winning this game 11 to 10, even though (1) Channel A has a show guaranteed to win and (2) Box Television is completely giving away its strategy.

In general, it is much more common for player A to play a pure strategy at Nash equilibrium than for player B. Our results on best-response pure strategies for A and B can be found at the ends of subsections 5.3, and 5.4, respectively. Towards those results, we now define some useful ideas.

Remark 5.2 (Modified strategies). *For a Derby game and strategy $P : [0, 1]^{m\times n}$, we can modify P to $P + \delta$, where $\delta : [-1, 1]^{m\times n}$, every row and column of δ sums to 0, and no cell of δ subtracts more than the corresponding cell in P. Then $P + \delta$ is also a strategy.*

Thus, in addition to saying that we are at a Nash equilibrium (P, Q) when neither player has a strategy which gives them a higher utility against their opponent's strategy, we will also say that we are at a Nash equilibrium when neither player has modification δ which produces a positive change in their utility when added to their current strategy: $\forall\delta.\, U_A(P + \delta, Q) - U_A(P, Q) \le 0$ and $\forall\delta.\, U_B(P, Q + \delta) - U_B(P, Q) \le 0$.

Definition 5.3 (Opposition). For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ and pair of strategies (P, Q), we define the opposition of an equivalence class with index x as the set of equivalence class indices that play against it in some round: $\mathrm{opp}(x, P, Q) = \{y \mid \exists i.\, P_{ix} > 0 \text{ and } Q_{iy} > 0\}$.

Theorem 5.4 (Narrow wins). *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash Equilibrium (P, Q), and for any equivalence class index $1 \le x \le n$, it holds that*

(1) *either* $\mathrm{supp}_A(P, x) \subseteq \mathrm{supp}_B(Q, x)$ *or for all* $y, x < y \le n$,
  (a) A_y *never plays* $B_x, ..., B_n$, *i.e.* $\mathrm{opp}(y, P, Q) \subseteq [x - 1]$
  (b) B_y *never plays* $A_x, ..., A_n$, *i.e.* $\mathrm{opp}(y, Q, P) \subseteq [x - 1]$
(2) *either* $\mathrm{supp}_B(Q, x) \subseteq \mathrm{supp}_A(P, x + 1)$ *or for all* $y, x < y \le n$,
  (a) B_y *never plays* $A_{(x+1)}, ..., A_n$, *i.e.* $\mathrm{opp}(y, Q, P) \subseteq [x]$
  (b) A_y *never plays* $B_{(x+1)}, ..., B_n$, *i.e.* $\mathrm{opp}(y, P, Q) \subseteq [x]$

Proof. See Section 5.2 for the complete proof. The general approach to the proof is to use the idea that at Nash equilibrium, neither player has a *modification* to their strategy with positive change in utility. For the narrow-wins theorem, we need to reason extensively about how possible swaps and cyclic shifts of probability would modify each player's utility. □

Many important results in this section use Theorem 5.4, which informally states that at a Nash Equilibrium, an equivalence class either has a chance to narrowly win against the equivalence class right after it in the victory relation, *or* all equivalence classes after it always lose (because they never have a chance to play against each other).

For example, we can rephrase the Narrow Wins Theorem to emphasize that, at equilibrium, both players are intentionally sacrificing some equivalence classes of resources, even when they could play those resources in rounds with some potential of victory:

Corollary 5.5 (Best-worst sacrificial behavior). *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash Equilibrium (P, Q) and any $K \in [n]$.*

1. *if $\mathrm{supp}_A(P, K) \not\subseteq \mathrm{supp}_B(Q, K)$, then for all $y > K$, A_y, B_y always lose (they only play against resources that are better according to the victory relation),*
2. *if $\mathrm{supp}_B(Q, K) \not\subseteq \mathrm{supp}_A(P, K+1)$, then for all $y > K$, A_y, B_y always lose.*

Note that this is the same best-worst behavior as described in Section 2, only *both* players are playing resources in ways that (with some probability) narrowly beat their opponent (via Theorem 5.4) and both are sacrificing resources after some bound (via Corollary 5.5). Moreover, unlike in the case of Tian Ji's horse-racing competition, the players are playing best-vs-worst with (1) weighted rounds, (2) a more general victory relation, and (3) without knowledge of their opponent's ordering!

Example 5.6. Continuing the running example, we note that the Nash equilibrium (P, Q) defined in Example 5.1 satisfies the conditions of Corollary 5.5, since $\mathrm{supp}_A(P, 2) \not\subseteq \mathrm{supp}_B(Q, 2)$ and A_3 always loses.

The Narrow Wins Theorem also lets us generalize Lemma 4.2 from the Binary Derby Games section:

Corollary 5.7. *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash equilibrium (P, Q), it holds that $\mathrm{supp}_A(P, 1) \subseteq \mathrm{supp}_B(Q, 1)$.*

Proof. Follows by contradiction. By Theorem 5.4, if $\mathrm{supp}_A(P, 1) \not\subseteq \mathrm{supp}_B(Q, 1)$, then A_2 would never play against $B_1, \ldots, B_n$. But then, what would A_2 play against? □

We will use the above two corollaries when we analyze Nash equilibria in Sections 5.3 and 5.4.

5.2 Proof of Narrow Wins Theorem

Throughout the proof of the narrow wins theorem, we rely on the idea that at Nash equilibrium, neither player has a *modification* to their strategy with positive change in utility. First, we strengthen the requirements of how equivalence classes must be played against one another at equilibrium in Lemmas 5.8 and 5.9 using arguments about modifications that swap probability of playing two resources. Next, we consider more complex modifications in Corollary 5.10: cycles of shifting probability, where each cell moves an equal amount of probability of playing a resource to the next, until the last cell in the cycle returns an equal probability back to the first. Finally, having proven Corollary 5.10, we make a few simple deductions in Corollary 5.11 before concluding with a proof of our Theorem 5.4.

We begin with the swap argument: intuitively, we may expect that at Nash equilibrium, when a resource from equivalence class A_x is being played on a round (with some probability), so must a resource from B_x (with some probability), otherwise A would get a higher utility by swapping the

A_x resource with an A_{x+1} resource on some round where a B_x resource is played – if such a round exists. Indeed, we show that the absence of such rounds is a necessary condition for equilibria in which A_x is played on rounds where B_x is not played, ensuring swapping A_x and A_{x+1} will not increase A's utility. Formally,

LEMMA 5.8. *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash equilibrium (P, Q), and any equivalence class index x,*

(1) *If there is a round where a resource in A_x may be played, but in which resources in B_x are never played, then B_x never plays against $A_{(x+1)}$:* $\mathrm{supp}_A(P, x) \not\subseteq \mathrm{supp}_B(Q, x) \implies (x + 1) \notin \mathrm{opp}(x, Q, P)$.
(2) *Likewise, if there exists a round where B_x is played, but in which resources in $A_{(x+1)}$ are never played, then $A_{(x+1)}$ never plays against $B_{(x+1)}$:* $\mathrm{supp}_B(Q, x) \not\subseteq \mathrm{supp}_A(P, x + 1) \implies (x + 1) \notin \mathrm{opp}(x + 1, P, Q)$.

PROOF. We show (1) by contradiction, (2) follows similarly. Suppose there is both a round i where A_x is played with some probability, but B_x isn't, and a round j where both B_x and $A_{(x+1)}$ are played with some probability. Since A_x isn't playing B_x at i, $A_{(x+1)}$ would do just as well as A_x in that round. Moreover, A_x would win (against B_x) where $A_{(x+1)}$ lost. Therefore, the modification δ which swaps some probability of playing A_x at round i with probability of playing $A_{(x+1)}$ at round j leads to a better strategy for A. This is a contradiction, since if there is a better strategy for A, we can't be at NE. □

Next we use use induction on equivalence classes of *both* players to extend from considering A_x, B_x, A_{x+1} to considering A_x, B_x, B_y, A_{y+1}, for $y > x$.

COROLLARY 5.9. *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash Equilibrium (P, Q), and any equivalence class index x,*

(1) *If there is a round where a resource in A_x may be played, but in which resources in B_x are never played, then for all $y \geq x$, B_y never plays against $A_{(y+1)}$:* $\mathrm{supp}_A(P, x) \not\subseteq \mathrm{supp}_B(Q, x) \implies \forall y \geq x.\, (y + 1) \notin \mathrm{opp}(y, Q, P)$.
(2) *Likewise, if there exists a round where B_x is played, but in which resources in $A_{(x+1)}$ are never played, then for all $y \geq x$, $A_{(y+1)}$ never plays against $B_{(y+1)}$:* $\mathrm{supp}_B(Q, x) \not\subseteq \mathrm{supp}_A(P, x+1) \implies \forall y \geq x.\, (y + 1) \notin \mathrm{opp}(y + 1, P, Q)$.

PROOF. Follows by induction using Lemma 5.8: note that Lemma 5.8 (1) for some x implies Lemma 5.8 (2) for the same x, and Lemma 5.8 (2) for some x implies Lemma 5.8 (1) for $x + 1$. □

And now we reason about cycles: just like Lemma 5.8 gives a necessary condition for equilibria in which there are rounds where A_x does not play B_x, *by ensuring there is no swap* that gives player A more payoff, we can derive a corollary that gives a gives a necessary condition for such equilibria, *by ensuring there is no sequence of swaps, forming a cycle,* that gives player A more payoff.

COROLLARY 5.10. *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ with Nash Equilibrium (P, Q), and for any pair of equivalence class indices $1 \leq x < y \leq n$ it holds that*

(1) *If there is a round where A plays a resource from A_x with some probability and B doesn't play a resource from B_x with any probability, then resources from A_y must only play resources from $B_1, \ldots, B_{(x-1)}$ or from $B_y, \ldots, B_n$. Formally,*

$$\mathrm{supp}_A(P, x) \not\subseteq \mathrm{supp}_B(Q, x) \implies \mathrm{opp}(y, P, Q) \subseteq [n] \setminus \{x, \ldots, y - 1\}.$$

(2) *Likewise, if there is a round where B plays B_x and A doesn't play $A_{(x+1)}$, then B_y must only play $A_1, ..., A_x, A_{(y+1)}, ..., A_n$. Formally,*

$$\mathrm{supp}_B(Q,x) \not\subseteq \mathrm{supp}_A(P,x+1) \implies \mathrm{opp}(y,Q,P) \subseteq [n] \setminus \{x+1,\dots,y\}.$$

Proof. We show (1) by contradiction, (2) follows similarly. Suppose there exists a round $i \in \mathrm{supp}_A(P,x)$ such that $i \notin \mathrm{supp}_B(Q,x)$. At the same time, we also suppose there exists a round j where a resource in A_y plays a resource in one of $B_x, ..., B_{(y-1)}$, i.e. $\exists j \in \mathrm{opp}(y,P,Q) \cap \{x, x+1,\dots,y-1\}$. By applying Corollary 5.9 to our round i we know many resources of player A never play against their corresponding resources of player B: $x \notin \mathrm{opp}(x,Q,P)$, $(x+1) \notin \mathrm{opp}(x+1,Q,P)$, and so on until $(y-1) \notin \mathrm{opp}(y-1,Q,P)$. Then consider the modification δ that rotates a small amount of the probabilities of A's strategy according to the cycle

$$\begin{pmatrix} P_{jy} & P_{*(y-1)} & P_{*(y-2)} & \cdots & P_{*(x+1)} & P_{ix} \end{pmatrix}$$

where P_{*z} represents the probability of playing equivalence class A_z on an arbitrary round $k \in \mathrm{supp}_A(P,z)$. Under δ, some small probability of A_y being played on round j is moved to a round where $A_{(y-1)}$ was previously played, displacing an equal probability of playing $A_{(y-1)}$, which is moved to a round where $A_{(y-2)}$ was played, and so on until an equal probability of playing A_x is moved to round j.

Notice that $P+\delta$ is a better strategy for A than P:

- For all $z, x < z <= y$, A_z does just as well as A_{z-1}, since Corollary 5.9 told us that A_{z-1} never played B_{z-1} in P
- On round j, A_x now wins where A_y previously lost, leading to a net gain in utility.

Since there is a better strategy for A, we cannot be at a Nash equilibrium. □

Rewriting and adding some final touches,

Corollary 5.11. *For a Derby game $(m,n,\vec{A},\vec{B},\vec{w})$ with Nash Equilibrium (P,Q), and for any equivalence class index $1 \le x \le n$, it holds that*

(1) *either* $\mathrm{supp}_A(P,x) \subseteq \mathrm{supp}_B(Q,x)$ *or for all* $y, x < y \le n$,
   (a) A_y *never plays* $B_x, ..., B_{(y-1)}$, *i.e.* $\mathrm{opp}(y,P,Q) \subseteq [n] \setminus \{x,\dots,y-1\}$
   (b) B_y *never plays* $A_{(x+1)}, ..., A_y$, *i.e.* $\mathrm{opp}(y,Q,P) \subseteq [n] \setminus \{x+1,\dots,y\}$

(2) *either* $\mathrm{supp}_B(Q,x) \subseteq \mathrm{supp}_A(P,x+1)$ *or for all* $y, x < y \le n$,
   (a) B_y *never plays* $A_{(x+1)}, ..., A_y$, *i.e.* $\mathrm{opp}(y,Q,P) \subseteq [n] \setminus \{x+1,\dots,y\}$
   (b) A_{y+1} *never plays* $B_{(x+1)}, ..., B_y$, *i.e.* $\mathrm{opp}(y,P,Q) \subseteq [n] \setminus \{x+1,\dots,y\}$

Proof. We show (1), (2) follows similarly. Note first that (1a) follows from Corollary 5.10 (1). For (1b), if $\mathrm{supp}_A(P,x) \subseteq \mathrm{supp}_B(Q,x)$, we are done. Otherwise, by Lemma 5.8, it holds that $(x+1) \notin \mathrm{opp}(x,Q,P)$. Then by Corollary 5.10 (2), it holds that $\forall y > x$, $\mathrm{opp}(y,Q,P) \cap \{x+1,\dots,y\} = \emptyset$. □

And here's our big theorem!

Theorem 5.4 (Narrow wins). *For a Derby game $(m,n,\vec{A},\vec{B},\vec{w})$ with Nash Equilibrium (P,Q), and for any equivalence class index $1 \le x \le n$, it holds that*

(1) *either* $\mathrm{supp}_A(P,x) \subseteq \mathrm{supp}_B(Q,x)$ *or for all* $y, x < y \le n$,
   (a) A_y *never plays* $B_x, ..., B_n$, *i.e.* $\mathrm{opp}(y,P,Q) \subseteq [x-1]$
   (b) B_y *never plays* $A_x, ..., A_n$, *i.e.* $\mathrm{opp}(y,Q,P) \subseteq [x-1]$

(2) *either* $\mathrm{supp}_B(Q,x) \subseteq \mathrm{supp}_A(P,x+1)$ *or for all* $y, x < y \le n$,
   (a) B_y *never plays* $A_{(x+1)}, ..., A_n$, *i.e.* $\mathrm{opp}(y,Q,P) \subseteq [x]$
   (b) A_y *never plays* $B_{(x+1)}, ..., B_n$, *i.e.* $\mathrm{opp}(y,P,Q) \subseteq [x]$

Proof. We show (1), (2) follows similarly. Consider an arbitrary equivalence class index x. If $\mathrm{supp}_A(P, x) \subseteq \mathrm{supp}_B(Q, x)$, we are done. Otherwise, consider an arbitrary $y_1 > x$: we must show that for all $y_2 \geq x$, it holds that $y_2 \notin \mathrm{opp}(y_1, P, Q)$.

Case 1: Suppose $y_2 < y_1$. Then by Corollary 5.11 (1a), it holds that $y_2 \notin \mathrm{opp}(y_1, P, Q)$.

Case 2: Suppose $y_1 \leq y_2 \leq n$. By Corollary 5.11 (1b), it holds that $y_1 \notin \mathrm{opp}(y_2, Q, P)$, since $(x + 1) \leq y_1 \leq y_2$. Thus $y_2 \notin \mathrm{opp}(y_1, P, Q)$.

This concludes (1). For (2), we proceed like above to show $\forall y > x$, $\mathrm{opp}(y, Q, P) \subseteq [x]$ and $\mathrm{opp}(y+1, P, Q) \subseteq [x]$. Then because $\forall y > x. \,(x+1) \notin \mathrm{opp}(y, Q, P)$, we know that $\mathrm{opp}(x+1, P, Q) \subseteq [x]$, also. □

5.3 Nash Equilibria where A_1 is fixed

In this subsection, we will give necessary and sufficient conditions for Nash Equilibria (P, Q) where in every round, P either plays a resource from A_1 with certainty, or will not play from A_1. (In general we say an equivalence class A_x is fixed when for each round i, $P_{ix} = 0$ or $P_{ix} = 1$, and similarly with B_x and Q.) Incidentally, these will also be necessary and sufficient conditions for Nash Equilibria where P is a pure strategy.

We begin with necessary conditions.

Remark 5.12. *At an NE (P, Q) if A_1 is fixed by P, then $\forall y \geq 2. \,\mathrm{opp}(y, P, Q) = \mathrm{opp}(y, Q, P) = \{1\}$.*

Proof. Suppose we have an NE (P, Q) where A_1 is fixed in P. By Corollary 5.7, we know that in all the rounds where A plays A_1, B must play B_1 with some probability. And since A_1 is fixed, we know A cannot play A_2 in those rounds. Thus, by Theorem 5.4 (2) where $x = 1$: $\forall y \geq 2. \,\mathrm{opp}(y, P, Q) = \mathrm{opp}(y, Q, P) = \{1\}$. □

In other words, if A_1 is fixed, then A_2 and worse are all playing against B_1 only, and B_2 and worse are playing A_1 only. In this situation, B has no incentive to switch strategies, since B is guaranteed to lose rounds where A_1 is played with certainty, and in this strategy B wins all other rounds. Likewise, A has no incentive to switch to any strategy that only changes the probabilities of playing $A_2, \ldots, A_n$: since they all continue to lose their rounds, A's utility is unchanged.

We will give necessary and sufficient conditions for A to have no incentive to change their strategy, and for this NE to exist.

Lemma 5.13. *Consider a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ and pair of strategies (P, Q), such that that A_1 is fixed in P and $\forall x > 1. \,\mathrm{opp}(x, P, Q) = \{1\}$. A has a better strategy against Q iff A has better strategy $P + \delta_1 + \delta_2$ where*

- *δ_1 only modifies probability in rounds where A_1 isn't played ($[m] \setminus \mathrm{supp}_A(1, P)$), and moreover ensures $p + \delta_1$ plays A_2 with some probability on the highest weight round where A_1 isn't played.*
- *δ_2 then swaps some probability of playing A_1 and A_2 between a round where A_1 is played and this highest weight round where A_1 isn't played.*

Proof. Consider any alternative strategy for A. We can convert from the alternative strategy to A's current strategy by first swapping A_1 to the correct rounds, and then permuting resource probabilities in all other rounds. Therefore, we can convert from the current strategy to the alternative through the reverse process: 1. we permute probabilities in the rounds where A_1 doesn't play (which doesn't change utility), and then 2. we swap probability of playing A_1 into the rounds where A_1 doesn't play (which may change utility).

Moreover, if an alternative strategy is better than A's current strategy, then it must be that at least one of the swaps in step 2 increases the utility of the strategy. We could do the same permutation and then do just that swap and still have a better strategy than A's current.

Therefore, there exists a better alternative strategy for A iff there exists a better performing strategy reachable by a permutation of resources in rounds A_1 isn't played followed by a single swap of resources between a round A_1 is always played and a round A_1 is never played.

Of these reachable alternative strategies, the best use a the permutation that moves A_2 to the highest weight round where A_1 isn't played, and then swaps A_2 with A_1: this maximizes the value of playing A_1 to beat B_1, and minimizes the loss of swapping A_1 out by replacing it with its best alternative. Thus, there is a better alternative strategy for A *iff* one of this group of strategies is better. □

Lemma 5.14. *Consider a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ and pair of strategies (P, Q), such that that A_1 is fixed in P and $\forall x > 1.\, \mathrm{opp}(x, P, Q) = \{1\}$. A has no incentive to change their strategy iff $\mathrm{supp}_A(P, 1) = [a_1]$ and*

$$\forall i \in [a_1].\, Q_{i1} \geq \frac{w_{(a_i+1)}}{w_i}.$$

Proof. Consider any alternative strategy $P + \delta_1 + \delta_2$ described by Lemma 5.13, where δ_1 moves A_2 to the highest weight round A_1 isn't playing and then δ_2 swaps A_2 with A_1. The change in utility due to δ_1 is 0, since it only rearranges always-losing rounds. A has a better strategy iff the change in utility due to δ_2 is positive.

Suppose in δ_2 we swap some probability p (we will see the exact probability is unimportant, since it only scales the change in utility).

In the round we replace A_1 with A_2, suppose it is round i, the loss in utility in this round is $p w_i Q_{i1}$.

In the round where we replace A_2 with A_1, the gain in utility is $p w_{\text{altmax}}$, where w_{altmax} is the maximum weight round where A_1 isn't played. (There's no term for probability of playing B_1 since B_1 is played with certainty.)

This swap is therefore better iff,

$$Q_{i1} < \frac{p w_{\text{altmax}}}{p w_i} = \frac{w_{\text{altmax}}}{w_i}.$$

Thus, there is a better alternative strategy for A iff there is some round indexed i where A_1 is certainly played, such that the above holds. Rephrasing, there is no better alternative strategy for A, iff for all rounds i where A_1 is played, $Q_{i1} \geq \frac{w_{\text{altmax}}}{w_i}$.

This cannot be the case when $w_{altmax} > w_i$, so we know that A_1 must be played on the highest weight rounds, $1, \ldots, a_i$, and so $w_{\text{altmax}} = w_{a_i+1}$ (recall the rounds are sorted by descending weight). □

Theorem 5.15 (NEs where A_1 fixed). *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ there exists an NE where A plays a strategy where A_1 is fixed iff*

$$b_1 \geq m - a_1 + \sum_{i=1}^{a_1} \frac{w_{(a_1+1)}}{w_i}$$

Proof. ($\Rightarrow$) Suppose there is an NE where A_1 is fixed. By Corollary 5.7 and Theorem 5.4, we know that A_2, A_3, and worse resources only play B_1.

Moreover, since we are at an NE, we know that A has no incentive to change their strategy, so by Lemma 5.14 we know that

$$\forall i \in [a_1].\, Q_{i1} \geq \frac{w_{(a_i+1)}}{w_i}.$$

Then

$$b_1 = \sum_{i=1}^{m} Q_{i1} \geq \sum_{i=1}^{a_i} \frac{w_{(a_i+1)}}{w_i} + \sum_{i=a_1+1}^{m} 1 = (m - a_1) + \sum_{i=1}^{a_i} \frac{w_{(a_i+1)}}{w_i}.$$

($\Leftarrow$) Suppose $b_1 \geq m - a_1 + \sum_{i=1}^{a_i} \frac{w_{(a_i+1)}}{w_i}$. We will construct an NE (P, Q) where A plays a strategy where A_1 is fixed.

We first require that A_1 is fixed by P, i.e. $\forall i \in [a_1]. P_{i1} = 1$. As for the other equivalence classes of A, they can be played arbitrarily on the remaining rounds (A can even play a pure strategy).

B plays a strategy restricting B_1: For $i \in [a_1]$, $Q_{i1} \geq w_{a_1+1}/w_i$, and for $a_1 + 1 \leq i \leq m$, $Q_{i1} = 1$. The other resources are played arbitrarily on the rounds $[a_1]$, since B_1 has filled the rest of the rounds. Under this strategy, B has no incentive to change strategies: B is only losing rounds where A_1 is played, and all strategies B could switch to would also lose those rounds (and potentially others). Also, by Lemma 5.14, we know that A has no incentive to change strategies. □

Corollary 5.16 (NEs where A plays a pure strategy). *For a Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ there exists an NE where A plays a* pure strategy *iff*

$$b_1 \geq m - a_1 + \sum_{i=1}^{a_1} \frac{w_{(a_1+1)}}{w_i}$$

We see that Derby games of various sizes m and various equivalence class counts n can have Nash equilibria where A plays a pure strategy, so long as there are enough resources in B_1 to be played with certainty on the rounds A_1 isn't played in, plus a little.

This leads to two natural follow-up questions: (1) are the Nash equilibria where B plays a pure strategy similarly general, and (2) are there any pure strategy Nash equilibria in the general weighted Derby game? In the next section, we answer both questions in the negative.

5.4 Nash Equilibria where B_1 is fixed

We begin by showing (1) there are no pure strategy Nash equilibria, (2) the Nash equilibria where B plays a pure strategy are limited to the case where the number of equivalence classes $n \leq 3$. Then we characterize the Nash equilibria where B plays a pure strategy for $n \leq 3$.

Remark 5.17. *For a nontrivial Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$, and any NE (P, Q) of the game, it cannot be that both A_1 is fixed by P and B_1 is fixed by Q.*

Proof. If A_1 is fixed and (P, Q) is a Nash equilibrium, we know that B_1 must be played with some probability in all rounds A_1 is played in and with certainty in all other rounds (by Corollary 5.7 and Theorem 5.4). Thus if B_1 is fixed, it must be played with certainty in all rounds, meaning the Derby game is trivial. □

Corollary 5.18. *For a nontrivial Derby game, there are no pure strategy Nash Equilibria.*

Theorem 5.19 (At NE, B_n prevents fixing B_1). *For a nontrivial Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$ where $n \geq 3$ with $a_1 > 0$ and $b_3 > 0$, there are no NEs (p, q) where B_1 is fixed by q.*

Proof. We proceed by contradiction. Suppose we have an NE where B's strategy fixes B_1.

Then it must be the case that there is some round proving $\mathrm{supp}_A(P, 2) \not\subseteq \mathrm{supp}_B(Q, 2)$. (If B_1 plays A_2 on some round, consider that round, otherwise apply Theorem 5.4 (2) for $x = 1$).

By Theorem 5.4 (1) where $x = 2$: $\forall y \geq 3. \mathrm{opp}(y, P, Q) = \mathrm{opp}(y, Q, P) = \{1\}$. Note specifically that this means that equivalence classes B_3 and worse play against A_1 only.

At the same time, by Corollary 5.7 and since B_1 is fixed, A_1 plays against B_1 only – contradicting the previous point. □ □

5.4.1 Nash Equilibria where B_1 is fixed and $n \leq 3$. Because the proofs below are relatively similar to cases where A_1 is fixed, we give proof sketches rather than complete proofs to save space.

Lemma 5.20. *Suppose we have a nontrivial Derby game $(m, 3, \langle A_1, A_2, A_3\rangle, \langle B_1, B_2\rangle, \vec{w})$, where potentially $A_3 = \emptyset$ (we deviate from the definition here because the change lets us analyze both $n = 2$ and $n = 3 \wedge B_3 = \emptyset$ at the same time). For any pair of both players' strategies (P, Q) such that Q is a pure strategy, Player B has no incentive to change their strategy iff $\mathrm{supp}_B(Q, 1) = [b_1]$ and $\forall i \in [b_1]. P_{i2} \geq \frac{w_{b_i+1}}{w_i}$.*

Proof. Follows very similarly to Lemma 5.14, but with B_1, A_2 rather than A_1, B_1. □

Lemma 5.21. *Suppose we have a nontrivial Derby game $(m, 3, \langle A_1, A_2, A_3\rangle, \langle B_1, B_2\rangle, \vec{w})$, where potentially $A_3 = \emptyset$. Suppose we also have a pair of strategies (P, Q) such that Q is a pure strategy, $2 \in \mathrm{opp}(1, Q, P)$, and $\mathrm{opp}(2, Q, P) = \{2\}$. Suppose also that player B has no incentive to change its strategy. Then A has no incentive to change its strategy iff (1) all the rounds where A_1 is played have the same weight, (2) they are the highest weight rounds where B_1 is played, and (3) there are at least $\left\lceil \frac{a_1}{1 - w_{(b_1+1)}/w_1} \right\rceil = \left\lceil \frac{a_1 w_1}{w_1 - w_{(b_1+1)}} \right\rceil$ such rounds.*

Proof. First consider the rounds where B_2 plays A_2, $\mathrm{supp}_B(Q, 2)$. We show that any alternate strategy of A's that plays a different equivalence class than A_2 in these rounds is not optimal:

- if A_2 is displaced by A_1, then A_2 will lose against B_1, where A_1 used to win, and A_1 will do no better than A_2 against B_2;
- if A_2 is displaced by A_3, then A_3 will lose against B_2, where A_2 used to win, and A_2 will do no better than A_3 against B_1.

Therefore, if A has a better strategy, then A must have a better strategy that doesn't affect the rounds where B_2 is played.

This leaves only the rounds where B_1 is played. For these rounds, A's best strategy is to play A_1 as much as possible on the highest weight rounds. Since B has no incentive to change its strategy, we know from Lemma 5.20 that in A's current strategy, A must play A_2 with some probability on each of the rounds where B_1 is played – even the round with highest weight. So in order for A not to have any incentive to replace probability of playing A_2 with probability of playing A_1, there must be enough rounds of the highest weight that A can use up all its probability of playing A_1.

If (from Lemma 5.20) we want each of the highest weight rounds to have at least w_{b_i+1}/w_1 probability of playing A_2, then that means that there must be enough highest weight rounds such that the probability of playing A_1 can be distributed among them without cutting into that probability: the number of rounds must be at least $\left\lceil \frac{a_1}{1 - w_{(b_i+1)}/w_1} \right\rceil$. □

Theorem 5.22. *For a nontrivial Derby game $(m, 3, [A_1, A_2, A_3], [B_1, B_2], \vec{w})$ where potentially $A_3 = \emptyset$, there exists an NE where B plays a strategy where B_1 is fixed iff it holds that*

(1) $a_2 \geq m - b_1 + \sum_{i=1}^{b_1} \frac{w_{(b_1+1)}}{w_i}$,

(2) $w_1 = w_2 = \cdots = w_l$ *where* $\left\lceil \frac{a_1}{1 - w_{(b_1+1)}/w_1} \right\rceil \leq l \leq b_1$.

Proof. ($\Rightarrow$) Suppose there is an NE where B_1 is fixed. We know (1) from Lemma 5.20 and (2) from Lemma 5.21.

($\Leftarrow$) Suppose (1), (2) hold. Then consider the pair of strategies (P, Q) where

- On rounds $i \in \{b_1 + 1, \ldots, m\}$, $P_{i2} = 1$. (This is the $m - b_1$ of (b).)

- On rounds $i \in [l]$, $P_{i1} = a_1/l$ (by (2) A is guaranteed to leave enough space for A_2 such that B has no incentive to switch).
- On rounds $i \in [b_1]$, A plays A_2 with probability at least w_{b_1+1}/w_i, and arbitrarily fills any probability not already taken by A_1 or A_2 with probability of playing A_2 or A_3.

For this pair of strategies, note that Lemmas 5.20 and 5.21 hold, and so we are at an NE. □

Theorem 5.22 lets us complete the circle on our earlier example: on one hand, we see that there are stringent requirements in order for Player B to play a pure strategy in a Nash equilibrium; on the other, these equilibria manage to exist for many different round counts, weights, and resource counts, even as player B deals with the fact that A's best resources outweigh B's best resources.

5.5 Nash Equilibria where A_x or B_x is fixed, $x > 1$

Theorem 5.23. *For a nontrivial Derby game $(m, n, \vec{A}, \vec{B}, \vec{w})$, if the game has a Nash Equilibrium (P, Q) where P fixes A_x or Q fixes B_x for $x > 1$, it must be that when A and B play strategies (P, Q), that equivalence class of resources (A_x or B_x) always loses.*

Proof. We show the case where A_x is fixed by contradiction, the case for B_x follows similarly. Suppose A_x is fixed and has a nonzero chance to win at least one round. By the contrapositive of Corollary 5.5, this means that

$$\mathrm{supp}_A(P, 1) \subseteq \cdots \subseteq \mathrm{supp}_A(P, x-1) \subseteq \mathrm{supp}_B(Q, x-1) \subseteq \mathrm{supp}_A(P, x).$$

Note that this means that A_{x-1} is only played on rounds where A_x is played. However, A_x is fixed, so A_x is played with certainty on the rounds where it is played. This leaves no room for A_{x-1}, causing a contradiction. □

This result explains some of our earlier questions – why is it possible for player A or player B to play a pure strategy, and specifically, why isn't the other player able to exploit weaknesses or respond more effectively? The reason, as Theorem 5.23 explains, is that the player playing the fixed equivalence class already intends to lose with the class they are fixing. Thus, the other player can do no better than to give them what they want.

6 RELATED WORK

We give a brief overview of some existing games related to Derby games.

Colonel Blotto. There has been much work on identifying Nash equilibria for Colonel Blotto and its variants [2, 4, 5, 11, 12] but no complete characterization has been found for many of these games (including Blotto itself). Of these variants, the closest relative to Derby games is Boolean Blotto [3], as described in Section 4. Also related are Gladiator games [10], a Blotto variant involving two teams of gladiators having 1v1 fights (like Derby games). However, in a Derby game, the players change the order of resources, whereas in Gladiator games the players change the strengths of each gladiator (ensuring the total strength is a constant).

Security games. A security game [7] is a game between two players (an attacker and a defender) who allocate resources over a given number of targets. Each player allocates one resource to a target to attack or defend it. Both Derby games and security games have multiple rounds (security games call them targets) and compute utility as a linear combination of round utilities. However they differ in that (1) security games need not be zero sum, and (2) players in Derby games can have more than two kinds of resources.

Ranking duels. The ranking duel [6], is a two-player zero sum game where two players (e.g. search engines) rank items (e.g. search results), with the payoff depending on which engine gives a higher rank to a given item. Ranking duels are similar to a special case of Derby games, where items are represented by rounds and ranks are represented by alternating resources belonging to each player (i.e. $f_1 > g_1 > f_2 > g_2 > \ldots$), but differ in that they allow ties.

Tian Ji's horse-racing strategy. There has been limited analysis of Tian Ji's horse-racing strategy. Weicheng shows the n-horse version of Tian Ji's competition — the special case of the unweighted game where the victory relation is restricted to alternating resources — has a Nash equilibrium where both players play resources at random [14]. Shu gives the probability of winning with a random strategy in the above game [13]. Leng and Parlar explore the three-horse game, but with probabilistic outcomes [8].

7 CONCLUSION AND FUTURE WORK

We have introduced Derby games, which can be used to analyze several forms of competition where the ordering of elements is the most significant factor. We completely characterize the equilibria in the unweighted version of the game and provide necessary and sufficient conditions for the existence of half-pure Nash equilibria in the general case.

One interesting question for future work would be a complete characterization of all equilibria. In this direction, we expect our Narrow Wins Theorem to be a significant first step. Like Blotto games, variants of Derby games are also apt for future study:

Other victory relations. The victory relation over resources in Derby games is a total order. A natural extension would be to consider more general victory relations that are not semiconnex (e.g. allowing ties) and/or are not transitive.

Sequential rounds with simultaneous play. In this version of the Derby game, each round is played with knowledge of previous rounds, so players can adjust their strategies accordingly. For example, consider two competing record labels with various albums, choosing which week to release which album. The album that is more preferred on a given week will top charts, getting related commendations, while the other will be overshadowed. Each label can use information from previous weeks to plan their current week's release, however because the music industry has a standard global release day (Friday), they don't know what the other label will release that week.

Sequential rounds with alternating play. This game is similar to the previous game, except player A plays before player B on each round. If all rounds have equal weight, this game reduces to the single-player Derby game (described in Appendix A) for player B: B can apply the best-worst strategy from Remark (2.3) to maximize B's payoff. In future work, it would be interesting to analyze the general case when rounds might have different weights.

Variants with incomplete information. Variants of the Derby game and the above games can be defined in which each side has only partial information about the other players' resources.

We expect the technical tools developed in our work to provide a good starting point to investigate such variants.

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A SINGLE PLAYER BEST RESPONSE

In Section 2, we remarked that for a Derby game, if all rounds have the same weight and player B's schedule is fixed, the following schedule for player A is optimal:

(1) For some constant $K \in [m]$, A plays their best resource against B's $(m-K+1)$th best resource, their second best resource against B's $(m-K+2)$th best resource, and so on until A plays their Kth best resource against B's worst (mth-best) resource,
(2) A wins these K rounds while losing the remaining $m-K$ rounds, and
(3) K is the largest such constant where (1) and (2) hold.

We will now formalize and prove this remark.

Definition A.1 (Matching). For an unweighted Derby game $(m, F, G, >, \vec{1})$, a matching $t: G \to F$ is a bijection from player B's resources to player A's resources. Then, knowing B's ordering $q: [m] \to G$, player A plays the ordering $p = t \circ q$.

Because all weights are equal, the payoff for A depends only on the matching t and not the particular ordering A knows B will play (for all q, $U_A(t \circ q, q)$ is constant). We define the payoff for A of matching t, $U_A(t) = \sum_{g \in G} \mathbb{1}[t(g) > g]$.

Lemma A.2 (Increasing wins). *For an unweighted Derby game $(m, F, G, >, \vec{1})$ and any matching $t: G \to F$, there is a matching $t': G \to F$ with the same payoff (i.e. $U_A(t) = U_A(t')$), such that t' is increasing on input-output pairings A wins, i.e.*

$$\forall g_1, g_2 \in G.\, t'(g_1) > g_1 \wedge t'(g_2) > g_2 \wedge g_1 > g_2 \implies t'(g_1) > t'(g_2).$$

Proof. We only need to show it is safe to swap one "out-of-order" pair of winning outputs of t: repeatedly performing such swaps will sort the winning pairings of t to be increasing. Let the out-of-order outputs be $t(g_1), t(g_2)$ for some $g_1, g_2 \in G$. We know A wins, i.e. $t(g_1) > g_1$ and

$t(g_2) > g_2$, and we know t is decreasing on these resources, i.e. $g_1 > g_2$ but $t(g_2) > t(g_1)$. Because $t(g_2) > t(g_1) > g_1$ and $t(g_1) > g_1 > g_2$, swapping $t(g_1)$ and $t(g_2)$ to produce a new matching preserves the payoff of t, but produces a matching one step closer to being increasing. □

Definition A.3 (k-best-vs-worst matching). For an unweighted Derby game $(m, F, G, >, \vec{1})$, let the resources of F and G be ranked $f_1 > f_2 > \ldots > f_m \in F$, and $g_1 > g_2 > \ldots > g_m \in G$ respectively. We say a matching $t\colon G \to B$ is k-best-vs-worst if it plays the k best resources of F against the k worst resources of G, ordered best-to-worst, and it wins all those k rounds. Formally, t is k-best-worst if $\forall i \in [k].\, f_i = t(g_{m-k+i}) > g_{m-k+i}$.

Lemma A.4 (k-best-worst wins equally). *For an weighted Derby game $(m, F, G, >, \vec{1})$ and any matching $t\colon G \to F$, there is a k-best-vs-worst matching t_{kbw} such that $k = U_A(t) = U_A(t_{kbw})$.*

Proof. First, from t, we use Lemma A.2 to create t_{inc}, a matching whose winning pairings are increasing. Let $k = U_A(t) = U_A(t_{inc})$, and suppose $g_1 > \cdots > g_k \in G$ are the losing resources of G (so $t(g_1), \ldots, t(g_k)$ win). Also, let $\hat{f}_1 > \cdots > \hat{f}_k$ be the k most preferred resources of F.

We show that there is a matching that pairs $g_1, \ldots, g_k$ to $\hat{f}_1, \ldots, \hat{f}_k$ (respectively) has the same payoff as t_inc. Then a similar argument can change the other half of the pairs to the least preferred resources of g to produce a k-best worst matching.

For any increasing matching t', we know for each $t'(g_i)$ that there are at least $i-1$ more preferred resources in F (specifically, $t'(g_1), \ldots, t'(g_{i-1})$). Thus, we know that $\forall i \in [k].\, t'(g_i) \le \hat{f}_i$. Therefore, consider the sequence of matchings, where $t'_0 = t_{inc}$ and $i \in [k]$,

$$t_i(g) = \begin{cases} \hat{f}_i & \text{if } g = g_i \\ t'_{i-1}(g_i) & \text{if } t'_{i-1}(g) = \hat{f}_i \\ t'_{i-1}(g) & \text{otherwise.} \end{cases} \quad ; \quad t'_i \text{ is the increasing matching corresponding with } t_i$$

Note that $U_A(t_i) = k$:

- If $\hat{f}_i$ was already a winning resource of A in t'_{i-1}, since for $j < i$, $t'_{i-1}(g_j) = \hat{f}_j$, this makes $\hat{f}_i$ the ith largest of the winning resources, meaning it is already paired with g_i (since t'_{i-1} is increasing) and does not need to move.
- If $\hat{f}_i$ was a losing resource of A, A plays $\hat{f}_i$ where A previously won with $t'_{i-1}(g_i) \le \hat{f}_i$, and A plays $t'_{i-1}(g_i) \le \hat{f}_i$ where A previously lost with $\hat{f}_i$, keeping the same payoff.

Then t_k plays the most preferred resources of F in order in the winning rounds of t_k, and $U_A(t_k) = k$. A similar argument can set $g_1, \ldots, g_k$ to the least preferred resources of G, creating the desired k-best-vs-worst strategy. □

Theorem A.5 (Best-vs-worst is optimal). *For an unweighted Derby game $(m, F, G, >, \vec{1})$, let $K = \max\{k \mid \exists t\colon G \to F \mid t \text{ is } k\text{-best-vs-worst}\}$. Then for any K-best-vs-worst matching t, $U_A(t) = K$ and there is no matching t' such that $U_A(t') > U_A(t)$.*

Proof. First note that for any k, there is no k-best-vs-worst matching with payoff less than k (by definition). Moreover, there cannot be a K-best-vs-worst matching with payoff $K' > K$, since by Lemma A.4 that would imply there is a K'-best-vs-worst matching and that K is not the maximum as assumed. □

B RELATING STRATEGIES AND EFFECTIVE STRATEGIES

Definition B.1 (Generalizing strategies to effective strategies). Suppose for an m-round Derby game, a player has set of resources F, broken into vector of equivalence classes $\vec{C}$. For any mixed strategy (distribution over schedules) $p \in ([m] \to F) \to [0, 1]$ of that player, we define the *generalization* of p to be the effective strategy P, where P_{ix} is the total probability of playing an ordering which plays a resource in equivalence class C_x on round i, i.e. $P_{ix} = \sum_{s:\,[m]\to F} p(s)\mathbb{1}[s(i) \in C_x]$.

Remark B.2 (Generalization preserves utility). *For any Derby game and strategies (p, q) of each player, if effective strategies P, Q are the generalizations of p, q respectively, then $U_A(P, Q) = U_A(p, q)$.*

Lemma B.3 (Effective strategies describe strategies). *Given any Derby game, for any effective strategy $P \in [0,1]^{m\times n}$ of a given player, there exists at least one (mixed) strategy $p\colon ([m] \to F) \to [0,1]$ over schedules of the player's resources F such that P is a generalization of p.*

Proof. For any effective strategy $P \in [0,1]^{m\times n} = [P_1, \dots, P_n]$ where P_j is the j^{th} column of P, we construct a square matrix $P' \in [0,1]^{m\times m}$ from P using the following procedure: for every column P_j that represents the distribution of resources in equivalence class A_j over the rounds, we add $|A_j|$ columns $[\frac{1}{|A_j|}P_j, \dots, \frac{1}{|A_j|}P_j]$ to P'. Note that the sum of entries in each column P_j, $\sum_i P_{ij} = |A_j|$. This transformation assigns each resource in an equivalence class the same probability of being placed in a given round, keeping the total probability of assigning some resource in an equivalence class to a round the same as in P. Note that P' is a doubly stochastic matrix: each entry $P'_{ij} \geq 0$, and each row and column sum up to 1. By the Birkhoff - von Neumann theorem, P' can be written as a convex combination of $m \times m$ permutation matrices, which represent pure strategies (resource orderings). Hence, P' corresponds to the mixed strategy $p\colon ([m] \to F) \to [0,1]$ with the coefficients of the permutation matrices giving the probabilities assigned to the associated pure strategies.

□

Theorem 3.10 (NE iff effectively uniform). *For any unweighted Derby game, $(m, n, \vec{A}, \vec{B}, \vec{1})$, the only Nash Equilibrium occurs when both players play the (effectively) uniform strategy, i.e. at $(Uniform(\vec{A}), Uniform(\vec{B}))$.*

Proof. First note that because an effective strategy keeps the utility of all strategies it generalizes, we know from Lemma 3.2 that there is a Nash equilibrium when both players play effectively uniform strategies. We show this is the only equilibrium by showing that no other effective strategy will form a Nash equilibrium with an effectively uniform strategy.

We proceed by contradiction. Consider an arbitrary effective strategy for player A, $P \in [0,1]^{m\times n}$, where P is not effectively uniform, and suppose P is part of a Nash Equilibrium. By the rectangle lemma, $(P, \text{Uniform}(\vec{B}))$ should be an equilibrium, meaning that all of B's pure effective strategies should have the same payoff against P. We will show this isn't the case.

Since P is not effectively uniform, there must be some equivalence class index $x > 1$, and round indices $i, j \in [m]$ such that $P_{ix} < \frac{a_x}{m} < P_{jx}$. Then consider two pure effective strategies for player B, Q, R such that Q and R differ only in that Q plays B_{x-1} on round i and B_x on round j, while R plays B_x on round i and B_{x-1} on round j.

Because Q, R are nearly identical, we can simplify their difference in utility with respect to P:

$$U_A(P, R) - U_A(P, Q) = P_{ix} - P_{jx} > 0$$

This means that against A's non-uniform effective strategy P, B is better off playing Q than R – causing a contradiction and showing that P is not part of a Nash Equilibrium. □

C PROOF OF THEOREM 4.5

THEOREM 4.5. *For a binary Derby game $(m, n, \langle A_1, A_2\rangle, \langle B_1, B_2\rangle, \vec{w})$, there is a half-pure Nash Equilibrium where A plays a pure strategy P iff*

$$b_1 \geq m - a_1 + \sum_{i=1}^{a_1} \frac{w_{(a_1+1)}}{w_i}.$$

where P places resources in A_1 in the highest weight rounds.

PROOF. We will prove that P is a best response pure strategy by showing that A's utility $U_A(P,Q) \geq U_A(P',Q)$ for an arbitrary pure strategy P', where $\mathrm{supp}_A(P,1) = S$ and $\mathrm{supp}_A(P',1) = T$. We denote the difference in the utility that A gets on playing the strategies P and P' by $\Delta U_A(P,P',Q) = U_A(P,Q) - U_A(P',Q)$. The difference in utility for a round i is denoted by $\Delta U_A(P,P',Q)_i$.

Since P is a best response to Q, we have

$$U_A(P,Q) - U_A(P',Q) \geq 0$$

Rewriting this using our notation for difference in utility, we get

$$\sum_{i\in[m]} \Delta U_A(P,P',Q)_i \geq 0$$

Writing $[m]$ as $([m] \setminus S \cup T) \cup (S \cap T) \cup (S \setminus T) \cup (T \setminus S)$, we get

$$\sum_{i\in[m]\setminus(S\cup T)} \Delta U_A(P,P',Q)_i + \sum_{i\in S\cap T} \Delta U_A(P,P',Q)_i + \sum_{i\in S\setminus T} \Delta U_A(P,P',Q)_i + \sum_{i\in T\setminus S} \Delta U_A(P,P',Q)_i \geq 0 \tag{1}$$

Simplifying using the definition of utility (Definition 2.1), we get

$$0 + 0 + \sum_{i\in S\setminus T} w_i - (1 - Q_{i1})w_i + \sum_{j\in T\setminus S} 0 - w_j \geq 0 \tag{2}$$

$$\sum_{i\in S\setminus T} Q_{i1}w_i + \sum_{j\in T\setminus S} -w_j \geq 0 \tag{3}$$

Since $|S| = |T| = a_1$, $|S \setminus T| = |S| - |S \cap T| = |T| - |S \cap T| = |T \setminus S|$. Let $i_1, \ldots, i_l$ and $j_1, \ldots, j_l$ where $l = |S \setminus T|$ be the rounds in $S \setminus T$ and $T \setminus S$ respectively, sorted in increasing order. Rearranging the terms in (3), we get

$$\sum_{k=1}^{l} Q_{i_k 1}w_{i_k} - w_{j_k} \geq 0$$

The condition $Q_{i_k1}w_{i_k} - w_{j_k} \geq 0$ for each $k \in [m]$ is sufficient for the above condition to hold. Since P' can be any pure strategy, $Q_{i_k1}w_i - w_j \geq 0$ for any $i \in S$ and $j \in [m] \setminus S$ is sufficient for $U_A(P,Q) \geq U_A(P',Q)$ for any pure strategy P'. Simplifying, we get

$$Q_{i1}w_i \geq w_j \;\forall i \in S \;\forall j \in S'$$

$$Q_{i1}w_i \geq max_{j\in S'} w_j \;\forall i \in S$$

$$Q_{i1} \geq \frac{w_{max}}{w_i} \quad \text{where } w_{max} = max_{j\in[m]\setminus S} w_j \tag{4}$$

Using the fact that $\sum_{i\in[m]} Q_{i1} = b_1$ and the necessary condition $Q_{i1} = 1$ for all $i \in [m] \setminus S$ (Lemma 4.4), we get the necessary and sufficient condition

$$b_1 = \sum_{i\in[m]\setminus S} 1 + \sum_{i\in S} Q_{i1}$$

$$b_1 = m - a_1 + \sum_{i\in S} Q_{i1}$$

Using (4), we get

$$b_1 \geq m - a_1 + \sum_{i\in S} \frac{w_{max}}{w_i}$$

Since $Q_{i1} \leq 1$ and $Q_{i1} \geq \frac{w_{max}}{w_i}$ for all $i \in S$ by (4), we get

$$1 \geq Q_{i1} \geq \frac{w_{max}}{w_i} \quad \forall i \in S$$

$$w_i \geq w_{max} \quad \forall i \in S$$

which implies that S is a set of highest weight rounds (as $w_{max} = max_{j\in[m]\setminus S} w_j$ is the highest weight round in the complement of S), and so $w_{max} = w_{a_1+1}$. □