Geometric control of hybrid systems

Benoît Legat[a], Raphaël M. Jungers[b]

[a] *LIDS, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA*
[b] *ICTEAM, UCLouvain, 4 Av. G. Lemaître, 1348 Louvain-la-Neuve, Belgium*

Abstract

In this paper, we present a geometric approach for computing controlled invariant sets for hybrid control systems. While the problem is well studied in the ellipsoidal case, this family is quite conservative for constrained or switched linear systems. We reformulate the invariance of a set as an inequality for its support function that is valid for any convex set. This produces novel algebraic conditions for the invariance of sets with polynomial or piecewise quadratic support functions.

Keywords: Controller Synthesis; Set Invariance; LMIs; Scalable Methods.

1. Introduction

Computing controlled invariant sets is paramount in many applications [7]. Indeed, the existence of a controlled invariant set is equivalent to the stabilizability[1] of a control system [38] and a (possibly nonlinear) stabilizable state feedback can be deduced from the controlled invariant set [6].

The stabilizability of a linear time-invariant (LTI) control system is equivalent to the stability of its uncontrollable subspace (which is readily accessible in its Controllability Form) [40, Section 2.4]. Indeed, the eigenvalues of its controllable subspace can be fixed to any value by a proper choice of linear state feedback. The resulting controlled system is stable hence an invariant ellipsoid can be determined by solving a system of linear equations [23]. This

[⋆] This paper extends our work on continuous-time controlled invariant sets presented at ADHS 2021 [21] to hybrid systems. Corresponding author B. Legat.

Email addresses: `blegat@mit.edu` (Benoît Legat), `raphael.jungers@uclouvain.be` (Raphaël M. Jungers)

[1] In the sense that the state variables can be controlled to remain bounded.

set is also controlled invariant for the control system. When a control system admits an ellipsoidal controlled invariant set, it is said to be *quadratically stabilizable*. When there exists a linear state feedback such that the resulting autonomous system admits an ellipsoidal invariant set, it is said to be *quadratically stabilizable via linear control*.

While the stabilizability of LTI control systems is equivalent to their quadratic stabilizability via linear control, it is no longer the case for *uncertain* or *switched* systems [30]. Furthermore, it is often desirable for *constrained* systems to find a controlled invariant set of maximal volume (or which is maximal in some direction [1]). For such problems, the method detailed above is not suitable as it does not take any volume consideration but more importantly, the maximal volume invariant set may not be an ellipsoid and may not be rendered stable via a linear control. For this reason, a Linear Matrix Inequality (LMI) was devised to encapsulate the controlled invariance of an ellipsoid via linear control [9, Section 7.2.2] and the conservatism of the choice of linear control was analysed [38]. As the linearity of the control was found to be conservative for uncertain systems [30], the LMI (9) (or (8) for discrete-time) was found to encapsulate controlled invariance of an ellipsoid via *any* state-feedback [6].

While these LMIs have had a tremendous impact on control, the approach is limited to ellipsoids due to its algebraic nature. Recent advances in control was enabled thanks to the introduction of new families of sets such as polynomial sublevel sets [27] (see Section 3.2) or polynomial zonotopes [14, 15]. An attempt to generalize the LMIs mentioned above to polynomials can be found in [31] but as detailed in [21, Section 2], it is quite conservative. The approach studied in [16] is complementary to our method as [16] computes outer bounds of the maximal controlled invariant sets while we compute actual controlled invariant sets (hence inner bounds to the maximal one).

In this paper, we reinterpret the controlled invariance in a geometric/behavioural framework, based on convex analysis, which allows us to formulate a general condition for the controlled invariance of arbitrary convex sets via any state-feedback in Theorem 1. While this condition reduces to (8) and (9) for the special case of ellipsoids, it provides a new method for computing convex controlled invariant sets with polynomial and piecewise quadratic support functions.

This paper genralizes [19, 20, 21] into a framework for computing convex controlled invariant sets for linear hybrid control systems. In [20], the authors treat the particular case where the continuous dynamic at each *mode* (see Definition 1) is trivial, i.e., $\dot{x} = 0$. In [19], the authors extends [20] to piecewise semi-ellipsoids. In [21], the authors handle the particular case where there is only one mode and no *transitions* (see Definition 1). While [20, 19] covers discrete-time systems and [21] covers continuous-time systems, we show in this paper that the two methods can be combined to compute controlled invariant sets for hybrid systems, exhibiting both discrete-time and continuous-time dynamics. Using the *set programming* framework (see Appendix C), this compatibility can be understood as a consequence of the fact that the controlled invariance conditions require the sets to be represented with their *support functions* (see Definition 6) both in discrete-time and continuous-time.

In Section 2, we show how to reduce the computation of controlled invariant sets for *hybrid control systems* to the computation of *weakly invariant* sets for *hybrid algebraic systems*. In Section 3, we develop a generic condition of control invariance for hybrid systems using our geometric approach. We particularize it for ellipsoids (resp. sets with polynomial and piecewise quadratic support functions) in Section 3.1 (resp. Section 3.2 and Section 3.3). We illustrate these new results with numerical examples in Section 4.

Reproducibility. The code used to obtain the results is published on codeocean [22]. The set programs are reformulated by SetProg [17] as described in Appendix C and then solved by Mosek v8 [4].

2. Controlled invariant set

In this section we define hybrid control and algebraic systems as well as the notion of invariance that will be studied in this paper. We then show how the invariance relations between the two different classes of systems.

Definition 1. A *Linear Control Hybrid Automaton (CHA)* is a system $S = (T, (A_q, B_q)_{q \in V}, (A_\sigma, B_\sigma)_{\sigma \in \Sigma}, (\mathcal{X}_q, \mathcal{U}_q)_{q \in V}, (\mathcal{U}_\sigma)_{\sigma \in \Sigma})$ where $T = (V, \Sigma, \to)$, V is a finite set of *modes*, Σ is a finite set of *signals* and $\to \subseteq V \times \Sigma \times V$ is a set of *transitions*. We denote $(q, \sigma, q') \in \to$ by $q \to_\sigma q'$.

Given a mode $q \in V$, we denote the state dimension as $n_{q,x}$ and the input dimension as $n_{q,u}$ Given a signal σ, we denote the input dimension as $n_{\sigma,u}$. The set $\mathcal{X}_q \subseteq \mathbb{R}^{n_{q,x}}$ is the *safe set* corresponding to mode q and the sets $\mathcal{U}_q \subseteq \mathbb{R}^{n_{q,u}}, \mathcal{U}_\sigma \subseteq \mathbb{R}^{n_{\sigma,u}}$ are the sets of allowed inputs. For any mode q, we have $A_q \in \mathbb{R}^{n_{q,x} \times n_{q,x}}$, $B_q \in \mathbb{R}^{n_{q,x} \times n_{q,u}}$. For any transition $q \to_\sigma q'$, we have $A_\sigma \in \mathbb{R}^{n_{q',x} \times n_{q,x}}$, $B_\sigma \in \mathbb{R}^{n_{q',x} \times n_{\sigma,u}}$.

A trajectory of S is an increasing sequence of times $t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1}$, transitions[2] $q_0 \to_{\sigma_1} q_1 \to_{\sigma_2} \cdots \to_{\sigma_N} q_N$, *reset map* inputs $\bar{u}_k \in \mathcal{U}_{\sigma_k}$ for $k \in \{1, \ldots, N\}$, and trajectories $x_k : [t_k, t_{k+1}] \to \mathcal{X}_{q_k} \in \mathcal{C}^1$ and $u_k : [t_k, t_{k+1}] \to \mathcal{U}_{q_k}$ for $k = 0, 1, \ldots, N$ satisfying:

$$\begin{aligned} \forall k \in \{1, \ldots, N\}, \quad & x_k(t_k) = A_{\sigma_k} x_{k-1}(t_k) + B_{\sigma_k} \bar{u}_k \\ \forall k \in \{0, 1, \ldots, N\}, \forall t \in [t_k, t_{k+1}], \quad & \dot{x}_k(t) = A_{q_k} x_k(t) + B_{q_k} u_k(t). \end{aligned}$$

The hybrid system defined in Definition 1 may be interpreted as a *hybrid automaton* [3] where the guard of each transition $q \to_\sigma q'$ is $\mathcal{X}_q$ or $\mathbb{R}^{n_{q,x}}$. In this context, the discrete-time dynamical system $x^+ = A_\sigma x + B_\sigma \bar{u}$ is commonly referred to as the *reset map*. We allow the state space of different modes to differ as our method naturally extends to different state spaces but the reader may consider them to have identical dimension for simplicity.

We define the *tangent cone* as follows [7, Definition 4.6].

Definition 2 (Tangent cone)**.** Consider a norm $\|\cdot\|$ and a distance function $d(\mathcal{S}, x)$ defined as:

$$d(\mathcal{S}, x) = \inf_{y \in \mathcal{S}} \|x - y\|.$$

Given a closed convex set $\mathcal{S}$, the *tangent cone* to $\mathcal{S}$ at x is defined as follows:

$$T_{\mathcal{S}}(x) = \left\{ y \mid \lim_{\tau \to 0} \frac{d(\mathcal{S}, x + \tau y)}{\tau} = 0 \right\}$$

The tangent cone is a closed convex cone and is independent of the norm used; see [12, Proposition 5.1.3].

We define below the *controlled invariance* of a collection of closed sets $\mathcal{S}_q$ for each mode q. Equation (1) encodes the controlled invariance for each transitions. Equation (2) is the *Nagumo condition* for each mode; see [7, Theorem 4.7]. The controlled invariant set is also known as *viability domain* [5].

Definition 3 (Controlled invariant sets for a CHA)**.** Consider a CHA S as defined in Definition 1. We say that closed sets $\mathcal{S}_q \subseteq \mathcal{X}_q$ for $q \in V$ are *controlled invariant* for S if

$$\forall q \to_\sigma q', \forall x \in \mathcal{S}_q, \exists u \in \mathcal{U}_\sigma \text{ such that } \quad A_\sigma x + B_\sigma u \in \mathcal{S}_{q'} \tag{1}$$

$$\forall q \in V, \forall x \in \mathcal{S}_q, \exists u \in \mathcal{U}_q \text{ such that } \quad A_q x + B_q u \in T_{\mathcal{S}_q}(x) \tag{2}$$

where $T_{\mathcal{S}_q}(x)$ denotes the tangent cone defined in Definition 2.

[2]Note that the transitions can occur arbitrarily often.

In view of Definition 3, the transitions are considered *autonomous* and not *controlled*; see details in [24, Section 1.1.3]. In the case of unconstrained controlled, i.e., $\mathcal{U}_q = \mathbb{R}^{n_{q,u}}, \mathcal{U}_\sigma = \mathbb{R}^{n_{\sigma,u}}$, the invariance condition can be reformulated geometrically using projections.

Lemma 1 ([21, Proposition 4])**.** Given a subset $\mathcal{S} \subseteq \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}$, the following holds:

$$A\mathcal{S} + B\mathbb{R}^m = \pi^{-1}_{\mathrm{Im}(B)^\perp} \pi_{\mathrm{Im}(B)^\perp} A\mathcal{S}$$

where $\pi_{\mathrm{Im}(B)^\perp} : \mathbb{R}^n \to \mathrm{Im}(B)^\perp$ is any orthogonal projection matrix onto the orthogonal subspace of $\mathrm{Im}(B)$, the linear span of the columns of B, and $\pi^{-1}_{\mathrm{Im}(B)^\perp} : \mathrm{Im}(B)^\perp \to \mathbb{R}^r$ is the preimage defined in Eq. (A.1).

Proof. Given $x \in \mathcal{S}$ and $y \in \mathbb{R}^r$, we have $y \in A\{x\} + B\mathbb{R}^m$ if and only if $y - Ax \in \mathrm{Im}(B)$. As $\pi_{\mathrm{Im}(B)^\perp}$ is orthogonal, its kernel is $\mathrm{Im}(B)$. Therefore $y - Ax \in \mathrm{Im}(B)$ is equivalent to $\pi_{\mathrm{Im}(B)^\perp} y = \pi_{\mathrm{Im}(B)^\perp} Ax$. □

2.1. Linear Hybrid Algebraic Automaton

In this section, we show the equivalence of the notion of invariance with another class of systems that directly models the geometric behaviours of the trajectories of a CHA with unconstrained input. Algebraic systems are also known as *descriptor systems*. The reduction of the computation of controlled invariant sets of CHA with constrained input to CHA of unconstrained input is detailed in [20, Section 2.2].

Definition 4. A *Linear Algebraic Hybrid Automaton (AHA)* is a system $S = (T, (C_q, E_q)_{q\in V}, (C_\sigma, E_\sigma)_{\sigma\in\Sigma}, (\mathcal{X}_q)_{q\in V})$ where $T = (V, \Sigma, \to)$, V is a finite set of modes, Σ is a finite set of signals and $\to \subseteq V \times \Sigma \times V$ is a set of transitions.

Given a mode $q \in V$, we denote the state dimension as $n_{q,x}$. The set $\mathcal{X}_q \subseteq \mathbb{R}^{n_{q,x}}$ is the *safe set* corresponding to mode q. For any mode q, there exists a $n_{q,p}$ such that, $C_q \in \mathbb{R}^{n_{q,p}\times n_{q,x}}$, $E_q \in \mathbb{R}^{n_{q,p}\times n_{q,x}}$. For any transition $q \to_\sigma q'$, there exists a $n_{\sigma,p}$ such that, $C_\sigma \in \mathbb{R}^{n_{\sigma,p}\times n_{q,x}}$, $E_\sigma \in \mathbb{R}^{n_{\sigma,p}\times n_{q',x}}$.

A trajectory of S is an increasing sequence of times $t_0 < t_1 < t_2 < \cdots t_N$, transitions $q_0 \to_{\sigma_1} q_1 \to_{\sigma_2} \cdots \to_{\sigma_N} q_N$, and trajectories $x_k : [t_{k-1}, t_k] \to \mathcal{X}_{q_k} \in \mathcal{C}^1$ for $k = 0, 1, \ldots, N$ satisfying:

$$\forall k \in \{1, \ldots, N\}, \quad E_{\sigma_k} x_k(t_k) = C_{\sigma_k} x_{k-1}(t_k)$$
$$\forall k \in \{0, 1, \ldots, N\}, \forall t \in [t_{k-1}, t_k], \quad E_{q_k} \dot{x}_k(t) = C_{q_k} x_k(t).$$

Note that the matrices C_q, E_q, C_σ and E_σ do not have to be square and no assumptions are needed on their rank.

Definition 5 (Weakly invariant sets for a AHA)**.** Consider a AHA S as defined in Definition 4. We say that closed sets $\mathcal{S}_q \subseteq \mathcal{X}_q$ for $q \in V$ are *weakly invariant* for S if

$$\forall q \to_\sigma q', \forall x \in \mathcal{S}_q, \quad C_\sigma x \in E_\sigma \mathcal{S}_{q'} \tag{3}$$

$$\forall q \in V, \forall x \in \mathcal{S}_q, \quad C_q x \in E_q T_{\mathcal{S}_q}(x). \tag{4}$$

We now show that the computation of controlled invariant sets for a CHA can be reduced to the computation of weakly invariant sets for a AHA. The following proposition generalizes both [20, Proposition 2] and [21, Proposition 5].

Proposition 1. The sets $\mathcal{S} = (\mathcal{S}_q)_{q\in V}$ are *controlled invariant* for the CHA $S = (T, (A_q, B_q)_{q\in V}, (A_\sigma, B_\sigma)_{\sigma\in\Sigma}, (\mathcal{X}_q, \mathbb{R}^{n_{q,u}})_{q\in V}, (\mathbb{R}^{n_{\sigma,u}})_{\sigma\in\Sigma})$ if and only if they are weakly invariant sets for the AHA

$$S' = (T, (\pi_{\mathrm{Im}(B_q)^\perp} A_q, \pi_{\mathrm{Im}(B_q)^\perp})_{q\in V}, (\pi_{\mathrm{Im}(B_\sigma)^\perp} A_\sigma, \pi_{\mathrm{Im}(B_\sigma)^\perp})_{\sigma\in\Sigma}, (\mathcal{X}_q)_{q\in V}).$$

Proof. By Lemma 1, (1) is equivalent to

$$\pi_{\mathrm{Im}(B_\sigma)^\perp} A_\sigma x \in \pi_{\mathrm{Im}(B_\sigma)^\perp} \mathcal{S}_{q'}$$

which is (3) for S'.

Similarly, by Lemma 1, (2) is equivalent to

$$\pi_{\mathrm{Im}(B_q)^\perp} A_q x \in \pi_{\mathrm{Im}(B_q)^\perp} T_{\mathcal{S}_q}(x)$$

which is (4) for S'. □

3. Computing controlled invariant sets

In this section we derive a characterization of the weak invariance of closed convex sets under the form of inequalities for their *support functions*.

Definition 6 ([34, p. 28])**.** Consider a convex set $\mathcal{S}$. The *support function* of $\mathcal{S}$ is defined as

$$\delta^*(y|\mathcal{S}) = \sup_{x\in\mathcal{S}} \langle y, x\rangle.$$

The *exposed face* (also called the *support set*, e.g., in [36, Section 1.7.1]) is defined as follows [12, Definition 3.1.3].

Definition 7 (Exposed face)**.** Consider a nonempty closed convex set $\mathcal{S}$. Given a vector $y \neq 0$, the *exposed face* of $\mathcal{S}$ associated to y is

$$F_{\mathcal{S}}(y) = \{\, x \in \mathcal{S} \mid \langle x, y \rangle = \delta^*(y|\mathcal{S}) \,\}.$$

The following theorem generalizes both [19, (27)] and [21, Theorem 7].

Theorem 1. Consider a AHA S as defined in Definition 4. Closed sets $\mathcal{S}_q \subseteq \mathcal{X}_q$ for $q \in V$ are weakly invariant for S if and only if

$$\forall q \to_\sigma q', \forall y \in \mathbb{R}^{n_{\sigma,p}}, \quad \delta^*(C_\sigma^\top y|\mathcal{S}_q) \leq \delta^*(E_\sigma^\top y|\mathcal{S}_{q'}) \tag{5}$$

$$\forall q \in V, \forall z \in \mathbb{R}^{n_{q,p}}, \forall x \in F_{\mathcal{S}_q}(E_q^\top z), \quad \langle z, C_q x \rangle \leq 0 \tag{6}$$

where $F_{\mathcal{S}}$ denotes the *exposed face* defined in Definition 7 and $\delta^*(y|\mathcal{S})$ denotes the *support function* defined in Definition 6.

Proof. We start by proving the equivalence between (3) and (5). By Proposition 5, Eq. (3) is equivalent to

$$\forall q \to_\sigma q', \forall y \in \mathbb{R}^{n_{\sigma,p}}, \quad \delta^*(y|C_\sigma \mathcal{S}_q) \leq \delta^*(y|E_\sigma \mathcal{S}_{q'})$$

which is equivalent to Eq. (5) by Proposition 4.

We now prove the equivalence between (4) and (6). Given any mode q, as $\mathcal{S}_q$ is convex, $T_{\mathcal{S}_q}(x)$ is a closed convex cone. By definition of the polar of a cone, $x \in E_q T_{\mathcal{S}_q}(x)$ if and only if $\langle y, x \rangle \leq 0$ for all $y \in [E_q T_{\mathcal{S}_q}(x)]^\circ$. By Proposition 3, $[E_q T_{\mathcal{S}_q}(x)]^\circ = E_q^{-\top} N_{\mathcal{S}_q}(x)$. Therefore, the set $\mathcal{S}_q$ is weakly invariant if and only if

$$\forall x \in \partial \mathcal{S}_q, \forall z \in E_q^{-\top} N_{\mathcal{S}_q}(x), \langle z, C_q x \rangle \leq 0.$$

By Proposition 2, we have

$$\{\, (x, z) \in \partial \mathcal{S}_q \times \mathbb{R}^r \mid E_q^\top z \in N_{\mathcal{S}_q}(x) \,\} = \{\, (x, z) \in \partial \mathcal{S}_q \times \mathbb{R}^r \mid x \in F_{\mathcal{S}_q}(E_q^\top z) \,\}.$$

□

Observe that for the trivial case $\mathrm{Im}(B_q) = \mathbb{R}^{n_{q,x}}$ for some node q, Proposition 1 produces a AHA with $n_{q,p} = 0$ hence the condition (6) would be trivially satisfied for any $\mathcal{S}_q$, which is expected. The same applies for (5) in case $\mathrm{Im}(B_\sigma) = \mathbb{R}^{n_{q',x}}$ for some transition $q \to_\sigma q'$.

As we show in the remainder of this section, Theorem 1 allows to reformulate the invariance as an inequality in terms of the support functions of the sets $\mathcal{S}_q$. This is already the case of Eq. (5) so it remains to reformulate Eq. (6). As shown in the following theorem, this is possible in case the support function is differentiable. We generalize this result with a relaxed notion of differentiability in Theorem 3. The following theorem generalizes both [19, (27)] and [21, Theorem 8].

Theorem 2. Consider a AHA S as defined in Definition 4 and nonempty closed convex sets $\mathcal{S}_q \subseteq \mathcal{X}_q$ for $q \in V$ such that $\delta^*(\cdot|\mathcal{S}_q)$ is differentiable for all $q \in V$. Then the sets are weakly invariant for S if and only if

$$\begin{aligned} \forall q \to_\sigma q', \forall y \in \mathbb{R}^{n_{\sigma,p}}, \delta^*(C_\sigma^\top y|\mathcal{S}_q) &\leq \delta^*(E_\sigma^\top y|\mathcal{S}_{q'}) \\ \forall q \in V, \forall z \in \mathbb{R}^{n_{q,p}}, \langle z, C_q \nabla \delta^*(E_q^\top z|\mathcal{S}_q)\rangle &\leq 0. \end{aligned} \tag{7}$$

Proof. By Proposition 6, $F_{\mathcal{S}_q}(E_q^\top z) = \{\nabla \delta^*(E_q^\top z|\mathcal{S}_q)\}$ hence (6) is equivalent to (7). □

As Theorem 2 formulates the invariance in terms of the support function of $\mathcal{S}_q$, it allows to combine the invariance constraint with other set constraints that can be formulated in terms of support functions. Moreover, for an appropriate family of sets, also called *template*, the set program can be automatically rewritten into a convex program combining all constraints using the *set programming* framework detailed in Appendix C. For this reason, we only focus on the invariance constraint and do not detail how to formulate the complete convex programs with the objective and all the constraints needed to obtain the results of Section 4 as these problems are decoupled.

3.1. Ellipsoidal controlled invariant set

In this section, we particularize Theorem 2 to the case of ellipsoids. Since the support function of an ellipsoid $\mathcal{E}_Q \triangleq \{\, x \mid x^\top Q x \leq 1 \,\}$ is $\delta^*(y|\mathcal{E}_Q) = \sqrt{y^\top Q^{-1} y}$, we have the following corollary of Theorem 2 that generalizes both [20, Theorem 2] and [21, Corollary 9].

Corollary 1. Consider a AHA S as defined in Definition 4 and positive semidefinite matrices Q_q such that the ellipsoid $\mathcal{E}_{Q_q} \subseteq \mathcal{X}_q$ for $q \in V$. Then the sets are weakly invariant for S if and only if

$$\forall q \to_\sigma q', C_\sigma Q_q^{-1} C_\sigma^\top \preceq E_\sigma Q_{q'}^{-1} E_\sigma^\top \tag{8}$$

$$\forall q \in V, C_q Q_q^{-1} E_q^\top + E_q Q_q^{-1} C_q^\top \preceq 0. \tag{9}$$

3.2. Polynomial controlled invariant set

In this section, we derive the algebraic condition for the controlled invariance of a set with polynomial support function. This template is referred to as *polyset*; see [17, Section 1.5.3]. The following corollary generalizes both [20, Theorem 5] and [21, Corollary 10].

Corollary 2. Consider a AHA S as defined in Definition 4, convex homogeneous[3] nonnegative polynomials $(p_q(x))_{q \in V}$ of degree $2d$ and the sets $\mathcal{S}_q$ defined by the support function $\delta^*(y|\mathcal{S}_q) = p_q(y)^{\frac{1}{2d}}$ for $q \in V$. Suppose that $\mathcal{S}_q \subseteq \mathcal{X}_q$ for all $q \in V$. Then the sets are weakly invariant for S if and only if

$$\forall q \to_\sigma q', \forall y \in \mathbb{R}^{n_{\sigma,p}}, p_q(C_\sigma^\top y) \leq p_{q'}(E_\sigma^\top y) \tag{10}$$

$$\forall q \in V, \forall z \in \mathbb{R}^{n_{q,p}}, z^\top C_q \nabla p_q(E_q^\top z) \leq 0. \tag{11}$$

Proof. We have

$$\nabla \delta^*(y|\mathcal{S}_q) = \frac{1}{p_q(y)^{1-\frac{1}{2d}}} \nabla p_q(y).$$

If $p_q(y)$ is identically zero, this is trivially satisfied. Otherwise, $p_q(y)^{1-\frac{1}{2d}}$ is nonnegative and is zero in an algebraic variety of dimension $n-1$ at most. Therefore, (7) is equivalent to (11). □

The conditions (10) and (11) require the nonnegativity of a multivariate polynomial. While verifying the nonnegativity of a polynomial is co-NP-hard, a sufficient condition can be obtained via the standard Sum-of-Squares programming framework; see Appendix B. Moreover, the theorem requires the convexity of the polynomials p_q. It is shown in [2] that the convexity or quasi-convexity of a multivariate polynomial of degree at least four is NP-hard to decide. However, the convexity constraint can be replaced by the tractable SOS-convexity constraint which is a sufficient condition for convexity [2].

[3] A polynomial is *homogeneous* if all its monomials have the same total degree

3.3. Piecewise semi-ellipsoidal controlled invariant set

In [13], the authors study the computation of piecewise quadratic Lyapunov functions for continuous-time autonomous piecewise affine systems. In [19], the authors present a convex programming approach to compute *piecewise semi-ellipsoidal* controlled invariant sets for discrete-time control systems. A similar approach is developed in [21] for continuous-time control system. In this section, we combine the two approaches into a condition for hybrid systems using Theorem 1. We recall [19, Definition 2] below.

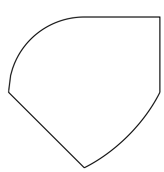
Definition 8. A *polyhedral conic partition* of $\mathbb{R}^n$ is a set of m polyhedral cones $\mathcal{P}_i \subseteq \mathbb{R}^n$ with nonempty interior for $i = 1, \dots, m$ such that for all $i \neq j$, $\dim(\mathcal{P}_i \cap \mathcal{P}_j) < n$ and $\cup_{i=1}^m \mathcal{P}_i = \mathbb{R}^n$.

A polyhedral conic partition defines the full-dimensional faces of a *complete fan*, as defined in [41, Section 7].

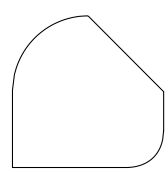
A piecewise semi-ellipsoid is defined as the closed convex set with support function

$$\delta^*(y|\mathcal{S}) = \sqrt{y^\top Q_i y}, \qquad y \in \mathcal{P}_i, \qquad i = 1, \dots, m \tag{12}$$

where $(\mathcal{P}_i)_{i=1}^m$ is a polyhedral conic partition and $(Q_i)_{i=1}^m$ are positive semidefinite matrices. The support function additionally has to satisfy [19, (2) and (3)] to ensure its continuity and convexity. Note that the convexity of $\delta^*(y|\mathcal{S})$ and $\delta^*(y|\mathcal{S})^2$ are equivalent by [34, Corollary 15.3.1].



(a) Set $\mathcal{S}$ whose support function is defined by (13).



(b) Set $\mathcal{S}^\circ$ whose Minkowski function is defined by (13).

Figure 1: Illustration for sets $\mathcal{S}$ and $\mathcal{S}^\circ$ defined in Example 1.

Example 1. The piecewise semi-ellipsoid defined by the following support function is represented by Fig. 1. See [19, Example 1] for more details on this example.

$$\delta^*(y|\mathcal{S}) = \begin{cases} |y_1+y_2| & \text{if } 0 \le y_1, y_2, \\ \sqrt{y_1^2+y_2^2} & \text{if } y_1 \le 0 \le y_2, \\ |y_1| & \text{if } y_1 \le y_2 \le 0, \\ |y_2| & \text{if } y_2 \le y_1, 2y_1+y_2 \le 0, \\ 2\sqrt{y_1^2+y_1y_2+y_2^2}/\sqrt{3} & \text{if } 2y_1+y_2 \ge 0, y_1+2y_2 \le 0, \\ |y_1| & \text{if } y_1+2y_2 \ge 0, y_2 \ge 0. \end{cases} \tag{13}$$

The following theorem generalizes both [19, (27)] and [21, Theorem 12].

Theorem 3. Consider a AHA S as defined in Definition 4, polyhedral conic partitions $(\mathcal{P}_{q,i})_{i=1}^{m_q}$ and nonempty closed convex sets $(\mathcal{S}_q)_{q\in V}$ defined by the support function

$$\delta^*(y|\mathcal{S}_q) = f_{q,i}(y) \qquad y \in \mathcal{P}_{q,i} \qquad i = 1, \ldots, m_q.$$

Suppose that $\mathcal{S}_q \subseteq \mathcal{X}_q$ for all $q \in V$. The sets $\mathcal{S}_q$ are weakly invariant for S if and only if

$$\forall q \to_\sigma q', \forall i \in [m_q], \forall j \in [m_{q'}],$$
$$\forall y \in C_\sigma^{-\top}\mathcal{P}_{q,i} \cap E_\sigma^{-\top}\mathcal{P}_{q',j}, \quad f_{q,i}(C_\sigma^\top y) \le f_{q',j}(E_\sigma^\top y) \tag{14}$$
$$\forall q \in V, \forall i \in [m_q], \forall z \in E_q^{-\top}\mathcal{P}_{q,i}, \quad \langle z, C_q \nabla f_{q,i}(E_q^\top z)\rangle \le 0. \tag{15}$$

Proof. If $y \in C_\sigma^{-\top}\mathcal{P}_{q,i} \cap E_\sigma^{-\top}\mathcal{P}_{q',j}$, then $\delta^*(C_\sigma^\top y|\mathcal{S}_q) = f_{q,i}(C_\sigma^\top y)$ and $\delta^*(E_\sigma^\top y|\mathcal{S}_{q'}) = f_{q',j}(E_\sigma^\top y)$ hence (5) is reformulated as (14).

We now prove the equivalence between (6) and (15). Consider a mode $q \in V$. Given $z \in \mathbb{R}^{n_{q,p}}$ such that $E_q^\top z$ is in the intersection of the boundary of $\mathcal{S}_q$ and the interior of $\mathcal{P}_{q,i}$, the support function is differentiable at $E_q^\top z$ hence, by Proposition 6, $F_{\mathcal{S}}(E_q^\top z) = \{\nabla f_{q,i}(E_q^\top z)\}$. The condition (6) is therefore reformulated as (15).

Given a subset I of $\{1, \ldots, m\}$ and $z \in \mathbb{R}^{n_{q,p}}$ such that $E_q^\top z$ is in the intersection of the boundary of $\mathcal{S}_q$ and $\cap_{i\in I}\mathcal{P}_{q,i}$, $F_{\mathcal{S}_q}(E_q^\top z)$ is the convex hull of $\nabla\delta^*(E_q^\top z|\mathcal{S}_q)$ for all $i \in I$. For any convex combination (i.e., nonnegative numbers summing to 1) $(\lambda_i)_{i\in I}$, (15) implies that

$$\langle z, C_q \sum_{i\in I} \lambda_i \nabla f_{q,i}(E_q^\top z)\rangle = \sum_{i\in I} \lambda_i \langle z, C_q \nabla f_{q,i}(E_q^\top z)\rangle \le 0.$$

□

The following corollary generalizes both [19, Theorem 4] and [21, Corollary 13].

Corollary 3. Consider a AHA S as defined in Definition 4 and piecewise semi-ellipsoids $\mathcal{S}_q \subseteq \mathcal{X}_q$ for $q \in V$. The sets are weakly invariant for S if and only if

$$\forall q \to_\sigma q', \forall i \in [m_q], \forall j \in [m_{q'}], \qquad \forall y \in C_\sigma^{-\top}\mathcal{P}_{q,i} \cap E_\sigma^{-\top}\mathcal{P}_{q',j}, y^\top C_\sigma Q_{q,i} C_\sigma^\top y \le y^\top E_\sigma Q_{q',j} E_\sigma^\top y \quad (16)$$

$$\forall q \in V, \forall i \in [m_q], \forall z \in E_q^{-\top}\mathcal{P}_{q,i}, \quad z^\top C_q Q_{q,i} E_q^\top z + z^\top E_q Q_{q,i} C_q^\top z \le 0. \quad (17)$$

The conditions (16) and (17) amount to verifying the positive semidefiniteness of a quadratic form when restricted to a polyhedral cone. When this cone is the positive orthant, this is called the *copositivity* which is co-NP-complete to decide [25]. However, a sufficient LMI is given in [19, Proposition 2] and a necessary and sufficient condition is given by a hierarchy of Sum-of-Squares programs [28, Chapter 5]. We use the sufficient LMI in the numerical examples of Section 4.

4. Numerical examples

4.1. Illustrative example

This example considers the CHA with one mode of continuous-time dynamics:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

with state constraint $x \in [-1,1]^2$ and input constraint $u \in [-1,1]$ and the following transition from the only mode to itself:

$$x_1^+ = -x_1 + u/8$$
$$x_2^+ = x_2 - u/8$$

with state constraint $x \in [-1,1]^2$ and input constraint $u \in [-1,1]$.

The union of controlled invariant sets is controlled invariant. Moreover, by linearity and convexity of the constraint sets, the convex hull of the unions of controlled invariant sets is controlled invariant. Therefore, there exists a *maximal* controlled invariant set, i.e., a controlled invariant set in which all controlled invariant sets are included, for any family that is closed under union (resp. convex hull); it is the union (resp. convex hull) of all controlled invariant sets included in $[-1, 1]^2$.

For this simple planar system, the maximal controlled invariant set can be obtained by hand. We represent it in yellow in Figure 3 and Figure 4.

As Proposition 1 requires the input to be unconstrained, it cannot be applied to this system directly. We follow the approach detailed in [20, Section 2.2] to reduce the computation of controlled invariant sets for this system to a system with unconstrained input. In this example, it corresponds to the projection onto the first two dimensions of controlled invariant sets for the following lifted system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= u(t) \end{aligned}$$

with state constraint $x \in [-1, 1]^3$; with a first transition to a temporary mode:

$$\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= x_2 \\ x_3^+ &= \bar{u} \end{aligned}$$

with state constraint $x \in [-1, 1]^3$ and unconstrained input; and a second transition back to the original mode:

$$\begin{aligned} x_1^+ &= -x_1 + x_3/8 \\ x_2^+ &= x_2 - x_3/8 \\ x_3^+ &= \bar{u}. \end{aligned}$$

Note that the input $\bar{u}$ chosen in the first transition is the input that will be used for the reset map and the input $\bar{u}$ chosen for the second transition is the input that will be used for the state x_3 of the continuous-time system.

As shown in Proposition 1, a set is controlled invariant for this system if and only if it is weakly invariant for the algebraic system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \end{aligned}$$

with state constraint $x \in [-1, 1]^3$; with a first transition to a temporary mode:

$$x_1^+ = x_1$$
$$x_2^+ = x_2$$

with state constraint $x \in [-1, 1]^3$ and a second transition back to the original mode:

$$x_1^+ = -x_1 + x_3/8$$
$$x_2^+ = x_2 - x_3/8.$$

We represent the safe set $[-1, 1]^2$ and its polar in green in Figure 3 and Figure 4.

While the *maximal* invariant set is well defined, it is not the case anymore when we restrict the set to belong to the family of ellipsoids, polysets or piecewise semi-ellipsoids for a fixed polyhedral conic partition as these families are not invariant under union nor convex hull. The objective used to determine which invariant set is selected depends on the particular application. For this toy example, the the goal is to determine how well visually a family is able to approximate the maximal controlled invariant set. Therefore, we consider $\mathcal{D}$ defined as the convex hull of $\{(-1+\sqrt{3}, -1+\sqrt{3}), (-1/2, 1), (-1, 3/4), (1-\sqrt{3}, 1-\sqrt{3}), (1/2, -1), (1, -3/4)\}$ and we maximize γ such that $\gamma\mathcal{D}$ is included in the projection of the invariant set onto the first two dimensions. We represent $\gamma\mathcal{D}$ in red in Figure 3 and Figure 4.

For the ellipsoidal template considered in Section 3.1, the optimal solution is shown in Figure 3 as ellipsoids corresponds to polysets of degree 2. The optimal objective value is $\gamma \approx 0.894$.

For the polyset template considered in Section 3.2, the optimal solution are represented in Figure 3. The optimal objective value for degree 4 (resp. 6 and 8) is $\gamma \approx 0.896$. (resp. $\gamma \approx 0.93$ and $\gamma \approx 0.96$).

For the piecewise semi-ellipsoidal template, we consider as polyhedral conic partitions the *face fan* [41, Example 7.2], i.e., the conic hull of each facet, of the polytope with extreme points

$$(\cos(\alpha)\cos(\beta), \sin(\alpha)\cos(\beta), \sin(\beta)) \tag{18}$$

where $\alpha = 0, 2\pi/m_1, 4\pi/m_1, \ldots, 2(m_1-1)\pi/m_1$ and $\beta = -\pi/2, \ldots, -2\pi/(m_2-1), -\pi/(m_2-1), 0, \pi/(m_2-1), 2\pi/(m_2-1), \ldots, \pi/2$.

The optimal objective value for $m = (4, 3)$ (resp. $(8, 5)$, $(16, 7)$) is $\gamma \approx 0.894$ (resp. $\gamma \approx 0.92$, $\gamma \approx 0.94$). The corresponding optimal solution is shown in Figure 4.

4.2. Truck with trailers

In this section, we benchmark the computation time of the set program for the different families considered in this paper. We consider the following example inspired from the cruise control example of [35]. The system is illustrated in Fig. 2 and the result of the benchmark is provided in Table 1.

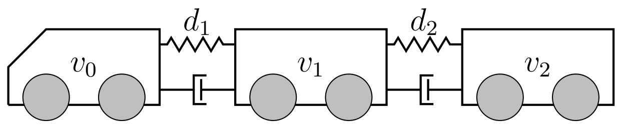


Figure 2: Illustration for Section 4.2 with two trailers.

We consider a truck with M trailers as represented by Figure 2. There is a truck with mass m_0 and speed v_0 followed by multiple trailers, each with mass m. The speed of the ith trailer is denoted v_i. There is a spring with stiffness k_s and elongation d_1 (resp. d_i) and a damper with coefficient k_d There is a mass-spring-damper system between the truck and the first trailer (resp. the $(i-1)$th trailer and the ith trailer). The scalar input u controls the speed v_0 of the truck by creating a force $m_0 u$. The possible modes are $V = \{0, 1, \ldots, M\}$. The dynamical system of mode q has q trailers and is given by the following equations:

The dynamics of the system is given by the following equations:

$$\begin{aligned}
\dot{v}_0 &= \frac{k_d}{m_0}(v_1 - v_0) - \frac{k_s}{m_0} d_1 + u \\
\dot{v}_i &= \frac{k_d}{m}(v_{i-1} - 2v_i + v_{i+1}) + \frac{k_s}{m}(d_i - d_{i+1}) & 1 \le i < q \\
\dot{v}_q &= \frac{k_d}{m}(v_{q-1} - v_q) + \frac{k_s}{m} d_q \\
\dot{d}_i &= v_{i-1} - v_i & 1 \le i \le q.
\end{aligned} \tag{19}$$

The spring elongation should always remain between $-0.5\,\mathrm{m}$ and $0.5\,\mathrm{m}$ and the speeds of the truck and trailers should remain below $36\,\mathrm{m\,s^{-1}}$.

There is a transition from mode q to mode $q+1$ defined by $v_i^+ = v_j$ and $d_i^+ = d_i$ for $i < q$. That is, the values of $v_q^+, d_q^+, v_{q+1}^+, d_{q+1}^+$ are free, as allowed

M	0	1	2	3
Ellipsoid	0.00251	0.00449	0.00784	0.0123
PolySet $d = 4$	0.00867	0.0336	0.315	2.80
PolySet $d = 6$	0.0152	0.268	14.62	435
Piecewise	0.00692	0.0584	1.60	90.1

Table 1: Computation time in seconds for Mosek v8 [4] on codeocean [22] to solve the set program described in Section 4.2 for computing controlled invariant sets for different families and hybrid automata corresponding to different values of M. The piecewise template uses the face fan of a hypercube as polyhedral conic partition.

by the transitions defined in Definition 4. Moreover there is a transition from mode q to mode $q - 1$ defined by $v_i^+ = v_i$ and $d_i^+ = d_i$ for $i < q$.

5. Conclusion

We proved a condition for controlled invariance of convex sets for a hybrid control system based on their support functions. We particularized the condition for three templates: ellipsoids, polysets and piecewise semi-ellipsoids. In the ellipsoidal case, it combines known LMIs for discrete-time and continuous-time systems. In the polyset case, it provides a condition significantly less conservative than [31]. Indeed, our condition is equivalent to invariance by Corollary 2 and, as shown in [21, Section 2], [31] is quite conservative. We leave as future work the convergence guarantee as the degrees of the polynomials defining the polysets increase, such as obtained in [27, 16]. In the piecewise semi-ellipsoidal case, it provides the first convex programming approach for the controlled invariance of hybrid control systems to the best of our knowledge.

As future work, we aim to apply this framework to other families such as the *piecewise polysets* defined in [17]. Moreover, instead of considering a uniform discretization of the hypersphere as in (18), a more adaptive methods could be considered. The sensitivity information provided by the dual solution of the optimization program could for instance determine which pieces of the partition should be refined. Alternatively, the polyhedral conic partition could be chosen as the face fan of a polyhedral outer approximation of the maximal controlled invariant set that would be obtained for instance with a few iterations of backward reachability analysis.

Finally, our definition of hybrid control system (Definition 1) does not support encoding a *guard* that would restrict the possible transitions depending on the current state. Integrating this additional feature to the framework would allow the method to handle *any* hybrid automaton with linear continuous-time dynamic at each mode and linear reset maps.

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Appendix A. Convex geometry

For a convex set $\mathcal{S}$, the *normal cone* is the *polar* of the tangent cone $N_{\mathcal{S}}(x) = T^{\circ}_{\mathcal{S}}(x)$.

Definition 9 (Polar set)**.** For any convex set $\mathcal{S}$ the polar of $\mathcal{S}$, denoted $\mathcal{S}^{\circ}$, is defined as

$$\mathcal{S}^{\circ} = \{\, y \mid \delta^*(y|\mathcal{S}) \le 1 \,\}.$$

The exposed faces (see Definition 7) and normal cones are related by the following property [12, Proposition C.3.1.4].

Proposition 2. Consider a nonempty closed convex set $\mathcal{S}$. For any $x \in \mathcal{S}$ and nonzero vector y, $x \in F_{\mathcal{S}}(y)$ if and only if $y \in N_{\mathcal{S}}(x)$.

Given a set $\mathcal{S}$ and a matrix A, let $A^{-\top}$ denote the preimage:

$$A^{-\top}\mathcal{S} \triangleq \{\, x \mid A^{\top}x \in \mathcal{S} \,\}. \tag{A.1}$$

Proposition 3 ([34, Corollary 16.3.2])**.**
For any convex set $\mathcal{S}$ and linear map A,

$$(A\mathcal{S})^{\circ} = A^{-\top}\mathcal{S}^{\circ}.$$

where $\mathcal{S}^{\circ}$ denotes the polar of the set $\mathcal{S}$.

Proposition 4 ([33, Corollary 11.24(c)] or [34, Corollary 16.3.1])**.** Given a matrix $A \in \mathbb{R}^{n_1 \times n_2}$ and a nonempty closed convex set $\mathcal{S} \subseteq \mathbb{R}^{n_2}$, the following holds for all $y \in \mathbb{R}^{n_1}$:

$$\delta^*(y|A\mathcal{S}) = \delta^*(A^{\top}y|\mathcal{S}). \tag{A.2}$$

Proposition 5 ([34, Corollary 13.1.1])**.** Consider two nonempty closed convex subsets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^n$. The inclusion $\mathcal{S}_1 \subseteq \mathcal{S}_2$ is equivalent to the inequality $\delta^*(x|\mathcal{S}_1) \le \delta^*(x|\mathcal{S}_2)$ for all $x \in \mathbb{R}^n$.

When the support function is differentiable at a given point, $F_{\mathcal{S}}$ is a singleton and may be directly obtained using the following result:

Proposition 6 ([34, Corollary 25.1.2])**.**
Given a nonempty closed convex set $\mathcal{S}$, if $\delta^*(y|\mathcal{S})$ is differentiable at y then $F_{\mathcal{S}}(y) = \{\nabla\delta^*(y|\mathcal{S})\}$.

In fact, for nonempty compact convex sets, the differentiability at y is even a necessary and sufficient conditions for the uniqueness of $F_{\mathcal{S}}(y)$ [36, Corollary 1.7.3].

Appendix B. Sum-of-Squares programming

This section briefly describes Sum-of-Squares programming; see [8] for more details.

Deciding whether a multivariate polynomial of degree $2d \geq 4$ is nonnegative is known to be co-NP-hard. However a sufficient condition for a polynomial to be nonnegative is easy to check. We say that a polynomial is a *Sum-of-Squares* (SOS) if there exist polynomials $q_1, \dots, q_r$ such that

$$p(x) = \sum_{k=1}^{r} q_k^2(x).$$

If a polynomial is SOS, then it is obviously nonnegative.

It is well known that if $p(x)$ is an homogeneous polynomial of degree $2d$ then each $q_k(x)$ must be an homogeneous polynomial of degree d; this can be shown easily using the Newton polytope of $p(x)$ and [32, Theorem 1]. We can check whether a polynomial is SOS using semidefinite programming thanks to the following theorem.

Theorem 4 ([10, 26, 28, 29, 37])**.** A homogeneous multivariate polynomial $p(x)$ of degree $2d$ is a sum of squares if and only if

$$p(x) = b^\top Q b$$

where Q is a symmetric positive semidefinite matrix and b is a basis of the space homogeneous polynomials of degree d.

Appendix C. Set programming

In this section, we give a brief introduction to set programming; see [17] for more details. A generic set program is defined as follows:

$$\begin{aligned} \min_{\mathcal{S}_1 \subseteq \mathbb{R}^{n_1}, \ldots, \mathcal{S}_N \subseteq \mathbb{R}^{n_N}} \quad & f(\mathcal{S}_1, \ldots, \mathcal{S}_N) \\ \text{subject to:} \quad & g_i(\mathcal{S}_1, \ldots, \mathcal{S}_N) \subseteq h_i(\mathcal{S}_1, \ldots, \mathcal{S}_N), \quad i = 1, 2, \ldots, M. \end{aligned} \tag{C.1}$$

where $\mathcal{S}_i \subseteq \mathbb{R}^{n_i}$ is a set decision variable, $f(\mathcal{S}_1, \ldots, \mathcal{S}_N)$ is the objective function and the inclusion constraints are given by $g_i(\mathcal{S}_1, \ldots, \mathcal{S}_N) \subseteq h_i(\mathcal{S}_1, \ldots, \mathcal{S}_N)$ for $i = 1, \ldots, M$. Note that this form also encapsulates membership constraints $x \in \mathcal{S}$ as it can be encoded as an inclusion constraint $\{x\} \subseteq \mathcal{S}$.

We use the following approach to reformulate set programs as Sum-of-Squares programs:

1. First, given the properties of gauge and support functions and the program constraints, we determine whether the set variables should be represented with the gauge or support function.
2. Second, we consider the different templates and analyze the program obtained by formulating the program in terms of its gauge or support function, depending on what was determined in the previous step.

The advantage of this approach is that the first step is independent of the actual template and hence it gives a generic geometric approach to the computation of sets that are solutions to the set program instead of the algebraic template-dependent approaches.

The reformulation as Sum-of-Squares program is done in the second step and, interestingly, the interdependence of the different constraints and the objective only appears in the choice of the representation, which is already carried out in a template-independent fashion in the first step. Therefore, the second-step can be done independently for the objective function and for each constraint; similary to the bridge mechanism in MathOptInterface [18]. For this reason, the reformulation of the constraints and the objective functions can be studied in isolation. Moreover, since each constraint can be reformulated independently, implementing only a few constraint reformulation enables the reformulation of programs made of any of their combinations, as long as they agree on the choice of reformulation between the gauge and support function for the set variables involved in the constraints.

The package SetProg [17] extends JuMP [11] to provide an algebraic modeling language for encoding set programs in the form Eq. (C.1). They are then automatically reformulated into Sum-of-Squares programs which are then reformulated into a semidefinite program by SumOfSquares [39]. The resulting program can be solved by any solver implementing MathOptInterface [18].

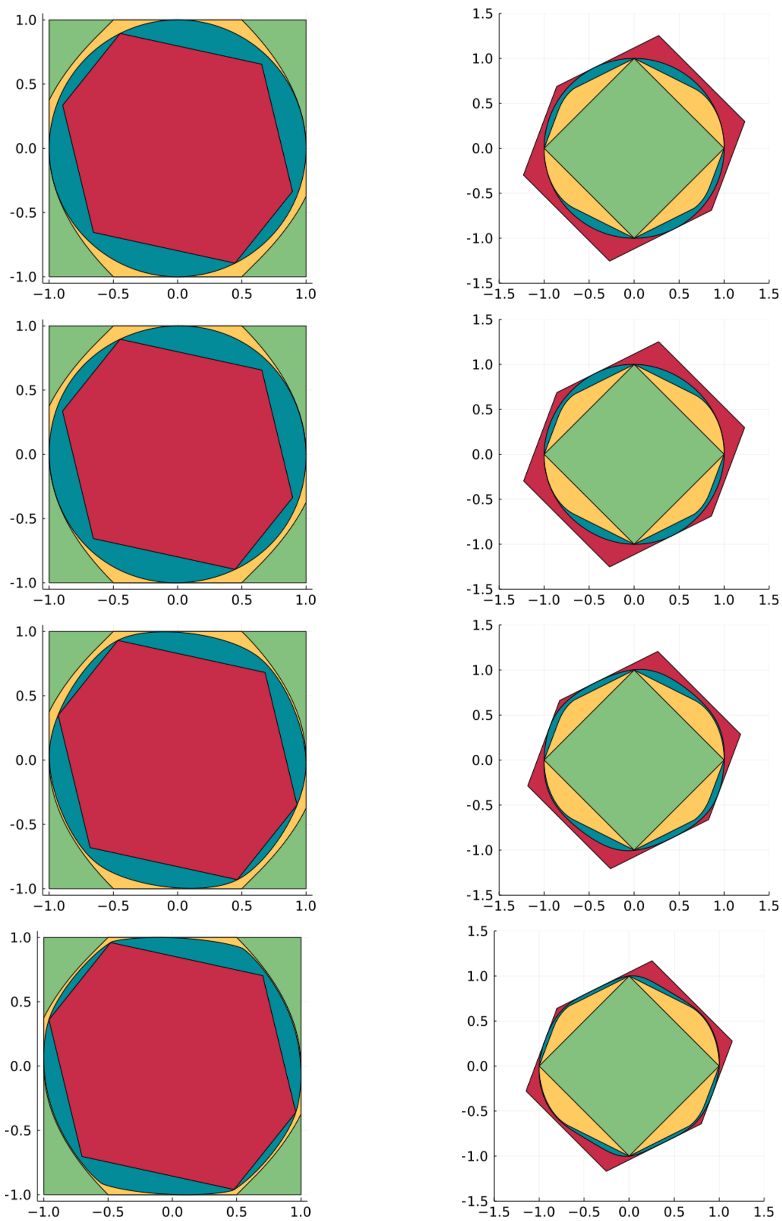

Figure 3: In blue are the solution for polysets of different degrees. The degrees from top to bottom are respectively 2, 4, 6 and 8. The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set and the red set is $\gamma\mathcal{D}$. The sets are represented in the primal space in left figures and in polar space in the right figures. The axis in primal space are x_1 (horizontal) and x_2 (vertical). In the dual space, the axis correspond to the coefficients a_1, a_2 of the halfspaces $\{\,(a_1, a_2) \mid a_1x_1 + a_2x_2 \le 1\,\}$ that contain the set.

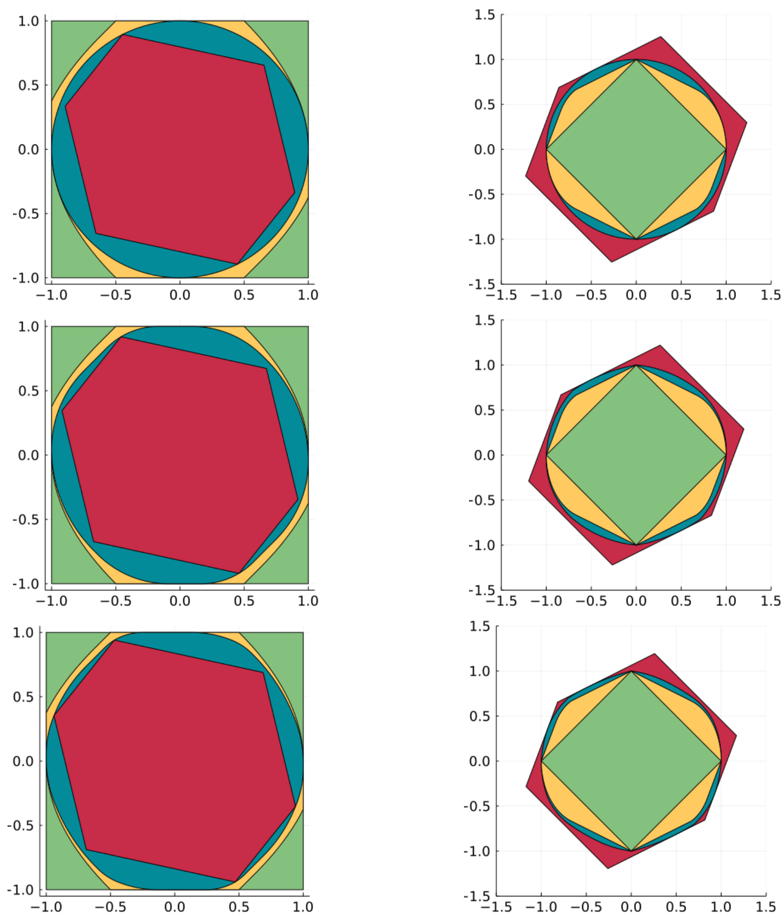

Figure 4: In blue are the solution for piecewise semi-ellipsoids for two different polyhedral conic partitions. The partitions from top to bottom are as described in (18) with $m = (4, 3)$ (resp. $(8, 5)$, $(16, 7)$). The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set and the red set is $\gamma\mathcal{D}$. The sets are represented in the primal space in left figures and in polar space in the right figures. The axis are the same as Fig. 3.