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THE EIGENSPLITTING OF THE FIBER OF THE CYCLOTOMIC TRACE FOR THE SPHERE SPECTRUM

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ABSTRACT. Let $p \in \mathbb{Z}$ be an odd prime. We show that the fiber sequence for the cyclotomic trace of the sphere spectrum $\mathbb{S}$ admits an "eigensplitting" that generalizes known splittings on K-theory and TC. We identify the summands in the fiber as the covers of $\mathbb{Z}_p$-Anderson duals of summands in the $K(1)$-localized algebraic K-theory of $\mathbb{Z}$. Analogous results hold for the ring $\mathbb{Z}$ where we prove that the $K(1)$-localized fiber sequence is self-dual for $\mathbb{Z}_p$-Anderson duality, with the duality permuting the summands by $i \mapsto p - i$ (indexed mod $p-1$). We explain an intrinsic characterization of the summand we call Z in the splitting $TC(\mathbb{Z})^\wedge_p \simeq j \vee \Sigma j' \vee Z$ in terms of units in the p-cyclotomic tower of $\mathbb{Q}_p$.

1. INTRODUCTION

The algebraic K-theory of the sphere spectrum, $K(\mathbb{S})$, is an object of basic and fundamental interest, relating geometric topology and arithmetic. Celebrated work of Waldhausen establishes a comparison between $K(\mathbb{S})$ and a stable space of h-cobordisms for the disk D^n. On the other hand, $K(\mathbb{S})$ is intimately related to $K(\mathbb{Z})$, the algebraic K-theory of the integers, which encodes arithmetic invariants (e.g., Bernoulli numerators and denominators). For instance, the natural map $K(\mathbb{S}) \to K(\mathbb{Z})$ is a rational equivalence, and the latter is understood rationally by old work of Borel.

At a prime p, our understanding of algebraic K-theory of ring spectra relies on trace methods. Bökstedt, Hsiang, and Madsen constructed a topological version of negative cyclic homology called topological cyclic homology (TC) and a Chern character $K \to TC$, the cyclotomic trace. Following earlier work of Rognes, in a previous paper we studied the homotopy groups of $K(\mathbb{S})$ in terms of the cyclotomic trace and linearization maps: a basic theorem of Dundas (building on work of Goodwillie and McCarthy) provides a homotopy cartesian square

$$\begin{array}{ccc} K(\mathbb{S})^\wedge_p & \longrightarrow & K(\mathbb{Z})^\wedge_p \\ \downarrow & & \downarrow \\ TC(\mathbb{S})^\wedge_p & \longrightarrow & TC(\mathbb{Z})^\wedge_p, \end{array}$$

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where the maps $K(\mathbb{S}) \to K(\mathbb{Z})$ and $TC(\mathbb{S}) \to TC(\mathbb{Z})$ are the linearization maps induced by the unit map of E_∞ ring spectra $\mathbb{S} \to \mathbb{Z}$.

For the rest of the paper, we restrict to the case of p an odd prime. In [3, 5.3], the authors showed that the fiber square above splits as a wedge of $p-1$ fiber squares of the form:

$$\begin{array}{ccc} \epsilon_i K(\mathbb{S})^\wedge_p & \longrightarrow & \epsilon_i K(\mathbb{Z})^\wedge_p \\ \downarrow & & \downarrow \\ \epsilon_i TC(\mathbb{S})^\wedge_p & \longrightarrow & \epsilon_i TC(\mathbb{Z})^\wedge_p \end{array}$$

for $i = 0, \dots, p-2$ (or better, numbered modulo $(p-1)$). We identify the spectra $\epsilon_i(-)$ more specifically below. These squares are exactly the summands that would result from an eigensplitting of the fiber square for an action of $\mathbb{F}_p^\times$ via the Teichmüller character ω for a conjectural action of p-adically interpolated Adams operations; see [ibid., §5]. We refer to the summands in the ith square above as the "ω^i eigenspectra" even though such Adams operations have not been constructed in this generality. (If they do exist, the ω^i eigenspectrum is the summand where the $\mathbb{F}_p^\times$-action in the stable category is given by the character $\omega^i \colon \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ and the action of $\mathbb{Z}_p^\times$.)

As a formal consequence, the fiber of the cyclotomic trace for $\mathbb{S}$, or equivalently, for $\mathbb{Z}$ also comes with an eigensplitting. In this paper, we identify the eigenspectra summands. In [4], the authors identified the fiber of the cyclotomic trace $\mathrm{Fib}(\tau)$ in K-theoretic terms as the (-3)-connected cover of $\Sigma^{-1} I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}))$, where $I_{\mathbb{Z}_p}$ denotes the $\mathbb{Z}_p$-Anderson dual. Taking the idea of eigenspectra seriously, the natural conjecture is that the ω^i eigenspectra of $\mathrm{Fib}(\tau)$ should be the corresponding eigenspectra of this Anderson dual. In the spirit of the identification in [ibid.], the ω^i eigenspectrum of $\mathrm{Fib}(\tau)$ should correspond to the $\mathbb{Z}_p$-Anderson dual of the ω^{p-i} eigenspectrum of $L_{K(1)}K(\mathbb{Z})$. Our main theorem establishes this conjecture.

Theorem 1.1. *There is a canonical weak equivalence between the fiber of the map $\epsilon_i\tau \colon \epsilon_i K(\mathbb{S})^\wedge_p \to \epsilon_i TC(\mathbb{S})^\wedge_p$ and the (-3)-connected cover of $\Sigma^{-1} I_{\mathbb{Z}_p} L_{K(1)}(\epsilon_{p-i}K(\mathbb{Z})^\wedge_p)$.*

We also describe how the duality map of [4] interacts with the fiber sequence; we discuss these additional results in Section 5, after reviewing terminology and notation.

Theorem 1.1 is consistent with an expansive picture of the behavior of the conjectural p-adically interpolated Adams operations on $K(\mathbb{S})$. In particular, it is natural to conjecture compatibility with the (known) p-adic Adams operations on $TC(\mathbb{S})$ as well as multiplicative properties. Given such operations on the ∞-category level, Theorem 1.1 would follow. However, while the existence of such Adams operations on the stable category level is enough to obtain a splitting on $\mathrm{Fib}(\tau)$, it would not be enough to deduce Theorem 1.1 without additional arguments like the ones below.

While this paper obviously builds on the authors' previous work [2–4], we have tried to make it as self-contained as possible, with specific citations to any facts needed from those papers.

Conventions. We use the term "stable category" to refer to the homotopy category of spectra with its structure as a tensor triangulated category. The symbol $\wedge$ denotes the smash product in the stable category.

Some of the statements and results below involve precise accounting for signs. For this we use the following conventions: suspension is $(-) \wedge S^1$ and cone is $(-) \wedge I$, where in the latter case we use 1 as the basepoint. Cofiber sequences are sequences isomorphic (in the stable category) to Puppe sequences formed in the usual way using this suspension and cone. A cofiber sequence leads to a long exact sequence of homotopy groups; we use the sign convention that for a map $f\colon A \to B$, the connecting map $\pi_n Cf \to \pi_{n-1}A$ in the long exact sequence of homotopy groups is $(-1)^n\sigma^{-1}$ composed with the Puppe sequence map $\pi_n Cf \to \pi_n \Sigma A$, where σ denotes the suspension isomorphism $\pi_{n-1}A \to \pi_n \Sigma A$. (Consideration of the example of standard cells explains the desirability of the sign.)

We form the homotopy fiber $\mathrm{Fib}(f)$ of a map f using the space of paths starting from the basepoint; we then have a canonical map $\Sigma\,\mathrm{Fib}(f) \to Cf$ in the usual way (using the suspension coordinate to follow the path and then follow the cone). We switch between fiber sequences and cofiber sequences at will, using the convention that for the fiber sequence

$$\Omega B \xrightarrow{\delta} \mathrm{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B,$$

the sequence

$$\mathrm{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B \xrightarrow{-\Sigma\delta\circ\epsilon^{-1}} \Sigma\,\mathrm{Fib}(f)$$

is a cofiber sequence where $\epsilon\colon \Sigma\Omega B \to B$ is the counit of the Σ, Ω adjunction. For the long exact sequence of homotopy groups associated to a fiber sequence, we use the long exact sequence of homotopy groups of the associated cofiber sequence. In terms of the fiber sequence displayed above, the connecting map $\pi_{n+1}B \to \pi_n\,\mathrm{Fib}(f)$ is the composite of the canonical isomorphism $\pi_{n+1}B \cong \pi_n\Omega B$ and the map $(-1)^n\pi_n\delta$.

For a cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and a fixed spectrum X, the sequence

$$\Omega F(A,X) \xrightarrow{-h^*} F(C,X) \xrightarrow{g^*} F(B,X) \xrightarrow{f^*} F(A,X)$$

is a fiber sequence and

$$F(C,X) \xrightarrow{g^*} F(B,X) \xrightarrow{f^*} F(A,X) \xrightarrow{h^*} \Sigma F(C,X)$$

is a cofiber sequence.

2. A review of the eigensplitting of the cyclotomic trace

In this section, we review the splitting constructed in [3] of the cofiber sequence

$$\mathrm{Fib}(\tau) \longrightarrow K(\mathbb{Z})^\wedge_p \xrightarrow{\tau} TC(\mathbb{Z})^\wedge_p \longrightarrow \Sigma\,\mathrm{Fib}(\tau). \tag{2.1}$$

It can be useful to express this in purely K-theoretic terms, using the Hesselholt-Madsen results that $TC(\mathbb{Z})^\wedge_p \to TC(\mathbb{Z}_p)^\wedge_p$ is a weak equivalence [11, Add. 6.2] and that $K(\mathbb{Z}_p)^\wedge_p \to TC(\mathbb{Z}_p)^\wedge_p$ is a connective cover [11, Th. D]. Thus, we can just as well identify the connective cover of $\mathrm{Fib}(\tau)$ as the fiber of the map $K(\mathbb{Z})^\wedge_p \to K(\mathbb{Z}_p)^\wedge_p$ induced by the completion map $\mathbb{Z} \to \mathbb{Z}_p$. As we will recall below, little information is lost by working in the $K(1)$-local category and hence by studying the cofiber sequence

$$\mathrm{Fib}(\kappa) \longrightarrow L_{K(1)}K(\mathbb{Z}) \xrightarrow{\kappa} L_{K(1)}K(\mathbb{Z}_p) \longrightarrow \Sigma\,\mathrm{Fib}(\kappa). \tag{2.2}$$

We will switch back and forth between discussing the localized sequence (2.2) and non-localized sequence (2.1).

We begin by reviewing notation for some of the basic building blocks of the splitting. Let $KU_p^\wedge$ denote p-completed complex periodic K-theory, and let L denote the p-complete Adams summand:

$$KU_p^\wedge \simeq L \vee \Sigma^2 L \vee \cdots \vee \Sigma^{2p-4} L.$$

Let $J = L_{K(1)}\mathbb{S}$. If we choose an integer l which multiplicatively generates the units of $\mathbb{Z}/p^2$, then J is weakly equivalent to the homotopy fiber of $1 - \psi^l \colon L \to L$ for the Adams operation ψ^l. In deducing p-complete results from $K(1)$-local results, we write $ku_p^\wedge$, ℓ, and j for the connective covers respectively of $KU_p^\wedge$, L, and J.

The main theorem of Dwyer-Mitchell [10] (as reinterpreted in [3, §2]) produces a canonical splitting of $L_{K(1)}K(\mathbb{Z})$ as a certain wedge of $K(1)$-local spectra

$$L_{K(1)}K(\mathbb{Z}) \simeq J \vee Y_0 \vee \cdots \vee Y_{p-2}, \tag{2.3}$$

where Y_i is characterized by the property that $L^*(Y_i)$ is concentrated in degrees congruent to $2i - 1 \bmod 2(p-1)$ with $L^{2i-1}(Y_i)$ defined as an L^0L-module in terms of a certain abelian Galois group. Because $L^{2i-1}(Y_i)$ is a finitely generated L^0L-module of projective dimension 1, Y_i is the homotopy fiber of a map between wedges of copies of $\Sigma^{2i-1}L$ (which on $L^{2i-1}(-)$ give a projective L^0L-resolution of $L^{2i-1}(Y_i)$). From this it follows that π_*Y_i is concentrated in degrees congruent to $2i-1$ and $2i-2 \bmod 2(p-1)$; moreover, it is free (as a $\mathbb{Z}_p$-module) in odd degrees. As π_*J is concentrated in degrees congruent to $-1 \equiv 2p-3 \bmod 2(p-1)$ and degree 0, any particular homotopy group of $L_{K(1)}K(\mathbb{Z})$ involves only at most a single Y_i and possibly J. The following two results from [3] (q.v. (2.8) and the preceding paragraph) simplify certain arguments.

Proposition 2.4. $Y_0 \simeq *$.

Proposition 2.5. *L^1Y_1 is a free L^0L-module of rank 1, and so Y_1 is (non-canonically) weakly equivalent to ΣL.*

We say more about the relationship of Y_1 and ΣL in Remark 7.7 in Section 7.

Let y_i be the 1-connected cover of Y_i for all i; then

$$K(\mathbb{Z})_p^\wedge \simeq j \vee y_0 \vee \cdots \vee y_{p-2}.$$

We have a similar canonical splitting of $L_{K(1)}K(\mathbb{Z}_p)$ that takes the form

$$L_{K(1)}K(\mathbb{Z}_p) \simeq J \vee \Sigma J' \vee Z_0 \vee \cdots \vee Z_{p-2},$$

where Z_i is non-canonically weakly equivalent to $\Sigma^{2i-1}L$ and J' is the $K(1)$-localization of the Moore spectrum $M_{\mathbb{Z}_p^\times}$ for the units of $\mathbb{Z}_p$, $J' := L_{K(1)}M_{\mathbb{Z}_p^\times}$. Alternatively, J' is canonically weakly equivalent to $(J \wedge M_{\mathbb{Z}_p^\times})_p^\wedge$; it is non-canonically weakly equivalent to J since $(M_{\mathbb{Z}_p^\times})_p^\wedge$ is non-canonically equivalent to $\mathbb{S}_p^\wedge$. The spectra Z_i were denoted $\Sigma^{-1}L_{TC}(i)$ in [3] for the non-canonical weak equivalence $Z_i \simeq \Sigma^{-1}L(i) = \Sigma^{2i-1}L$. The Z_i admit a canonical description in terms of the units of cyclotomic extensions of $\mathbb{Q}_p$, which we review in Section 7.

Let j' be the connective cover of J', and for $i \neq 0$, let z_i be the 1-connected cover of Z_i; let z_0 be the (-2)-connected cover of Z_0. Then

$$TC(\mathbb{Z})_p^\wedge \simeq j \vee \Sigma j' \vee z_0 \vee \cdots \vee z_{p-2}.$$

A key result proved in [3, 3.1] is that the cyclotomic trace and the completion map are diagonal on the corresponding pieces.

Theorem 2.6 ([3, 3.1])**.** *In the notation above, the cyclotomic trace* $\tau\colon K(\mathbb{Z})^\wedge_p \to TC(\mathbb{Z})^\wedge_p$ *decomposes as the wedge of the identity map* $j \to j$ *and maps* $y_i \to z_i$ *for* $i = 0, \dots, p-2$*; the completion map* $\kappa\colon L_{K(1)}K(\mathbb{Z}) \to L_{K(1)}K(\mathbb{Z}_p)$ *decomposes as the wedge of the identity map* $J \to J$ *and maps* $Y_i \to Z_i$ *for* $i = 0, \dots, p-2$*. The composite with the projection to the summands* $\Sigma j'$ *and* $\Sigma J'$ *is the trivial map for* τ *and* κ*, respectively.*

It follows that the cofiber sequences of equations (2.1) and (2.2) decompose into wedges of cofiber sequences. To explain this, we need to introduce some notation.

Definition 2.7. Let X_i and x_i denote the homotopy fiber of the maps $Y_i \to Z_i$ and $y_i \to z_i$, respectively.

We have the following analogues of Propositions 2.4 and 2.5 for X_1 and X_0. The first is an immediate consequence of Proposition 2.4; the second is a restatement of [3, 4.4], which asserts (in the notation here) that $Y_1 \to Z_1$ is a weak equivalence.

Proposition 2.8. *The connecting map* $Z_0 \to \Sigma X_0$ *is a weak equivalence; in particular,* X_0 *is (non-canonically) weakly equivalent to* $\Sigma^{-2}L$*.*

Proposition 2.9. $X_1 \simeq *$

As a consequence of Definition 2.7 we have that

$$\mathrm{Fib}(\tau) \simeq j' \vee x_0 \vee \cdots \vee x_{p-2} \qquad \text{and} \qquad \mathrm{Fib}(\kappa) \simeq J' \vee X_0 \vee \cdots \vee X_{p-2}.$$

Because the maps in the definition are only defined in the stable category, the splitting of the fibers do not automatically give a canonical splitting of the fiber sequences. However, looking at the cofiber sequence

$$\Sigma^{-1}Z_i \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i$$

we see that X_i can have L-cohomology only in degrees congruent to $2i-1$ and $2i-2$ mod $2(p-1)$, or equivalently:

Proposition 2.10. *For* $i = 0, \dots, p-2$*,* $[X_i, \Sigma^j L] = 0$ *unless* $j \equiv 2i-1$ *or* $j \equiv 2i-2$ *mod* $2(p-1)$*.*

From this we see that $[X_i, \Sigma^{-1}Z_j] = 0$ for $j \neq i$ and $[x_i, \Sigma^{-1}z_j] = 0$ for $j \neq i$. The latter is clear from Proposition 2.8 in the case $i = 0$ and in the remaining cases follows from the isomorphisms

$$[x_i, \Sigma^{-1}z_j] \cong [x_i, \Sigma^{-1}Z_j] \cong [X_i, \Sigma^{-1}Z_j].$$

The first isomorphism holds since x_i is 0-connected for $i = 1, \dots, p-2$ while $\Sigma^{-1}z_j \to \Sigma^{-1}Z_j$ is a weak equivalence on 0-connected covers for all j. The second isomorphism holds because X_i is the $K(1)$-localization of x_i and $\Sigma^{-1}Z_j$ is $K(1)$-local. We also note that $[J', \Sigma^{-1}Z_j] = 0$ and $[j', \Sigma^{-1}Z_j] = 0$ for all $j \neq 1$ since $\pi_0\Sigma^{-1}Z_j = 0$. We have that $[J', \Sigma^{-1}Z_1] \cong \mathbb{Z}_p$ (non-canonically) and so $[j', \Sigma^{-1}Z_1[0,\infty)] \cong \mathbb{Z}_p$ with the isomorphism induced by π_0:

$$[j', \Sigma^{-1}Z_1[0,\infty)] \longrightarrow \mathrm{Hom}(\pi_0 j', \pi_0\Sigma^{-1}Z_1) = \mathrm{Hom}((\mathbb{Z}_p^\times)^\wedge_p, \mathbb{Z}_p) \cong \mathbb{Z}_p.$$

Since $\Sigma^{-1}z_1$ is the fiber of the map $\Sigma^{-1}Z_1[0,\infty) \to H\mathbb{Z}_p$ that induces the identification $\pi_1 Z_1 = \mathbb{Z}_p$, we see that $[j', \Sigma^{-1}z_1] = 0$. In particular, we have shown that

all of the indeterminacy in the map of fiber sequences is zero (in the non-localized case), and we get the following consequence.

Corollary 2.11. *The cofiber sequence* (2.1) *splits canonically as a wedge of the cofiber sequences*

$$\begin{array}{ccccccc}
* & \longrightarrow & j & \xrightarrow{=} & j & \longrightarrow & * \\
j' & \longrightarrow & * & \longrightarrow & \Sigma j' & \xrightarrow{=} & \Sigma j' \\
x_0 & \longrightarrow & y_0 & \longrightarrow & z_0 & \longrightarrow & \Sigma x_0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
x_{p-2} & \longrightarrow & y_{p-2} & \longrightarrow & z_{p-2} & \longrightarrow & \Sigma x_{p-2}.
\end{array}$$

The cofiber sequence (2.2) *splits canonically as an analogous wedge in terms of J, J', X_i, Y_i, and Z_i.*

3. Review of arithmetic duality in algebraic K-theory

In [4], we identified the fiber of the cyclotomic trace using a spectral lift of global arithmetic duality. Relating this identification to the splittings described above requires some of the details of the global duality map and a closely related local duality map. We give a short and mostly self-contained review of the construction in this section.

In arithmetic, local duality is an isomorphism

$$H^i_{\text{ét}}(k; M) \xrightarrow{\cong} (H^{2-i}_{\text{ét}}(k; M^*(1)))^*,$$

where k is the field of fractions of a complete discrete valuation ring whose residue field is finite (e.g., a finite extension of $\mathbb{Q}_p$), M is a finite Galois module, and $(-)^*$ denotes the Pontryagin dual. A version of this duality holds in algebraic K-theory where it takes the following form. (For a proof, see [4, 1.4].)

Theorem 3.1 (K-theoretic local duality)**.** *Let k be the field of fractions of a complete discrete valuation ring whose residue field is finite. The map*

$$L_{K(1)}K(k) \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq I_{\mathbb{Z}_p}(L_{K(1)}K(k))$$

adjoint to the composite map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

described below is a weak equivalence.

We have stated the theorem in a way that emphasizes the parallel with the algebraic result. The functor $I_{\mathbb{Q}/\mathbb{Z}}(-)$ denotes Brown-Comenetz duality, the spectral analogue of Pontryagin duality; it can be constructed in terms of the Brown-Comenetz dual of the sphere spectrum $I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$ as the derived function spectrum $F(-, I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S})$. The functor $I_{\mathbb{Z}_p}(-)$ denotes $\mathbb{Z}_p$-Anderson duality; it can be constructed as

$$I_{\mathbb{Z}_p}(-) = I_{\mathbb{Q}/\mathbb{Z}}(- \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \cong F(-, F(M_{\mathbb{Q}_p/\mathbb{Z}_p}, I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S})) \cong F(-, I_{\mathbb{Z}_p}\mathbb{S}).$$

The duality map in Theorem 3.1 is constructed as follows. The E_∞ multiplication on $K(k)$ induces a map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}$$

and we have a canonical map

$$L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S} \tag{3.2}$$

essentially induced by the Hasse invariant (see [4, (1.2)] for details).

In terms of Anderson duality, the weak equivalence

$$L_{K(1)}K(k) \xrightarrow{\simeq} I_{\mathbb{Z}_p}(L_{K(1)}K(k))$$

in Theorem 3.1 is adjoint to the map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \longrightarrow I_{\mathbb{Z}_p}\mathbb{S}$$

induced by the multiplication and the map

$$v_k \colon L_{K(1)}K(k) \longrightarrow I_{\mathbb{Z}_p}\mathbb{S}. \tag{3.3}$$

adjoint to (3.2).

The local duality theorem relates to our work in this paper when we consider the case $k = \mathbb{Q}_p$. In this case, Quillen's localization theorem and Quillen's calculation of the K-theory of $\mathbb{F}_p$ together say that the fiber of the map $K(\mathbb{Z}_p)^\wedge_p \to K(\mathbb{Q}_p)^\wedge_p$ is $H\mathbb{Z}_p$ and it follows that the map $L_{K(1)}K(\mathbb{Z}_p) \to L_{K(1)}L(\mathbb{Q}_p)$ is a weak equivalence. We then have the following corollary.

Corollary 3.4 (Local duality for $\mathbb{Z}_p$)**.** *Let $v_{\mathbb{Z}_p} \colon L_{K(1)}K(\mathbb{Z}_p) \to I_{\mathbb{Z}_p}\mathbb{S}$ be the composite of the map $L_{K(1)}K(\mathbb{Z}_p) \xrightarrow{\simeq} L_{K(1)}K(\mathbb{Q}_p)$ and the map $v_{\mathbb{Q}_p} \colon L_{K(1)}K(\mathbb{Q}_p) \to I_{\mathbb{Z}_p}\mathbb{S}$ of* (3.3)*. The map $L_{K(1)}K(\mathbb{Z}_p) \to I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}_p))$ adjoint to the composite of multiplication and $v_{\mathbb{Z}_p}$*

$$L_{K(1)}K(\mathbb{Z}_p) \wedge L_{K(1)}K(\mathbb{Z}_p) \longrightarrow L_{K(1)}K(\mathbb{Z}_p) \xrightarrow{v_{\mathbb{Z}_p}} I_{\mathbb{Z}_p}\mathbb{S}$$

is a weak equivalence.

The K-theoretic analogue of global duality identifies $\mathrm{Fib}(\kappa)$ in (2.2) in terms of the $\mathbb{Z}_p$-Anderson dual of $L_{K(1)}K(\mathbb{Z})$. Rather than stating it in the full generality proved in [4], we state it just in this case. In [4, (1.7)], we construct a map

$$u_{\mathbb{Q}} \colon \mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}, \tag{3.5}$$

from the Albert-Brauer-Hasse-Noether sequence for $\mathbb{Q}$. It is compatible with the map $v_{\mathbb{Q}_p}$ above in the sense that $v_{\mathbb{Q}_p}$ is the composite

$$L_{K(1)}K(\mathbb{Q}_p) \simeq L_{K(1)}K(\mathbb{Z}_p) \longrightarrow \Sigma\,\mathrm{Fib}(\kappa) \xrightarrow{\Sigma u_{\mathbb{Q}}} \Sigma\Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S} \cong I_{\mathbb{Z}_p}\mathbb{S},$$

where $L_{K(1)}K(\mathbb{Z}_p) \to \Sigma\,\mathrm{Fib}(\kappa)$ is the connecting map in the fiber sequence (2.2). The map (3.5) induces the following K-theoretic global duality theorem.

Theorem 3.6 (K-Theoretic Tate-Poitou duality for $\mathbb{Z}$)**.** *The map*

$$\mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}L_{K(1)}K(\mathbb{Z})$$

adjoint to the map

$$L_{K(1)}K(\mathbb{Z}) \wedge \mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$$

induced by the $L_{K(1)}K(\mathbb{Z})$-module structure map $L_{K(1)}K(\mathbb{Z}) \wedge \mathrm{Fib}(\kappa) \to \mathrm{Fib}(\kappa)$ and the map $u_{\mathbb{Q}} \colon \mathrm{Fib}(\kappa) \to \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$ is a weak equivalence.

4. The eigensplitting of the fiber of the cyclotomic trace

Combining Theorem 3.6 with the canonical wedge decomposition of $K(\mathbb{Z})$ described above in equation (2.3), we obtain the following decomposition of $\mathrm{Fib}(\kappa)$.

$$\begin{aligned}\mathrm{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) &\simeq \Sigma^{-1} I_{\mathbb{Z}_p}(J \vee Y_0 \vee \cdots \vee Y_{p-2}) \\ &\simeq \Sigma^{-1} I_{\mathbb{Z}_p} J \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_0 \vee \cdots \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-2}.\end{aligned}$$

Our goal in this section is to identify this wedge decomposition with (a permutation of) the wedge decomposition

$$\mathrm{Fib}(\kappa) \simeq J' \vee X_0 \vee \cdots \vee X_{p-2}$$

constructed above in Corollary 2.11. This is accomplished in Theorem 4.1 together with an observation on the J' summand stated in Theorem 4.3 below.

Theorem 4.1. *The canonical isomorphism in the stable category*

$$\mathrm{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} J \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_0 \vee \cdots \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-2}$$

identifies J' as $\Sigma^{-1} I_{\mathbb{Z}_p} J$ and X_i as $\Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-i}$ (for $i \neq 0, 1$) or $\Sigma^{-1} I_{\mathbb{Z}_p} Y_{1-i}$ (for $i = 0, 1$). Moreover:

(i) $[X_i, \Sigma^{-1} I_{\mathbb{Z}_p} Y_j] = 0$ *unless* $i + j \equiv 1$ *mod* $(p-1)$.
(ii) $[X_i, \Sigma^{-1} I_{\mathbb{Z}_p} J] = 0$ *for all* i.
(iii) $[J', \Sigma^{-1} I_{\mathbb{Z}_p} Y_j] = 0$ *for all* j.

In later formulas we will simply write $X_i \simeq \Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-i}$ and understand the indexing to be mod $(p-1)$.

Proof. To simplify notation, we write D for $\Sigma^{-1} I_{\mathbb{Z}_p}$ inside this proof. The multiplication $L \wedge L \to L$ together with the canonical identification of $\pi_0 L$ as $\mathbb{Z}_p$ induces a map $L \to I_{\mathbb{Z}_p} L$ that is easily seen to be an isomorphism in the stable category, q.v. [13, 2.6] (this is essentially due to Anderson [1]). This gives us a canonical identification of DL as $\Sigma^{-1} L$, which is the main tool we use.

All statements follow from verification of (i), (ii), and (iii). As discussed above, Y_j fits in a cofiber sequence of the form

$$\bigvee \Sigma^{2j-2} L \longrightarrow Y_j \longrightarrow \bigvee \Sigma^{2j-1} L \longrightarrow \bigvee \Sigma^{2j-1} L$$

(for some finite wedges of copies of $\Sigma^n L$); it follows that DY_j fits into a cofiber sequence of the form

$$\bigvee \Sigma^{-2j} L \longrightarrow \bigvee \Sigma^{-2j} L \longrightarrow DY_j \longrightarrow \bigvee \Sigma^{-2j+1} L.$$

Applying Proposition 2.10, we see that $[X_i, DY_j] = 0$ unless $-2j \equiv 2i - 2 \bmod 2(p-1)$, or equivalently $i + j \equiv 1 \bmod (p-1)$. This proves (i).

Writing J as the fiber of a self-map of L, DJ fits then into a cofiber sequence of the form

$$\Sigma^{-1} L \longrightarrow DJ \longrightarrow L \longrightarrow L$$

and again applying Proposition 2.10, it follows that $[X_i, DJ] = 0$ unless $2i - 2 \equiv 0$ or $2i - 1 \equiv -1 \bmod 2(p-1)$. In the first case, $X_1 \simeq *$ by Proposition 2.9. In the second case $i = 0$; by Proposition 2.8, X_0 is non-canonically weakly equivalent to $\Sigma^{-2} L$, and $[\Sigma^{-2} L, DJ] = 0$. This proves (ii).

Finally, to prove (iii), we note that DY_j is $K(1)$-local. Since J' is non-canonically weakly equivalent to $J \simeq L_{K(1)} \mathbb{S}$, to see that $[J', DY_j] = 0$ it suffices to see that

$\pi_0 DY_j = 0$. Since $\pi_* Y_j$ is concentrated in degrees congruent to $2j-1$ and $2j-2$ mod $2(p-1)$, we have that $\pi_0 DY_j$ can only possibly be non-zero for $j = 0$ but by Proposition 2.4, $Y_0 \simeq *$. □

While we have defined the X_i solely in terms of the fiber sequence, we have defined J' intrinsically, and so the equivalence of J' with $\Sigma^{-1} I_{\mathbb{Z}_p} J$ under the isomorphism in the stable category $\mathrm{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} K(\mathbb{Z})$ constitutes additional information. In fact, we have a canonical weak equivalence

$$J' \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p} J$$

that we call the *standard weak equivalence*, constructed as follows. Since J' is defined as the $K(1)$-localization of the Moore spectrum $M_{\mathbb{Z}_p^\times}$, and $(\mathbb{Z}_p^\times)_p^\wedge$ is a projective $\mathbb{Z}_p$-module, maps in the stable category from J' into $K(1)$-local spectra are in canonical one-to-one correspondence with homomorphisms from $(\mathbb{Z}_p^\times)_p^\wedge$ into π_0. We note that $\Sigma^{-1} I_{\mathbb{Z}_p} J$ is $K(1)$-local, and to calculate $\pi_0 \Sigma^{-1} I_{\mathbb{Z}_p} J$, we use the fundamental short exact sequence for the $\mathbb{Z}_p$-Anderson dual: For any spectrum X, there is a canonical natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{-n-1} X, \mathbb{Z}_p) \longrightarrow \pi_n I_{\mathbb{Z}_p} X \longrightarrow \mathrm{Hom}(\pi_{-n} X, \mathbb{Z}_p) \longrightarrow 0. \tag{4.2}$$

For finitely generated $\mathbb{Z}_p$-modules, $\mathrm{Hom}(-, \mathbb{Z}_p)$ and $\mathrm{Ext}(-; \mathbb{Z}_p)$ coincide with $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$ and $\mathrm{Ext}_{\mathbb{Z}_p}(-; \mathbb{Z}_p)$. In the case of $X = J$, since $\pi_{-2} J = 0$, we then have a canonical identification of $\pi_0 \Sigma^{-1} I_{\mathbb{Z}_p} J$ as $\mathrm{Hom}(\pi_{-1} J, \mathbb{Z}_p)$. The Morava Change of Rings Theorem identifies $\pi_{-1} J$ canonically in terms of continuous group cohomology:

$$\pi_{-1} J \cong H_c^1(\mathbb{Z}_p^\times; \mathbb{Z}_p) \cong \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \mathrm{Hom}((\mathbb{Z}_p^\times)_p^\wedge, \mathbb{Z}_p),$$

q.v. [9, (1.1)], for the continuous action of $\mathbb{Z}_p^\times$ on $\pi_* KU_p^\wedge$ arising from the p-adic interpolation of the Adams operations. This then gives a canonical isomorphism

$$\pi_0 \Sigma^{-1} I_{\mathbb{Z}_p} J \cong \mathrm{Hom}(\mathrm{Hom}((\mathbb{Z}_p^\times)_p^\wedge, \mathbb{Z}_p), \mathbb{Z}_p).$$

Since $(\mathbb{Z}_p^\times)_p^\wedge$ is projective of rank 1, the double dual map is an isomorphism, giving us a canonical isomorphism

$$(\mathbb{Z}_p^\times)_p^\wedge \longrightarrow \pi_0 \Sigma^{-1} I_{\mathbb{Z}_p} J$$

specifying the standard weak equivalence.

On the other hand, we have a canonical map $J' \to \mathrm{Fib}(\kappa)$ arising from the fiber sequence (2.2) and the $\Sigma J'$ summand of $L_{K(1)} K(\mathbb{Z}_p) \simeq L_{K(1)} K(\mathbb{Q}_p)$. In terms of maps from $(\mathbb{Z}_p^\times)_p^\wedge$ into $\pi_0 \mathrm{Fib}(\kappa)$, we can therefore identify this map $J' \to \mathrm{Fib}(\kappa)$ as coming from the canonical identification of the cokernel of

$$\pi_1 L_{K(1)} K(\mathbb{Z}) \longrightarrow \pi_1 L_{K(1)} K(\mathbb{Z}_p)$$

as $(\mathbb{Z}_p^\times)_p^\wedge$ (the p-completion of the cokernel of the map $(\mathbb{Z}[1/p])^\times \to \mathbb{Q}_p^\times$). Theorem 4.3 compares the two maps.

Theorem 4.3. *The composite map $J' \to \mathrm{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} K(\mathbb{Z}) \to \Sigma^{-1} I_{\mathbb{Z}_p} J$ is the standard weak equivalence.*

We postpone the proof to Section 6.

5. Self-duality of the fiber sequence of the cyclotomic trace

In this section, we extend the analysis from the previous section by showing that the fiber sequence defining $\mathrm{Fib}(\kappa)$ is self-dual. This requires the compatibility of K-theoretic local and global duality discussed in Section 3.

Theorem 5.1. *The following diagram commutes up to the indicated sign*

$$\begin{array}{ccccccc}
\mathrm{Fib}(\kappa) & \xrightarrow{\rho} & L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\kappa} & L_{K(1)}K(\mathbb{Z}_p) & \xrightarrow{\partial} & \Sigma\,\mathrm{Fib}(\kappa) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & (-1) & \simeq\downarrow \\
\Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) & \xrightarrow[\Sigma^{-1}I_{\mathbb{Z}_p}\rho]{} & \Sigma^{-1}I_{\mathbb{Z}_p}(\mathrm{Fib}(\kappa)) & \xrightarrow[I_{\mathbb{Z}_p}\partial]{} & I_{\mathbb{Z}_p}L_{K(1)}K(\mathbb{Z}_p) & \xrightarrow[I_{\mathbb{Z}_p}\kappa]{} & I_{\mathbb{Z}_p}L_{K(1)}K(\mathbb{Z})
\end{array}$$

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining $\mathrm{Fib}(\kappa)$*, the bottom sequence is the* $\mathbb{Z}_p$*-Anderson dual of its rotation, and the vertical maps are induced by the* K*-theoretic Tate-Poitou duality theorem for* $\mathbb{Z}$ *(Theorem* 3.6*) and the* K*-theoretic local duality theorem for* $\mathbb{Z}_p$ *(Theorem* 3.4*).*

Proof. The assertion is that $\Sigma\rho$ is $\mathbb{Z}_p$-Anderson dual to ρ and κ is $\mathbb{Z}_p$-Anderson dual to ∂. Given pairings

$$\epsilon_i\colon A_i \wedge B_i \longrightarrow I_{\mathbb{Z}_p}\mathbb{S}$$

whose adjoints $\eta_i\colon B_i \to F(A_i, I_{\mathbb{Z}_p}\mathbb{S})$ are weak equivalences, then for maps $f\colon A_1 \to A_2$ and $g\colon B_2 \to B_1$, $\eta_1 \circ g \circ \eta_2^{-1}$ is $\mathbb{Z}_p$-Anderson dual to f exactly when the diagram

$$\begin{array}{ccc}
A_1 \wedge B_2 & \xrightarrow{\mathrm{id}\wedge g} & A_1 \wedge B_1 \\
{\scriptstyle f\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \epsilon_1} \\
A_2 \wedge B_2 & \xrightarrow[\epsilon_2]{} & I_{\mathbb{Z}_p}\mathbb{S}
\end{array}$$

commutes. In this case, when the weak equivalences η_1, η_2 are fixed and understood, we say that g is $\mathbb{Z}_p$-Anderson dual to f. By construction, the following diagram commutes

$$\begin{array}{ccccc}
L_{K(1)}K(\mathbb{Z}) \wedge L_{K(1)}K(\mathbb{Z}_p) & \xrightarrow{\mathrm{id}\wedge\partial} & L_{K(1)}K(\mathbb{Z}) \wedge \Sigma\,\mathrm{Fib}(\kappa) & \xrightarrow{\xi} & \Sigma\,\mathrm{Fib}(\kappa) \\
{\scriptstyle \kappa\wedge\mathrm{id}}\downarrow & & & \nearrow_{\partial} & \downarrow{\scriptstyle \Sigma u_{\mathbb{Z}}} \\
L_{K(1)}K(\mathbb{Z}_p) \wedge L_{K(1)}K(\mathbb{Z}_p) & \xrightarrow[\mu]{} & L_{K(1)}K(\mathbb{Z}_p) & \xrightarrow[v_{\mathbb{Z}_p}]{} & I_{\mathbb{Z}_p}\mathbb{S}
\end{array}$$

where μ denotes the multiplication, ξ denotes the $L_{K(1)}K(\mathbb{Z})$-module action map, and u and v are the maps in the global and local duality theorems, respectively. This gives the duality between ∂ and κ. To compare ρ and $\Sigma\rho$, consider the diagram

$$\begin{array}{ccc}
\mathrm{Fib}(\kappa) \wedge \Sigma\,\mathrm{Fib}(\kappa) & \xrightarrow{\mathrm{id}\wedge\Sigma\rho} & \mathrm{Fib}(\kappa) \wedge \Sigma L_{K(1)}K(\mathbb{Z}) \\
{\scriptstyle \rho\wedge\mathrm{id}}\downarrow & & \downarrow \\
L_{K(1)}K(\mathbb{Z}) \wedge \Sigma\,\mathrm{Fib}(\kappa) & \longrightarrow & I_{\mathbb{Z}_p}\mathbb{S}
\end{array}$$

where the unlabeled maps are induced by the duality pairing. The down-then-right composite is $u_{\mathbb{Z}}$ composed with the suspension of the non-unital multiplication on $\mathrm{Fib}(\kappa)$, whereas the right-then-down composite is $u_{\mathbb{Z}}$ composed with the suspension

of the opposite of the non-unital multiplication on $\mathrm{Fib}(\kappa)$. Since the non-unital multiplication on $\mathrm{Fib}(\kappa)$ is E_∞ and in particular commutative in the stable category, the diagram commutes. □

As an immediate consequence, we get duality between the cofiber sequences in Corollary 2.11. Theorem 4.3 indicates the relationship between the j and j' sequences. The relationship between the remaining ones is summarized in the following Corollary.

Corollary 5.2. *For each i in $\mathbb{Z}/(p-1)$, the following diagram commutes up to the indicated sign*

$$\begin{array}{ccccccc} X_i & \xrightarrow{\rho_i} & Y_i & \xrightarrow{\kappa_i} & Z_i & \xrightarrow{\partial_i} & \Sigma X_i \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & (-1) & \simeq\downarrow \\ \Sigma^{-1}I_{\mathbb{Z}_p}(Y_{p-i}) & \xrightarrow[\Sigma^{-1}I_{\mathbb{Z}_p}\rho_{p-i}]{} & \Sigma^{-1}I_{\mathbb{Z}_p}(X_{p-i}) & \xrightarrow[I_{\mathbb{Z}_p}\partial_{p-i}]{} & I_{\mathbb{Z}_p}(Z_{p-i}) & \xrightarrow[-I_{\mathbb{Z}_p}\kappa_{p-i}]{} & I_{\mathbb{Z}_p}(Y_{p-i}) \end{array}$$

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining X_i, the bottom sequence is the $\mathbb{Z}_p$-Anderson dual of its rotation, and the vertical maps are induced by the maps

$$\begin{aligned} X_i &\longrightarrow \mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(Y_{p-i}) \\ Y_i &\longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(\mathrm{Fib}(\kappa)) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(X_{p-i}) \\ Z_i &\longrightarrow L_{K(1)}K(\mathbb{Z}_p) \longrightarrow I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}_p)) \longrightarrow I_{\mathbb{Z}_p}(Z_{p-i}) \end{aligned}$$

arising from local and global K-theoretic duality.

In the case of primes that satisfy the Kummer-Vandiver condition, we can be a bit more specific. A prime p *satisfies the Kummer-Vandiver condition* when p does not divide the order of the class group of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ for $\zeta_p = e^{2\pi i/p}$. In this case, Dwyer-Mitchell [10, 12.2] identifies the homotopy type of the spectra Y_i in terms of the Kubota-Leopoldt p-adic L-function: given any power series f in the p-adic integers, there is a unique self-map ϕ_f on L in the stable category such that on $\pi_{2(p-1)n}$, ϕ_f is multiplication by $f((1+p)^n - 1)$ (cf. [16, 2.4]). A celebrated theorem of Iwasawa [12] implies in this context that for $i = 2, 4, \ldots, p-3$, there exists a self-map of $\Sigma^{2i-1}L$ which on π_{2n+1} is multiplication by the value of the Kubota-Leopoldt p-adic L-function $L_p(-n, \omega^i)$. The spectrum Y_i is non-canonically weakly equivalent to the homotopy fiber of this map. The $\mathbb{Z}_p$-Anderson self-duality of L then identifies $I_{\mathbb{Z}_p}Y_i$ as (non-canonically) weakly equivalent to the fiber of the self-map of $\Sigma^{2-2i}L$ that on π_{2n} is multiplication by $L_p(n, \omega^i)$. In particular, for $j = 3, 5, \ldots, p-2$, $X_j \simeq \Sigma^{-1}I_{\mathbb{Z}_p}Y_{p-j}$ is then non-canonically weakly equivalent to the homotopy fiber of the self-map of $\Sigma^{2j-1}L$ that on π_{2n-1} is multiplication by $L_p(n, \omega^{1-j})$, or equivalently, on π_{2n+1} is multiplication by $L_p(n+1, \omega^{1-j})$. For i odd and for j even, Y_i and X_j are non-canonically weakly equivalent to $\Sigma^{2i-1}L$ and $\Sigma^{2j}L$, respectively. (Independently of the Kummer-Vandiver condition $Y_0 \simeq *$ and $X_1 \simeq *$ by Propositions 2.4 and 2.9.)

In the case of an odd regular prime, the relevant values of the Kubota-Leopoldt p-adic L-functions are units, and the spectra X_j and x_j are trivial for j odd. This is consistent with Rognes' computation [19, 3.3] of the homotopy fiber of

the cyclotomic trace as (non-canonically) weakly equivalent to $j \vee \Sigma^{-2} ko_p^\wedge$ in this case. More generally, we have the following corollary.

Corollary 5.3. *Let p be an odd prime that satisfies the Kummer-Vandiver condition. The cofiber sequence*

$$\mathrm{Fib}(\tau) \longrightarrow K(\mathbb{Z})_p^\wedge \longrightarrow TC(\mathbb{Z})_p^\wedge \longrightarrow \Sigma\,\mathrm{Fib}(\tau)$$

is (non-canonically) weakly equivalent to the wedge of the cofiber sequences

$$\begin{aligned}
&* \longrightarrow j \xrightarrow{=} j \longrightarrow * \\
&j' \longrightarrow * \longrightarrow \Sigma j' \xrightarrow{=} \Sigma j' \\
&x_i \longrightarrow \Sigma^{2i-1}\ell \xrightarrow{\lambda_i^o} \Sigma^{2i-1}\ell \longrightarrow \Sigma x_i && i = 3, 5, \dots, p-2 \\
&\Sigma^{2i}\ell \longrightarrow y_i \longrightarrow \Sigma^{2i-1}\ell \xrightarrow{\Sigma\lambda_i^e} \Sigma^{2i-1}\ell && i = 2, 4, \dots, p-3 \\
&* \longrightarrow \Sigma^{2p-1}\ell \xrightarrow{=} \Sigma^{2p-1}\ell \longrightarrow * && (i = 1) \\
&\Sigma^{-2}\ell \longrightarrow * \longrightarrow \Sigma^{-1}\ell \xrightarrow{=} \Sigma^{-1}\ell && (i = 0),
\end{aligned}$$

where λ_i^o is the unique self-map of $\Sigma^{2i-1}\ell$ that on π_{2n+1} is multiplication by the value $L_p(n+1, \omega^{p-i})$, λ_i^e is the unique self-map of $\Sigma^{2i-1}\ell$ that on π_{2n+1} is multiplication by the value $L_p(-n, \omega^i)$, and L_p denotes the Kubota-Leopoldt p-adic L-function.

6. Proof of theorem 4.3

As discussed above the statement of Theorem 4.3, maps from J' to $\mathrm{Fib}(\kappa)$ are determined by maps from $(\mathbb{Z}_p^\times)_p^\wedge$ to $\pi_0\,\mathrm{Fib}(\kappa)$; we have two isomorphisms of $(\mathbb{Z}_p^\times)_p^\wedge$ with $\pi_0\,\mathrm{Fib}(\kappa)$ and we need to show that they are the same. It is slightly easier to work with the $\mathbb{Z}_p$-duals instead. We can canonically identify the $\mathbb{Z}_p$-dual of $\pi_0\,\mathrm{Fib}(\kappa)$ as $\pi_{-1}L_{K(1)}K(\mathbb{Z})$: the fundamental short exact sequence (4.2) for $\pi_0\Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z})$ gives an isomorphism

$$\pi_0\Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z}) \cong \mathrm{Hom}(\pi_{-1}L_{K(1)}K(\mathbb{Z}), \mathbb{Z}_p)$$

since $\pi_{-2}L_{K(1)}K(\mathbb{Z}) = 0$ (as $Y_0 \simeq *$). The identification of $\pi_{-1}J$ as $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ above and the canonical map $J \to L_{K(1)}K(\mathbb{Z})$ gives one isomorphism of $\pi_{-1}L_{K(1)}(\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, which we denote as α. The isomorphism of $\pi_0\,\mathrm{Fib}(\tau)$ with $(\mathbb{Z}_p^\times)_p^\wedge$ as the quotient of $\pi_1 L_{K(1)}K(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)_p^\wedge$ gives another isomorphism of $\pi_{-1}L_{K(1)}(\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, which we denote as η. We need to prove that $\eta = \alpha$.

We have an intrinsic identification of $\pi_{-1}L_{K(1)}K(\mathbb{Z})$ coming from Thomason's descent spectral sequence [23, 4.1], which in this case canonically identifies

$$\pi_{-1}L_{K(1)}K(\mathbb{Z}) \cong \pi_{-1}L_{K(1)}K(\mathbb{Z}[1/p])$$

as $H^1_{\text{ét}}(\mathrm{Spec}\,\mathbb{Z}[1/p]; \mathbb{Z}_p)$ (continuous étale cohomology). Let $\mathbb{Q}_S$ denote the maximal algebraic extension of $\mathbb{Q}$ that is unramified except over $S = \{p\}$, and let $G_S = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. The abelianization G_S^{ab} of G_S corresponds to the maximal abelian extension of $\mathbb{Q}$ that is unramified except over p, which is $\mathbb{Q}(\mu_{p^\infty})$ (where μ_{p^∞} denotes the group of p^nth roots unity for all n). We have the standard identification of the continuous étale cohomology $H^1_{\text{ét}}(\mathrm{Spec}\,\mathbb{Z}[1/p]; \mathbb{Z}_p)$ as the Galois cohomology $H^1_{\mathrm{Gal}}(\mathbb{Q}_S/\mathbb{Q}; \mathbb{Z}_p)$ [15, II.2.9], which we can identify as the abelian group

of continuous homomorphisms

$$\mathrm{Hom}_c(G_S, \mathbb{Z}_p) \cong \mathrm{Hom}_c(G_S^{\mathrm{ab}}, \mathbb{Z}_p).$$

We have a further isomorphism

$$G_S^{\mathrm{ab}} = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \lim_n \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong \lim_n (\mathbb{Z}/p^n)^\times \cong \mathbb{Z}_p^\times,$$

where the first isomorphism is inverse to the isomorphism $(\mathbb{Z}/p^n)^\times \cong \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ taking an element $u \in (\mathbb{Z}/p^n)^\times$ to the automorphism of $\mathbb{Q}(\mu_{p^n})$ induced by the automorphism $\zeta \mapsto \zeta^u$ on μ_{p^n}. This then constructs an isomorphism

$$\gamma\colon \pi_{-1}L_{K(1)}K(\mathbb{Z}) \longrightarrow \mathrm{Hom}_c(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

First we show $\gamma = \alpha$. Choose a prime number l such that l is a topological generator of $\mathbb{Z}_p^\times$, or equivalently, a generator of $\mathbb{Z}/p^2$, and consider the quotient map $\mathbb{Z}[1/p] \to \mathbb{Z}/l = \mathbb{F}_l$. By celebrated work of Quillen [18], the composite map

$$j \longrightarrow K(\mathbb{Z})^\wedge_p \longrightarrow K(\mathbb{F}_l)^\wedge_p$$

is a weak equivalence and an embedding of $\bar{\mathbb{F}}_l^\times$ in $\mathbb{C}^\times$ induces a weak equivalence $K(\bar{\mathbb{F}}_l)^\wedge_p \to ku^\wedge_p$ with the automorphism Φ on $K(\bar{\mathbb{F}}_l)^\wedge_p$ induced by the Frobenius automorphism of $\bar{\mathbb{F}}_l$ mapping to the Adams operation Ψ^l on $ku^\wedge_p$ (independently of the choice of embedding). We will also write Φ for the corresponding automorphism of $L_{K(1)}K(\bar{\mathbb{F}}_l)$. For any functorial model of $L_{K(1)}K(-)$, the induced map from $L_{K(1)}K(\mathbb{F}_l)$ into the homotopy fixed points of Φ (the homotopy equalizer of Φ and the identity on $L_{K(1)}K(\bar{\mathbb{F}}_l)$) is a weak equivalence. Writing $L_{K(1)}K(\bar{\mathbb{F}}_l)^{h\Phi}$ for the homotopy fixed points of Φ, the map

$$L_{K(1)}K(\mathbb{F}_l) \longrightarrow L_{K(1)}K(\bar{\mathbb{F}}_l)^{h\Phi}$$

is the unique map that takes the unit element of $\pi_0(L_{K(1)}K(\mathbb{F}_l))$ to the unique element of $\pi_0(L_{K(1)}K(\bar{\mathbb{F}}_l)^{h\Phi})$ that maps to the unit element of $\pi_0(L_{K(1)}K(\bar{\mathbb{F}}_l))$. This gives a canonical identification of $\pi_{-1}(L_{K(1)}K(\mathbb{F}_l))$ as $H^1(\langle\Phi\rangle; \mathbb{Z}_p)$, where $\langle\Phi\rangle$ denotes the cyclic group generated by Φ; we have used the canonical isomorphism $\pi_0(L_{K(1)}K(\bar{\mathbb{F}}_l)) \cong \mathbb{Z}_p$ induced by the unit and we note that this isomorphism is consistent with the canonical isomorphism $\pi_0 KU^\wedge_p \cong \mathbb{Z}_p$ under the weak equivalence $L_{K(1)}K(\bar{\mathbb{F}}_l) \to KU^\wedge_p$ (independently of the choice of the embedding $\bar{\mathbb{F}}_l^\times \to \mathbb{C}^\times$). Under the identification of $\pi_{-1}J$ as $H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p)$ above, the composite map

$$J \longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow L_{K(1)}K(\mathbb{F}_l)$$

induces on π_{-1} the map

$$H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p) \longrightarrow H^1(\langle\Phi\rangle; \mathbb{Z}_p)$$

induced by the inclusion of l in $\mathbb{Z}_p^\times$ (the inclusion of Ψ^l in the group of p-adically interpolated Adams operations). This gives us information about α. In terms of the identification of $\pi_{-1}(L_{K(1)}K(\mathbb{F}_l))$ as $H^1_{\text{ét}}(\mathbb{F}_l; \mathbb{Z}_p)$ from Thomason's descent spectral sequence, the map

$$H^1_{\text{ét}}(\mathbb{F}_l; \mathbb{Z}_p) \cong H^1_{\mathrm{Gal}}(\bar{\mathbb{F}}_l/\mathbb{F}_l; \mathbb{Z}_p) \longrightarrow H^1(\langle\Phi\rangle; \mathbb{Z}_p)$$

is induced by the inclusion of the Frobenius in $\mathrm{Gal}(\bar{\mathbb{F}}_l/\mathbb{F}_l)$. By naturality, the composite map

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{p}]; \mathbb{Z}_p) \cong H^1_c(G_S; \mathbb{Z}_p) \longrightarrow H^1_c(\mathrm{Gal}(\bar{\mathbb{F}}_l; \mathbb{Z}_p)) \longrightarrow H^1(\langle\Phi\rangle; \mathbb{Z}_p)$$

is induced by the inclusion of Φ in G_S as the automorphism of (the p-integers in) $\mathbb{Q}_S$ that does the automorphism $\zeta \mapsto \zeta^l$ on μ_{p^∞}. This then shows that $\gamma = \alpha$.

We now compare γ and η. Here it is easiest to work first in terms of $L_{K(1)}K(\mathbb{Q}_p)$. Using the standard identification of $\pi_1 L_{K(1)}K(\mathbb{Q}_p)$ as the p-completion of the units, we have a $\mathbb{Q}_p$-analogue of η using local duality: let

$$\eta_p \colon \pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \longrightarrow \mathrm{Hom}((\mathbb{Q}_p^\times)^\wedge_p, \mathbb{Z}_p) \cong \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p)$$

be the isomorphism derived from the isomorphism $\pi_1 L_{K(1)}K(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^\wedge_p$ by Anderson duality. We then have a commutative diagram

$$\begin{array}{ccccc}
\pi_{-1}L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\cong} & \pi_{-1}L_{K(1)}K(\mathbb{Z}[\tfrac{1}{p}]) & \longrightarrow & \pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \\
\downarrow{\scriptstyle \eta}\ \cong & & & & \cong\ \downarrow{\scriptstyle \eta_p} \\
\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & & \xrightarrow[q^*]{} & & \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p)
\end{array}$$

by the compatibility of local and global duality. (Here q^* is the map induced by the standard quotient isomorphism $\mathbb{Q}_p^\times/\langle p\rangle \cong \mathbb{Z}_p^\times$.) To produce a local analogue of γ, we use the Artin symbol θ in local class field theory [21, §3.1]. The Artin symbol gives an isomorphism between the finite completion of the units of $\mathbb{Q}_p$ and the Galois group of the maximal abelian extension $\bar{\mathbb{Q}}_p^{\mathrm{ab}}$ of $\mathbb{Q}_p$: For $x \in \mathbb{Q}_p^\times$, writing $x = ap^m$ for $a \in \mathbb{Z}_p^\times$, the Artin symbol takes x to the unique element $\theta(x)$ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p^{\mathrm{ab}}/\mathbb{Q}_p)$ that acts on the p^nth roots of unity μ_{p^n} by raising to the $1/a$ power and acts on the maximal unramified extension $(\mathbb{Q}_p)^{\mathrm{nr}}$ of $\mathbb{Q}_p$ by the mth power of a lift of the Frobenius. Using the isomorphism

$$\pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \cong H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p)$$

from Thomason's descent spectral sequence and the canonical isomorphism

$$H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p) \cong H^1_{\mathrm{Gal}}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p; \mathbb{Z}_p) \cong \mathrm{Hom}_c(\mathrm{Gal}(\bar{\mathbb{Q}}_p^{\mathrm{ab}}/\mathbb{Q}_p), \mathbb{Z}_p)$$

(as above), the Artin symbol induces an isomorphism

$$-\gamma_p \colon \pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \longrightarrow \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p).$$

We have implicitly defined an isomorphism γ_p: The formula for the Artin symbol implies that the following diagram commutes

$$\begin{array}{ccccc}
\pi_{-1}L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\cong} & \pi_{-1}L_{K(1)}K(\mathbb{Z}[\tfrac{1}{p}]) & \longrightarrow & \pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \\
\downarrow{\scriptstyle \gamma}\ \cong & & & & \cong\ \downarrow{\scriptstyle \gamma_p} \\
\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & & \xrightarrow[q^*]{} & & \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p) \\
=\ \downarrow & & & & \downarrow\ \cong \\
\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & & \xrightarrow[(\mathrm{id},0)]{} & & \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \times \mathbb{Z}_p
\end{array}$$

(where the bottom right vertical isomorphism is induced by the ap^m decomposition of $\mathbb{Q}_p^\times$ as $\mathbb{Z}_p^\times \times \mathbb{Z}$). In other words, omitting notation for the isomorphism arising from Thomason's descent spectral sequence and the usual isomorphism $H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p) \cong \mathrm{Hom}(\mathrm{Gal}(\bar{\mathbb{Q}}_p^{\mathrm{ab}}/\mathbb{Q}_p), \mathbb{Z}_p)$, γ_p is the $\mathbb{Z}_p$-dual of $-\theta$. The Artin

symbol has a cohomological characterization [21, §2.3, Prop. 1]: for a character $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ and $x \in \mathbb{Q}_p^\times$,

$$\rho(\theta(x)) = -inv(x \cup \rho)$$

(cf. [17, p. 386]) where on the right we interpret x as an element of $H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Z}_p)$ and ρ as an element of $H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Q}/\mathbb{Z})$, while the symbol $\cup$ denotes the cup product on étale cohomology

$$H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^2_{\text{ét}}(\mathbb{Q}_p;\mathbb{Q}/\mathbb{Z}(1)),$$

and inv denotes the map induced by the Hasse invariant (q.v. [22, XIII§3, Prop. 6ff]). Because

$$inv(x \cup y)\colon H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the local duality pairing, restricting to

$$H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Z}/p^n(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p;\mathbb{Z}/p^n) \longrightarrow \mathbb{Z}/p^n,$$

taking the inverse limit, and applying the isomorphism from Thomason's descent spectral sequence gives us the Anderson duality pairing

$$\pi_1 L_{K(1)} K(\mathbb{Q}_p) \otimes \pi_{-1} L_{K(1)} K(\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p.$$

We conclude that $\eta_p = \gamma_p$. Since the map $q^*\colon \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \to \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p)$ is an injection, we conclude that $\eta = \gamma$.

7. The eigenspectra of $TC(\mathbb{Z})^\wedge_p$

As shown in [3, (2.4)], $TC(\mathbb{Z})^\wedge_p$ admits a canonical splitting with summands j, $\Sigma j'$, and the spectra denoted in Section 2 as z_i, for $i = 0, \dots, p-2$. As indicated there, z_i is non-canonically weakly equivalent to $\Sigma^{2i-1}\ell$ (for $i \neq 1$) or $\Sigma^{2p-1}\ell$ (for $i = 1$). The purpose of this section is to give an identification of these summands in intrinsic terms. We work in terms of the $K(1)$-localizations Z_i, and our main result is to explain the perspective that Z_i is $\Sigma^{2i-1}L$ tensored over $\Lambda \cong [L, L]$ with a free Λ-module and to identify that Λ-module intrinsically in number theoretic terms; see Corollary 7.5 for a precise statement. The remainder of the section discusses the problem of finding a generator for this free module. The number theory literature discusses several approaches, which we review. The content of this section is independent of the other sections.

Let

$$Z = Z_0 \vee \cdots \vee Z_{p-2} = L_{K(1)} z_0 \vee \cdots \vee L_{K(1)} z_{p-2}.$$

Since z_i is the 1-connected cover (for $i \neq 0$) or (-2)-connected cover (for $i = 0$) of $Z_i = L_{K(1)} z_i$, to identify z_i, it suffices to identify Z_i. Since Z_i is non-canonically weakly equivalent to $\Sigma^{2i-1}L$ (for L the Adams summand of $KU^\wedge_p$), we have that $[Z_i, Z_j] = 0$ for $i \neq j$. The decomposition of Z into the summands Z_i is therefore unique, and so it suffices to identify Z. It follows from Hesselholt-Madsen [11, Th. D, Add 6.2] that TC of the p-completion map and the cyclotomic trace

$$TC(\mathbb{Z}) \longrightarrow TC(\mathbb{Z}_p) \longleftarrow K(\mathbb{Z}_p)$$

are $K(1)$-equivalences. These maps induce a weak equivalence from Z to a summand of $L_{K(1)}K(\mathbb{Z}_p)$, which Dwyer-Mitchell [10, §13] identifies in terms of units of cyclotomic extensions of $\mathbb{Q}_p$. We now review this identification.

Let $F_n = \mathbb{Q}_p(\mu_{p^{n+1}})$ (with $F_{-1} = \mathbb{Q}_p$), and let $E_n = (F_n^\times/\text{torsion})^\wedge_p$ (where $\mu_{p^{n+1}}$ denotes the p^{n+1} roots of unity in some algebraic closure of $\mathbb{Q}_p$). The norm (in Galois theory) gives maps $E_n \to E_{n-1}$; let E_∞ be the inverse limit. The Galois group $\mathrm{Gal}(F_n/\mathbb{Q}_p)$ is canonically isomorphic to $(\mathbb{Z}/p^{n+1})^\times$ and this makes E_n a p-complete $\mathbb{Z}_p[(\mathbb{Z}/p^{n+1})^\times]$-module. The norm $E_n \to E_{n-1}$ is a $\mathbb{Z}_p[(\mathbb{Z}/p^{n+1})^\times]$-module map and so in the inverse limit E_∞ is a module over the Iwasawa algebra

$$\Lambda' = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] = \lim \mathbb{Z}_p[(\mathbb{Z}/p^{n+1})^\times].$$

As a completed group ring, Λ' comes with an (anti)involution χ that sends the group elements to their inverses; for a Λ'-module M, we denote by M^χ the Λ'-module obtained via this involution. Coincidentally, $\Lambda' \cong [KU_p^\wedge, KU_p^\wedge]$ via the map that takes an element a of $\mathbb{Z}_p^\times$ to the (p-adically interpolated) Adams operation ψ^a (which exists for p-completed topological K-theory), and in particular, we have a canonical action of Λ' on $KU_p^\wedge$ (in the stable category). Dwyer-Mitchell [10, §13] then shows that E_∞ (denoted there as $E'_\infty(\mathrm{red})$) is free of rank 1 as a Λ'-module, observes that

$$[Z, KU_p^\wedge] = (KU_p^\wedge)^0(Z) = 0$$

(cf. [ibid., 6.11]), and constructs a canonical isomorphism

$$[\Sigma^{-1}Z, KU_p^\wedge] = (KU_p^\wedge)^1(Z) \cong \mathrm{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi \tag{7.1}$$

of Λ'-modules (cf. [ibid., 8.10]). For formal reasons, this characterizes Z in the stable category via the Yoneda Lemma (see [ibid., 4.12]).

Theorem 7.2 (Dwyer-Mitchell [10, 9.1])**.** *For any spectrum X, the natural map*

$$\begin{aligned}[X, \Sigma^{-1}Z] \longrightarrow \mathrm{Hom}_{[KU_p^\wedge, KU_p^\wedge]}([\Sigma^{-1}Z, KU_p^\wedge], [X, KU_p^\wedge]) \\ \cong \mathrm{Hom}_{\Lambda'}(\mathrm{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi, [X, KU_p^\wedge]) \cong E_\infty^\chi \otimes_{\Lambda'} [X, KU_p^\wedge]\end{aligned}$$

is an isomorphism.

The last isomorphism follows from the fact that E_∞ is free of finite rank, using the isomorphism

$$\mathrm{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi \cong \mathrm{Hom}_{\Lambda'}(E_\infty^\chi, \Lambda')$$

adjoint to the χ-twisted evaluation map

$$\mathrm{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi \otimes E_\infty^\chi \longrightarrow (\Lambda')^\chi \xrightarrow[\chi^{-1}]{\cong} \Lambda'. \tag{7.3}$$

Plugging $X = KU_p^\wedge$ into Theorem 7.2, we get an isomorphism of Λ'-modules

$$[KU_p^\wedge, \Sigma^{-1}Z] \xrightarrow{\cong} E_\infty^\chi \otimes_{\Lambda'} [KU_p^\wedge, KU_p^\wedge] \cong E_\infty^\chi.$$

Concisely, this isomorphism and the isomorphism of (7.1) identify the χ-twisted evaluation map (7.3) with the composition in the stable category

$$[\Sigma^{-1}Z, KU_p^\wedge] \otimes [KU_p^\wedge, \Sigma^{-1}Z] \longrightarrow [KU_p^\wedge, KU_p^\wedge] \cong \Lambda'.$$

We have a canonical identification of endomorphism rings $\Lambda' \cong [KU_p^\wedge, KU_p^\wedge]$ and $[\Sigma^{-1}Z, \Sigma^{-1}Z]$ induced by choosing any weak equivalence $KU_p^\wedge \simeq \Sigma^{-1}Z$: because Λ' is commutative every choice induces the same isomorphism. Commutativity also implies that the Λ'-module structure on $[KU_p^\wedge, \Sigma^{-1}Z]$ from $[\Sigma^{-1}Z, \Sigma^{-1}Z]$ coincides with the Λ'-structure from $[KU_p^\wedge, KU_p^\wedge]$ and we see that the isomorphism in Theorem 7.2 is an isomorphism of Λ'-modules for the Λ'-module structure on $[X, \Sigma^{-1}Z]$ from $[\Sigma^{-1}Z, \Sigma^{-1}Z]$. Using the duality of the invertible Λ'-modules $[\Sigma^{-1}Z, KU_p^\wedge]$

and $[KU_p^\wedge, \Sigma^{-1}Z]$, Theorem 7.2 then implies the following slightly less complicated isomorphism.

Corollary 7.4. *For any spectrum X, the natural map*

$$E_\infty^\chi \otimes_{\Lambda'} [X, KU_p^\wedge] \cong [KU_p^\wedge, \Sigma^{-1}Z] \otimes_{[KU_p^\wedge, KU_p^\wedge]} [X, KU_p^\wedge] \longrightarrow [X, \Sigma^{-1}Z]$$

is an isomorphism.

We can identify the spectra Z_i using the eigensplitting of E_∞^χ. To explain this, let Λ be the completed group ring

$$\Lambda = \mathbb{Z}_p[[U^1]] = \lim \mathbb{Z}_p[U^1/U^n],$$

where U^n denotes the subgroup of $\mathbb{Z}_p^\times$ of elements congruent to 1 mod p^n. Then Λ is a (topological) subring of Λ' and because $\mathbb{Z}_p^\times \cong \mu_{p-1} \times U^1$, Λ' is isomorphic to the group algebra $\Lambda[\mu_{p-1}]$. We prefer to write $\Lambda' \cong \Lambda[\mathbb{F}_p^\times]$, using the Teichmüller character $\omega\colon \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ for the isomorphism $\mathbb{F}_p^\times \to \mu_{p-1}$. Because $p-1$ is invertible in $\mathbb{Z}_p$ and $\mu_{p-1} \subset \mathbb{Z}_p$, we get orthogonal idempotent elements

$$\epsilon_i = \frac{1}{p-1} \sum_{\alpha \in \mathbb{F}_p^\times} \omega(\alpha)^{-i} \psi^{\omega(\alpha)}$$

in Λ' for $i = 0, \dots, p-2$ (where we have written the group elements (in $\mathbb{Z}_p^\times$) using Adams operation notation to distinguish them from the coefficient elements (in $\mathbb{Z}_p$) of the completed group algebra Λ'). These give a Cartesian product decomposition of Λ',

$$\Lambda' = \epsilon_0\Lambda' \times \cdots \times \epsilon_{p-2}\Lambda',$$

where the elements of $\epsilon_i\Lambda$ can be characterized as the elements of Λ' on which $\mathbb{F}_p^\times$ acts by multiplication by ω^i; we call this the ω^i eigenspace of Λ'. Since $\omega^i = \omega^{i+p-1}$, it makes sense and can be convenient to index the eigenspaces on elements of $\mathbb{Z}/(p-1)$ rather than the specific representatives $0, \dots, p-2$. The inclusion $\Lambda \to \Lambda'$ induces an isomorphism of (topological) $\mathbb{Z}_p$-algebras between Λ and $\epsilon_i\Lambda'$ for each i. Every Λ'-module admits a corresponding decomposition into ω^i eigenspaces, which we can regard as Λ-modules.

The Cartesian product decomposition of Λ' above corresponds exactly to the Cartesian product decomposition of $[KU_p^\wedge, KU_p^\wedge]$,

$$[KU_p^\wedge, KU_p^\wedge] \cong [L, L] \times [\Sigma^2 L, \Sigma^2 L] \times \cdots \times [\Sigma^{2p-4}L, \Sigma^{2p-4}L]$$

induced by the Adams splitting $KU_p^\wedge \simeq L \vee \Sigma^2 L \vee \cdots \vee \Sigma^{2p-4}L$: the decomposition isomorphism takes $[\Sigma^{2i}L, \Sigma^{2i}L]$ to precisely the subset of $[KU_p^\wedge, KU_p^\wedge]$ of elements on which $\psi^{\omega(\alpha)}$ acts by multiplication by $\omega(\alpha)^i$ for all $\alpha \in \mathbb{F}_p^\times$. The isomorphism $\Lambda' \cong [KU_p^\wedge, KU_p^\wedge]$ induces an isomorphism $\Lambda \cong [L, L]$ for the inclusion of $[L, L]$ as the diagonal in $\prod[L, L] \cong \prod[\Sigma^{2i}L, \Sigma^{2i}L]$. Since the identification of $[KU_p^\wedge, \Sigma^{-1}Z]$ as E_∞^χ above is an isomorphism of Λ'-modules, we get an isomorphism of Λ-modules

$$\epsilon_i[KU_p^\wedge, \Sigma^{-1}Z] \cong \epsilon_i E_\infty^\chi$$

on the ω^i eigenspaces for all i. The idempotent ϵ_i of $[KU_p^\wedge, KU_p^\wedge]$ is the composite of the projection and inclusion

$$KU_p^\wedge \longrightarrow \Sigma^{2i}L \longrightarrow KU_p^\wedge,$$

so we see that

$$\epsilon_i[KU_p^\wedge, \Sigma^{-1}Z] \cong [\Sigma^{2i}L, \Sigma^{-1}Z] \cong [\Sigma^{2i}L, \Sigma^{-1}Z_{i+1}]$$

(numbering mod $p-1$) as $[\Sigma^{2i}L, \Sigma^{-1}Z_j] = 0$ unless $j \equiv i+1 \pmod{p-1}$. We then get the following characterization of Z_i.

Corollary 7.5. *Let $i \in \mathbb{Z}/(p-1)$. For any spectrum X, the natural map*

$$\epsilon_i E_\infty^\chi \otimes_\Lambda [X, \Sigma^{2i}L] \cong [\Sigma^{2i}L, \Sigma^{-1}Z_{i+1}] \otimes_{[\Sigma^{2i}L, \Sigma^{2i}L]} [X, L] \longrightarrow [X, \Sigma^{-1}Z_{i+1}]$$

is an isomorphism.

More can be said about choosing weak equivalences $Z_i \simeq \Sigma^{2i-1}L$. From the discussion above, choosing such a weak equivalence is equivalent to choosing a generator of the free rank one Λ-module $\epsilon_{i-1}E_\infty^\chi$, and this is equivalent to choosing a generator of the free rank one Λ-module $\epsilon_{p-i}E_\infty$.

We treat the case of $\epsilon_0 E_\infty$ separately, but all cases require a choice of a system of primitive pth roots of units $\zeta_n \in \mu_{p^{n+1}}$ with $\zeta_n^p = \zeta_{n-1}$, which we now fix. (For example, choosing an embedding of $\mathbb{Q}_p$ in $\mathbb{C}$, one could take $\zeta_n = e^{2\pi i/p^{n+1}}$.) Then $\zeta_n - 1$ is a uniformizer for $\mathcal{O}_{F_n}$ for each n: we have $N_{F_{n+1}/F_n}(\zeta_{n+1}-1) = \zeta_n - 1$ and $N_{F_0/\mathbb{Q}_p}(\zeta_0 - 1) = p$. This argument also shows that the system $(\!(\zeta_n - 1)\!)$ specifies an element of E_∞, and we can consider its ω^0 eigenfactor $\epsilon_0(\!(\zeta_n - 1)\!)$.

Proposition 7.6. *In the Λ'-module E_∞, $\epsilon_0(\!(\zeta_n - 1)\!)$ generates $\epsilon_0 E_\infty$*

Proof. The valuation $F_n^\times \to \mathbb{Z}$ is a homomorphism that sends roots of unity to zero and so extends to a homomorphism $E_n \to \mathbb{Z}_p$. It commutes with the norm and so assembles to a homomorphism $E_\infty \to \mathbb{Z}_p$. Giving $\mathbb{Z}_p$ the trivial Λ'-action, the homomorphism $E_\infty \to \mathbb{Z}_p$ is Λ'-linear and factors through $\epsilon_0 E_\infty$. Because $\zeta_n - 1 \in F_n^\times$ has valuation 1, Λ is a local ring, and $\epsilon_0 E_\infty$ is a free rank one Λ-module, it follows by Nakayama's Lemma that $\epsilon_0(\!(\zeta_n - 1)\!)$ is a generator of $\epsilon_0 E_\infty$. □

Remark 7.7. We note that the element $\epsilon_0(\!(\zeta_n - 1)\!)$ of $\epsilon_0 E_\infty$ is in the image of the corresponding Λ'-module defined in terms of $\mathbb{Z}[1/p]$ in place of $\mathbb{Q}_p$, that is, the inverse limit over norm maps of the p-completion of the units of $\mathbb{Z}[\zeta_n, 1/p]$ modulo torsion. This element can be used to construct a weak equivalence $\Sigma^{-1}Y_1 \simeq L$ by an argument analogous to the one for Theorem 7.2 (but for the ϵ_0 piece only). Since the resulting weak equivalence is just the composite of the weak equivalence $\Sigma^{-1}Y_1 \to \Sigma^{-1}Z_1$ induced by the cyclotomic trace and the weak equivalence $\Sigma^{-1}Z_1 \simeq L$ coming from the previous proposition (and Corollary 7.5), we omit the details.

For other eigenspaces, $\epsilon_i(\!(\zeta_n - 1)\!)$ is generally not a generator. If $i \not\equiv 0 \bmod p-1$, then $\epsilon_i(\zeta_n - 1) \in F_n^\times$ is a *cyclotomic unit* [14, 3§5]. In particular, it is a real multiple of a root of unity [14, p. 84], and as such, $\epsilon_i(\zeta_n - 1)$ becomes the identity element in $\epsilon_i E_n$ for i odd. For i even, $\epsilon_i(\!(\zeta_n - 1)\!)$ is a generator of $\epsilon_i E_\infty$ if and only if the Bernoulli number B_i is relatively prime to p (see the argument for Theorem 1.4 in Chapter 7 of Lang [14]).

The authors know of two (for $i = 1$) or three (for $i = 2, \ldots, p-2$) distinct ways of producing generators for the other eigenspaces. For the constructions, let U_n^1 denote the subgroup of $F_n^\times$ congruent to 1 mod $\zeta_n - 1$ and let $U_\infty^1 = \lim U_n^1$ under the norm maps. Since U_n^1 is p-complete, the Galois action on U_∞^1 makes it a Λ'-module. Let $T_p(\mu) = \lim \mu_{p^{n+1}}$, a Λ'-submodule of U_∞^1. Since $\mu_{p^{n+1}}$ is the

torsion of U_n^1 and $\mu_{(p-1)p^{n+1}}$ is the torsion in $F^\times$, we have an exact sequence of Λ'-modules

$$0 \longrightarrow T_p(\mu) \longrightarrow U_\infty^1 \longrightarrow E_\infty \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

where $\mathbb{Z}_p$ has the trivial Galois action and the map $E_\infty \to \mathbb{Z}_p$ is induced by the valuation as in the proof of Proposition 7.6 above. In particular $U_\infty^1 \to E_\infty$ is an isomorphism on ω^i eigenspaces for $i \neq 0, 1$ and an epimorphism on ω^1.

Given an element $(\!(u_n)\!) \in U_\infty^1$, we can detect whether $\epsilon_i(\!(u_n)\!)$ maps to a generator of $\epsilon_i E_\infty$ using the *Kummer homomorphisms* (see [14, 7§1–2]) $\phi_i \colon U_0^1 \to \mathbb{F}_p$. Let $D \colon \mathbb{Z}_p[[X]] \to \mathbb{Z}_p[[X]]$ be the homomorphism of $\mathbb{Z}_p$-modules $Df = (1+X)f'(X)$. Given $u \in U_0^1$, $u = f_u(\zeta_0 - 1)$ for some (non-unique) $f_u \in \mathbb{Z}_p[[X]]$ with leading coefficient congruent to 1 mod p; for $i = 1, \ldots, p-2$, define

$$\phi_i(u) = D^i(\log(f_u))|_{X=0} \pmod p \in \mathbb{F}_p,$$

where $\log(f) = (f-1) - (f-1)^2/2 + \cdots$ (or more generally, for any power series f with constant term a unit in $\mathbb{Z}_p$, we can equivalently interpret $D^i \log(f)$ as $D^{i-1}((1+X)f'/f)$). The power series f_u is well defined mod (p, X^{p-1}) since $\mathcal{O}_{F_0}$ has ramification index $p-1$ over $\mathbb{Z}_p$. As a consequence, $f_u(X)f_v(X) \equiv f_{uv}(X)$ mod (p, X^{p-1}). The formal power series identity for $\log(1+X)$ (or the product rule applied to $(gh)'/(gh)$) implies

$$D^i \log(f_u f_v) = D^i \log(f_u) + D^i \log(f_v)$$

and ϕ_i is a well-defined homomorphism. The (easily checked) formula

$$D(f((X+1)^a - 1)) = a(X+1)^a f'((X+1)^a - 1) = aDf|_{(X+1)^a - 1}$$

shows that $\phi_i(\psi^a u) = a^i \phi_i(u)$ for any $u \in U_0^1$, $a \in \mathbb{Z}_p^\times$. It follows that $\phi_i(\epsilon_i u) = \phi_i(u)$. Since $\epsilon_i E_\infty$ is a free rank one Λ-module and Λ is a local ring, if $\phi_i(u_0) \neq 0$ for some $(\!(u_n)\!) \in U_\infty^1$, and either $i \in \{2, \ldots, p-2\}$ or $i = 1$ and $\epsilon_1 u_0 \notin \mu_p$, then the image of $\epsilon_i(\!(u_n)\!)$ generates $\epsilon_i E_\infty$. For the first construction of generators, we use the following proposition from [14, 7§3].

Proposition 7.8. *Fix $i \in 1, \ldots, p-2$. There exists $\lambda \in \mu_{p-1}, \lambda \neq 1$ such that $\phi_i(\omega(\lambda-1)^{-1}(\lambda - \zeta_0)) \neq 0$.*

The proof is an analysis of $\phi_i(u_0(\lambda))$ for $u_0(\lambda) = \omega(\lambda-1)^{-1}(\lambda - \zeta_0)$. Clearly, we can take $f_{u_0(\lambda)} = \omega(\lambda-1)^{-1}(\lambda - 1 - X)$. We then have $D(\log(f_{u_0}))|_{X=0} = 1/(1-\lambda)$, and in general, a little bit of algebra (q.v. [14, p. 182]) shows that $D^i(\log f_{u_0(\lambda)})|_{X=0}$ is a rational polynomial in λ whose numerator is a monic polynomial of degree $i - 1 < p - 2$; thus, for at least one of the $p-2$ values of $\lambda \in \mu_{p-1} \setminus \{1\}$, $\phi_i(u_0(\lambda))$ must be non-zero mod p.

For the ω^1 eigenspace, we also need the following observation.

Proposition 7.9. *Let $\lambda \in \mu_{p-1} \setminus \{1\}$ and let $u_0(\lambda) = \omega(\lambda-1)^{-1}(\lambda - \zeta_0)$. Then $\epsilon_1(u_0(\lambda))$ is not a pth root of unity.*

Proof. Let $\pi = \zeta_0 - 1 \in \mathcal{O}_{F_0}$; we note $(p) = (\pi^{p-1})$. It suffices to check that

$$\epsilon_1(u_0(\lambda))^p \not\equiv 1 \pmod{\pi^{p+1}}.$$

Since $\omega(\lambda-1)^{-1}(\lambda-1)$ is in $\mathbb{Z}_p$, and is by construction congruent to 1 mod p (*a fortiori* congruent to 1 mod π in $\mathcal{O}_{F_0}$), it is in $\epsilon_0 U_0^1$ and ϵ_1 takes it to the identity. We then have

$$\epsilon_1(u_0(\lambda)) = \epsilon_1\left(\frac{u_0(\lambda)}{\omega(\lambda-1)^{-1}(\lambda-1)}\right) = \epsilon_1\left(1 - \frac{\pi}{\lambda - 1}\right),$$

and

$$\left(1-\frac{\pi}{\lambda-1}\right)^p \equiv 1-\frac{p\pi}{\lambda-1} \pmod{\pi^{p+1}}.$$

Moreover, since

$$\psi^a\left(1-\frac{p\pi}{\lambda-1}\right) \equiv 1-\frac{ap\pi}{\lambda-1} \equiv \left(1-\frac{p\pi}{\lambda-1}\right)^a \pmod{\pi^{p+1}},$$

it follows that $\epsilon_1(u_0(\lambda))^p \equiv 1 - p\pi/(1-\lambda) \not\equiv 1 \pmod{\pi^{p+1}}$. □

Taking λ as in Proposition 7.8, let $u_n = \omega(\lambda-1)^{-1}(\lambda-\zeta_n)$. Because $\lambda \in \mu_{p-1}$, we have $N_{F_{n+1}/F_n}(u_{n+1}) = u_n$, and the system $\epsilon_i(\!(u_n)\!)$ maps to a generator of $\epsilon_i E_\infty$.

Proposition 7.10. *Let $i \in 1,\dots,p-2$, and choose $\lambda \in \mu_{p-1} \setminus \{1\}$ such that $\phi_i(\omega(\lambda-1)^{-1}(\lambda-\zeta_0)) \neq 0$. Then $\epsilon_i(\omega(\lambda-1)^{-1}(\lambda-\zeta_n)) \in \epsilon_i U^1_\infty$ maps to a generator of $\epsilon_i E_\infty$.*

The next construction of generators, due to Coates-Wiles [6, Theorem 4], avoids the indefiniteness of the unspecified choice λ in the previous construction. Let $\beta \in \mathbb{Z}_p$ denote the unique $(p-1)$th root of $1-p$ which is congruent to 1 mod p. Let Γ be the unique Lubin-Tate formal group law over $\mathbb{Z}_p$ with $[p]_\Gamma[X] = X^p + pX$, and let θ be the unique strict isomorphism from the multiplicative formal group law G_m to Γ. (For an introduction to this Lubin-Tate theory, see for example [14, 8§1], particularly Theorems 1.1 and 1.2.) Because $\pi_n = \zeta_n - 1$ satisfies

$$[p]_{G_m}(\pi_{n+1}) = (\pi_{n+1}+1)^p - 1 = \pi_n,$$

for $n \geq 0$, $x_n = \theta(\pi_n)$ satisfies

$$x_{n+1}^p + px_{n+1} = x_n$$

for $n \geq 0$, and $x_0^p + px_0 = 0$. Let $u_n = \beta - x_n$. We have $N_{F_{n+1}/F_n}(u_{n+1}) = u_n$, and it is easy to calculate $\phi_i(u_0)$ as follows. Because Γ agrees with the additive formal group law to order $p-1$, $\theta(X) \equiv \log(1+X) \pmod{X^p}$. Calculating

$$D^i(\beta - \log(1+X)) = \frac{-(i-1)!}{(\beta-\log(1+X))^i},$$

we get $\phi_i(u_0) = -(i-1)! \neq 0 \in \mathbb{F}_p$. In the case $i = 1$, $\epsilon_1 u_0$ is not a pth root of unity (see [20, 2.5]). This proves that $\epsilon_i(\!(u_n)\!)$ maps to a generator of $\epsilon_i E_\infty$ for all $i = 1,\dots,p-2$.

Proposition 7.11. *Let $i \in 1,\dots,p-2$. In the notation of the previous paragraph, $\epsilon_i(\!(\beta - \theta(\zeta_n - 1))\!) \in \epsilon_i U^1_\infty$ maps to a generator of $\epsilon_i E_\infty$.*

Finally, Coleman [7,8] constructs an extremely nice isomorphism $\epsilon_i E_\infty \to \Lambda$ for $i \neq 0, 1$. The Coleman map $\mathcal{L}\colon U^1_\infty \to \Lambda'$ fits into an exact sequence of Λ'-modules

$$0 \longrightarrow T_p(\mu) \longrightarrow U^1_\infty \xrightarrow{\mathcal{L}} \Lambda' \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0$$

(see [5, 3.5.1] or [20, Theorem 2]), where $\mathbb{Z}_p(1)$ is the Λ'-module with underlying $\mathbb{Z}_p$-module $\mathbb{Z}_p$ where $\mathbb{Z}_p^\times$ acts by multiplication (it is isomorphic to $T_p(\mu)$ under the isomorphism $a \mapsto (\!(\zeta_n^a)\!)$). In particular, the Coleman map is an isomorphism on ω^i eigenspaces for $i \neq 1$. Choosing a generator of $\epsilon_i\Lambda'$ for $i \neq 0, 1$, $\mathcal{L}^{-1}$ gives a generator of $\epsilon_i U^1_\infty$ which maps to a generator of $\epsilon_i E_\infty$. (This generator has ϕ_i equal to 1 by [5, 3.5.2].)

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