EPJ manuscript No.
(will be inserted by the editor)

A rigorous multi-population multi-lane hybrid traffic model and its mean-field limit for dissipation of waves via autonomous vehicles

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Abstract. In this paper, a multi-lane multi-population microscopic model, which presents stop and go waves, is proposed to simulate traffic on a ring-road. Vehicles are divided between human-driven and autonomous vehicles (AV). Control strategies are designed with the ultimate goal of using a small number of AVs (less than 5% penetration rate) to represent Lagrangian control actuators that can smooth the multilane traffic flow and dissipate the stop-and-go waves. This in turn may reduce fuel consumption and emissions. The lane-changing mechanism is based on three components that we treat as parameters in the model: safety, incentive and cool-down time. The choice of these parameters in the lane-change mechanism is critical to modeling traffic accurately, because different parameter values can lead to drastically different traffic behaviors. In particular, the number of lane-changes and the speed variance are highly affected by the choice of parameters. Despite this modeling issue, when using sufficiently simple and robust controllers for AVs, the stabilization of uniform flow steady-state is effective for any realistic value of the parameters, and ultimately bypasses the observed modeling issue. Our approach is based on accurate and rigorous mathematical models, which allows a limit procedure that is termed, in gas dynamic terminology, mean-field. In simple words, from increasing the human-driven population to infinity, a system of coupled ordinary and partial differential equations are obtained. Moreover, control problems also pass to the limit, allowing the design to be tackled at different scales.
Finally, we explore collaborative driving by assuming that a fraction of human drivers is instructed to drive smoothly to stabilize traffic. We show that this approach also leads to dissipations of waves.
Keywords: autonomous vehicles, stop-and-go waves, multi-lane traffic, hybrid models, mean-field.

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1 Introduction

Traffic flow displays various instabilities at high densities, and this is known as a congested phase. Such instabilities may grow into persistent stop-and-go waves and travel upstream to the flow of traffic. This phenomenon is especially observed on highways, and was reproduced in experiments [37,22]. Waves may be generated by network features (bottlenecks, ramps etc.) as well as by drivers' behavior (lane changing, strong breaking, etc.). These waves are responsible for traffic inefficiencies and increased fuel consumption.

Traditional traffic management techniques include variable speed advisory and variable speed limits. However, the technological advancements in terms of autonomy allows the use of a Lagrangian approach using autonomous vehicles as Lagrangian actuators that are sparse along the road network. A number of studies addressed the problem of dampening waves to smooth traffic using autonomous vehicles, both in simulation [8,38,18,44] as well as in experiments [46,36]. The achieved results also showed that at low penetration (around 5%), traffic can be smoothed to a great extent in terms of fuel economy (reduction up to 40%). The results from the experiments were mostly in a confined setting and only used one lane, with controls designed from first principles and control-theoretic methods, [6,10].

The present paper aims at designing rigorous mathematical models and control algorithms for a multi-lane setting in a ring road. More precisely, we design a multi-population model with human-driven and autonomous vehicles. The microscopic dynamics is described by a Bando-Follow-the-Leader model, proven to generate stop-and-go waves and tuned to experimental data. The lane-changing mechanism is mainly based on MOBIL [43] and includes: safety, incentive and cool-down time. Safety poses constraints on acceleration/deceleration of vehicles, incentive is based on the potential for higher acceleration in a new lane and cool-down time allows lane-changing only after a certain amount of time from the last lane change. The resulting dynamics is of a hybrid nature and heavily depends on the choice of parameters for these three mechanisms.

In particular, we show in this article that the quantitative but also qualitative behavior of the dynamics (stop-and-go waves or not; number of lane-change; speed variance) highly depends on the parameters of the lane changing mechanism. Despite the variability of traffic patterns, we show that we can still design simple control algorithms, which are robust and can stabilize traffic with a low penetration rate for any choice of parameters (in a physically relevant parameters' space). Finally, we show that a collaborative driving approach, where a minority of vehicles would have "good human behavior", would also bring some stability to the system.

The hybrid model and control strategies are based on accurate mathematical analysis. This in turn allows a limiting procedure, called mean-field, with the population of human-driven cars sent to infinity. The limiting controlled dynamics couples a partial differential equation for the human-driven car density with a controlled hybrid system for the autonomous vehicles. Optimal control problems are also compatible with the limiting procedure, and thus control strategies can be designed for the limiting dynamics and used for the microscopic model.

1.1 Existing traffic models in the literature

A multilane model is typically composed of two components : longitudinal dynamics for each lane and a lane-change mechanism. Due to the different scales that can be represented in vehicular traffic, one can also classify traffic models by modelling longitudinal dynamics for each lane into two typical categories: micro-models and macro-models. For general discussions about traffic models at different scales, we refer to the survey papers [32,4,2].

a) Micro-model

There are many different micro scale traffic models. We show several continuous time models here that are well-understood and used regularly: the Intelligent Driver Model (IDM), the Bando model, and the Follow The Leader (FTL) model. For the IDM, the longitudinal dynamics for a lane are written:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = a\left(1 - \left(\frac{v_i}{v_0}\right)^{\delta} - \left(\frac{s^*(v_i, \Delta v_i)}{s_i}\right)^2\right), \end{cases}$$

where $s^*(v_i, \Delta v_i) = s_0 + v_i T + \frac{v_i \Delta v_i}{2\sqrt{ab}}$, given model parameters v_0, s_0, T, a, b.

For the Bando model, the longitudinal dynamics for a lane are written:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \alpha\left(V(\Delta x_i) - v_i\right), \end{cases}$$

where $V(\cdot)$ is the optimal velocity function that depends on the space headway, $\Delta x_i = x_{i+1} - x_i$, in front of the i^{th} vehicle (see (1)).

For the FTL model, the longitudinal dynamics for a lane are written:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \beta \frac{v_{i+1} - v_i}{(x_{i+1} - x_i - l_v)^2} \end{cases}$$

Not all models recreate stop and go traffic waves. Regarding this phenomena, the IDM or a combination of the Bando model and FTL, so called "Bando-FTL model", are used. For the Bando-FTL, the longitudinal dynamics for a lane are written:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \alpha(V(x_{i+1} - x_i) - v_i) + \beta \frac{v_{i+1} - v_i}{(x_{i+1} - x_i - l_v)^2}, \end{cases}$$

where v_i is the velocity of the i^{th} car and x_i is its location. The constant α is the weight for the Bando model and β is the weight of the Follow-the-leader model. V is still the optimal velocity function given by

$$V(x) = V_{\max} \frac{\tanh(\frac{x - l_v}{d_0} - 2) + \tanh(2)}{1 + \tanh(2)}. \tag{1}$$

where l_v is the length of the car, and d_0 is the minimal distance for the optimal velocity model (see Table 1). The Bando-FTL model is primarily studied in this paper. This model has been used in the past, for instance in [10,7], and has several advantages :

– the FTL model represents the competing dynamics between drivers and deals with the safety issues by applying a large braking value when a vehicle is too close to the leading vehicle. This portion is at the origin of the stop and go waves.
– the Bando model enables realistic uniform flow steady-states: for the density of cars on the road, there is a unique uniform flow equilibrium $(h, v^*(h))$, where h is the equilibrium headway and $v^*(h)$ is the equilibrium speed, which decreases with h.

– Because the FTL portion already incorporates a safety criteria, given reasonable initial conditions, the Bando-FTL model usually does not require an additional fail/safe condition. The model is also generally more robust than the widely used Intelligent Driver's Model (IDM) [42].

Moreover, if we consider N vehicles on a multilane ring-road with only human-driven vehicles, the Bando-FTL model reads:

$$\begin{cases} \dot{x}_i^j = v_i^j, \quad i = 1, \ldots, n^j, \\ \dot{v}_i^j = \alpha(V(x_{i+1}^j - x_i^j) - v_i^j) + \beta \frac{v_{i+1}^j - v_i^j}{(x_{i+1}^j - x_i^j - l_v)^2}, \end{cases}$$

where j is the lane number, n^j is the number of vehicles in lane j, and we set $v_{n^j+1}^j := v_1^j$ and $x_{n^j+1}^j := x_1^j$ to take into account the ring-road geography. To take into account physical limitations of real cars, we also cap the acceleration to $2.5\ m{\cdot}s^{-2}$ and the deceleration to $4\ m{\cdot}s^{-2}$. It has been shown that this model produces stop and go waves in lane j if [7]

$$\frac{\alpha}{2} + \frac{L^2 \beta}{(n^j)^2} < V' \left(\frac{n^j}{L} \right). \tag{2}$$

where V' is the derivative of function $V(x)$ in equation (1), and L is the length of the road.

b) Macro-model

In this paper we consider a micro-model, which gives us a better understanding of the dynamics and behavior of individual cars, and thus a more accurate measure of fuel consumption. However, when the number of cars becomes high, the analysis for optimization and optimal control can become computationally unfeasible. Therefore many macroscopic models (macro-model) have been derived to study the behavior of traffic flow at a larger scale. In these models, the dynamics are distributed and represented by partial differential equations. The first models were scalar, such as the celebrated but limited Lighthill-Whitham-Richards model, where the density of cars on the road is the only variable and the speed is a decreasing function of density [28,34]. These models regained interest with the emergence of more realistic second-order models [3,25,12]. These models included two equations where both the density and speed were included as variables. The first equation often represents a transport density, while the second equation represents the effect from acceleration. Second order macro-models can also represent traffic waves more easily. One can cite in particular the study of "jamitons" waves [13]. A harder question when dealing with macro-models is the question of the interactions between the AVs and the regular traffic flow. While for micro-models, this interaction is relatively easy to represent accurately (one only needs to give the AV a different acceleration law than the other vehicles), the interaction between the AVs and the rest of the traffic flow in a macro-model raises several issues:

– Should the AVs also obey there own macro-model and, if so, how are the two macro-models coupled?
– Should the AVs be represented as individual cars and how should the microscale and macroscale be coupled?

One proposition to interact these two models is in the form of an ODE-PDE system, which is given in [9]. Several works were even developed to show that this system makes sense mathematically (i.e. are well-posed) and exhibits the expected behavior

[26,27,20]. For the above reasons, we restricted ourselves to a micro-model in this paper, even though a macro-scale model may be promising to design efficient controllers that can dissipate stop-and-go waves. Another approach consists of trying to obtain "the best of each scale" by starting from a microscopic model and using a mean-field limit to obtain a PDE that represents the well-designed behavior of the microscopic system at a larger scale. The last section of this paper is devoted to this approach.

c) Lateral dynamics and lane-change mechanism
Regarding the Bando-FTL model, we include a lane changing mechanism suggested by Treiber et al. in [43]. Several models of lane changing dynamics have been explored [48], such as 1. Gipps-type lane changing [16,47,21], and 2. Utility theory based lane changing [1,39]. In Gipps-type lane changing, the driver's behavior is governed by maintaining a desired speed and being in the correct lane for an intended maneuver. These types of lane changes depend on parameters corresponding to an incentive and an acceptable level of risk for a collision, where some differentiate between cooperative and forced lane changes. Characteristics distinguishing utility theory based lane changing are a hierarchical decision-making process, desirability versus necessity, and the consideration of multiple driver types (driver behavior heterogeneity).

A regular vehicle changes lane if and only if

- It is safe to do so: changing lane does not imply a huge braking for the vehicle behind.
- It has an acceleration incentive: the expected acceleration after changing lane is higher than the expected acceleration from not changing lane.
- A certain amount of time has passed from the time of the vehicle's last lane change to the current time. We refer to this as the lane change "cooldown time."

In mathematical form, if we denote i as the vehicle changing lane and j as the potential new lane, we have

$$\begin{aligned}
&\tilde{a}_i^j > a_i + \Delta_I \ \text{ (incentive)}, \\
&\tilde{a}_i^j > -\Delta_s, \ \ \tilde{a}_{\mathrm{fol}}^j(i) > -\Delta_s \ \text{ (safety)}, \\
&t > t_0 + \tau \ \text{ (cooldown time)}.
\end{aligned} \tag{3}$$

Here a_i is the acceleration of the vehicle changing lane in the original lane, $\tilde{a}_i$ is the expected acceleration in the new lane, $\tilde{a}_{\mathrm{fol}}^j(i)$ is the expected acceleration of its follower in the new lane, Δ_I is a constant representing the threshold incentive and Δ_s represents the threshold safety. For the cooldown time equation, t_0 represents the last time a lane change occurred for the considered vehicle, and shows that the time of the next lane change should be greater by a threshold value τ.

There are two main advantages to using acceleration, instead of speed, to model lane-change: (1) the lane-change decision-making process is dramatically simplified; (2) one can readily calculate accelerations with an underlying microscopic longitudinal traffic model, see [48]. We also point out that the lane-change mechanisms lead to discrete dynamics of the vehicles. The presence of both continuous dynamics and discrete dynamics of vehicles motivate us to consider a hybrid system, see [5,15,17,31,40].

A natural question is to wonder about the influence of the lane-changing mechanisms on the stability of the system and whether such a model reduces stop and go waves when adding the lane-changing mechanisms, or on the contrary, whether it produces even stronger stop and go waves. We show in the next section that the model heavily depends on the parameters of the lane-changing mechanisms. Besides this, the possible behaviors are extensive. As the lanes are coupled, there are scenarios where one lane can produce stop and go waves while another lane does not.

2 Strong influence of the lane-changing parameters on the traffic behavior

In this section we study the effect of the threshold parameters Δ_I and Δ_s and we show that different values of these parameters can lead to radically different behaviors for the traffic flow. To illustrate this phenomena, we fix a given initial condition where all lanes have the same number of cars (in this case 24 cars for the middle lane of length 240m, see Table 1 for a summary of the parameters used), and all the cars are initially located within $1m$ from their steady-state location (the steady-state location corresponds to a uniform spacing). Then we perform traffic simulations over $1000s$ with Δ_I ranging from 0.6 to 3 $m{\cdot}s^{-2}$ and Δ_s ranging from 0.5 to 5 $m{\cdot}s^{-2}$. Note that in our model, the maximum acceleration allowed by the car is $2.5m{\cdot}s^{-2}$ and the maximum deceleration allowed by the car is $4m{\cdot}s^{-2}$. The explanation for the choice of the range on Δ_I and Δ_s is as follows: requiring an incentive Δ_I of $3m{\cdot}s^{-2}$ to change lane means that you can only change lane when your lane is decelerating and you can accelerate strongly in the neighboring lane, whereas a safety threshold Δ_s of $5m{\cdot}s^{-2}$ means that we do not require any safety since the vehicles cannot brake more strongly anyway. We expect that the higher the request on the incentive is, the lower the number of lane changes. Similarly the higher the security threshold (hence the lower security required), the higher the number of lane changes. These expectations were confirmed in Figure 1, where we plot the number of lane changes over the total length of the simulation for each combination of parameters Δ_I and Δ_s.

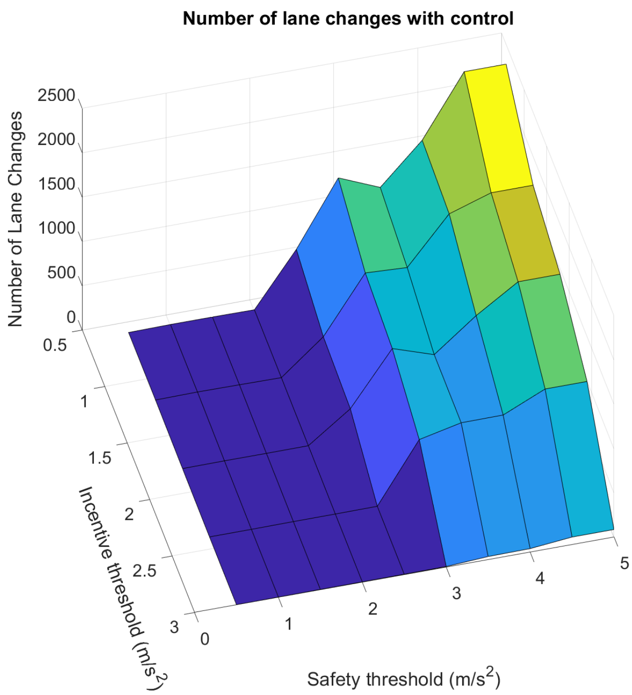

Fig. 1. Number of lane changes without control in the system given different threshold values for incentive and safety

In Figure 2, for each simulation, we compute the speed variance for each lane and at each time-step, and find the average of the value over 1000 s and across the three lanes. We see that in some cases the speed variance is close to 0, which suggests that the system reached equilibrium. In other cases the speed variance has a high value, suggesting that the system still undergoes some stop and go waves.

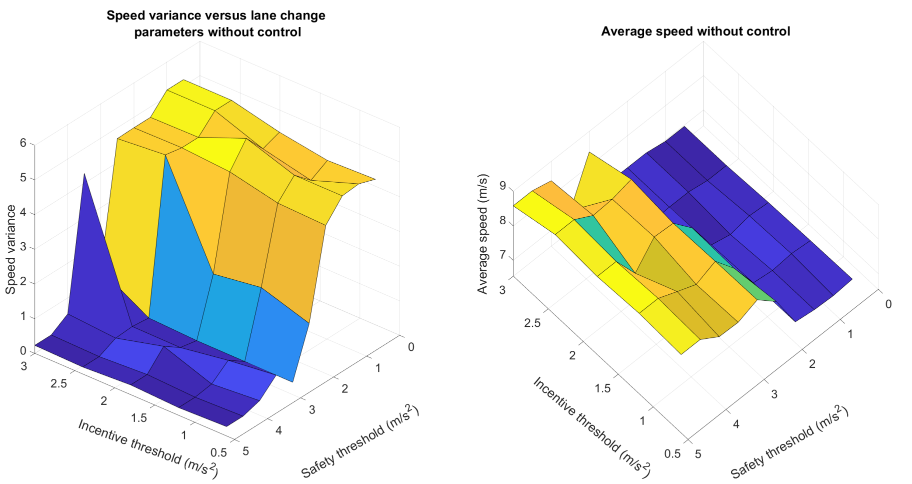

Fig. 2. Speed variance and average speed of the system without control given different threshold values for incentive and safety

These speculations seems to be clear when looking at the instantaneous speed variance over time for Fig 3 with parameter values $\Delta_s = 4\ m{\cdot}s^{-2}$, $\Delta_I = 0.6\ m{\cdot}s^{-2}$ and $\Delta_s = 0.5\ m{\cdot}s^{-2}$ $\Delta_I = 3\ m{\cdot}s^{-2}$ respectively. We see that for $\Delta_s = 4\ m{\cdot}s^{-2}$ and $\Delta_I = 0.6\ m{\cdot}s^{-2}$, the system approaches a uniform flow after 200 s, while for $\Delta_s = 0.5$ $m{\cdot}s^{-2}$ and $\Delta_I = 3\ m{\cdot}s^{-2}$, it does not and stop and go waves persist in the system.

3 Using autonomous vehicles to smooth stop and go waves.

Traffic flow is very particular in that a single individual can have a global effect on the entire dynamic of the flow. This is found in both micro and macro models and can be understood from a simple example: a single individual can be a bottleneck and thus influence the traffic across the entire system. Given this, the section serves to investigate the following: is it possible to dissipate and prevent stop and go waves by simply adding a single AV that follows a prescribed acceleration? And if so, what prescribed acceleration should be given to these cars in order to smooth traffic efficiently?

When adding an AV to the system, the equations are modified as follows: the lane of the AV is denoted by j and the car's number is denoted by 1. From this, we have

$$\begin{aligned} \dot{x}_1(t) &= v_1(t), \\ \dot{v}_1(t) &= u(t)(t), \end{aligned} \tag{4}$$

where u is a control law that can be chosen. Using AVs to smooth traffic flow has already been studied in both theory and experiments in a single lane context

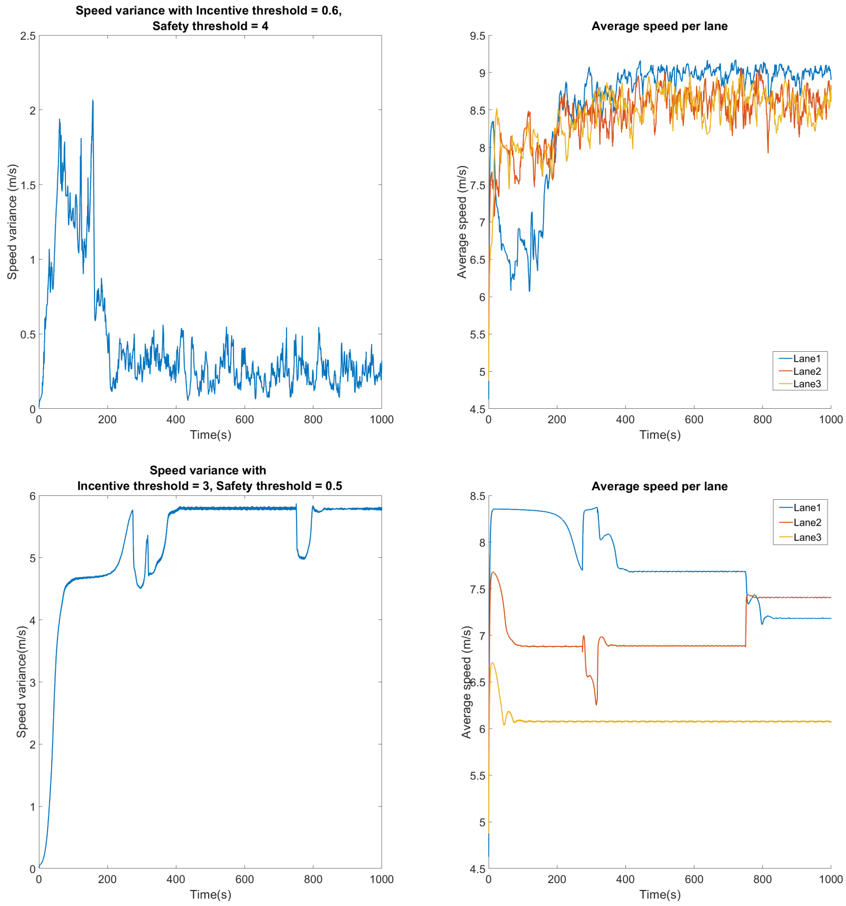

Fig. 3. Speed variance and average speed over time for different threshold values Up left: speed variance with incentive threshold $= 0.6m.s^{-2}$ and safety threshold $= 4m.s^{-2}$, Up right: the average speed per lane with the same incentive and safety thresholds. Down left: speed variance with incentive threshold $= 3m.s^{-2}$ and safety threshold $= 0.5m.s^{-2}$, Down right: the average speed per lane with the same incentive and safety thresholds.

[10]. In particular, from [10], the author uses two very simple controllers, one proportional and one slightly proportional integral controller. From the theoretical analysis and experiments, the author demonstrated the efficiency of such simple controllers to smooth stop and go waves in a single lane ring-road, with a reduction of fuel consumption of up to 40%. However, when the traffic is multi-lane, the problem becomes more difficult for several reasons:

- The lane-changes add complexity to the dynamics of the system and impacts the stability of the stop and go waves, potentially making them harder to smooth.
- The AV only belongs to one lane but can dissipate and prevent waves on all three lanes. Hence, on two lanes, the lane-changes are represented by the coupling between waves.
- The model is very sensitive to errors from the parameters, as shown in the previous section, and these errors could lead to simulations that are far from the ground truth.

The controller we use in our setting is a proportional controller where the ideal command is given as follows:

$$
\begin{aligned}
u(t) &= -k\left(v_1 - v_{\text{target}}\right), \\
v_{\text{target}} &= v^*\left(\frac{n^j + l_v}{L^j}\right),
\end{aligned} \tag{5}
$$

where l_v is the average length of a car, k is a constant design parameter, L^j is the length of lane j and n^j is the total number of cars in the j-th lane. Recall that $v^*(h)$ is the steady-state speed of the system corresponding to the steady-state headway h. v_{target} is the speed of the uniform flow steady-state we would like to reach. This ideal command does not take into account to prevent the AV from crashing into another car. To tackle this, one could add a safety mechanism where the AV would brake if it is too close to its leader. However, even with a safety mechanism, the AV can still get stuck in stop and go waves. This is because v_{target} would be too high compared to the current velocity of the cars in front of the AV. The AV would then try to increase its speed until it is too close to the vehicle in front, then it would brake, and then increase its speed again, thus maintaining a stop and go wave. Due to this, we add the following features to our control:

- (quasi-stationary steady-state strategy) As mentioned, we are trying to make the AV not get stuck in a stop and go wave as it tries to reach an ideal steady-state speed that is higher than the speed of its leader. To deal with this, we start by stabilizing a smaller speed, and then we slowly raise the stabilizing speed to the ideal steady state speed. In control literature, this is referred to as following a continuous path of a steady-state. This is only possible because adding the AV allows the number of possible steady-states to go from a single steady-state to a continuous range. In mathematical terms, the control law becomes

$$
u(t) = -k(v_{N+1} - \bar{v}_d(t)), \tag{6}
$$

where v_d is given by

$$
\begin{cases}
v_d(t) = v_{\min} + (\bar{v} - v_{\min})\dfrac{t}{t_{tr}}, \text{ for } t \in [0, t_{tr}], \\
v_d(t) = \bar{v}^*(h^*),
\end{cases} \tag{7}
$$

t_{tr} is the time of transition and $v^*(h^*)$ is the ideal steady-state speed.

- (safety mechanism) When the AV starts to get close to its leading vehicle we change the target speed to the speed of the leading vehicle for safety.

a) Lateral controller

In a multilane framework, another interesting means of control for the AV is having the ability to change lanes, and this is referred to as a lateral controller. Given the results from [10], traffic can be stabilized with one AV per lane in the case that the AVs cannot change lanes. However, if AVs can change lanes and have good lateral controllers, then traffic can be stabilized in multilane ring-roads with potentially even a single AV. Our lateral controller is the following: the AV changes lane if and only if

- the safety conditions (3) are satisfied (just like for a regular vehicle).
- the speed variance in another lane averaged on the last t_1 seconds, is higher than the speed variance in the AV's lane, also averaged on the last t_1 seconds. This difference has to be larger than a threshold (noted c_1 in Table 1).

Parameter	Value	Description
N	24	number of vehicles per lane
J	3	number of lanes
lv	4.5	length of a car $[m]$
d_0	2.5	minimal distance for optimal velocity model
β	20	weight FTL
α	0.5	weight OV
dt	0.02	timestep size for the simulation
t_f	1000	final time of the simulation $[s]$
max-dec	4	maximum deceleration $[m{\cdot}s^{-2}]$
max-acc	2.5	maximum acceleration $[m{\cdot}s^{-2}]$
iter-lc	50	iteration for lane changing, dependent on dt
τ	5	cool down duration after lane change [s]
k	1	constant in control law AV
c_1	0.5	speed variance threshold for AV changing lane
d_1	10	time to average speed variance for AV changing lane [s]
d_2	10	AV cool down duration after lane change [s]

Table 1. Parameters used for the simulations

– the AV has not been changing lanes in the last t_2 seconds.

We denote the AV's lane by j_0, and the last time the AV changed lane as t_0 ($t_0 = 0$ if the AV never changed lane). From this, we have

– $t > t_1$ and there exists $j \in \{1, ..., 3\} \setminus \{j_0\}$ such that

$$\begin{aligned}\int_{t-t_1}^{t} \frac{1}{N}\sum_{i=1}^{n}(v_i^j)^2(s) - \frac{1}{N^2}\left(\sum_{i=1}^{n} v_i^j(s)\right)^2 ds \\ > c_1 + \int_{t-t_1}^{t} \frac{1}{N}\sum_{i=1}^{n}(v_i^{j_0})^2(s) - \frac{1}{N^2}\left(\sum_{i=1}^{n} v_i^{j_0}(s)\right)^2 ds.\end{aligned} \tag{8}$$

– $t > t_2 + t_0$.
– the safety condition (3) is satisfied with $i = i_0$ and $j = j_0$.

The main difference between the regular vehicles and the AV is that the incentive for the AV is to go in the lane with the highest speed variance. This is different to an incentive that is based on acceleration. Averaging and threshold values are included to account for the stochastic nature of the measurements, and to avoid the AV changing lanes constantly, which could destabilize the system.

b) Results
In this section we show that, similar simple controllers not only manage to smooth stop and go waves in a multi-lane setting, but also hold a large range of parameters Δ_I and Δ_s. We run two batches of simulations with a fixed initial condition and different parameters of Δ_I and Δ_s. The first batch in the experiment is similar to the previous section in that there are no AVs. In the second batch of the experiment, we turn an AV on. The AV is initially in the middle lane. In Figure 4, for simulations with and without the AV respectively, we represent the speed variance averaged over time and the three lanes, for each pair of parameters Δ_I and Δ_s. We can see that the two figures very roughly follows the same trend, but the speed variance when adding the AV is four times smaller. Note that the reduction of speed variance and dissipation

of waves is effective over all the range of parameters Δ_I and Δ_s. Moreover, there is a single AV in this simulation, therefore the penetration rate (fraction of AV in the total traffic) is below 2%. To illustrate what is going on, we represent in Figure 5 the speed variance over time in all three lanes, where $\Delta_I = 3\ m{\cdot}s^{-2}$ and $\Delta_s = 0.5\ m{\cdot}s^{-2}$, both with and without a controller. As expected, we see that the AV is stabilizing mostly one lane that reaches uniform flow, but despite the very weak coupling of the lanes (due to the very small number of lane changes), it is still enough to roughly dissipate the waves that form in the other lanes. On the other hand, when there is no AV, the speed variance remains high. Note that the y-axis in the figure with the control only goes to 4.5 $m{\cdot}s^{-1}$ while the axis of the figure without the control goes to 9 $m{\cdot}s^{-1}$.

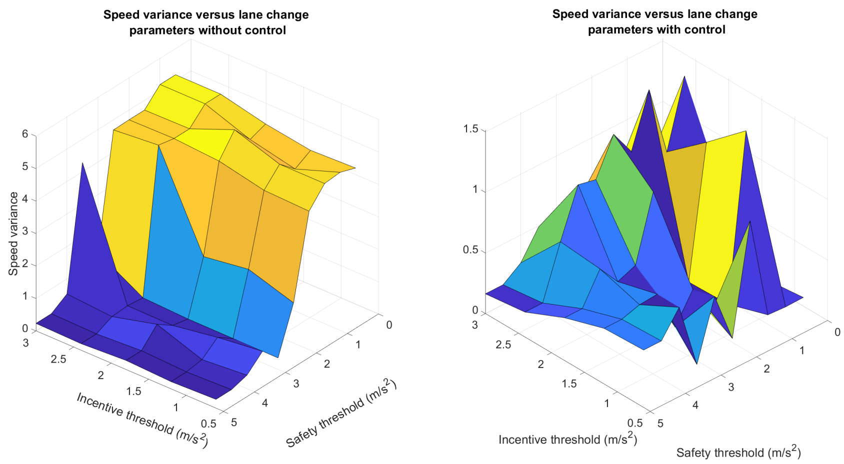

Fig. 4. Speed variance for different safety and incentive thresholds. Left: without control, Right: with control.

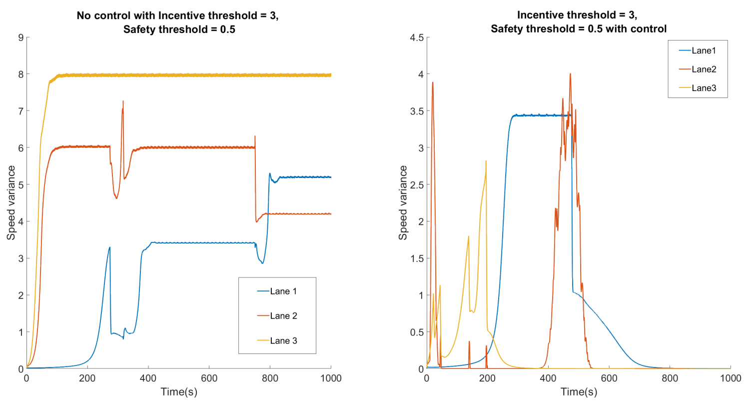

Fig. 5. Speed variance over time for different threshold parameters. Left: without control, Right: with control.

4 Collaborative driving

Collaborative driving (CD), also combined with autonomy [11,45], is an important emerging aspect of Intelligent Transportation Systems (ITS). CD promises to highly impact traffic inefficiencies, including dissipation of stop-and-go waves. CD is often times based on communication with various possible approaches proposed in the literature [19,23,29,30,33,35], and needs to take into account human behavior [24,41].

Here, we take the simple approach of assuming that a fraction of the human drivers is instructed to target specific preferred speed while keeping a smooth and safe driving. This represents an offline centralized control mechanism, with decentralized human actuators. The target speed may be communicated daily or four times of day. More precisely, denoting S the number of vehicles in collaborative driving and I the rest of the vehicles in the lane, we have

$$\begin{cases} \dot{x}_i = v_i, \quad i = 1, \ldots, n, \\ \dot{v}_i = \alpha_i (V(x_{i+1} - x_i) - v_i) + \beta_i \dfrac{v_{i+1} - v_i}{(x_{i+1} - x_i - l_v)^2}, \end{cases} \tag{9}$$

where n is the number of vehicles, $\alpha_i = \alpha$ and $\beta_i = \beta$ if $i \in I$, $\alpha_i = \alpha_S$ and $\beta_i = \beta_S$ if $i \in S$, with α and β such that (2) holds and α_S and β_S satisfy the opposite inequality

$$\frac{\alpha}{2} + \frac{L^2 \beta}{(n)^2} > V'\left(\frac{n}{L}\right). \tag{10}$$

We present here numerical simulations suggesting that such a collaborative behavior allows to recover some stability of the flow and decreases speed variance and car accelerations, and hence energy consumption.

In Figure 6 we present simulations where the proportion of cars p varies from no collaborative behavior to 100% of drivers following this collaborating behavior. As expected, when $p = 0$ (no collaborative behavior) the system has a large speed variance, while when $p = 1$ the system is stable and hence the speed variance is close to 0. However, what is interesting to see is that as soon as $p > 0$, that is to say as soon as some vehicles starts to have a collaborative behavior, the global speed variance of the system diminishes. In Figure 6 we ran simulations on a single lane ring-road of $258m$ with 25 cars during $1000s$ an starting close to the steady-state equilibrium with random initial conditions. Among the 25 cars the number of cars with a collaborative behavior was 25 ($p = 1$), 12 ($p = 0.48$), then 8 ($p = 0.32$), 6 ($p = 0.24$), 5 ($p = 0.20$), 4 ($p = 0.16$), 3 ($p = 0.12$), 2 ($p = 0.08$), 1 (p=0.04), 0 ($p = 0$). For each of these proportions we ran 40 simulations for which we computed the instantaneous spatial speed variance between cars averaged over the last 100s of the simulation, and then averaged it over the 40 simulations. This observation is an incentive to look more in details at collaborating behaviors. For instance, in these simulations the cars with collaborative behaviors are as evenly distributed in the traffic, and it would be interesting to see if there is any difference when they are clustered.

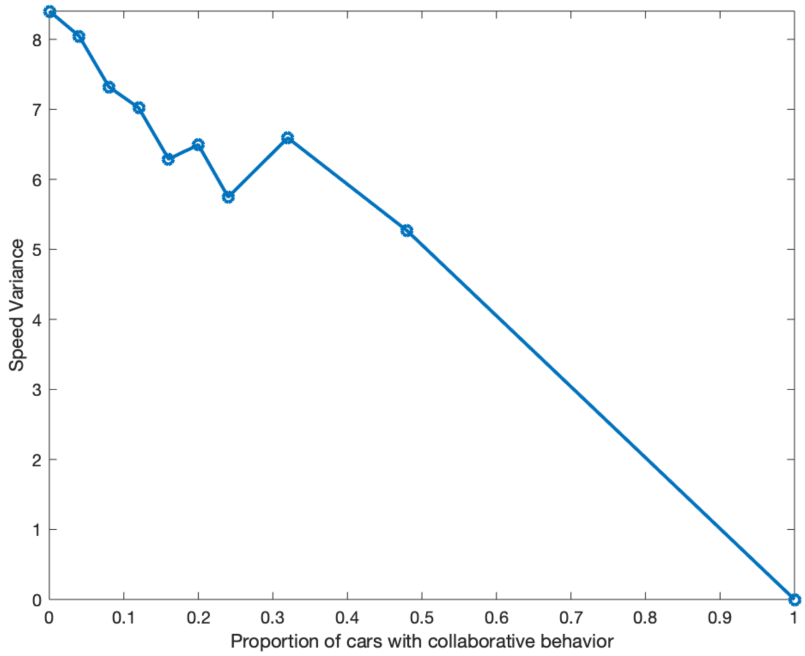

Fig. 6. Speed variance with respect to proportion of cars with a collaborative behavior

5 Going further and mean-field models

In this subsection, we go further and talk about the mean-field models on vehicular traffic.

Microscopic models describe the details of the traffic flow by studying each individual vehicle's microscopic properties like its position and velocity. The trajectories of the vehicles are predicted by means of ordinary differential equations (ODEs). Macroscopic models, assume a sufficiently large number of vehicles on the road and treats vehicular traffic as fluid flow. In particular, the evolution of the traffic density μ is governed by partial differential equations (PDEs). Thus, by capturing and predicting the main phenomenology of microscopic dynamics, macroscopic models can provide an overall and statistical view of traffic. One can also use a coupled ODE-PDE system to model the dynamics of a small number of AVs and a large number of regular vehicles on a single lane. This is clearly a combination of the microscopic and macroscopic models using multiple scales together.

The relationship between the two different scale models, microscopic and macroscopic models, can be both formally and rigorously established via mean-field approach by taking the number of vehicles N to go to infinity. Let (x_i, v_i) be the position-velocity vector of the i-th vehicle and μ be the density distribution of infinitely many vehicles in the space of position and velocity. The dynamics of the finitely many vehicles can be described by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = H * \mu_N(x_i, v_i), \quad i = 1, \dots, N, \end{cases} \tag{11}$$

where $H\colon \mathbb{R}\times\mathbb{R}^+ \mapsto \mathbb{R}$ is a convolutional kernel and $\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{(x_i(t),v_i(t))}$ is a probability measure. The dynamics of the infinitely many vehicles can be described by

$$\partial_t\mu + v\cdot\nabla_x\mu = \nabla_v\cdot[(H*\mu)\mu]. \tag{12}$$

Furthermore, one can rigorously derive the mean-field limit of the finite-dimensional ODE system (11), and the infinite dimensional mean-field limit (12) (a Vlasov-Poisson type PDE), see [14]. We also want to point out that the above mean-field approach is different with the so called "mean-field games" approach. For the mean-field games approach, one assumes many agents with perfect knowledge of the system and gives a strategy to solve a game, then passes it to the limit as in the mean-field approach.

For the multi-lane and multi-class traffic that includes AVs and regular vehicles, one can only consider a mean-field limit for the dynamics of the regular vehicles compared to the finitely many AVs governed by control dynamics. The lane changing maneuvers of the infinitely many regular vehicles lead to a source term of the Vlasov-Poisson type PDE. The limit process from a finite-dimensional controlled ODE system to an infinite-dimensional controlled coupled ODE-PDE system can be established in generalized Wasserstein distance. Additionally, one can also consider optimal control problems associated to the controlled ODE and coupled ODE-PDE systems where the cost functions represent, for instance, fuel consumption. Moreover, we have the following theorem, see [14].

Theorem 51 *The optimal solution to the optimal control problem of the ODE system converges to the optimal solution of the optimal control problem of the coupled ODE-PDE system as the number of regular vehicles N goes to infinity.*

Note that Theorem 51 implies that one can design controls in the microscopic level and be able to pass the limit to get the control in the mean-field limit level.

6 Conclusion

In this paper, we presented a hybrid multi-lane micro-model for traffic flow in a ring-road. This model exhibits stop and go waves, and we show that the safety and inventive thresholds in lane changing conditions highly impact the behavior of the system. We use a single AV as a means of control for dissipating stop and go waves, and we show that even simple controllers can be very efficient in reducing traffic, whatever the thresholds of the lane changing conditions. Additionally, this can be shown to be a very good basis to derive a controller for mean-field models, when we consider a mean-field limit for the dynamics of infinitely many regular vehicles and control dynamics for finitely many AVs.

7 Authors contribution statement

N.K. and A.H. performed research and simulations on a project designed by B.P. with A.B. and A.H.. S.T and P.A. wrote the simulation code, N.K., A.H., S. M., X. G. and B.P. wrote the paper

8 Acknowledgements

This research is based upon work supported by the U.S. Department of Energy's Office of Energy Efficiency and Renewable Energy (EERE) under the Vehicle Technologies Office award number CID DE-EE0008872. The views expressed herein do not necessarily represent the views of the U.S. Department of Energy or the United States Government. B.P. acknowledges the support of USDOT award # 69A3551747124 via the NYU-C2SMART project "Collaborative Driving, Ramp Metering and Mean-field Controls".

References

1. K Ahmed, M Ben-Akiva, H Koutsopoulos, and R Mishalani. Models of freeway lane changing and gap acceptance behavior. *Transportation and traffic theory*, 13:501–515, 1996.
2. G Albi, N Bellomo, L Fermo, S-Y Ha, J Kim, L Pareschi, D Poyato, and J Soler. Vehicular traffic, crowds, and swarms: From kinetic theory and multiscale methods to applications and research perspectives. *Mathematical Models and Methods in Applied Sciences*, 29(10):1901–2005, 2019.
3. AATM Aw and Michel Rascle. Resurrection of" second order" models of traffic flow. *SIAM journal on applied mathematics*, 60(3):916–938, 2000.
4. Nicola Bellomo and Christian Dogbe. On the modeling of traffic and crowds: A survey of models, speculations, and perspectives. *SIAM review*, 53(3):409–463, 2011.
5. M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, 1998.
6. S. Cui, B. Seibold, R. Stern, and D. B. Work. Stabilizing traffic flow via a single autonomous vehicle: Possibilities and limitations. In *2017 IEEE Intelligent Vehicles Symposium (IV)*, pages 1336–1341, 2017.
7. Shumo Cui, Benjamin Seibold, Raphael Stern, and Daniel B Work. Stabilizing traffic flow via a single autonomous vehicle: Possibilities and limitations. In *2017 IEEE Intelligent Vehicles Symposium (IV)*, pages 1336–1341. IEEE, 2017.
8. L. C. Davis. Effect of adaptive cruise control systems on traffic flow. *Phys. Rev. E*, 69:066110, Jun 2004.
9. Maria Laura Delle Monache and Paola Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *Journal of Differential equations*, 257(11):4015–4029, 2014.
10. Maria Laura Delle Monache, Thibault Liard, Anaïs Rat, Raphael Stern, Rahul Bhadani, Benjamin Seibold, Jonathan Sprinkle, Daniel B Work, and Benedetto Piccoli. Feedback control algorithms for the dissipation of traffic waves with autonomous vehicles. In *Computational Intelligence and Optimization Methods for Control Engineering*, pages 275–299. Springer, 2019.
11. Zheng Dong, Weisong Shi, Guangmo Tong, and Kecheng Yang. Collaborative autonomous driving: Vision and challenges. In *2020 International Conference on Connected and Autonomous Driving (MetroCAD)*, pages 17–26, 2020.
12. Shimao Fan, Ye Sun, Benedetto Piccoli, Benjamin Seibold, and Daniel B Work. A collapsed generalized aw-rascle-zhang model and its model accuracy. *arXiv preprint arXiv:1702.03624*, 2017.
13. Morris R Flynn, Aslan R Kasimov, J-C Nave, Rodolfo R Rosales, and Benjamin Seibold. Self-sustained nonlinear waves in traffic flow. *Physical Review E*, 79(5):056113, 2009.
14. Massimo Fornasier, Benedetto Piccoli, and Francesco Rossi. Mean-field sparse optimal control. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20130400, 2014.
15. Mauro Garavello and Benedetto Piccoli. Hybrid necessary principle. *SIAM Journal on Control and Optimization*, 43(5):1867–1887, 2005.
16. Peter G Gipps. A model for the structure of lane-changing decisions. *Transportation Research Part B: Methodological*, 20(5):403–414, 1986.
17. R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.

18. Maxime Guériau, Romain Billot, Nour-Eddin [El Faouzi], Julien Monteil, Frédéric Armetta, and Salima Hassas. How to assess the benefits of connected vehicles? a simulation framework for the design of cooperative traffic management strategies. *Transportation Research Part C: Emerging Technologies*, 67:266 – 279, 2016.
19. S. Halle, J. Laumonier, and B. Chaib-Draa. A decentralized approach to collaborative driving coordination. In *Proceedings. The 7th International IEEE Conference on Intelligent Transportation Systems (IEEE Cat. No.04TH8749)*, pages 453–458, 2004.
20. Amaury Hayat, Thibault Liard, Francesca Marcellini, and Benedetto Piccoli. A multiscale second order model for the interaction between av and traffic flows: analysis and existence of solutions. *Preprint*, 2021.
21. Peter Hidas. Modelling lane changing and merging in microscopic traffic simulation. *Transportation Research Part C: Emerging Technologies*, 10(5-6):351–371, 2002.
22. Rui Jiang, Mao-Bin Hu, HM Zhang, Zi-You Gao, Bin Jia, Qing-Song Wu, Bing Wang, and Ming Yang. Traffic experiment reveals the nature of car-following. *PloS one*, 9(4):e94351, 2014.
23. Nevin Vunka Jungum, Razvi M. Doomun, Soulakshmee D. Ghurbhurrun, and Sameerchand Pudaruth. Collaborative driving support system in mobile pervasive environments. In *2008 The Fourth International Conference on Wireless and Mobile Communications*, pages 358–363, 2008.
24. Rinta Kridalukmana, Hai Yan Lu, and Mohsen Naderpour. A supportive situation awareness model for human-autonomy teaming in collaborative driving. *Theoretical Issues in Ergonomics Science*, 21(6):658–683, 2020.
25. Jean-Patrick Lebacque, Salim Mammar, and Habib Haj Salem. Generic second order traffic flow modelling. In *Transportation and Traffic Theory 2007. Papers Selected for Presentation at ISTTT17Engineering and Physical Sciences Research Council (Great Britain) Rees Jeffreys Road FundTransport Research FoundationTMS ConsultancyOve Arup and Partners, Hong KongTransportation Planning (International) PTV AG*, 2007.
26. Thibault Liard and Benedetto Piccoli. On entropic solutions to conservation laws coupled with moving bottlenecks. Technical report, June 2019. working paper or preprint, 2019.
27. Thibault Liard and Benedetto Piccoli. Well-posedness for scalar conservation laws with moving flux constraints. *SIAM Journal on Applied Mathematics*, 79(2):641–667, 2019.
28. M. J. Lighthill and G. B. Whitham. On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads. *Royal Society of London Proceedings Series A*, 229:317–345, May 1955.
29. Shou-Pon Lin and Nicholas F. Maxemchuk. The fail-safe operation of collaborative driving systems. *Journal of Intelligent Transportation Systems*, 20(1):88–101, 2016.
30. Alberto Petrillo, Alessandro Salvi, Stefania Santini, and Antonio Saverio Valente. Adaptive multi-agents synchronization for collaborative driving of autonomous vehicles with multiple communication delays. *Transportation Research Part C: Emerging Technologies*, 86:372–392, 2018.
31. Benedetto Piccoli. Hybrid systems and optimal control. In *Proceedings of the 37th IEEE Conference on Decision and Control (Cat. No. 98CH36171)*, volume 1, pages 13–18. IEEE, 1998.
32. Benedetto Piccoli and Andrea Tosin. Vehicular traffic: A review of continuum mathematical models. *Encyclopedia of Complexity and Systems Science*, 22:9727–9749, 2009.
33. Shou pon Lin and Nicholas F. Maxemchuk. An architecture for collaborative driving systems. In *2012 20th IEEE International Conference on Network Protocols (ICNP)*, pages 1–2, 2012.
34. P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.
35. Adrian Sannier, Ole Balling, Mark Knight, and Bryan Walter. Collaborative driving simulation. In *SAE 2002 World Congress & Exhibition*. SAE International, mar 2002.
36. Raphael E. Stern, Shumo Cui, Maria Laura Delle Monache, Rahul Bhadani, Matt Bunting, Miles Churchill, Nathaniel Hamilton, R'mani Haulcy, Hannah Pohlmann, Fangyu Wu, Benedetto Piccoli, Benjamin Seibold, Jonathan Sprinkle, and Daniel B. Work. Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments. *Transportation Research Part C: Emerging Technologies*, 89:205 – 221, 2018.

37. Yuki Sugiyama, Minoru Fukui, Macoto Kikuchi, Katsuya Hasebe, Akihiro Nakayama, Katsuhiro Nishinari, Shin ichi Tadaki, and Satoshi Yukawa. Traffic jams without bottlenecks – experimental evidence for the physical mechanism of the formation of a jam. *New Journal of Physics*, 10(3):033001, 2008.
38. Alireza Talebpour and Hani S. Mahmassani. Influence of connected and autonomous vehicles on traffic flow stability and throughput. *Transportation Research Part C: Emerging Technologies*, 71:143 – 163, 2016.
39. Tomer Toledo, Haris N Koutsopoulos, and Moshe E Ben-Akiva. Modeling integrated lane-changing behavior. *Transportation Research Record*, 1857(1):30–38, 2003.
40. C. Tomlin, G.J. Pappas, and S. Sastry. Conflict resolution for air traffic management: a study in multiagent hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):509–521, apr 1998.
41. Duy Tran, Jianhao Du, Weihua Sheng, Denis Osipychev, Yuge Sun, and He Bai. A human-vehicle collaborative driving framework for driver assistance. *IEEE Transactions on Intelligent Transportation Systems*, 20(9):3470–3485, 2019.
42. Martin Treiber, Ansgar Hennecke, and Dirk Helbing. Congested traffic states in empirical observations and microscopic simulations. *Physical review E*, 62(2):1805, 2000.
43. Martin Treiber and Arne Kesting. Modeling lane-changing decisions with mobil. In *Traffic and Granular Flow'07*, pages 211–221. Springer, 2009.
44. M. Wang, W. Daamen, S. P. Hoogendoorn, and B. van Arem. Cooperative car-following control: Distributed algorithm and impact on moving jam features. *IEEE Transactions on Intelligent Transportation Systems*, 17(5):1459–1471, 2016.
45. Junqing Wei and John M. Dolan. A multi-level collaborative driving framework for autonomous vehicles. In *RO-MAN 2009 - The 18th IEEE International Symposium on Robot and Human Interactive Communication*, pages 40–45, 2009.
46. Fangyu Wu, Raphael E. Stern, Shumo Cui, Maria Laura [Delle Monache], Rahul Bhadani, Matt Bunting, Miles Churchill, Nathaniel Hamilton, R'mani Haulcy, Benedetto Piccoli, Benjamin Seibold, Jonathan Sprinkle, and Daniel B. Work. Tracking vehicle trajectories and fuel rates in phantom traffic jams: Methodology and data. *Transportation Research Part C: Emerging Technologies*, 99:82 – 109, 2019.
47. Qi Yang and Haris N Koutsopoulos. A microscopic traffic simulator for evaluation of dynamic traffic management systems. *Transportation Research Part C: Emerging Technologies*, 4(3):113–129, 1996.
48. Zuduo Zheng. Recent developments and research needs in modeling lane changing. *Transportation research part B: methodological*, 60:16–32, 2014.