2022 American Control Conference (ACC)
Atlanta, USA, June 8-10, 2022

Understanding the Interplay Between Herd Behaviors and Epidemic Spreading Using Federated Evolutionary Games

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Abstract— **The recent COVID-19 pandemic has led to an increasing interest in the modeling and analysis of infectious diseases. Our social behaviors in the daily lives have been significantly affected by the pandemic. In this paper, we propose a federated evolutionary game-theoretic framework to study the coupling of herd behaviors changes and epidemics spreading. Our framework extends the classical degree-based mean-field epidemic model over complex networks by integrating it with the evolutionary game dynamics. The statistically equivalent individuals in a population choose their social activity intensities based on the fitness or the payoffs that depend on the state of the epidemics. Meanwhile, the spread of infectious diseases over the complex network is reciprocally influenced by the players' social activities. We address the challenge of federated dynamics by breaking the analysis into the studies of the stationary properties of the epidemic for given herd behavior and the structural properties of the game for a given epidemic process. We use numerical experiments to show that our framework enables the prediction of the historical COVID-19 statistics.**

I. Introduction

The COVID-19 pandemic has unprecedentedly impacted our society in many ways. Many people work at home, shop online, and communicate over zoom. During the past years, we have witnessed a litany of policies regarding social distancing, mask-wearing, and vaccination to prevent and mitigate the spreading of the pandemic. The pandemic has made a significant impact on the way we behave and interact in our daily life. We have observed a strong interplay between people's behaviors and the pandemic. When the number of COVID cases goes down, reopening policies enable more social activities to return to normal. If not done carefully, they would create additional infection waves, which we have witnessed recently in many countries.

Herd behavior describes the collective behavioral pattern of a population resulting from the behaviors of individuals in the same fashion. It plays an important role in the pandemic since it is often driven by policies or individual incentives. For example, cities will design incentives for individuals to be vaccinated to reach targeted herd immunity. Many countries have enforced the policies of mask-wearing in public spaces to create herd behavior that reduces the risk of mass infection.

Research on herd behaviors has focused mainly on topics related to financial markets and economics [1]. At the same time, epidemic processes are often studied as stand-alone dynamical processes without incorporating individual behaviors into the model [2]. There is a need for an integrated framework that gives a holistic understanding of the pandemic together with herd behaviors.

This work is partially supported by grants ECCS-1847056, CNS-2027884, and BCS-2122060 from National Science Foundation (NSF), grant 20-19829 from DOE-NE, and grant W911NF-19-1-0041 from Army Research Office (ARO).

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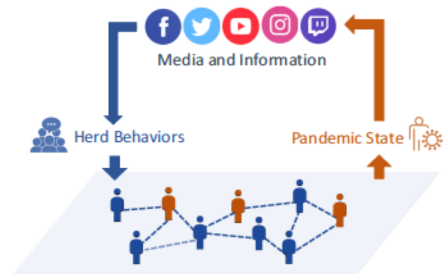


Fig. 1. The federated evolutionary game framework. The herd behaviors influence epidemic spreading among the population. The media reports about the epidemic states stimulate strategy revisions, which reshape herd behaviors.

In this paper, we propose a federated evolutionary game-theoretic framework to model the herd behaviors that are coupled with the spreading of epidemics. Motivated by the vaccination game of [3], we consider evolutionary games played by populations of players. However, instead of considering the convergently stable Nash equilibria, we focus directly on the evolutionary dynamics. The evolutionary game dynamics [4] are population-level or mean-field dynamics that describe the evolution or the adaptive revision of the strategies when the populations interact with each other. They explicitly describe the changes in herd behaviors.

One critical component of the evolutionary game framework is the modeling of infectious disease. In this work, we consider a class of mean-field epidemic models over complex networks [2], [5], [6], [7] to capture the social interactions among the individuals. The individuals over the network are assumed to be statistically equivalent within the same sub-population. The mean-field dynamics forthrightly describe the influences of herd behavior on the spreading of the epidemic. We use a complex network model characterized by a degree distribution to represent the social interactions of the populations. Each individual in the network is associated with a degree of connections that determines the probability of infection and thus the spreading of the disease.

The epidemic model is consolidated into the evolutionary game as illustrated in Fig. 1. The spreading of the epidemic among the populations is affected by the social activity intensities of the individuals. As the information and the policies concerning the epidemic are communicated to the population through public media, individuals respond to them

978-1-6654-5196-3/$31.00 ©2022 AACC

and adapt their social activities, constituting herd behavior at the population level. It is clear from Fig. 1 that the state of the epidemics and the herd behaviors are interdependent.

The federated framework in Fig. 1 can be mathematically described by a system of coupled differential equations. One set of differential equations represents the mean-field evolutionary dynamics of the game strategies. The other set of differential equations describes the epidemic process. It is critical to examine the structural properties of the federated dynamics, including the stability and the steady state. To this end, we first discuss the stability of the epidemic dynamics under fixed herd behavior and then analyze the structural properties of the evolutionary game under the steady states. We find out that, under certain conditions, there is a unique nontrivial globally asymptotically stable steady state given the herd behaviors. The players' decisions in the game turn out to be strategic substitutes. This property makes the evolutionarily stable strategies (ESS) [4] or the Nash equilibria achievable even when the individuals revise their strategies myopically on their own. We also formulate a unified optimization problem to compute the Nash equilibrium based on an equivalent representation of the population game as a finite-player game problem, where each population is viewed as a player. We use numerical experiments to compare the simulated infection curve with the COVID-19 statistics of the infected in New York City. The prediction of two peaks in the pandemic over a time interval of interest provides a promising analytical and policy design tool for the pandemic.

In the full-length online version of our paper [8], we extend the analysis of structural properties of the game to different time scales and provide techniques for approximation purposes. We also investigate the role of information in shaping human behavior and study network structures as the consequences of behavioral patterns.

We introduce the general framework in Section II. In Section III, we present analytical results when the epidemic evolves at a faster time-scale. Section IV presents the numerical experiments. We conclude the paper in Section V.

II. Problem Setting

In this section, we describe our federated evolutionary game framework in detail. We first introduce the general framework and then turn to the setting under epidemic. We will start by considering a finite number of players and show later that the number can go to infinity.

A. Description of the General Framework

Consider N players (nodes) over a network. Each player belongs to a subset in the set $\mathcal{D} := \{1, 2, ..., D\}$ representing its degree of connectivity, *i.e.*, a player in $d \in \mathcal{D}$ has degree d. The number of players in $d \in \mathcal{D}$ is $Nm^d \in \mathbb{N}_+$. The degree distribution is then denoted by $[m^d]_{d\in\mathcal{D}}$. Let Ξ be the finite state space of all the players and $\mathcal{S}^d \subset [0,1]^{n^d}$ be the finite strategy space of players with degree d, with $|\Xi| = L$ and $|\mathcal{S}^d| = n^d$, respectively. We use $s_i^d \in \mathcal{S}^d$ to denote a typical strategy. Let $\mathcal{S} := \prod_{d\in\mathcal{D}} \mathcal{S}^d$ with $|\mathcal{S}| = \sum_{d\in\mathcal{D}} n^d = n$. Under the epidemic context, elements in Ξ describe the health status of individuals in an epidemic, such as 'Susceptible', 'Infected', and 'recovered' in the classic SIR model [5]. Elements in X are the individual actions that affect the spreading of the virus, such as willingness to wear a mask and the probability to take the vaccine.

Let $w_\xi^d(t)$ be the fraction of players with degree d who are in state ξ at time t. Let $x_i^d(t)$ be the fraction of players with degree d playing strategy s_i^d at time t. We use $w(t) = (w^1(t), ..., w^D(t)) \in \mathcal{W} \subset \mathbb{R}_+^{DL}$ and $(t) = (x^1(t), ..., x^D(t)) \in \mathcal{X} = \prod_{d\in\mathcal{D}} \mathcal{X}^d \subset \mathbb{R}_+^{n^1\times\cdots\times n^D}$ to denote the concatenations of $w_\xi^d(t)$ and $x_i^d(t)$, respectively.

Suppose that players are constantly interacting physically on the network. Physical interactions cause potential changes in players' states. We use the degree-based mean-field approach [2] to capture the federated dynamical system on the large network. The players are assumed to be statistically equivalent if they have the same degree and the same strategy. In other words, in the large population, the players are distinguishable only based on their degree and their strategy.

Let $Q_\xi^d : \mathcal{W} \times \mathcal{X} \times [0,1]^P \to \mathbb{R}$ be a Lipschitz function describing the dynamical evolution of the fraction of players with degree d that are in state ξ. The federated dynamics of players' state transitions are as follows:

$$\dot{w}_\xi^d(t) = Q_\xi^d\left(w(t), x(t), [m^d]_{d\in\mathcal{D}}\right), \quad \forall d \in \mathcal{D}, \forall \xi \in \Xi. \tag{1}$$

Note that in (1), the dependence on $w(t)$ emphasizes the coupling of players' state transitions, and the dependence on $[m^d]_{d\in\mathcal{D}}$ illustrates the effect of the network. We use $\bar{w}(x)$ to denote the steady-state value of $w(x)$.

As the epidemic evolves, the players receive information from public media containing samples of the epidemic state at times. We assume that the times between information broadcasts are independent, and they follow a rate τ exponential distribution. We use a strategic interaction to represent a round where players update their strategies based on the current information broadcast. In a strategic interaction, players update their strategies by considering only the current information broadcast. The information broadcast contains w and x at the time of sampling. Let $R_{ij}^d(w(t), x(t)) : \mathcal{W} \times \mathcal{X} \to [0,1]$ denote the probability of a player with degree d switching from strategy s_i^d to strategy s_j^d. Next, we illustrate the evolution of the fraction of players with degree d playing strategy s_i^d. Consider a small time period dt. There will be τdt expected informational interactions during this period. The change of the number of players with degree d playing strategy s_i^d is:

$$N\left[\sum_{j\in\mathcal{S}^d} x_j^d(t) R_{ji}^d(w(t), x(t)) - \sum_{j\in\mathcal{S}^d} x_i^d(t) R_{ij}^d(w(t), x(t))\right] \tau dt. \tag{2}$$

By considering fractions of players in (2), we obtain the mean dynamic as follows:

$$\begin{aligned} \frac{1}{\tau}\dot{x}_i^d(t) = &\sum_{j\in\mathcal{S}^d} x_j^d(t) R_{ji}^d(w(t), x(t)) \\ &- \sum_{j\in\mathcal{S}^d} x_i^d(t) R_{ij}^d(w(t), x(t)), \forall i \in \mathcal{S}^d, \forall d \in \mathcal{D}. \end{aligned} \tag{3}$$

The evolutions of (1) and (3) constitute a system of federated differential equations.

Since the evolutions in (1) and (3) consider the fractions of the players, we naturally interpret the above setting as the interactions among populations when the total number of players goes to infinity, *i.e.*, $N \to +\infty$.

Let $F_i^d : \mathcal{W} \times \mathcal{X} \to \mathbb{R}$ be the payoff function of a player with degree d who plays strategy s_i^d. In general, F_i^d depends on $w(t)$ and $x(t)$. The dependence on $x(t)$ characterizes the game-theoretic aspect of the framework. The dependence on $w(t)$ reveals the coupling of all players' state evolutions. Let $F = (F^1, ..., F^D) \in \mathbb{R}^{n^1 \times \cdots \times n^D}$.

Connecting with the standard definition of evolutionary games [4], we call x a social state and call, with a slight abuse of terminology, $R := (R^1, ..., R^D)$ a revision protocol. We refer to the game defined by the payoff function F, the evolutionary dynamics (3), and the federated state dynamics (1) as a federated evolutionary game.

Definition 1. *Let $NE(F)$ denote the set of NE of the game defined by the payoff F and the federated state transitions (1). A social state $x \in \mathcal{X}$ is a Nash Equilibrium (NE), i.e., $x \in NE(F)$, if for all $d \in \mathcal{D}$, $x^d \in m^d BR^d(x)$, where the set $BR^d(\cdot)$ denotes the set of best responses, i.e., $BR^d(x) := \{z \in \mathbb{R}_+^{n^d} : \mathbf{1}^T z = 1, z_i > 0$ if $s_i^d \in \arg\max_{j \in \mathcal{S}^d} F_j^d(x, \bar{w}(x))\}$.*

The difference between Definition 1 and the standard NE is that x is the best response to the payoffs generated by x together with the steady-state value of (1) given x. This definition integrates the coupling of (1) and (3) into the definition of NE of our framework.

In the following, we assume that the strategies in each set $\mathcal{S}^d$ are listed in an increasing order, *i.e.*, $s_i^d > s_j^d$ if $i > j$.

B. The framework Under Epidemics

Consider the states of the players described by the Susceptible-Infected compartmental model on a degree-based network [5] with degree distribution $[m^d]_{d \in \mathcal{D}}$. We use $I_i^d(t)$ to denote the fraction of the infected players in population d adopting strategy s_i^d at time t. We regard a strategy s_i^d as a player's social inactivity level, which is the opposite of the social activity level (SAL). The SAL, represented by $1 - s_i^d$, describes the probability of a player behaving actively in face-to-face social activities through all of her connections with other players. A strategy s_i^d close to 1 means that the player is highly cautious when interacting physically through her connections with others. Given social state x, the evolution of $I_i^d(t)$ is also affected by a recovery rate $\gamma \in \mathbb{R}_+$ and a contagion rate $\lambda \in \mathbb{R}_+$. The dynamical system, analogous to (1), describing the time evolution of $I_i^d(t)$ is

$$\dot{I}_i^d(t) = -\gamma I_i^d(t) + \lambda_i^d \left(1 - I_i^d(t)\right) d\Theta(t), \tag{4}$$

where $\lambda_i^d = \lambda(1 - s_i^d)$ denotes the activity-aware contagion rate of a player with degree d and SAL $1 - s_i^d$. The second term on the right-hand side of (4) describes the events related to contagion, whose probability is proportional to the activity-aware contagion rate λ_i^d, the density of susceptible players $1 - I_i^d(t)$, the degree of connections d, and the probability $\Theta(t)$ that a link is connected to an infected player. This probability can be expressed as follows:

$$\Theta(t) = \frac{\sum_{d \in \mathcal{D}} \left(\sum_{i \in \mathcal{S}^d} d x_i^d I_i^d(t) \right)}{\sum_{d \in \mathcal{D}} d m^d}. \tag{5}$$

Since we have assumed the statistical equivalence of players with the same degree and the same strategy, the contributions of players to the probability $\Theta(t)$ are lumped together according to their degrees and strategies. The probability of a link connecting to an infected player with degree d choosing strategy s_i^d is proportional to $d x_i^d I_i^d(t)$. Hence, by normalizing using the average degree, we obtain (5). The consistency of (4) and (5) follows from a similar reasoning as discussed in [5], since the effect of SAL has already been considered in λ_i^d.

We remark that (5) couples the dynamics in (4) of each strategy in each population. Let $I(t) = (I^1(t), ..., I^D(t)) \in [0,1]^{n^1 \times \cdots \times n^D}$. We use $\bar{I}_i^d$ and $\bar{\Theta}$ to denote the steady-states of $I_i^d(t)$ and $\Theta(t)$. The concatenations are $\bar{I}$ and $\bar{I}^d$.

The payoff F_i^d depends on the information broadcast at the time of sampling. In general, it takes the form

$$F_i^d = s_i^d \mathcal{U}_{ina} + (1 - s_i^d) \mathcal{U}_{act}^{d,i}, \tag{6}$$

where $\mathcal{U}_{act}^{d,i} : [0,1] \to \mathbb{R}$ and $\mathcal{U}_{ina} := r_{ina} \in \mathbb{R}_-$. In (6), $s_i^d \mathcal{U}_{ina}$ represents the expected utility of being socially inactive. The term $\mathcal{U}_{ina}$ corresponds to isolating oneself from others. Hence, we assume that $\mathcal{U}_{ina}$ is a negative constant reward for all players. The term $(1 - s_i^d)\mathcal{U}_{act}^{d,i}$ in (6) contributes to expected utility of being socially active, where $\mathcal{U}_{act}^{d,i}$ corresponds to the reward from getting infected through physical interactions on the network. Therefore, we set $\mathcal{U}_{act}^{d,i}$ to be a decreasing function of η_i^d, which represents the probability that a player in population d playing strategy s_i^d is infected. The probability that a player is infected can be equivalently understood as the fraction of infected players. Then, we obtain $\eta_i^d = \mathcal{O}_i^d(I_i^d(t))$, where $\mathcal{O}_i^d$ is a player's observation of the infected fraction of players at the time of an information broadcast. Note that the case of imperfect observations will be considered later. For now, we consider perfect observations, *i.e.*, $\eta_i^d = I_i^d(t)$. Since the evolution (4) is coupled and the term (5) depends on x, the payoff satisfies the definition in Section II-A. We remark that the rate parameter τ determines the rate of the federated system of differential equations. Therefore, the sampled epidemic status is at a steady state if $\tau \to \infty$ and is time-dependent otherwise. In this paper, we let $\mathcal{U}_{act}^{d,i} = -r_{act} \eta_i^d$ for all players with reward parameter $r_{act} \in \mathbb{R}_+$, for simplicity reasons.

By defining $r = \frac{r_{ina}}{r_{act}} \in \mathbb{R}_-$ to be the relative reward of being socially inactive against being socially active, we obtain the payoff function suitable under (4) as follows:

$$F_i^d = s_i^d r - (1 - s_i^d) \eta_i^d. \tag{7}$$

Note that we drop the dependence of F_i^d and η_i^d on the state I when we analyze equilibrium behaviors since I is a function of the social state x, as can be observed in Definition 1.

III. Long-Term Behavior

In this section, we study the behavior of our model when $\frac{1}{\tau} \to \infty$. This setting means that information broadcasts take place at the steady states of (4). The scenario where a information broadcast takes place at time $T < +\infty$ is considered in [8], where we adopt a different style of analysis and obtain extensions of results to be presented in the following.

A. Steady States of Epidemic Dynamics Given Social States

From (4) and (5), we express the steady-state values as

$$\bar{I}_i^d = \frac{\theta_i^d \bar{\Theta}}{\gamma + \theta_i^d \bar{\Theta}}, \tag{8}$$

where $\theta_i^d = \lambda d(1 - s_i^d)$, and

$$\bar{\Theta} = \frac{\sum_{d\in\mathcal{D}} \left(d \sum_{i\in\mathcal{S}^d} x_i^d \bar{I}_i^d \right)}{\sum_{d\in\mathcal{D}} dm^d}. \tag{9}$$

Let $\bar{d} := \sum_{d\in\mathcal{D}} dm^d$ denote the average degree of the network. By combining (8) and (9), we obtain the equation containing only $\bar{\Theta}$ as follows:

$$\bar{\Theta} = \bar{d}^{-1} \sum_{d\in\mathcal{D}} \left(d \sum_{i\in\mathcal{S}^d} \frac{x_i^d \theta_i^d \bar{\Theta}}{\gamma + \theta_i^d \bar{\Theta}} \right). \tag{10}$$

From (10), we observe that $\bar{\Theta}_0 = 0$ is always a solution. Accordingly, $\bar{I}_{i,0}^d = 0$ for all d and i. We call the steady state $(\bar{I}_0, \bar{\Theta}_0)$ the zero steady state (or disease-free steady state), which refers to the proportion of the infected in players playing all strategies in all populations are all zero and that all links have zero probability to lead to an infected node. The zero steady state is often referred to as the disease-free state [6]. Meanwhile, positive steady states arise from dividing $\bar{\Theta}$ from both sides of (10) when $\bar{\Theta} \neq 0$:

$$1 = \bar{d}^{-1} \sum_{d\in\mathcal{D}} \left(d \sum_{i\in\mathcal{S}^d} \frac{x_i^d \theta_i^d}{\gamma + \theta_i^d \bar{\Theta}} \right). \tag{11}$$

In a positive steady state pair $(\bar{I}_+(x), \bar{\Theta}_+(x))$, we have $\bar{\Theta}_+(x) \in (0, 1]$. It shows that a link possesses a positive probability to connect to an infected node. In addition, $\bar{I}_{i,+}^d(x) = 0$ if and only if $s_i^d = 1$. This fact explains that a player can be safe from the epidemic only if she lives a totally isolated life. We remark that the positive steady state pair depends on the social state x since (10) contains x. The next result presents the conditions on the stability of the zero steady state and the positive steady state. We refer the readers to [8] for the proof.

Theorem 2. *Given a social state x, the zero steady state $(\bar{I}_0, \bar{\Theta}_0)$ is globally asymptotically stable if $\frac{\lambda d(1-s_i^d)}{\gamma} < 1$ for all $i \in \mathcal{S}^d$ and all $d \in \mathcal{D}$; the unique positive steady state $(\bar{I}_+(x), \bar{\Theta}_+(x))$ is globally asymptotically stable if $\frac{\lambda d(1-s_i^d)}{\gamma} \geq 1$ for all $i \in \mathcal{S}^d$ and all $d \in \mathcal{D}$.*

We will focus on the positive steady state pair $(\bar{I}_+(x), \bar{\Theta}_+(x))$ in the sequel, *i.e.*, the scenario where $\frac{\lambda d(1-s_i^d)}{\gamma} \geq 1$ for all $i \in \mathcal{S}^d$ and all $d \in \mathcal{D}$. Note that the uniqueness of positive steady state follows from the fact that the right-hand-side of (11) is strictly decreasing in $\bar{\Theta}$.

B. Numerical Computation of Steady States

Define $M(z) = \bar{d}^{-1} \sum_{d\in\mathcal{D}} \left(d \sum_{i\in\mathcal{S}^d} \frac{x_i^d \theta_i^d z}{\gamma + \theta_i^d z} \right)$ with $z \in [0, 1]$. The computation method to obtain a steady state relies on the next result, whose proof is omitted.

Theorem 3. *$M(\cdot)$ is a contraction mapping on $[0, 1]$.*

Theorem 3 indicates that the steady state $\bar{\Theta}$ can be obtained by the fixed-point iterations using the mapping $M(\cdot)$.

C. Equilibrium Analysis

Before focusing on the NE, we first introduce an equivalent D-player game associated with the population game. In this equivalent game, a player with degree d plays a weighted-mixed strategy x^d from the set $\mathcal{S}^d$. By weighted-mixed strategy, we refer to the restriction that $\mathbf{1}^T x^d = m^d$. Given a social state $x = (x^d, x^{-d})$, where x^{-d} denotes the population states of populations other than population d, the expected payoff of player d playing x^d is $EF^d(x^d, x^{-d}) := (x^d)^T F^d(x^d, x^{-d}) = \sum_{i\in\mathcal{S}^d} x_i^d F_i^d(x^d, x^{-d})$. The following result on the NE is inspired by this equivalent game. The proof can be found in the full-length paper [8].

Theorem 4. *A social state x^* is an NE of the game defined in Section II if and only if it solves:*

$$\begin{aligned} \min_{x\in\mathcal{X}, y\in\mathbb{R}^D} \quad & \sum_{d\in\mathcal{D}} -EF^d(x^d, x^{-d}) + \sum_{d\in\mathcal{D}} y^d m^d \\ \text{s.t.} \quad & -F^d(x) \geq -y^d \mathbf{1}_{n^d}, \quad \forall d \in \mathcal{D}, \\ & x^d \geq 0, \quad \mathbf{1}^T x^d = m^d, \quad \forall d \in \mathcal{D}. \end{aligned} \tag{12}$$

Gradient-based algorithms can be used to numerically solve (12). At each iteration of the algorithm, the descent direction consists of the gradient vector $\frac{\partial F_i^d}{\partial x}(x)$ for all $i \in \mathcal{S}^d$ and for all $d \in \mathcal{D}$. We provide below the explicit expression of the gradient vector given a social state.

With a slight abuse of notation, we specify the dependence on x by writing the steady-state quantities using $\bar{\Theta}(x)$ and $\bar{I}_i^d(x)$. We express the gradient using the chain rule as $\frac{\partial F_i^d}{\partial x}(x) = -\frac{(1-s_i^d)\gamma\theta_i^d}{(\gamma+\theta_i^d\bar{\Theta}(x))^2} \cdot \frac{\partial \bar{\Theta}}{\partial x}(x)$. Next, we derive the term $\frac{\partial \bar{\Theta}}{\partial x}(x)$ leveraging (11). Define $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by $H(\Theta, x) = (\bar{d})^{-1} \sum_{d\in\mathcal{D}} \left(d \sum_{i\in\mathcal{S}^d} \frac{x_i^d \theta_i^d}{\gamma + \theta_i^d \Theta} \right) - 1$. It is clear from the definition that H is continuously differentiable with respect to both arguments. Suppose that, given x^*, the pair (Θ^*, x^*) solves (11), *i.e.*, $H(\Theta^*, x^*) = 0$. The Jacobian of H with respect to the first argument at (Θ^*, x^*) is $J_\Theta(\Theta^*, x^*) := \frac{\partial H}{\partial \Theta}((\Theta^*, x^*)) \in \mathbb{R}$. It can be observed from (10) that $\Theta^* > 0$ if x^* is a social state. Hence, $J_\Theta(\Theta^*, x^*) < 0$. Invoking the implicit function theorem, we observe that there exists a neighborhood $\mathcal{V}_\Theta$ of Θ^* and a neighborhood $\mathcal{V}_x$ of x^*, such that there is a unique continuously differentiable function $h : \mathcal{V}_x \to \mathcal{V}_\Theta$ satisfying $h(x^*) = \Theta^*$ and $H(h(x^*), x^*) = 0$. Furthermore, the derivative of $h(\cdot)$ can be expressed as

$$\frac{\partial h}{\partial x_i^d}(x^*) = -(J_\Theta(h(x^*), x^*))^{-1} \frac{\partial H}{\partial x_i^d}(h(x^*), x^*). \tag{13}$$

Thus, the term $\frac{\partial \bar{\Theta}}{\partial x}(x)$ can be obtained directly using (13) at the given social state x. Therefore, the explicit gradient vectors are of the form

$$\frac{\partial F_i^d}{\partial x}(x) = \frac{(1 - s_i^d)\gamma\theta_i^d}{(\gamma + \theta_i^d \bar{\Theta}(x))^2} \left((J_\Theta(\bar{\Theta}(x), x))^{-1} \frac{\partial H}{\partial x}(\bar{\Theta}(x), x) \right), \tag{14}$$

where $\bar{\Theta}(x)$ is obtained from the fixed-point iterations using the mapping $M(\cdot)$ and

$$\frac{\partial H}{\partial x}(\bar{\Theta}(x),x)=\frac{1}{\bar{d}}\begin{pmatrix}\frac{1\cdot\theta_1^1}{\gamma+\theta_1^1\bar{\Theta}(x)} & \cdots & \frac{D\cdot\theta_{n^D}^D}{\gamma+\theta_{n^D}^D\bar{\Theta}(x)}\end{pmatrix}. \quad (15)$$

In general, the optimization problem (12) is nonconvex. However, gradient-based algorithms are still promising methods for finding stationary points, *i.e.*, points with sufficiently small gradients. Moreover, we know from the proof of Theorem 4 that the global optimal point yields a zero objective value. Therefore, we can test the stationary point obtained by the algorithms using the objective value to verify whether it is a potential global optimal point, *i.e.*, an NE social state.

D. Property of the Game

Stability studies the structural properties of the games under which sequential plays following specific revision protocols converge to an NE. In this section, we analyze players' incentives to change strategies when the game is played sequentially.

Let $DF(x) := \frac{d}{dx}F(x)\in\mathbb{R}^{n\times n}$ denote the derivative of the payoffs with respect to the social state. From (14), we can express DF as

$$DF(x) = \frac{\gamma}{\bar{d}}(J_\Theta(\bar{\Theta}(x),x))^{-1}\begin{pmatrix}\frac{(1-s_1^1)\theta_1^1}{(\gamma+\theta_1^1\bar{\Theta}(x))^2}\\ \vdots \\ \frac{(1-s_{n^D}^D)\theta_{n^D}^D}{(\gamma+\theta_{n^D}^D\bar{\Theta}(x))^2}\end{pmatrix}\begin{pmatrix}\frac{1\cdot\theta_1^1}{\gamma+\theta_1^1\bar{\Theta}(x)}\\ \vdots \\ \frac{D\cdot\theta_{n^D}^D}{\gamma+\theta_{n^D}^D\bar{\Theta}(x)}\end{pmatrix}^T. \quad (16)$$

In some classes of games, such as potential games and stable games, various evolutionary dynamics show global stability. These games require special structures of the derivative matrix $DF(x)$. Next, we study the structural properties of (16).

Theorem 5. *Under the assumption that every information broadcast takes place at the steady state of (4),* i.e., $\tau\to\infty$*, the game defined in Section II is a submodular game.*

The proof can be found in [8]. A more transparent equivalent explanation of the above result is that the decisions in our evolutionary game are strategic substitutes, *i.e.*, increases in strategies of other players result in a relatively lower strategy of a given player. This fact is straightforward in the following sense. People are less likely to choose outdoors if the streets are crowded owing to a higher likelihood of infection. Otherwise, people tend to choose outdoors.

The counterpart to a submodular game is a supermodular game [4], where decisions of players are strategic complements. In supermodular games, increases in strategies of other players result in a relatively higher strategy of a given player. This isotone property of the payoff function makes the best response correspondences of players well-behaved and the best-response dynamics with stochastic perturbation converge to perturbed NE of the game [4]. The behavior of learning dynamics in submodular games is more involved [9]. However, following [10] and [11], we can obtain guarantees on the stability of simple learning processes.

Consider the best-response dynamics of the form

$$x_{[k+1]}^d = m^d BR^d(x_{[k]}), \forall d\in\mathcal{D}, \quad (17)$$

where the subscript $[k]$ represents iteration k. Let $x_{\min}^d = (m^d,0,...,0)^T$ and $x_{\max}^d=(0,...,0,m^d)^T$ denote the minimal and maximal state of population d. Let $x_{\min} = (x_{\min}^1,...,x_{\min}^D)$ and $x_{\max}=(x_{\max}^1,...,x_{\max}^D)$ denote the minimal and maximal social state. The following result [10] characterizes the stability of the learning process (17).

Corollary 6. *There exists a minimal point $x_{\min}^*\in NE(F)$ and a maximal point $x_{\max}^*\in NE(F)$. The best-response dynamics (17) generate a monotonically increasing sequence which converges to $x_{\min}^*$ when the initial point is $x_{\min}$; (17) generates a monotonically decreasing sequence which converges to $x_{\max}^*$ when the initial point is $x_{\max}$.*

The results in Corollary 6 have the following interpretations. The initialization at $x_{\min}$ corresponds to the situation where players pay little attention to potential infections caused by the epidemic. In this scenario, players are at high SAIs and interact actively over the network. Through sequential revisions of strategies, players gradually become aware of the potential risks from physical interactions and they become increasingly careful about their physical interactions with others. Hence, the sequence generated by (17) starting from $x_{\min}$ is increasing. The convergence to $x_{\min}^*$ shows that by naively best-responding to current payoffs, the population can eventually reach a point where no one has an incentive to further revise her strategies. On the contrary, the scenario where the starting point is $x_{\max}$ indicates cautious plays at the beginning, since players have no information about the potential consequences of the epidemic. Through sequential plays, players become better informed and behave more audacious, *i.e.*, the subsequent social states generated by (17) after $x_{\max}$ are decreasing. Finally, there is a point where no one is willing to take more risks (e.g., going to the supermarket without wearing a mask).

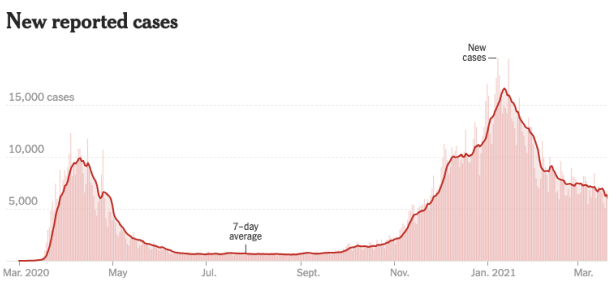
Note that the maximal and the minimal points $x_{\max}^*$ and $x_{\min}^*$ do not, in general, correspond to the equilibrium points where the payoffs of players reach the maximum and the minimum, respectively [11]. Corollary 6 enables the monotone convergence to the NE of the evolutionary dynamics of the form:

$$\dot{x}^d = \min\{m^d BR^d(x)\} - x^d, \quad \forall d\in\mathcal{D}, \quad (18)$$

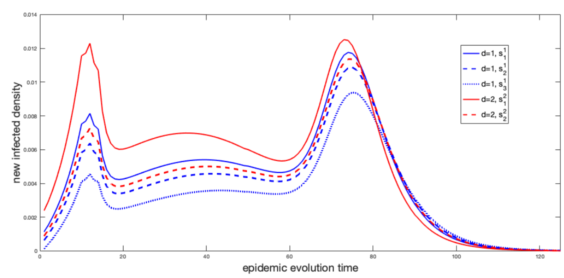
where $\min\{\cdot\}$ selects the least element from a set. The reason lies in the discretization of (18): $x^d(t+\delta) = \delta\min\{m^d BR^d(x(t))\} + (1-\delta)x^d(t)$, which has the interpretation that in a small period δ, only δ portion of the population revises their strategies to the one obtained using the best-responses. The updates (17) correspond to $\delta=1$. Suppose that t_k and t_{k+1} are two time instances corresponding to iteration $[k]$ and $[k+1]$ in (17). Since starting from $x_{\min}$, (17) yields $x_{[k]}^d \le x_{[k+1]}^d$, and for any $\delta\in(0,1)$, $x^d(t_k+\delta) = \delta\min\{m^d BR^d(x(t_k))\} + (1-\delta)x^d(t_k) = \delta x_{[k+1]}^d + (1-\delta)x_{[k]}^d$. Then, $x_{[k]}^d \le x^d(t_k+\delta)\le x_{[k+1]}^d$. If we pick $\delta,\delta_1,\cdots,\delta_{end}\in[0,1]$ such that $0<\delta_1<\delta_2<\cdots<\delta_{end}<1$, the same relation follows: $x_{[k]}^d\le x^d(t_k+\delta_1)\le\cdots\le x^d(t_k+\delta_{end})\le x_{[k+1]}^d$. Therefore, the discretization of (18) is monotone between t_k and t_{k+1} for arbitrary choices of an increasing sequence of δ. This fact shows the monotonicity of (18) and its convergence to the NE from $x_{\min}$, when we let $\delta \to 0$. The scenario where the starting point is $x_{\max}$ follows the similar reasoning.

IV. Numerical Experiments

In this section, we present the results of numerical experiments. For presentation purposes, we use 2 populations with the number of strategies being 3 and 2, respectively. We set the parameters r, λ, and γ to values such that the unique positive steady state appears under a given social state and that there are no dominant strategies.



(a)



(b)

Fig. 2. Multiple peaks of new infections. (a) Statistics of COVID-19 in New York [12]. (b) Predicted new infected density curve using our framework.

In Fig. 2, we compare the curve of the reported cases using the historical COVID-19 data and the simulated curve of the infected density using our framework. The multi-peak curves in Fig. 2(a) and Fig. 2(b) correspond to different waves of epidemic outbreak. The first wave is the natural outbreak of an epidemic when it first starts to spread among infectious individuals. The decreases of new cases between July 2020 and November 2020 in Fig. 2(a), and between time 20 to 50 in Fig. 2(b) correspond to the period when people start to avoid close contacts and the policies are enforced to mitigate the epidemic, such as wearing masks all the time. In Fig. 2(a), the second infection wave is a consequence of relaxed social guidance [13] and the violations of existing quarantine policies. At $t = 50$ in Fig. 2(b), we set the behaviors of the populations to change. This change captures the populations' overconfidence in the epidemic status as the number of new cases decreases. When the populations become less careful, *i.e.*, more people play strategies close to s_1^d in the set $\mathcal{S}^d$, the newly infected density curve increases and shows a second peak. This second peak is an indication of the influence of herd behaviors on the epidemic evolution.

V. Conclusion

We have proposed a federated evolutionary game framework that couples the epidemic state dynamics with the evolution of strategies to study the herd behaviors over complex networks. We have designed the mechanism containing physical interactions and information broadcasts to combine federated state transitions over a large network and sequential strategy revisions in the populations. Taking the epidemic model as a special case, we have found a unique nontrivial steady state when the epidemic evolves faster. We have characterized the Nash equilibrium of the game at the steady-state. In addition, we have shown that decisions in the game are strategic substitutes, which have enabled simple learning processes to reach equilibrium. Our numerical examples have indicated the predictive power of our framework by comparing the simulated dynamics to the historical COVID-19 statistics. The multi-peak pattern observed in the case study has shown that the herd behaviors have been attributed to multiple epidemic outbreaks.

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